

## LA GACETA DE LA RSME - CHALLENGES (I)

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315. If  $a, b \in \mathbb{R}; a < b$  then:

$$\frac{e^b - e^a}{b - a} \geq \frac{e^a}{24}(b - a)^2 + \sqrt{e^{a+b}}$$

*Proof.* Let be  $f : [a, b] \rightarrow \mathbb{R}; f(x) = e^x - \frac{x^2 e^a}{2}$

$$f'(x) = e^x - xe^a; f''(x) = e^x - e^a \geq 0; x \in [a, b]$$

By Hermite - Hadamard inequality:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ e^{\frac{a+b}{2}} - \frac{e^a}{2} \left(\frac{a+b}{2}\right)^2 &\leq \frac{1}{b-a} \int_a^b e^x dx - \frac{e^a}{2(b-a)} \cdot \frac{x^3}{3} \Big|_a^b \\ \sqrt{e^{a+b}} - \frac{e^a}{2} \left(\frac{a+b}{2}\right)^2 + \frac{e^a}{2(b-a)} \cdot \frac{b^3 - a^3}{3} &\leq \frac{e^b - e^a}{b-a} \\ \sqrt{e^{a+b}} + \frac{e^a}{2} \left(\frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}\right) &\leq \frac{e^b - e^a}{b-a} \\ \sqrt{e^{a+b}} + \frac{e^a}{2} \cdot \frac{4b^2 + 4ab + 4a^2 - 3a^2 - 6ab - 3b^2}{12} &\leq \frac{e^b - e^a}{b-a} \\ \frac{e^b - e^a}{b-a} &\geq \sqrt{e^{a+b}} + \frac{e^a}{24}(b^2 - 2ab + a^2) \\ \frac{e^b - e^a}{b-a} &\geq \frac{e^a}{24}(b - a)^2 + \sqrt{e^{a+b}} \end{aligned}$$

□

323. If  $m \in [0, \infty); a, b, x, y, z \in (0, \infty)$  then:

$$\begin{aligned} \frac{x^{2m+2}}{(ay + bz)^{2m+2} \sec^{2m} \frac{\pi}{18}} + \frac{y^{2m+2}}{(az + bx)^{2m+2} \csc^{2m} \frac{\pi}{9}} + \frac{z^{2m+2}}{(ax + by)^{2m+2} \csc^{2m} \frac{\pi}{9}} &\geq \\ \geq \frac{3}{4^m (a + b)^{2m+2}} \end{aligned}$$

*Proof.*

$$LHS = \frac{\left(\left(\frac{x}{ay + bz}\right)^2\right)^{m+1}}{\left(\sec^2 \frac{\pi}{18}\right)^m} + \frac{\left(\left(\frac{y}{az + bx}\right)^2\right)^{m+1}}{\left(\csc^2 \frac{\pi}{9}\right)^m} + \frac{\left(\left(\frac{z}{ax + by}\right)^2\right)^{m+1}}{\left(\csc^2 \frac{2\pi}{9}\right)^m} \geq$$

$$\begin{aligned}
& \overbrace{\geq}^{\text{Radon}} \frac{\left(\sum\left(\frac{x}{ay+bz}\right)^2\right)^{m+1}}{\left(\sec^2 \frac{\pi}{18} + \csc^2 \frac{\pi}{9} + \csc^2 \frac{2\pi}{9}\right)^m} \overbrace{\geq}^{\text{Bergstrom}} \\
(1) \quad & \geq \frac{\left(\sum\frac{x}{ay+bz}\right)^{2m+2}}{3^{m+1}(\sec^2 \frac{\pi}{18} + \csc^2 \frac{\pi}{9} + \csc^2 \frac{2\pi}{9})^m}
\end{aligned}$$

By Larry Glasser's identity (La Gaceta de la RSME, vol 19 (2016), no. 3, pp 594-595, problem 283):

$$(2) \quad \sec^2 \frac{\pi}{18} + \csc^2 \frac{\pi}{9} + \csc^2 \frac{\pi}{9} = 12$$

