SSMA MATH CHALLENGES-(V)

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5531. If $x, y, z > 1$; $xyz = 2\sqrt{2}$ then:

$$x^y + y^z + z^x + y^x + z^y + x^z > 9$$

Proof.

$$x^y = \big(1 + (x-1)\big)^y \overset{Bernoulli}{\geq} 1 + (x-1)y \qquad (1)$$
$$y^x = \big(1 + (y-1)\big)^x \overset{Bernoulli}{\geq} 1 + (y-1)x \qquad (2)$$

By adding $(1); (2)$:

$$x^y + y^x \geq 1 + (x-1)y + 1 + (y-1)x =$$
$$= 1 + xy - x - y + 1 + xy =$$
$$= 1 + xy + (x-1)(y-1) \geq 1 + xy$$
$$x^y + y^x > 1 + xy \qquad (1)$$

Analogous:

$$y^z + z^y > 1 + yz \qquad (2)$$
$$z^x + x^z > 1 + zx \qquad (3)$$

By adding $(1); (2); (3)$:

$$x^y + y^z + z^x + y^x + z^y + x^z > 1 + 1 + 1 + xy + yz + zx \overset{AM-GM}{\geq} 3 + 3\sqrt[3]{(xyz)^2} =$$
$$= 3 + 3\big(\sqrt[3]{2\sqrt{2}}\big)^2 = 3 + 3 \cdot \big(\sqrt{2}\big)^2 = 3 + 3 \cdot 2 = 9$$

□

5562. If $a, b, c \geq 1$ then:

$$e^{ab} + e^{bc} + e^{ca} > 3 + \frac{c}{a} + \frac{b}{c} + \frac{a}{b}$$

Proof. Let be $f:[0,\infty)\to\mathbb{R}; f(x)=\frac{e^{ax}}{x^2+a^2}$

$$f'(x)=\frac{ae^{ax}(x^2+a^2)-2xe^{ax}}{(x^2+a^2)^2}=\frac{e^{ax}(ax^2-2x+a^3)}{(x^2+a^2)^2}$$

$$f'(x)=0\Rightarrow ax^2-2x+a^3=0$$

$$\Delta=4-4a^4=4(1-a)(1+a)(1+a^2)\le 0$$

because $a\ge 1$. Hence $f'(x)\ge 0, (\forall)x\ge 0$.

$$\min f(x)=f(0)=\frac{1}{a^2}$$

$$f(x)\ge f(0)\Rightarrow \frac{e^{ax}}{x^2+a^2}\ge\frac{1}{a^2}\Rightarrow e^{ax}\ge\frac{x^2+a^2}{a^2}$$

(1) $$e^{ax}>1+\left(\frac{x}{a}\right)^2;(\forall)x\ge 1$$

(2) $$\text{Take in (1)}: x=b\Rightarrow e^{ab}>1+\left(\frac{b}{a}\right)^2$$

(3) $$\text{Analogous: } e^{bc}>1+\left(\frac{c}{b}\right)^2$$

(4) $$e^{ca}>1+\left(\frac{a}{c}\right)^2$$

By adding (2); (3); (4):

$$e^{ab}+e^{bc}+e^{ca}>3+\left(\frac{b}{a}\right)^2+\left(\frac{c}{b}\right)^2+\left(\frac{a}{c}\right)^2\ge 3+\frac{b}{a}\cdot\frac{c}{b}+\frac{c}{b}\cdot\frac{a}{c}+\frac{a}{c}\cdot\frac{b}{a}=$$

$$=3+\frac{c}{a}+\frac{b}{c}+\frac{a}{b}$$

□

5568. If $A\in M_5(\mathbb{R}); \det(A^5+I_5)\ne 0; A^{20}-I_5=A^5(A^5+I_5)$ then: $\sqrt[4]{\det A}\in\mathbb{R}$.

Proof.

$$A^{20}=A^{10}+A^5+I_5\Rightarrow A^{20}+A^5=A^{10}+2A^5+I_5$$

$$A^{20}+A^5=(A^5+I_5)^2$$

$$(A^{15}+I_5)A^5=(A^5+I_5)^2$$

$$\left((A^5)^3+I_5^3\right)A^5=\left(A^5+I_5\right)^2$$

$$(A^5+I_5)(A^{10}-A^5+I_5)A^5=(A^5+I_5)^2$$

Multiplying by $(A^5+I_5)^{-1}$:

$$(A^{10}-A^5+I_5)A^5=A^5+I_5$$

$$A^{15}-A^{10}+A^5=A^5+I_5$$

$$A^{15}-A^{10}=I_5\Rightarrow A^{10}(A^5-I_5)=I_5$$

$$(\det A)^{10}\det(A^5-I_5)=\det I_5=1>0\Rightarrow$$

(1) $$\det(A^5-I_5)>0$$

$$A^{20}-I_5=A^{10}+A^5$$

$$(A^{10}+I_5)(A^{10}-I_5)=A^5(A^5+I_2)$$

$$(A^{10}+I_5)(A^5-I_5)(A^5+I_5)=A^5(A^5+I_5)$$

Multiplying by $(A^5+I_5)^{-1}$:

$$(A^{10}+I_5)(A^5-I_5)=A^5$$

$$\det(A^5)=\det(A^{10}+I_5)\cdot(A^5-I_5)=\det(A^{10}+I_5)\cdot\det(A^5-I_3)\geq 0$$

$$(\det A)^5\geq 0\Rightarrow \det A\geq 0\Rightarrow \sqrt[4]{\det A}\geq 0$$

□

5574. If $0<a\leq b\leq c$ then:

$$\frac{1}{1+e^{a-b+c}}+\frac{1}{1+e^b}\leq\frac{1}{1+e^a}+\frac{1}{1+e^c}$$

Proof.

