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ABOUT RECURRENCE RELATIONSHIPS
FOR REAL NUMBER SEQUENCES-(II)

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Abstract: *In this paper are presented few techniques for study the convergence of real numbers sequences.*

1. Introduction.

Let $(x_n)_{n\in\mathbb{N}}$ be sequence of real numbers. We define the relationship homogeneity for second order of the type:

$$x_n = ax_{n-1} + bx_{n-2}, n > 2; (1.1)$$

where $a, b \in \mathbb{R}$ and x_1, x_2 are the initial conditions.
If $a = q$ and $b = 0$ we deduce that $x_{n-1} = x_{n-2} + r$. So, we get $x_n = x_1 \cdot q^n, n \geq 1; (1.2)$ and $x_n = 2x_{n-1} + x_{n-2}, n \geq 2; (1.3)$ with $x_n = x_1 + (n-1)r, n \geq 1$.
Now, from (1.2) we admits that x_n has the form $x_n = c \cdot q^{n-1}, n \geq 1$, where $c \in \mathbb{R}$.
For $n > 3$, the relationship (1.1)can be written as:

$$cq^{n-1} = ac \cdot q^{n-2} + bc \cdot q^{n-3} \text{ or } cq^{n-3}(q^2 - aq - b) = 0$$

which means that (1.1) exist when q check the algebraic equation

$$q^2 - aq - b = 0; (1.3)$$

which is named characteristic equation associate to the recurrence relationship.

If q_1, q_2 are the roots of the characteristic equation (1.3), namely $x_n' = cq_1^{n-1}, x_n'' = cq_2^{n-1}$.

In these conditions, we find that

$$\begin{cases} q_1 = \frac{1}{2}\left(a - \sqrt{a^2 + 4b}\right) \\ q_2 = \frac{1}{2}\left(a + \sqrt{a^2 + 4b}\right) \end{cases}; (1.4)$$

Theorem 1.1

If $a^2 + 4b > 0$ then $(x_n)_{n\geq 1}$ satisfy relationship (1.1) if and only if $\exists c, d \in \mathbb{R}$ such that

$$x_n = cq_1^{n-1} + dq_2^{n-1}, n \geq 1; (1.5)$$

Proof.

Admits that $(x_n)_{n\geq 1}$ is verified (1.1) and then, for $n=1, n=2$, we get $\begin{cases} c+d=x_1 \\ q_1c+q_2d=x_2 \end{cases}$.

Hence, $q_2-q_1=\sqrt{a^2+4b}$ and

$$c=\frac{q_2x_1-x_2}{q_2-q_1}, d=\frac{x_2-q_1x_1}{q_2-q_1}; (1.6)$$

Suppose that (1.5) is verify for all $1,2,\dots,n$ then for $n+1$, we have:

$$x_{n+1}=ax_n+bx_{n-1}=a(cq_1^{n-1}+dq_2^{n-1})+b(cq_1^{n-2}+dq_2^{n-2})=$$
$$=cq_1^{n-2}(aq_1+b)+dq_2^{n-2}(aq_2+b)=$$
$$=cq_1^{n-2}q_1^2+dq_2^{n-2}q_2^2=cq_1^n+dq_2^n.$$

Reciprocal, admitting that exists $c,d\in\mathbb{R}$ such that (1.5) is verified, hence for $n>2$,

$$ax_{n-1}+bx_{n-2}=a(cq_1^{n-2}+dq_2^{n-2})+b(cq_1^{n-3}+dq_2^{n-3})=$$
$$=cq_1^{n-3}(aq_1+b)+dq_2^{n-3}(aq_2+b)=cq_1^{n-1}+dq_2^{n-1}=x_n.$$

Theorem 1.2

If $a^2+4b=0$ and $a\neq 0$ then $(x_n)_{n\geq 1}$ check it out (1.1) if and only if

$\exists c,d\in\mathbb{R}$ such that

$$x_n=cq^{n-1}+d(n-1)q^{n-1}, n\geq 1; (1.7)$$

Proof.