By (1); (2):

$$(3) \quad LHS \geq \frac{\left(\sum\left(\frac{x}{ay+bz}\right)\right)^{2m+2}}{3^{m+1} \cdot 12^m}$$

But:

$$\begin{aligned}
& \sum \frac{x}{ay+bz} = \sum \frac{x^2}{axy+bzx} \overbrace{\geq}^{\text{Bergström}} \\
& \geq \frac{(x+y+z)^2}{\sum(axy+bzx)} = \frac{(x+y+z)^2}{(a+b)(xy+yz+zx)} \geq \\
(4) \quad & \geq \frac{3(xy+yz+zx)}{(a+b)(xy+yz+zx)} = \frac{3}{a+b}
\end{aligned}$$

By (3); (4):

$$\begin{aligned}
LHS & \geq \left(\frac{3}{a+b}\right)^{2m+2} \cdot \frac{1}{3^{m+1} \cdot 12^m} = \frac{3^{2m+2}}{3^{2m+1} \cdot 4^m(a+b)^{2m+2}} = \\
& = \frac{3}{4^m(a+b)^{2m+2}}
\end{aligned}$$

Equality holds when  $x = y = z$ . □

338. Prove that in  $ABC$  triangle the following relationship holds:

$$\left(\sum \frac{m_a^2}{m_b^2}\right) \left(\sum \frac{m_a^4}{m_b^4}\right) \left(\sum \frac{m_a^8}{m_b^8}\right) \geq 8S^3 \left(\sum \frac{1}{am_b}\right)^3$$

*Proof.*

$$\begin{aligned}
& \frac{4m_a^2}{6m_b^2} + \frac{m_b^2}{6m_c^2} + \frac{m_c^2}{6m_a^2} = \frac{m_a^2}{6m_b^2} + \frac{m_a^2}{6m_b^2} + \frac{m_a^2}{6m_b^2} + \frac{m_a^2}{6m_c^2} + \frac{m_b^2}{6m_c^2} + \frac{m_c^2}{6m_a^2} \geq \\
& \overbrace{\geq}^{AM-GM} \frac{1}{6} \cdot 6 \sqrt[6]{\left(\frac{m_a^2}{m_b^2}\right)^4 \cdot \frac{m_b^2}{m_c^2} \cdot \frac{m_c^2}{m_a^2}} = \sqrt[6]{\frac{m_a^6}{m_b^6}} = \frac{m_a}{m_b} \\
(1) \quad & \frac{4m_a^2}{6m_b^2} + \frac{m_b^2}{6m_c^2} + \frac{m_c^2}{6m_a^2} \geq \frac{m_a}{m_b}
\end{aligned}$$

Analogous:

$$(2) \quad \frac{m_a^2}{6m_b^2} + \frac{4m_b^2}{6m_c^2} + \frac{m_c^2}{6m_a^2} \geq \frac{m_b}{m_a}$$

$$(3) \quad \frac{m_a^2}{6m_b^2} + \frac{m_b^2}{6m_c^2} + \frac{4m_a^2}{6m_b^2} \geq \frac{m_c}{m_b}$$

By adding the relationships (1); (2); (3):

$$\frac{6m_a^2}{6m_b^2} + \frac{6m_b^2}{6m_c^2} + \frac{6m_c^2}{6m_a^2} \geq \frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a}$$

$$(4) \quad \sum \frac{m_a^2}{m_b^2} \geq \sum \frac{m_a}{m_b}$$

Replacing in (4)  $(m_a, m_b, m_c)$  with  $(m_a^2, m_b^2, m_c^2)$  then with  $(m_a^4, m_b^4, m_c^4)$ .

$$(5) \quad \sum \frac{m_a^4}{m_b^4} \geq \sum \frac{m_a^2}{m_b^2} \stackrel{(4)}{\geq} \sum \frac{m_a}{m_b}$$

$$(6) \quad \sum \frac{m_a^8}{m_b^8} \geq \sum \frac{m_a^4}{m_b^4} \stackrel{(5)}{\geq} \sum \frac{m_a}{m_b}$$

