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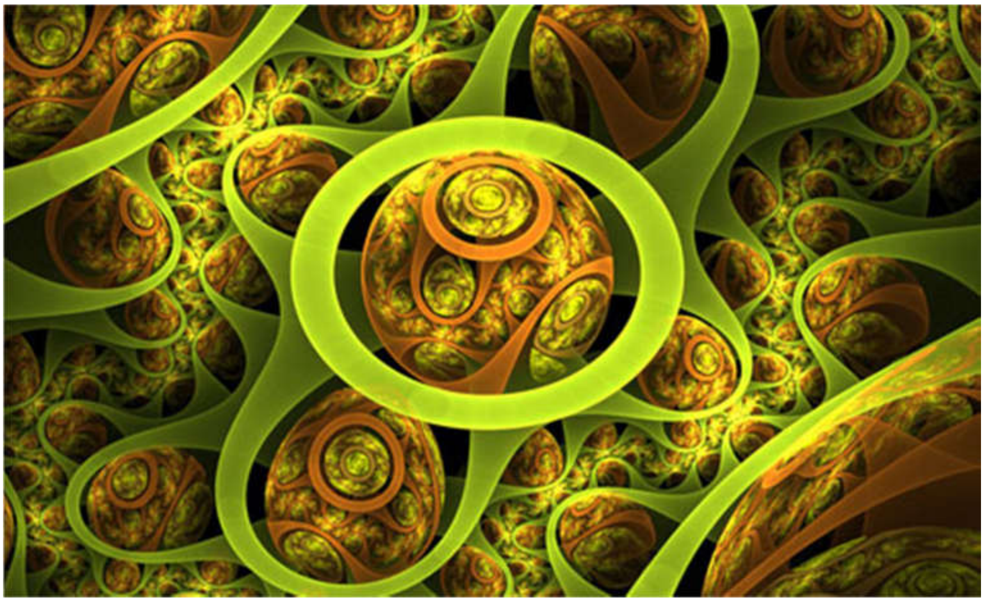

If $0 < a \le b < 1$ then:

$$\int_a^b \frac{\sin x \cdot \tan^{-1}(x^2)}{x \cdot \tan^{-1} x} dx \ge \cos a - \cos b$$

Proposed by Daniel Sitaru-Romania

Solution by Tapas Das-India

We know that:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$x \cdot \tan^{-1} x = x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \frac{x^{10}}{9} - \cdots$$

Now,

$$\tan^{-1}(x^2) = x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \frac{(x^2)^7}{7} + \frac{(x^2)^9}{9} - \cdots =$$

$$= x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \frac{x^{18}}{9} - \cdots$$

Hence,

$$\tan^{-1}(x^2) - x \cdot \tan^{-1} x = \frac{1}{3}(x^4 - x^6) + \frac{1}{5}(x^{10} + x^6) + \frac{1}{7}(x^8 - x^{14}) + \cdots$$

For $x \in (0,1)$: $x^4 > x^6 > x^8 > x^{10} > x^{14} \ldots$

Clearly, $\tan^{-1}(x^2) - x \cdot \tan^{-1} x \ge 0 \Leftrightarrow \tan^{-1}(x^2) \ge x \cdot \tan^{-1} x$, thus

$$\frac{\tan^{-1}(x^2)}{x\cdot\tan^{-1}x}\geq 1$$

Therefore,

$$\int_a^b \frac{\sin x\cdot\tan^{-1}(x^2)}{x\cdot\tan^{-1}x}dx\geq\int_a^b \sin x\,dx$$

$$\int_a^b \frac{\sin x\cdot\tan^{-1}(x^2)}{x\cdot\tan^{-1}x}dx\geq\cos a-\cos b$$