Admits that $(x_n)_{n\geq 1}$ verified (1.1) and for $n=1, n=2$ let $c=x_1, qc+qd=x_2$. Therefore,

$$c=x_1 \text{and } d=\frac{1}{q}(x_2-qx_1); (1.8)$$

So, the numbers c and d verified the relationships (1.8) and suppose that these relationships are verified for all the integer numbers $1,2,\dots,n$, then for $n+1$, it follows

$$x_{n+1}=cq^{n-2}(aq+b)+dq^{n-2}[a(n-1)q+b(n-2)]=$$
$$=cq^{n-2}q^2+dq^{n-2}[(n-1)(aq+b)-b]=$$
$$=cq^n+dq^{n-2}[(n-1)q^2+q^2]=cq^n+dnq^n.$$

Reciprocal, admitting that exists $a,d\in\mathbb{R}$ such that (1.7) has verified, from (1.5) we get

$$ax_{n-1}+bx_{n-2}=cq^{n-3}(aq+b)+dq^{n-3}[aq(n-2)+b(n-3)]=$$

$$= cq^{n-1} + dq^{n-3}[(n-2)q^2 + q^2] = cq^{n-1} + d(n-1)q^{n-1} = x_n.$$

Theorem 1.3

If $a^2 + 4b < 0$ then the roots of the equation (1.1) has not real numbers and are as

$$\begin{cases} q_1 = r(\cos\theta - i\sin\theta) \\ q_2 = r(\cos\theta + i\sin\theta) \end{cases}, n \geq 1, \text{where } r = \sqrt{-b} \text{ and } \cos\theta = \frac{a}{2\sqrt{-b}}, 0 < u < \pi.$$

Then the sequence $(x_n)_{n\geq1}$ satisfy relationship (1.1) if and only if $\exists c, d \in \mathbb{R}$ such that

$$x_n = r^{n-1}[c \cdot \cos(n-1)\theta + d \cdot \sin(n-1)\theta], n \geq 1.; (1.9)$$

Proof. We establish the auxiliary relationships:

$$\begin{cases} ar \cdot \cos(n-1)\theta + b \cdot \cos(n-2)\theta = r^2\cos(n\theta) \\ ar \cdot \sin(n-1)\theta + b\sin(n-2)\theta = r^2\sin(n\theta) \end{cases}, n \geq 2; (2.0)$$

For $n = 2$, we have $q_2^2 = aq_2 + b$ which written as:

$r^2(\cos 2\theta + i\sin 2\theta) = ar(\cos\theta + i\sin\theta)$, therefore,

$$\begin{cases} r^2\cos(2\theta) = ar \cdot \cos\theta + b \\ r^2\sin(2\theta) = ar \cdot \sin\theta \end{cases}$$

From (1.9) we have:

$$r^2\cos(n+1)\theta = r^2(\cos(n\theta) \cdot \cos\theta - \sin(n\theta) \cdot \sin\theta) =$$

$$= \cos\theta\,[ar \cdot \cos(n-1)\theta + b \cdot \cos(n-2)\theta] - \sin\theta\,[ar \cdot \sin(n-1)\,\theta + b \cdot \sin(n-2)\,\theta]$$

$$= ar[\cos(n-1)\,\theta \cdot \cos\theta - \sin(n-1)\,\theta \cdot \sin\theta]$$

$$+ b[\cos(n-2)\,\theta \cdot \cos\theta - \sin(n-2)\,\theta \cdot \sin\theta] =$$

$$= ar \cdot \cos(n\theta) + b \cdot \cos(n-1)\theta$$

So, if $(x_n)_{n\geq1}$ verified relationships (1.1), for $n = 1,2$ in (2.0), we can find the system:

$$\begin{cases} c = x_1 \\ r(c \cdot \cos\theta + d \cdot \sin\theta) = x_2 \end{cases} \text{ which has unique solution}$$

$$\begin{cases} c = x_1 \\ d = \dfrac{1}{\sin\theta}\left(\dfrac{1}{r}x_2 - x_1 \cdot \cos\theta\right) \end{cases}$$

Suppose that these relationships are verified for $2, 3, \ldots, n$, then we will prove that are true for $n + 1$.