We multiply (4); (5); (6):

$$\begin{aligned} & \left( \sum \frac{m_a^2}{m_b^2} \right) \left( \sum \frac{m_a^4}{m_b^4} \right) \left( \sum \frac{m_a^8}{m_b^8} \right) \geq \\ & \geq \left( \sum \frac{m_a}{m_b} \right)^3 \geq \left( \sum \frac{2S}{am_b} \right)^3 = 8S^3 \left( \sum \frac{1}{am_b} \right)^3 \end{aligned}$$

□

357. Si  $a, b$  y  $c$  son números reales positivos y  $x \in (0, \frac{\pi}{2})$ , probar que:

$$\frac{a^2 \sin^6 x}{x^6} + \frac{b^2 \sin^4 x}{x^4} + \frac{c^2 \sin^2 x}{x^2} + 3 \sqrt[3]{(abc)^2} \frac{\tan^2 x}{x^2} > 6 \sqrt[3]{(abc)^2}$$

*Solution 1 by proposer.*

$$\begin{aligned} & \frac{a^2 \sin^6 x}{x^6} + \frac{b^2 \sin^4 x}{x^4} + \frac{c^2 \sin^2 x}{x^2} = \\ & = \frac{\left( \frac{a \sin^3 x}{x^3} \right)^2}{1} + \frac{\left( \frac{b \sin^2 x}{x^2} \right)^2}{1} + \frac{\left( \frac{c \sin x}{x} \right)^2}{1} \stackrel{\text{BERGSTROM}}{\geq} \\ & \geq \frac{\left( \frac{a \sin^3 x}{x^3} + \frac{b \sin^2 x}{x^2} + \frac{c \sin x}{x} \right)^2}{3} \stackrel{\text{AM-GM}}{\geq} \\ & \geq \frac{\left( 3 \sqrt[3]{abc} \frac{\sin^3 x}{x^3} \cdot \frac{\sin^2 x}{x^2} \cdot \frac{\sin x}{x} \right)^2}{3} = \end{aligned}$$

$$\begin{aligned}
(1) \quad &= 3\sqrt[3]{(abc)^2} \sqrt[3]{\frac{\sin^{12}x}{x^{12}}} = 3\sqrt[3]{(abc)^2} \cdot \frac{\sin^4x}{x^4} \\
&a^2 \cdot \frac{\sin^6x}{x^6} + b^2 \cdot \frac{\sin^4x}{x^4} + c^2 \cdot \frac{\sin^2x}{x^2} + 3\sqrt[3]{(abc)^2} \cdot \frac{\tan^2x}{x^2} \geq \\
&\stackrel{(1)}{\geq} 3\sqrt[3]{(abc)^2} \cdot \frac{\sin^4x}{x^4} + 3\sqrt[3]{(abc)^2} \cdot \frac{\tan^2x}{x^2} = \\
&= 3\sqrt[3]{(abc)^2} \left( \frac{\left(\frac{\sin^2x}{x^2}\right)^2}{1} + \frac{\left(\frac{\tan x}{x}\right)^2}{1} \right) \stackrel{\text{BERGSTROM}}{\geq} \\
&\geq 3\sqrt[3]{(abc)^2} \cdot \frac{\left(\frac{\sin^2x}{x^2} + \frac{\tan x}{x}\right)^2}{1+1} \stackrel{\text{WILKER}}{\geq} \frac{3}{2} \sqrt[3]{(abc)^2} \cdot 2^2 = \\
&= 6\sqrt[3]{(abc)^2}
\end{aligned}$$

