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If $x, y, z > 0$ then

$$(3x+y)\sqrt{\frac{y}{x+y}}+(3y+z)\sqrt{\frac{z}{y+z}}+(3z+x)\sqrt{\frac{x}{z+x}}\le 2\sqrt{2}(x+y+z)$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Marian Ursărescu-Romania, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 3 by Vivek Kumar-India, Solution 4 by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Marian Ursărescu-Romania

We must show:

$$\sum_{cyc}(3x+y)\sqrt{\frac{2y}{x+y}}\le 4(x+y+z);(1)$$

$$\sqrt{\frac{2y}{x+y}}=\sqrt{\frac{2y}{x+y}\cdot 1}\overset{AGM}{\le}\frac{\frac{2y}{x+y}+1}{2}=\frac{x+3y}{2(x+y)};(2)$$

From (1) and (2) we must show that:

$$\sum_{cyc}\frac{(3x+y)(3y+x)}{2(x+y)}\le 4(x+y+z)$$

$$\sum_{cyc}\frac{3x^2+10xy+3y^2}{x+y}\le 8(x+y+z)$$

$$\sum_{cyc}\frac{3(x+y)^2+4xy}{x+y}\le 8(x+y+z)$$

$$\sum_{cyc}3(x+y)+\sum_{cyc}\frac{4xy}{x+y}\le 8(x+y+z)$$

$$6(x+y+z)+4\sum_{cyc}\frac{xy}{x+y}\le 8(x+y+z)$$

$$\sum_{cyc}\frac{2xy}{x+y}\le x+y+z \text{ true because } \frac{2xy}{x+y}\le\frac{x+y}{2}$$

$$\sum_{cyc}\frac{2xy}{x+y}\le\sum_{cyc}\frac{x+y}{2}=x+y+z$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

From AM-GM Inequality, we have :

$$(3x+y)\cdot(3x+y)\cdot 4y\le\left(\frac{(3x+y)+(3x+y)+4y}{3}\right)^3=8(x+y)^3$$

$$\text{Therefore,}\qquad (3x+y)\sqrt{\frac{y}{x+y}}\le\sqrt{2}(x+y)$$

Similarly, we have :

$$(3y+z)\sqrt{\frac{z}{y+z}}\le\sqrt{2}(y+z)\quad and\quad (3z+x)\sqrt{\frac{x}{z+x}}\le\sqrt{2}(z+x).$$

Adding these inequalities, we obtain :

$$(3x+y)\sqrt{\frac{y}{x+y}}+(3y+z)\sqrt{\frac{z}{y+z}}+(3z+x)\sqrt{\frac{x}{z+x}}\le 2\sqrt{2}(x+y+z).$$

Equality holds iff $x=y=z$.

Solution 3 by Vivek Kumar-India

$$(3x+y)\sqrt{\frac{y}{x+y}}=\frac{(2x+x+y)\sqrt{y}}{\sqrt{x+y}}=\frac{2x\sqrt{y}+(x+y)\sqrt{y}}{\sqrt{x+y}}=$$

$$=\frac{2\sqrt{x}\cdot\sqrt{y}\cdot\sqrt{x}}{\sqrt{x+y}}+\sqrt{y}\cdot\sqrt{x+y}\overset{AGM}{\leq}\frac{(x+y)\sqrt{x}}{\sqrt{x+y}}+\sqrt{y}\cdot\sqrt{x+y}=$$

$$=\sqrt{x}\cdot\sqrt{x+y}+\sqrt{y}\cdot\sqrt{x+y}=\left(\sqrt{x}+\sqrt{y}\right)\sqrt{x+y}\overset{AGM}{\leq}$$

$$\leq\sqrt{2(x+y)}\cdot\sqrt{x+y}=\sqrt{2}(x+y)$$

Similarly,

$$(3y+2)\sqrt{\frac{z}{y+z}}\leq\sqrt{2}(y+z)\text{ and }(3z+x)\sqrt{\frac{x}{z+x}}\leq\sqrt{2}(z+x)$$

