A Limit summation problem

Let $\mathcal{S}_n$ denote the sum of product of inverse of squares of natural numbers taken n at a time where $n \in \mathbb{N}$ and define $\mathcal{S}_0 = 1$
Then find $\mathcal{S}_n$ and henceforth prove that :

$$\sum_{n=0}^{\infty} \mathcal{S}_n = \frac{\sinh(\pi)}{\pi}$$

solution

$$\mathcal{P} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right) = \lim_{n\to\infty}\prod_{k=1}^{n}\frac{(k-x)(k+x)}{k^2} = \lim_{n\to\infty}\frac{\Gamma(n+1+x)\Gamma(n+1-x)}{\Gamma(1+x)\Gamma(1-x)\Gamma^2(n+1)}$$

where we've used :

$$\prod_{k=1}^{n}(x+k) = \frac{\Gamma(n+1+x)}{\Gamma(x+1)}$$

Now using the fact that

$$\lim_{n\to\infty}\frac{\Gamma(n+a)}{\Gamma(n+b)} = n^{a-b}$$

we get

$$\mathcal{P} = \frac{1}{\Gamma(1-x)\Gamma(1+x)} = \frac{\sin(\pi x)}{(\pi x)}$$

Now after substituting $x^2 = t$,we expand the product as :

$$\mathcal{P} = \frac{\sin(\pi\sqrt{t})}{\pi\sqrt{t}} = \prod_{k=1}^{\infty}\left(1 - \frac{t}{k^2}\right) = \frac{1}{(n!)^2}\prod_{k=1}^{\infty}(k^2 - t)$$

$$= \lim_{n\to\infty} = (-1)^n t^n + a_1 t^{n-1} + \cdots + \cdots a_{n-1}t + a_n$$

where the coefficients of each term (that is all the a_i's) is nothing but sum of product of roots taken i at a time divided by $(n!)^2$.
Rewriting the same using Vieta's formula (for the sign of each term) and replacing the coefficients we get that :

$$\frac{\sin(\pi\sqrt{t})}{\pi\sqrt{t}} = 1 - \mathcal{S}_1 t + \mathcal{S}_2 t^2 - \mathcal{S}_3 t^3 + \cdots - \cdots (-1)^n x^n \qquad \text{(i)}$$

Therefore from here we deduce that $\mathcal{S}_k$ is the sum of product of inverse of all perfect squares taken k at a time.

Now we use the Taylor expansion of the same as

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots - \cdots$$

Replacing z with $\pi\sqrt{t}$ and dividing by z we have

$$\frac{\sin(\pi\sqrt{t})}{\pi\sqrt{t}} = 1 - \frac{\pi^2 t}{3!} + \frac{\pi^4 t^2}{5!} - \frac{\pi^6 t^3}{7!} + \cdots - \cdots \tag{ii}$$

Now as (i) and (ii) represent the same sum therefore by comparing the coefficients we have

$$\mathcal{S}_k = \frac{\pi^{2k}}{(2k+1)!}$$

This completes the proof.
Therefore summing all the coefficient gives (we've replaced k with n),

$$\sum_{n=0}^{\infty} \mathcal{S}_n = \sum_{n=0}^{\infty} \frac{\pi^{2k}}{(2k+1)!} = \frac{\sinh(\pi)}{\pi}$$

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