□

*Solution 2 by Soumava Chakraborty - SoftWebTechnologies - Kolkata - India.*

$$\begin{aligned}
&\frac{a^2 \sin^6x}{x^6} + \frac{b^2 \sin^4x}{x^4} + \frac{c^2 \sin^2x}{x^2} \stackrel{\text{A-G}}{\stackrel{(1)}{\geq}} 3\sqrt[3]{(abc)^2} \left( \frac{\sin^4x}{x^4} \right) \\
(1) \Rightarrow LHS &\geq 3\sqrt[3]{(abc)^2} \left( \frac{\sin^4x}{x^4} + \frac{\tan^2x}{x^2} \right) \\
&\stackrel{\text{A-G}}{\geq} 3\sqrt[3]{(abc)^2} \cdot 2 \cdot \frac{\sin^2x \tan x}{x^3} \stackrel{?}{>} 6\sqrt[3]{(abc)^2} \\
&\Leftrightarrow \sin^2x \tan x \stackrel{?}{>} x^3 \Leftrightarrow \sin^3x \stackrel{?}{>} x^3 \cos x \\
&\Leftrightarrow \sin x \stackrel{?}{>} x \sqrt[3]{\cos x} \Leftrightarrow (\sin x)(\cos x)^{-\frac{1}{3}} - x \stackrel{?}{\geq} 0
\end{aligned}$$

Let  $f(x) = (\sin x)(\cos x)^{-\frac{1}{3}} - x$ ;  $\forall x \in [0, \frac{\pi}{2}]$   
Then,  $f'(x) = \frac{\sin^2x}{3(\cos x)^{\frac{4}{3}}} + (\cos x)^{\frac{2}{3}} - 1$  and  $f''(x) = \frac{4\sin^3x}{9(\cos x)^{\frac{7}{3}}} \geq 0$ ;  $\forall x \in [0, \frac{\pi}{2}]$   
 $\Rightarrow f'(x) \geq f'(0)$ ;  $\forall x \in [0, \frac{\pi}{2}]$   $= 0 \Rightarrow f(x) \geq f(0)$ ;  $\forall x \in [0, \frac{\pi}{2}]$   $= 0$   
 $\therefore \forall x \in [0, \frac{\pi}{2}], f(x) \geq 0$  equality at  $x = 0$   
 $\Rightarrow$  if  $0 < x < \frac{\pi}{2}$ ,  $f(x) > 0 \Rightarrow (2)$  is true. □

366. Si denotamos  $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ , probar que, para cada  $a > 1$ ,

$$\frac{1}{H_n} \int_1^a \sum_{k=1}^n \frac{1}{k+x^{2k}} dx < 1 - \frac{1}{a}.$$

*Propuesto por Daniel Sitaru - Rumanía*

*Solución enviada por Alberto Stadler, Herrliberg, Suiza.*

En primer lugar debemos observar que  $k + x^{2k} \geq kx^2$ , para  $k \geq 1$ . En efecto, el caso  $k = 1$  es obvio y si  $k > 1$ , usando la desigualdad entre la media aritmética y la media geométrica, se tiene que

$$k + x^{2k} = (k-1)\frac{k}{k-1} + x^{2k} \geq k\sqrt[k]{\left(\frac{k}{k-1}\right)^{k-1}x^{2k}} > kx^2.$$

Por tanto,

$$\int_1^a \sum_{k=1}^n \frac{1}{k+x^{2k}} dx < \sum_{k=1}^n \frac{1}{k} \int_1^a \frac{dx}{x^2} = H_n \left(1 - \frac{1}{a}\right)$$

y el resultado se sigue de manera inmediata. □

381. If  $1 < a \leq b$  then:

$$4 \int_a^b \int_a^b (x^y + y^x) dx dy \geq (b-a)^2 (4 + (b-a)^2)$$

*Daniel Sitaru - Romania*

*Solution 1 by Seán M. Stewart, Bomaderry, NSW, Australia.*

A partir de la desigualdad de Bernoulli

$$(1+t)^r \geq 1 + rt, \quad r, t \in \mathbb{R}, \quad r \geq 1, t \geq -1,$$

tomando, respectivamente,  $(t, r) = (x-1, y)$  y  $(t, r) = (y-1, x)$ , deducimos las desigualdades

$$x^y \geq 1 + xy - y \quad y \quad y^x \geq 1 + xy - x,$$

válidas ambas para  $x, y \geq 1$ . Así, puesto que  $(x-1)(y-1) > 0$  cuando  $x, y > 1$ , llegamos a que

$$x^y + y^x \geq 1 + xy + (x-1)(y-1) > 1 + xy, \quad x, y > 1,$$

y, por tanto,

$$\int_a^b \int_a^b (x^y + y^x) dx dy > \int_a^b \int_a^b (1 + xy) dx dy = \frac{(b-a)^2}{4} (4 + (b+a)^2).$$