$$x_{n+1} = ar^{n-1}[c \cdot \cos(n-1)\theta + d \cdot \sin(n-1)\theta]$$

$$+ br^{n-2}[c \cdot \cos(n-2)\theta + d \cdot \sin(n-2)\theta] =$$

$$= cr^{n-2}[ar \cdot \cos(n-1)\theta + b \cdot \cos(n-2)\theta] + dr^{n-2}[ar \cdot \sin(n-1)\theta + b \cdot \sin(n-2)\theta]$$

$$= cr^{n-2}r^2 \cdot \cos(n\theta) + dr^{n-2}r^2 \cdot \sin(n\theta) = r^n(c \cdot \cos(n\theta) + d \cdot \sin(n\theta))$$

Reciprocal, admits that $\exists c, d \in \mathbb{R}$ such that (1.9) are verified, from (2.0) and (1.1), we have:

$$ax_{n-1} + bx_{n-2} =$$

$$= ar^{n-2}[c \cdot \cos(n-2)\theta + d \cdot \sin(n-2)\theta] + br^{n-3}[c \cdot \cos(n-3)\theta + d \cdot \sin(n-3)\theta]$$

$$= cr^{n-3}[ar \cdot \cos(n-2)\theta + b \cdot \cos(n-3)\theta]$$

$$+ dr^{n-3}[ar \cdot \sin(n-2)\,\theta + b \cdot \sin(n-3)\,\theta]$$

$$= cr^{n-3}r^2 \cdot \cos(n-1)\,\theta + dr^{n-3}r^2 \cdot \sin(n-1)\,\theta$$

$$= r^{n-1}[c \cdot \cos(n-1)\,\theta + d \cdot \sin(n-1)\,\theta] = x_n.$$

2. Applications.

Ap. 2.1)

Let $(x_n)_{n\geq 0}$ be sequence of real numbers defined by $x_0 = x_1 = 1$ and $x_{n+2} = x_{n+1} + x_n$. Find x_n.

Solution. The characteristic equation associate for the sequence $(x_n)_{n\geq 1}$ is $r^2 = r + 1$ which has the roots

$$r_1 = \frac{1-\sqrt{5}}{2}, r_2 = \frac{1+\sqrt{5}}{2}$$

Using Theorem 1.1, we get, $x_n = c\left(\frac{1-\sqrt{5}}{2}\right)^n + d\left(\frac{1+\sqrt{5}}{2}\right)^n$ and from the initial conditions $x_0 = x_1 = 1$, it follows that

$$\begin{cases} c = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}-1}{2} \\ d = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}+1}{2} \end{cases}$$

Therefore,

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \;(Binet's\ formula)$$

Observation 2.1 The sequence $(x_n)_{n\geq 0}$ is known as $Fibonacci's$ sequence and is noticed $(F_n)_{n\geq 0}$

Ap. 2.2)

Let $(x_n)_{n\geq 0}$ be sequence of real numbers defined by $x_0=2, x_1=1$ and $x_{n+2}=x_{n+1}+x_n$. Find x_n.

Solution. Using application 2.1, we get

$$x_n=\left(\frac{1-\sqrt{5}}{2}\right)^n+\left(\frac{1+\sqrt{5}}{2}\right)^n$$

Observation 2.2 The sequence $(x_n)_{n\geq 0}$ is known as $Lucas'$ sequence and is noticed $(L_n)_{n\geq 1}$.

Ap. 2.3)

Let $(x_n)_{n\geq 0}$ be sequence of real numbers defined by $x_0=0, x_1=1$ and

$$x_{n+2}=\sqrt{2}x_{n+1}-x_n$$

Prove that $x_{n+4k}=x_n, k\in\mathbb{N}$.

Solution. The characteristic equation associate for the sequence $(x_n)_{n\geq 1}$ is $r^2=\sqrt{2}r-1$

$$\Delta=-2 \text{ and } r_{1,2}=\frac{1}{\sqrt{2}}(1\pm i)=\cos\frac{\pi}{4}\pm i\sin\frac{\pi}{4}$$

Using Theorem 1.3, we get $x_n=c\cdot\cos\frac{n\pi}{4}+d\cdot\sin\frac{n\pi}{4}$

From $x_0=1$ and $x_1=1$, it follows that $c=0, d=\sqrt{2}$ and $x_n=\sqrt{2}\sin\frac{n\pi}{4}$.

