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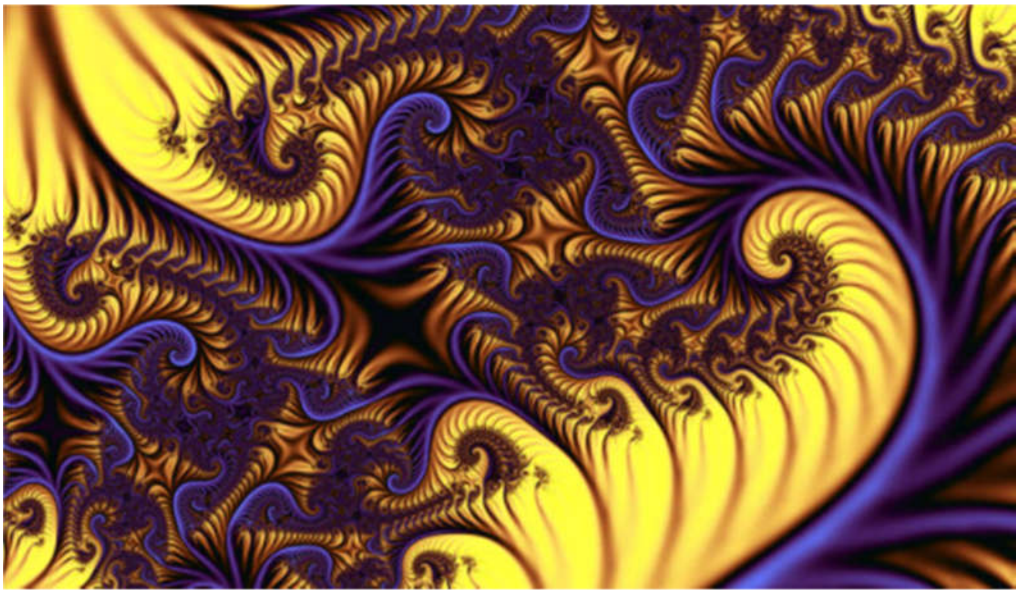

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}{\sin^4\frac{A}{2}+\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{B}{2}}\geq\frac{16r}{R}$$

Proposed by Marin Chirciu-Romania

Solution 1 by George Florin Șerban-Romania, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by George Florin Șerban-Romania

Lemma. If $x,y,z>0$ then holds:

$$\sum_{cyc}\frac{x+y}{x^2+xy+y^2}\geq\frac{4\sum x}{\sum x^2+\sum xy}$$

Let us denote: $x=\sin^2\frac{A}{2}, y=\sin^2\frac{B}{2}, z=\sin^2\frac{C}{2}$, then

$$\sum_{cyc}\frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}{\sin^4\frac{A}{2}+\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{B}{2}}\geq\frac{4\sum\sin^2\frac{A}{2}}{\sum\sin^4\frac{A}{2}+\sum\sin^2\frac{A}{2}\sin^2\frac{B}{2}}=$$

$$=\frac{4\left(1-\frac{r}{2R}\right)}{\frac{8R^2+r^2-s^2}{8R^2}+\frac{s^2+r^2-8Rr}{16R^2}}=\frac{4\left(1-\frac{r}{2R}\right)}{\frac{16R^2-8Rr+3r^2-s^2}{16R^2}}=$$

$$=\frac{64R^2\left(1-\frac{r}{2R}\right)}{16R^2-8Rr+3r^2-s^2}\overset{Gerretsen}{\geq}\frac{64R^2\left(1-\frac{r}{2R}\right)}{16R^2-8Rr+3r^2-16Rr+5r^2}=$$

$$=\frac{64R^2\left(1-\frac{r}{2R}\right)}{16R^2-24Rr+8r^2}=\frac{8\left(1-\frac{r}{2R}\right)}{2-\frac{3r}{R}+\left(\frac{r}{R}\right)^2}\overset{(1)}{\geq}\frac{16r}{R}$$

$$Let\ \frac{r}{R}=x\leq\frac{1}{2},then\ (1)\Leftrightarrow\frac{1-\frac{x}{2}}{2-3x+x^2}\geq 2x\Leftrightarrow$$