Finalmente, la desigualdad propuesta se sigue inmediatamente usando que  $b+a > b-a$ . □

*Solution 2 by proposer.*

$$x^y = (1 + (x-1))^y \stackrel{\text{Bernoulli}}{\geq} 1 + y(x-1) \quad (1)$$

$$y^x = (1 + (y-1))^x \stackrel{\text{Bernoulli}}{\geq} 1 + x(y-1) \quad (2)$$

By adding (1); (2) :

$$\begin{aligned} x^y + y^x &\geq 1 + y(x-1) + 1 + x(y-1) = \\ &= 1 + xy - y + 1 + xy - x = 1 + xy + (x-1)(y-1) \geq 1 + xy \end{aligned}$$

$$\begin{aligned}
& \int_a^b \int_a^b (x^y + y^x) dx dy \geq \int_a^b \int_a^b (1 + xy) dx dy = \\
& = \int_a^b \int_a^b dx dy + \int_a^b \int_a^b xy dx dy = (b-a)^2 + \frac{(b-a)^2}{2} \cdot \frac{(b-a)^2}{2} = (b-a)^2 \left(1 + \frac{(b-a)^2}{4}\right) \\
& 4 \int_a^b \int_a^b (x^y + y^x) dx dy \geq (b-a)^2 (4 + (b-a)^2)
\end{aligned}$$

□

388. In  $\Delta ABC$  the following relationship holds:

$$\frac{(3a+b)(3b+c)(3c+a)}{\sqrt{(a+b)^3(b+c)^3(c+a)^3}} \leq \frac{1}{r\sqrt{s}}$$

$r$  - inradii;  $s$  - semiperimeter

*Proof.* We prove that:

$$\frac{3a+b}{a+b} \leq \sqrt{2} \cdot \sqrt{\frac{a+b}{b}}$$

By squaring:

$$\begin{aligned}
& \frac{(3a+b)^2}{(a+b)^2} \leq \frac{2(a+b)}{b} \\
& b(3a+b)^2 \leq 2(a+b)^3 \\
& b(9a^2 + 6ab + b^2) \leq 2(a^3 + 3a^2b + 3ab^2 + b^3) \\
& 9a^2b + 6ab^2 + b^3 \leq 2a^3 + 6a^2b + 6ab^2 + 2b^3 \\
& 2a^3 - 3a^2b + b^3 \geq 0 \\
& 2a^3 + 2a^2b - a^2b + b^3 \geq 0 \\
& 2a^2(a-b) - b(a^2 - b^2) \geq 0 \\
& 2a^2(a-b) - b(a-b)(a+b) \geq 0 \\
& (a-b)(2a^2 - ba - b^2) \geq 0 \\
& (a-b)(2a^2 - 2ab + ab - b^2) \geq 0 \\
& (a-b)(2a(a-b) + b(a-b)) \geq 0 \\
& (a-b)^2(2a+b) \geq 0
\end{aligned}$$

which is true.

$$(1) \quad \frac{3a+b}{a+b} \leq \sqrt{2} \cdot \sqrt{\frac{a+b}{b}}$$

Analogous:

$$(2) \quad \frac{3b+c}{b+c} \leq \sqrt{2} \cdot \sqrt{\frac{b+c}{c}}$$

$$(3) \quad \frac{3c+a}{c+a} \leq \sqrt{2} \cdot \sqrt{\frac{c+a}{a}}$$

By multiplying (1); (2); (3):

$$\frac{(3a+b)(3b+c)(3c+a)}{(a+b)(b+c)(c+a)} \leq 2\sqrt{2} \cdot \sqrt{\frac{(a+b)(b+c)(c+a)}{abc}}$$

$$\begin{aligned} \frac{(3a+b)(3b+c)(3c+a)}{\sqrt{(a+b)^3(b+c)^3(c+a)^3}} &\leq \sqrt{\frac{8}{abc}} = \sqrt{\frac{8}{4RF}} = \\ &= \sqrt{\frac{2}{Rrs}} \stackrel{\text{EULER}}{\leq} \sqrt{\frac{2}{2r \cdot r \cdot s}} = \frac{1}{r\sqrt{s}} \end{aligned}$$