Therefore, $x_{n+8}=\sqrt{2}\sin\frac{(n+4k)\pi}{4}=\sqrt{2}\sin\left(\frac{n\pi}{4}+2k\pi\right)=x_n$.

Observation 2.3

If $(x_n)_{n\geq 1}$ are defined by $(*)$: $x_{n+k}=a_1x_{n+k-1}+a_2x_{n+k-2}+\dots+a_kx_n, (a_i)_{i=\overline{1,n}}\in\mathbb{R}$ and $x_0=p_0, x_1=p_1,\dots,x_{k-1}=p_{k-1}$ we can determine existence of the general terms.

Let $r^k=a_1r^{k-1}+a_2r^{k-2}+\dots+a_{k-1}r+a_k$ be the characteristic equation of the sequence, then the general terms of $(x_n)_{n\geq 1}$ are $x_n=c_1\alpha_1^n+c_2\alpha_2^n+\dots+c_k\alpha_n^k$, where $(c_i)_{i=\overline{1,n}}$ are determined from the initial conditions.

The characteristic equation has α_1 by the order $q, \alpha_1 \in \mathbb{R}$, thus the sequences with general terms $n \cdot \alpha_1^n, n^2 \cdot \alpha_1^n, \dots, n^{q-1} \cdot \alpha_1^n$ verified the relationships $(*)$.

Ap. 2.4)

Let $(x_n)_{n\geq 1}$ be sequence of real numbers defined by $x_0 = 3, x_1 = 2, x_3 = 6$ and $x_{n+3} = 2x_{n+2} + x_{n+1} - 2x_n$. Find x_n.

Solution.

The equation characteristic associate of the sequence is $r^3 = 2r^2 + r - 2$ which has the roots $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 2$. Using the Observation 2.3, we get $x_n = c_1 + c_2(-1)^n + c_3 \cdot 2^n$ and from the initial conditions, we obtain

$$x_n = 1 + (-1)^n + 2^n.$$

3. Non-homogeneous relationships of recurrence by two ordered

Let $(x_n)_{n\geq 1}$ be sequence of real numbers defined by

$$(3.0): \begin{cases} x_{n+2} = ax_{n+1} + bx_n + f(n) \\ x_0 = p_0, x_1 = p_1 \end{cases}; a, b, p_0, p_1 \in \mathbb{R}, b \neq 0 \text{ and } f: \mathbb{N} \to \mathbb{R}.$$

Theorem 3.1

If the characteristic equation $r^2 = ar + b$ has two distinct roots α, β and if the sequence $(z_n)_{n\geq 1}$ satisfy the relationships (3.0), thus the sequence $(x_n)_{n\geq 1}$ has the general terms $x_n = c\alpha^n + d\beta^n + z_n$.

Proof.

Because $\begin{cases} \alpha^2 = a\alpha + b| \cdot c\alpha^n \\ \beta^2 = a\beta + b| \cdot d\beta^n \end{cases}$, it follows that

$c\alpha^{n+2} + d\beta^{n+2} = a(c\alpha^{n+1} + d\beta^{n+1}) + b(c\alpha^n + d\beta^n)$ and how

$z_{n+2} = az_{n+1} + bz_n + f(n)$, by adding we get

$$c\alpha^{n+2} + d\beta^{n+2} + z_{n+2} = a(c\alpha^{n+1} + d\beta^{n+1} + z_{n+1}) + b(c\alpha^n + d\beta^n + z_n) + f(n), n \in \mathbb{N}$$

Therefore, $x_n = c\alpha^n + d\beta^n + z_n$ and from $x_0 = p_0, x_1 = p_1$ we have

$\begin{cases} c + d + z_0 = p_0 \\ c\alpha + d\beta + z_1 = p_1 \end{cases}$ the unique solution.

