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Problem proposed by: *Vasile Mircea Popa, Romania*
Prove that

$$\sqrt[3]{\cos\left(\frac{2\pi}{13}\right)\cos\left(\frac{3\pi}{13}\right)}-\sqrt[3]{\cos\left(\frac{4\pi}{13}\right)\cos\left(\frac{6\pi}{13}\right)}-\sqrt[3]{\cos\left(\frac{5\pi}{13}\right)\cos\left(\frac{\pi}{13}\right)}=\frac{\sqrt[3]{14-6\sqrt[3]{13}}}{2}$$

Solutions proposed by: *Surjeet Singhania*, *Himachal Pradesh, India*

Denote $\alpha=\cos\left(\frac{2\pi}{13}\right)\cos\left(\frac{3\pi}{13}\right)$, $\beta=-\cos\left(\frac{4\pi}{13}\right)\cos\left(\frac{6\pi}{13}\right)$ and

$\gamma=-\cos\left(\frac{\pi}{13}\right)\cos\left(\frac{5\pi}{13}\right)$ Now we will use $\prod_{k=1}^{n-1}\cos\left(\frac{k\pi}{n}\right)=2^{1-n}\sin\left(\frac{\pi n}{2}\right)$

$$\alpha\beta\gamma=\sqrt{2^{-12}\sin\left(\frac{13\pi}{2}\right)}=\frac{1}{64}$$

Now $\cos(A)\cos(B)=\frac{\cos(A-B)+\cos(A+B)}{2}$ applying this formula forα,β and γ

$\alpha+\beta+\gamma=\frac{1}{2}\sum_{k=1}^{6}(-1)^{k-1}\cos\left(\frac{k\pi}{13}\right)=\frac{1}{4}\sum_{k=1}^{12}(-1)^{k-1}\cos\left(\frac{k\pi}{13}\right)=\frac{1}{4}$

Similarly $\alpha\beta+\alpha\gamma+\gamma\beta=-\frac{1}{4}$ Now we can find a cubic polynomial with real roots α,β and γ polynomial we have $x^3-\frac{x^2}{4}-\frac{x}{4}-\frac{1}{64}=0$ this is well known polynomial RCP

RCP or *Ramanujan cubic polynomials* defined by *Vladimir Shevelev* in his *article*

The cubic equation $x^3+px^2+qx+r=0$ is RCP if it has real roots and satisfy the equation

$$p\sqrt[3]{r}+3\sqrt[3]{r^2}+q=0$$

if x_1,x_2 and x_3 are real roots then $\sqrt[3]{x_1}+\sqrt[3]{x_2}+\sqrt[3]{x_3}=\sqrt[3]{-p-6\sqrt[3]{r}+3\sqrt[3]{(9r-pq)}}$

we have $p=q=-\frac{1}{4}$ and $r=-\frac{1}{64}$ this satisfying equation of RCP hence we can apply result

$$\sqrt[3]{\cos\left(\frac{2\pi}{13}\right)\cos\left(\frac{3\pi}{13}\right)}-\sqrt[3]{\cos\left(\frac{4\pi}{13}\right)\cos\left(\frac{6\pi}{13}\right)}-\sqrt[3]{\cos\left(\frac{5\pi}{13}\right)\cos\left(\frac{\pi}{13}\right)}=\frac{\sqrt[3]{14-6\sqrt[3]{13}}}{2}$$

References

[1] V. Shevelev, On Ramanujan Cubic Polynomials, preprint, http://arxiv.org/abs/0711.3420, 2007 .

[2] Roman Witula Full Description of Ramanujan Cubic Polynomials Journal of Integer Sequences, https://cs.uwaterloo.ca/journals/JIS/VOL13/Witula/witula30, 2010 .