( $F$  - area;  $R$  - circumradii)

Equality holds for an equilateral triangle:  $a = b = c$ .  $\square$

396. If  $x, y, z, t \in (0, 1); 3\sqrt{3}(xyz + yzt + ztx + txy) = 4$  then:

$$\frac{yzt}{x(1-x^2)} + \frac{ztx}{y(1-y^2)} + \frac{txy}{z(1-z^2)} + \frac{xyz}{t(1-t^2)} \geq 2$$

*Proof.*

$$\begin{aligned} 2x^2 \cdot (1-x^2)(1-x^2) &\stackrel{\text{AM-GM}}{\leq} \left( \frac{2x^2 + 1 - x^2 + 1 - x^2}{3} \right)^3 \\ 2x^2(1-x^2)^2 &\leq \frac{8}{27} \Rightarrow x^2(1-x^2)^2 \leq \frac{4}{3\sqrt{3}} \\ x(1-x^2) &\leq \frac{2}{3\sqrt{3}} \Rightarrow \frac{1}{x(1-x^2)} \geq \frac{3\sqrt{3}}{2} \end{aligned}$$

$$(1) \quad \frac{yzt}{x(1-x^2)} \geq \frac{\sqrt{3}}{2} yzt$$

Analogous:

$$(2) \quad \frac{ztx}{y(1-y^2)} \geq \frac{3\sqrt{3}}{2} ztx$$

$$(3) \quad \frac{txy}{z(1-z^2)} \geq \frac{3\sqrt{3}}{2} txy$$

$$(4) \quad \frac{xyz}{t(1-t^2)} \geq \frac{3\sqrt{3}}{2} xyz$$

By adding (1); (2); (3); (4):

$$\begin{aligned} \frac{yzt}{x(1-x^2)} + \frac{ztx}{y(1-y^2)} + \frac{txy}{z(1-z^2)} + \frac{xyz}{t(1-t^2)} &\geq \\ \geq \frac{3\sqrt{3}}{2} (yzt + ztx + txy + xyz) &= \frac{3\sqrt{3}}{2} \cdot \frac{4}{3\sqrt{3}} = 2 \end{aligned}$$

Equality holds for  $x = y = z = t = \frac{1}{\sqrt{3}}$ .  $\square$

407. In any triangle  $ABC$  the following relationship holds:

$$\frac{\tan^2 1^\circ \cdot \tan^2 2^\circ \cdot \tan^2 3^\circ}{h_a} + \frac{\tan^2 2^\circ}{h_b} + \frac{\tan^2 1^\circ}{h_c} > \frac{(2\sqrt{2}+1)\tan^2 3^\circ}{7s}$$

$h_a, h_b, h_c$  - altitudes;  $s$  - semiperimeter.

*Proof.* Lemma 1.

$$\tan 3^\circ - \tan 2^\circ - \tan 1^\circ = \tan 3^\circ \cdot \tan 2^\circ \cdot \tan 1^\circ$$

*Proof.* Denote  $\tan 1^\circ = x$ . We must prove that:

$$\begin{aligned} \frac{3x - x^3}{1 - 3x^2} - \frac{2x}{1 - x^2} - x &= \frac{3x - x^3}{1 - 3x^2} \cdot \frac{2x}{1 - x^2} \cdot x \\ (3x - x^3)(1 - x^2) - 2x(1 - 3x^2) - x(1 - x^2)(1 - 3x^2) &= 2x^2(3x - x^3) \\ (3 - x^2)(1 - x^2) - 2(1 - 3x^2) - (1 - x^2)(1 - 3x^2) &= 2x(3x - x^3) \\ 3 - 3x^2 - x^2 + x^4 - 2 + 6x^2 - 1 + 3x^2 + x^2 - 3x^4 &= 6x^2 - 2x^4 \\ 3 - 3 - 2x^4 + 6x^2 &= 6x^2 - 2x^4 \\ 0 &= 0 \end{aligned}$$

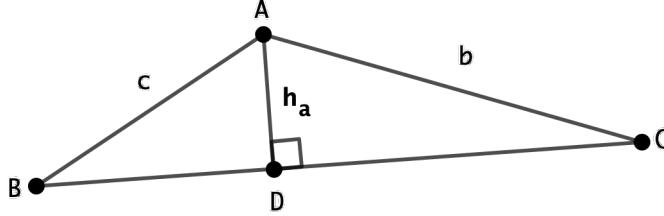
□

Lemma 2.