Theorem 3.2

If the characteristic equation $r^2 = ar + b$ has double roots and $(z_n)_{n\geq 1}$ satisfy the relationships (3.0), thus the sequence $(x_n)_{n\geq 1}$ has the general term $x_n = c\alpha^n + dn \cdot \alpha^n + z_n, ; c, d \in \mathbb{R}$

Proof. Using Theorem 3.1 for $c = p_0 - z_0$ and $d = \frac{1}{\alpha}(p_1 - z_1) - c$, thus we find that

exist P polynomial function with $degree(P) = p$ and an constant $\gamma \in \mathbb{R}$ such that

$$f(n) = \gamma^n \cdot P(n).$$

Ap. 2.5)

Let $(x_n)_{n\geq 1}$ be sequence of real numbers defined by

$$x_{n+2} = 2x_{n+1} - x_n + 17 \cdot 5^n + 7\cos\frac{n\pi}{3}. \text{ Find } x_n.$$

Solution. Let $y_n = c + nd$ solution of the homogenous relationship and let $f_1(n) = 17 \cdot 5^n$.

We find an particular solution $\bar{z}_n = \bar{c} \cdot 5^n$, where $\bar{c} = 1$.

Let $f_2(n) = 7\cos\frac{n\pi}{3}$ and we will to find $\lambda, \mu \in \mathbb{R}$ such that

$z_n = \lambda\cos\frac{n\pi}{3} + \mu\sin\frac{n\pi}{3}$ which verify the relationships

$$z_{n+2} = 2z_{n+1} - z_n + 7\cos\frac{n\pi}{3}$$

From the initial conditions, we get $\begin{cases} \lambda = -\frac{7}{2} = -7\cos\frac{\pi}{3} \\ \mu = -\frac{7\sqrt{3}}{2} = -7\sin\frac{\pi}{3} \end{cases}$ and then

$z_n = -7\cos\frac{(n-1)\pi}{3}$. Therefore,

$$x_n = c + dn + 5^n - 7\cos\frac{(n-1)\pi}{3}$$

Ap. 2.6)

For $a, b, x_1, x_2 \in \mathbb{R}$ and $(b_n)_{n\geq 1}$ sequence of real numbers, find the general term of the sequence $(x_n)_{n\geq 1}$ defined by

$$x_{n+2} - (a+b)x_{n+1} + abx_n = b_n, n \geq 1.$$

Solution. Let $y_n = x_{n+1} - ax_n, n \geq 1$ then we get

$$y_{n+1} - by_n = b_n, n \geq 1 \text{ or} y_{n+1} = by_{n-1} + b_{n-1}, n \geq 1.$$

Therefore, $(y_n)_{n\geq 1}$verify the relationship

$$y_2 = by_1 + b_1$$

$$y_3 = by_2 + b_2 = b^2 y_1 + bb_1 + b_2$$

$$y_4 = by_3 + b_3 = b^3 y_1 + b^2 b_1 + bb_2 + b_3$$

$$\dots \dots \dots \dots (by\ induction)$$

$$y_n = b^{n-1} y_1 + \sum_{k=1}^{n-2} b^{n-k-1} b_k + b_{n-1}, n > 1$$

Now, from $x_n = ax_{n-1} + y_{n-1}, n > 1$, we obtain

$$x_n = a^{n-1} x_1 + \sum_{k=1}^{n-2} a^{n-k-1} y_k + y_{n-1}, n > 1$$

Therefore, from $y_1 = x_2 - ax_1$ we get

$$x_n = a^{n-1} x_1 + \sum_{k=1}^{n-2} a^{n-k-1} \left[b^{k-1}(x_2 - ax_1) + \sum_{i=1}^{k-2} b^{k-i-1} b_i + b_{k-1} \right] + b^{n-2}(x_2 - ax_1) +$$

$$+ \sum_{j=1}^{n-3} b^{n-j-2} b_j + b_{n-2}, n > 2$$

Theorem 3.3

If $(a_i)_{i=\overline{1,k}} \in \mathbb{R}, a_k \neq 0$ and $(x_n)_{n\geq 0}$ is an sequence positive of real numbers such that $a_0 + a_1 x_n + a_2 x_n x_{n+1} + \dots + a_k x_n x_{n+1} \dots x_{n+k-1} = x_n x_{n+1} \dots x_{n+k}, \forall n \in \mathbb{N}$; then exist an $(u_n)_{n\geq 0}$ sequence of real numbers such that $x_n = \frac{u_{n+1}}{u_n}, \forall n \in \mathbb{N}$ and

