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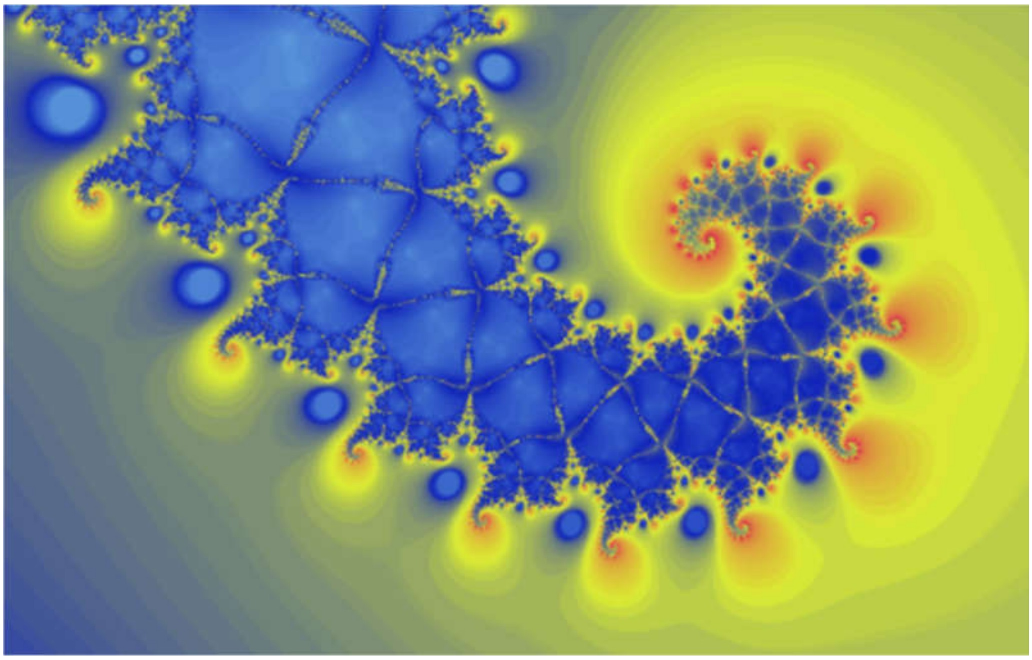

If $a \geq 0$ then:

$$\int_0^a (2x-a)\log(1+x+x^2)\,dx \geq 0$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohammed Diai-Morocco, Solution 2 by Tapas Das-India

Solution 1 by Mohammed Diai-Morocco

$$\int_0^a (2x-a)\log(1+x+x^2)\,dx \geq 0;\ (*)$$

$$\int_0^a (2x-a)\log(1+x+x^2)\,dx =$$

$$= \int_0^a (2x+1)\log(1+x+x^2)\,dx - \int_0^a (a+1)\log(1+x+x^2)\,dx$$

$$\int_0^a (2x+1)\log(1+x+x^2)\,dx = [(1+x+x^2)\log(1+x+x^2)-(x+x^2)]|_0^a =$$

$$= (1+a+a^2)\log(1+a+a^2)$$

Therefore,

$$(*) \Leftrightarrow \int_0^a \log(1+x+x^2)\,dx \le \frac{(1+a+a^2)\log(1+a+a^2)}{1+a}$$

The function $x \to \log(1+x+x^2)$ is increasing in $[0,a]$, so:

$$\int_0^a \log(1+x+x^2)\,dx \le \int_0^a \log(1+a+a^2)\,dx = a\log(1+a+a^2) \le$$

$$\le \frac{(1+a+a^2)\log(1+a+a^2)}{1+a}$$

Solution 2 by Tapas Das-India

We know that:

If $f,g:[a,b] \to \mathbb{R}$ are two monotonic functions of same monotonically

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \ge \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)$$

Let us denote: $f(x) = 2x-a$ and $g(x) = \log(1+x+x^2)\,;x \in [0,a]$

$f'(x) = 2 > 0$, then f –is an increasing function.

$g'(x) = \frac{2x+1}{x^2+x+1} > 0$, then g –is an increasing function.

Now, we have:

$$\frac{1}{a-0}\int_0^a (2x-a)\log(1+x+x^2)\,dx \ge$$

$$\ge \left(\frac{1}{a-0}\int_0^a (2x-a)\,dx\right)\cdot\left(\frac{1}{a-0}\int_0^a \log(1+x+x^2)\,dx\right)$$

Clearly, $\int_0^a (2x-a)\,dx = (x^2-ax)\,\Big|_0^a = 0$.

Therefore,

$$\int_0^a (2x-a)\log(1+x+x^2)\,dx \ge 0$$