In any triangle  $ABC$  the following relationship holds:

$$h_a + h_b + h_c < \frac{7s}{2\sqrt{2} + 1}$$

*Proof.*



$$\begin{aligned} \cos B &= \frac{BD}{AB} = \frac{BD}{c} \Rightarrow BD = c \cos B \\ \cos C &= \frac{DC}{AC} = \frac{DC}{b} \Rightarrow DC = b \cos C \\ (1) \quad a &= BD + DC = c \cos B + b \cos C \\ \sin B &= \frac{AD}{AB} = \frac{h_a}{c} \Rightarrow h_a = c \sin B \\ \sin C &= \frac{AD}{AC} = \frac{h_a}{b} \Rightarrow h_a = b \sin C \end{aligned}$$

By adding:

$$(2) \quad 2h_a = h_a + h_a = c \sin B + b \sin C$$

By (1); (2):

$$(3) \quad a + 2h_a = c(\sin B + \cos B) + b(\sin C + \cos C)$$

$$\begin{aligned} \sin B + \cos C &= \sin B + 1 \cdot \cos B = \sin B + \tan \frac{\pi}{4} \cdot \cos B = \\ &= \sin B + \frac{\sin \frac{\pi}{4}}{\cos \frac{\pi}{4}} \cdot \cos B = \frac{1}{\cos \frac{\pi}{4}} \left( \sin B \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos B \right) = \\ &= \frac{1}{\frac{\sqrt{2}}{2}} \sin \left( B + \frac{\pi}{4} \right) = \sqrt{2} \sin \left( B + \frac{\pi}{4} \right) \leq \sqrt{2} \end{aligned}$$

Analogous:  $\sin C + \cos C \leq \sqrt{2}$

By (3):

$$\begin{aligned} a + 2h_a &= c(\sin B + \cos B) + b(\sin C + \cos C) \leq c \cdot \sqrt{2} + b \cdot \sqrt{2} = (b+c)\sqrt{2} \\ a + 2h_a &\leq (b+c)\sqrt{2}; a + 2h_b \leq (c+a)\sqrt{2} \\ c + 2h_c &= (a+b)\sqrt{2} \end{aligned}$$

Equality case can't be true in all three inequalities. By adding:

$$\begin{aligned} a + b + c + 2(h_a + h_b + h_c) &< 2(a + b + c)\sqrt{2} \\ 2s + 2(h_a + h_b + h_c) &< 4s\sqrt{2} \\ 2(h_a + h_b + h_c) &< 2s(2\sqrt{2} - 1) \\ h_a + h_b + h_c &< s(2\sqrt{2} - 1) = \frac{s(8-1)}{2\sqrt{2}+1} = \frac{7s}{2\sqrt{2}+1} \end{aligned}$$

□

Back to the problem:

$$\begin{aligned} \frac{\tan^2 1^\circ \cdot \tan^2 2^\circ \cdot \tan^2 3^\circ}{h_a} + \frac{\tan^2 2^\circ}{h_b} + \frac{\tan^2 1^\circ}{h_c} &> \\ \text{BERGSTROM} \quad & \frac{(\tan 1^\circ \cdot \tan 2^\circ \cdot \tan 3^\circ + \tan 2^\circ + \tan 1^\circ)^2}{h_a + h_b + h_c} > \\ & > \frac{\tan^2 3^\circ}{h_b + h_c + h_a} \stackrel{\text{LEMMA 1}}{>} \stackrel{\text{LEMMA 2}}{>} \frac{7s}{2\sqrt{2}+1} \end{aligned}$$

□

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