$$\sum_{i=0}^{k} a_i u_{n+i} = u_{n+k+i}; \forall n \in \mathbb{N}$$

Proof. Because $x_n \neq 0, \forall n \in \mathbb{N}$ then $\exists (u_n)_{n\geq 1}$ such that $x_n = \frac{u_{n+1}}{u_n}$. Let $u_0 = 1, u_1 = x_0$

and

$u_n = x_0 x_1 \dots x_n, \forall n \geq 2$, hence, replacing $x_n = \frac{u_{n+1}}{u_n}$ in

$$a_0 + a_1 x_n + a_2 x_n x_{n+1} + \cdots + a_k x_n x_{n+1} \dots x_{n+k-1} = x_n x_{n+1} \dots x_{n+k}, \forall n \in \mathbb{N}$$

It follows that:

$$a_0 + a_1 \cdot \frac{u_{n+1}}{u_n} + a_2 \cdot \frac{u_{n+2}}{u_{n+1}} \cdot \frac{u_{n+1}}{u_n} + \cdots + a_k \cdot \frac{u_{n+k}}{u_{n+k-1}} \cdot \frac{u_{n+k-1}}{u_{n+k-2}} \cdot \dots \cdot \frac{u_{n+2}}{u_{n+1}} \cdot \frac{u_{n+1}}{u_n} =$$

$$= \frac{u_{n+k+1}}{u_{n+k}} \cdot \frac{u_{n+k}}{u_{n+k-1}} \cdot \frac{u_{n+k-1}}{u_{n+k-2}} \cdot \dots \cdot \frac{u_{n+2}}{u_{n+1}} \cdot \frac{u_{n+1}}{u_n}$$

Therefore,

$$\sum_{i=0}^{k} a_i u_{n+i} = u_{n+k+i}; \forall n \in \mathbb{N}$$

Ap. 2.7)

Let $(x_n)_{n\geq 0}$ sequence of real numbers defined by $x_0 \neq \frac{1}{2}, x_{n+1} = \frac{3x_n}{1-2x_n}$, $\forall n \in \mathbb{N}$. Prove that $(x_n)_{n\in\mathbb{N}}$ is convergent.

Solution. If $x_0 = 0$ then $x_n = 0$.

Let $x_0 \in \mathbb{R} - \left\{0, \frac{1}{2}\right\}$, thus $x_n \in \mathbb{R} - \left\{0, \frac{1}{2}\right\}$ and denoting $y_n = 1 - 2x_n, \forall n \in \mathbb{N}$ we get

$y_n \in \mathbb{R}^*$ and $y_{n+1} y_n - 4y_n + 3 = 0, \forall n \in \mathbb{N}$.

Using Theorem 3.3, let us denote $y_n = \frac{u_{n+1}}{u_n}, n \in \mathbb{N}$, and hence

$$u_{n+2} - 4u_{n+1} + 3u_n = 0, \forall n \in \mathbb{N}$$

The characteristic equation $r^2 - 4r + 3 = 0$ has the roots $r_1 = 1$ and $r_2 = 3$.

So, $u_n = c + d \cdot 3^n, \forall n \in \mathbb{N}$. Therefore,

$$y_n = \frac{c + d \cdot 3^{n+1}}{c + d \cdot 3^n}, \forall n \in \mathbb{N}$$

How $(y_n)_{n\geq 0}$ is convergent, hence $(x_n)_{n\geq 0}$ is convergent.

Ap. 2.8)

Let $(x_n)_{n\geq 0}$ be sequence of real numbers strictly positive defined by $x_1 = 1, x_{n+1} = \frac{x_n}{2+x_n}$

Prove that $(y_n)_{n\geq 1}$, $y_n = \sum_{k=1}^{n} x_k$ is convergent.