Therefore,

$$\sum_{cyc}(x+y)\sqrt{\frac{y}{x+y}}\leq 2\sqrt{2}(x+y+z)$$

Solution 4 by Nguyen Van Canh-Ben Tre-Vietnam

Using Cauchy-Schwarz's Inequality, we have:

$$\left((3x+y)\sqrt{\frac{y}{x+y}}+(3y+z)\sqrt{\frac{z}{y+z}}+(3z+x)\sqrt{\frac{x}{z+x}}\right)^2$$

$$=\left(\sqrt{\frac{(3x+y)^2y}{x+y}}+\sqrt{\frac{(3y+z)^2z}{y+z}}+\sqrt{\frac{(3z+x)^2x}{z+x}}\right)^2$$

$$\leq 3\left(\frac{(3x+y)^2y}{x+y}+\frac{(3y+z)^2z}{y+z}+\frac{(3z+x)^2x}{z+x}\right)\overset{(1)}{\leq}8(x+y+z)^2;$$

$$(1)\leftrightarrow 3\left[y(y+z)(z+x)(3x+y)^2+z(y+x)(z+x)(3y+z)^2+x(x+y)(y+z)(3z+x)^2\right]$$

$$\leq 8(x+y)(y+z)(z+x)(x+y+z)^2;$$

$$\leftrightarrow 3\left[\sum(x^4y+xy^4)+6\sum x^2y^3+10\sum x^3y^2+16xyz\sum x^2+30xyz\sum xy\right]$$

$$\leq 8\left[\sum(x^4y+xy^4)+3\sum x^2y^3+3\sum x^3y^2+6xyz\sum x^2+10xyz\sum xy\right];$$

$$\leftrightarrow 5\sum(x^4y+xy^4)+6\sum x^2y^3\ge 6\sum x^3y^2+10xyz\sum xy;$$

By AM-GM we have:

- $x^4y+x^4y+x^4y+x^4y+xy^4+xy^4\ge 6\sqrt[6]{x^{18}y^{12}}=6x^3y^2;$
- $y^4z+y^4z+y^4z+y^4z+yz^4+yz^4\ge 6\sqrt[6]{y^{18}z^{12}}=6y^3z^2;$
- $z^4x+z^4x+z^4x+z^4x+zx^4+zx^4\ge 6\sqrt[6]{z^{18}x^{12}}=6z^3x^2;$

$$\to 4\sum x^4y+2\sum xy^4\ge 6\sum x^3y^2;\quad (2)$$

By Cauchy-Schwarz's Inequality we have:

- $x^4y+y^4z+z^4x=\frac{(x^2)^2}{\frac{1}{y}}+\frac{(y^2)^2}{\frac{1}{z}}+\frac{(z^2)^2}{\frac{1}{x}}\ge\frac{(\sum x^2)^2}{\frac{\sum xy}{xyz}}=\frac{xyz(\sum x^2)^2}{\sum xy}\overset{\sum x^2\ge\sum xy}{\ge} xyz\sum xy$

$$\to \sum x^4y\ge xyz\sum xy;\quad (3)$$

- $xy^4+yz^4+zx^4=\frac{(y^2)^2}{\frac{1}{x}}+\frac{(z^2)^2}{\frac{1}{y}}+\frac{(x^2)^2}{\frac{1}{z}}\ge\frac{(\sum x^2)^2}{\frac{\sum xy}{xyz}}=\frac{xyz(\sum x^2)^2}{\sum xy}\overset{\sum x^2\ge\sum xy}{\ge} xyz\sum xy$

$$\to 3\sum xy^4\ge 3xyz\sum xy;\quad (4)$$

- $x^2y^3+y^2z^3+z^2x^3=\frac{(xy)^2}{\frac{1}{y}}+\frac{(yz)^2}{\frac{1}{z}}+\frac{(zx)^2}{\frac{1}{x}}\ge\frac{(\sum xy)^2}{\frac{\sum xy}{xyz}}=xyz\sum xy$

$$\to 6\sum x^2y^3\ge 6xyz\sum xy;\quad (5)$$

$$\overset{(2)+(3)+(4)+(5)}{\Rightarrow}\quad 5\sum(x^4y+xy^4)+6\sum x^2y^3\ge 6\sum x^3y^2+10xyz\sum xy;$$

$$\to (1)\ \text{true.}$$

$$\to (3x+y)\sqrt{\frac{y}{x+y}}+(3y+z)\sqrt{\frac{z}{y+z}}+(3z+x)\sqrt{\frac{x}{z+x}}\le 2\sqrt{2}(x+y+z);$$

Proved.