$$4x^3-12x^2+9x-2\leq 0\Leftrightarrow(x-2)(2x-1)^2\leq 0\ true\ \forall x\leq\frac{1}{2}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$AI^2=bc-4Rr\Leftrightarrow\left(\frac{r}{\left(\frac{r}{4R}\right)}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2$$

$$=16R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B}{2}\cos\frac{C}{2}-16R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\Leftrightarrow\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{A}{2}\Leftrightarrow\cos\frac{B+C}{2}=\sin\frac{A}{2}\to\text{true}$$

$$\therefore AI^2\overset{(i)}{\triangleq}bc-4Rr\ and\sum\frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}{\sin^4\frac{A}{2}+\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{B}{2}}\geq\frac{16r}{R}$$

$$\Leftrightarrow\sum\frac{\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}\right)+\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\right)}{\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}\right)\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\right)}$$

$$\geq\frac{32r}{R}$$

$$\Leftrightarrow\sum\frac{1}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}}+\sum\frac{1}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}}\overset{(\bullet)}{\geq}\frac{32r}{R}$$

$$\text{Now, }\sum\frac{1}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}}$$

$$+\sum\frac{1}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}}\overset{\text{Bergstrom}}{\geq}\frac{9}{2x+y}$$

$$+\frac{9}{2x-y}\left(x=\sum\sin^2\frac{A}{2}\ and\ y=\sum\sin\frac{A}{2}\sin\frac{B}{2}\right)=\frac{36x}{4x^2-y^2}\overset{?}{\geq}\frac{32r}{R}$$

$$\Leftrightarrow 9Rx\overset{?}{\geq}8r(4x^2-y^2)\Leftrightarrow 9Rx-32rx^2+8ry^2\underset{(\bullet\bullet)}{\overset{?}{\geq}}0$$

$$x=\frac{1}{2}\sum 2\sin^2\frac{A}{2}=\frac{1}{2}\sum(1-\cos A)\Rightarrow x\overset{(*)}{=}\frac{2R-r}{2R}\ and\ y^2=\left(\sum\sin\frac{A}{2}\sin\frac{B}{2}\right)^2$$

$$=\sum\sin^2\frac{A}{2}\sin^2\frac{B}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\left(\sum\sin\frac{A}{2}\right)$$

$$=\left(\frac{\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{r^2}\right)\sum AI^2$$

$$+\left(\frac{r}{4R}\right)\sum\frac{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}\overset{\because\, 0<\cos\frac{B-C}{2}\leq 1\ and\ analogs}{\geq}\frac{1}{16R^2}\sum(bc$$

$$-4Rr)+\left(\frac{r}{4R}\right)\sum(\cos B+\cos C)\ \ (via\ (i)\ and\ analogs)$$

$$=\frac{s^2+4Rr+r^2-12Rr}{16R^2}+\left(\frac{r}{2R}\right)\sum\cos A$$

$$=\frac{s^2-8Rr+r^2}{16R^2}+\left(\frac{r}{2R}\right)\left(\frac{R+r}{R}\right)\overset{via\ (*)}{\Rightarrow}\text{LHS of }(\bullet\bullet)$$

$$\geq 9R.\frac{2R-r}{2R}-32r.\left(\frac{2R-r}{2R}\right)^2+8r\left(\frac{s^2-8Rr+r^2}{16R^2}+\left(\frac{r}{2R}\right)\left(\frac{R+r}{R}\right)\right)$$

$$=\frac{9(2R-r)}{2}-\frac{8r(2R-r)^2}{R^2}$$

$$+\frac{(s^2+9r^2)r}{2R^2}\overset{\text{Gerretsen}}{\geq}\frac{9(2R-r)R^2-16r(2R-r)^2+r(16Rr+4r^2)}{2R^2}\overset{?}{\geq}0$$

$$\Leftrightarrow 18t^3-73t^2+80t-12\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((17t+1+(t-2))(t-2)+4\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \text{LHS of } (\bullet\bullet) \geq 0$$

$$\Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true} \Rightarrow \sum \frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}{\sin^4\frac{A}{2}+\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{B}{2}} \geq \frac{16r}{R} \ \text{(QED)}$$