Solution. For $x_n > 0, \forall n \in \mathbb{N}$ let us denote $u_n = \frac{1}{x_n}, n \in \mathbb{N}^*$ with $u_1 = 1, u_2 = 3$. Hence,

$u_{n+2} - 3u_{n+1} + 2u_n = 0, \forall n \in \mathbb{N}^*$ with general solution $u_n = c + d \cdot 2^n, \forall n \in \mathbb{N}^*$.

From $u_1 = 1, u_2 = 3$, we get $u_n = 2^n - 1$. So, $x_n = \frac{1}{2^n-1}, \forall n \in \mathbb{N}^*$.

$$y_{n+1} - y_n = x_{n+1} = \frac{1}{2^{n+1}-1} \geq 0, \text{then } (y_n)_{n\geq 0} \text{ increasing.}$$

$$y_n < \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} < 1 \Rightarrow (y_n)_{n\geq 1} - \text{bounded.}$$

Therefore, $(y_n)_{n\geq 1}$ is convergent.

Ap. 2.9)

For $a > 1$, let $(x_n)_{n\geq 0}$ be sequence of real numbers strictly positive defined by $x_0 = 1, x_2 = a$ and $x_{n+1} = \frac{x_n^{a+1}}{x_{n-1}^a}$. Find $\lim\limits_{n\to\infty} \frac{1}{x_n}$.

Solution. For $x_n > 0$ and $a > 1$, we have:

$$\log_a x_{n+1} = (a+1)\log_a x_n - a\log_a x_{n-1} \Leftrightarrow$$
$$\log_a x_{n+1} - (a+1)\log_a x_n + a\log_a x_n = 0$$

Let $y_n = \log_a x_n$ and from the initial conditions we get $y_0 = 0$ and $y_1 = 1$. Thus,

$$y_{n+1} - (a+1)y_n + ay_{n-1} = 0$$

The characteristic equation is:

$r^2 - (a+1)r + a = 0 \Leftrightarrow r_1 = 1, r_2 = a$. Hence, $y_n = c + d \cdot a^n$

$$\begin{cases} c + d = y_0 = 0 \\ c + d \cdot a = y_1 = 1 \end{cases} \Rightarrow \begin{cases} c = -\dfrac{1}{a-1} \\ d = \dfrac{1}{a-1} \end{cases}$$

$$y_n = \frac{1}{a-1}(a^n - 1) = a^{n-1} + a^{n-2} + \dots + a + 1$$

$x_n = a^{y_n} = a^{a^{n-1}+a^{n-2}+\dots+a+1}$. Therefore,

$$\lim_{n\to\infty} \frac{1}{x_n} = 0.$$

Ap.10)

For $a, b, c \in \mathbb{R}, a > b > c > 0$ let $(x_n)_{n\geq 0}$ such that $ax_{n+2} + bx_{n+1} + cx_n = 1, \forall n \in \mathbb{N}$. Find: $\lim\limits_{n\to\infty} x_n$.

Solution. We have:

$$\begin{cases} ax_{n+3} + bx_{n+2} + cx_{n+1} = 1 \\ ax_{n+2} + bx_{n+1} + cx_n = 1 \end{cases}; \forall n \in \mathbb{N}$$

$$ax_{n+3}+(b-a)x_{n+2}+(c-b)x_{n+1}+cx_n=0, \forall n\in\mathbb{N}$$

The characteristic equation is $ar^3+(b-a)r^2+(c-b)r-a=0$ with $r_1=1$ or

$$ar^2+br+c=0; (1)$$

We distinguish the following cases:

Case I) If $\Delta=b^2-4ac>0$, then the equation (1) has two roots $r_2\neq r_3$ and from Vieta's theorem, $r_2+r_3=-\frac{b}{a}\in(-1,0), r_2r_3=\frac{c}{a}\in(0,1)$, then $r_2\in(-1,0)$ and $r_3\in(-1,0)$.

Therefore, $x_n=A\cdot r_1^n+B\cdot r_2^n+C\cdot r_3^n$ and from $r_2,r_3\in(-1,0)$ it follows that

$$\lim_{n\to\infty}x_n=A.$$

Case II) If $\Delta=b^2-4ac=0$, then $r_2=r_3=-\frac{b}{a}\in(-1,0)$. Therefore,

$x_n=A\cdot r_1^n+(B+nC)r_2^n$ and from $r_2\in(-1,0)$ it follows that

$$\lim_{n\to\infty}x_n=A.$$

Case III) If $\Delta=b^2-4ac<0$, then $r_2=r(\cos\theta+i\sin\theta)$ and $r_3=r(\cos\theta-i\sin\theta)$, $\theta\in\mathbb{R}-\{k\pi|k\in\mathbb{Z}\}$. From $r_1r_2=r^2=\frac{c}{a}<1$, then $r\in(0,1)$.

But $x_n=A\cdot 1^n+(B\cos n\theta+C\sin n\theta)r^n; A,B,C\in\mathbb{R}$, where

$\left((B\cos n\theta+C\sin n\theta)r^n\right)_{n\in\mathbb{N}}$ is bounded. Hence,

$$\lim_{n\to\infty}x_n=A.$$

Now, using $\lim_{n\to\infty}x_n=A$ and $ax_{n+2}+bx_{n+1}+cx_n=1, \forall n\in\mathbb{N}$, we get

$$A(a+b+c)=1\Rightarrow\lim_{n\to\infty}x_n=\frac{1}{a+b+c}.$$

Ap.11)

For $a,b\in\mathbb{R}^*, a\neq\pm b$, let $(x_n)_{n\geq 0}$ be sequence of positive real numbers defined by $x_0=1, x_1=\frac{1}{a+b}$ and $abx_{n+2}x_{n+1}+(a+b)x_{n+2}x_n+x_{n+1}x_n=0$. Find: $\lim_{n\to\infty}\frac{x_n}{x_{n+1}}$.

Solution. For $x_0=1, x_1=\frac{1}{a+b}$ and $x_n>0, \forall n\in\mathbb{N}$ we have:

$$abx_{n+2}x_{n+1}+(a+b)x_{n+2}x_n+x_{n+1}x_n=0\Leftrightarrow$$

$$\frac{1}{x_{n+2}}+(a+b)\cdot\frac{1}{x_{n+1}}+ab\cdot\frac{1}{x_n}=0$$

Let us denote $y_n=\frac{1}{x_n}$, then $y_0=1, y_1=a+b$ and hence

$$y_{n+2}+(a+b)y_{n+1}+aby_n=0$$

The characteristic equation $r^2+(a+b)r+ab=0$ has the roots $r_1=-a, r_2=-b$, thus $y_n=c\cdot(-a)^n+d\cdot(-b)^n$ and from the initial conditions $y_0=1, y_1=a+b$, it follows that $c=\frac{2b+a}{b-a}$ and $d=\frac{2a+b}{a-b}$. Therefore,

$$y_n=\frac{2b+a}{b-a}\cdot(-a)^n+\frac{2a+b}{a-b}\cdot(-b)^n \text{ and}$$

$$\lim_{n\to\infty}\frac{x_n}{x_{n+1}}=\lim_{n\to\infty}\frac{y_{n+1}}{y_n}=\lim_{n\to\infty}\frac{\frac{2b+a}{b-a}\cdot(-a)^{n+1}+\frac{2a+b}{a-b}\cdot(-b)^{n+1}}{\frac{2b+a}{b-a}\cdot(-a)^n+\frac{2a+b}{a-b}\cdot(-b)^n}=$$

$$=-\lim_{n\to\infty}\frac{\frac{2b+a}{b-a}\cdot a^{n+1}+\frac{2a+b}{a-b}\cdot b^{n+1}}{\frac{2b+a}{b-a}\cdot a^n+\frac{2a+b}{a-b}\cdot b^n}=$$

$$=-a\cdot\lim_{n\to\infty}\frac{\frac{2b+a}{b-a}+\frac{2a+b}{a-b}\cdot\left(\frac{b}{a}\right)^{n+1}}{\frac{2b+a}{b-a}+\frac{2a+b}{a-b}\cdot\left(\frac{b}{a}\right)^n}=-a$$

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