$$Let\ be\ f:(0,\infty)\to\mathbb{R};\ f(x)=\frac{1}{1+e^x};$$

$$f'(x)=-\frac{e^x}{(1+e^x)^2};\ f"(x)=-\frac{e^x(1-e^x)}{(1+e^x)^3}>0$$

$$f\ convexe;\ Suppose\ that\ a,b,c\ are\ different\ in\ pairs.$$

$$f(b)=f\Big(\frac{c-b}{c-a}\cdot a+\big(1-\frac{c-b}{c-a}\big)\cdot c\Big)\leq\frac{c-b}{c-a}f(a)+\Big(1-\frac{c-b}{c-a}\Big)f(c)$$

$$f(b)\leq\frac{c-b}{c-a}f(a)+\frac{b-a}{c-a}f(c)\qquad(1)$$

We used the identity:

$$b=\frac{c-b}{c-a}\cdot a+\Big(1-\frac{c-b}{c-a}\Big)\cdot c$$

$$(c-a)b=(c-b)a+(c-a-c+b)c$$

$$bc-ab=ac-ab-ac+bc$$

$$0=0$$

$$f(a-b+c)=f\Big(\frac{b-a}{c-a}a+\frac{c-b}{c-a}c\Big)\leq\frac{b-a}{c-a}f(a)+\frac{c-b}{c-a}f(c)\qquad(2)$$

We used the identity:

$$a-b+c=a-\Big(\frac{c-b}{c-a}a+\frac{b-a}{c-a}c\Big)+c=\frac{b-a}{c-a}a+\frac{c-b}{c-a}c$$

By adding $(1);(2)$:

$$f(a-b+c)+f(b)\leq f(a)\Big(\frac{b-a}{c-a}+\frac{c-b}{c-a}\Big)+f(c)\Big(\frac{c-b}{c-a}+\frac{b-a}{c-a}\Big)f(c)$$

$$f(a-b+c)+f(b)\leq f(a)+f(c)$$

$$\frac{1}{1+e^{a-b+c}}+\frac{1}{1+e^b}\leq\frac{1}{1+e^a}+\frac{1}{1+e^c}$$

Equality holds if $a = b = c$. □

5579. If $a, b \in \mathbb{R}; a \leq b$ then:

$$\log 5 \cdot \int_a^b 5^{x^2} dx + \log 5 \cdot \int_a^b 5^{x^4} dx \geq 5^b - 5^a$$

Proof. By absurdum suppose $5^x \geq 5^{x^2} + 5^{x^4}$

$$1 \geq 5^{x^2 - x} + 5^{x^4 - x} > 5^{x^2 - x}$$

$$5^{x^2 - x} < 1 \Rightarrow x^2 - x < 0 \Rightarrow x \in (0, 1)$$

$$x < 1 \Rightarrow 5 > 5^x \geq 5^{x^2} + 5^{x^4} > 1 + 1 = 2$$

$5 > 2$. False. Hence: $5^x < 5^{x^2} + 5^{x^4}$

$$\int_a^b 5^{x^2} dx + \int_a^b 5^{x^4} dx \geq \int_a^b 5^x dx =$$

$$= \frac{5^b}{\log 5} - \frac{5^a}{\log 5} = \frac{1}{\log 5}(5^b - 5^a)$$

$$\log 5 \cdot \int_a^b 5^{x^2} dx + \log 5 \cdot \int_a^b 5^{x^4} \geq 5^b - 5^a$$

□

5585. In ΔABC the following relationship holds:

$$\sin^4 A + \sin^4 B + \sin^4 C + \sin^4\left(\frac{\pi}{3} + A\right) + \sin^4\left(\frac{\pi}{3} + B\right) + \sin^4\left(\frac{\pi}{3} + C\right) \leq \frac{27}{8}$$

Proof.

$$(\sin A - \sqrt{3}\cos A)^4 \geq 0$$

$$\sin^4 A + 9\cos^4 A + 18\sin^2 A\cos^2 A - 12\sqrt{3}\sin A\cos^3 A - 4\sqrt{3}\sin^3 A\cos A \geq 0$$

$$16\sin^4 A + 9\cos^4 A + \sin^4 A + 6\sin^2 A\cos^2 A + 6\sin^2 A\cos^2 A + 12\sqrt{3}\sin A\cos^3 A +$$
$$+ 4\sqrt{3}\sin^3 A\cos A \leq 18\sin^4 A + 18\cos^4 A + 36\sin^2 A\cos^2 A$$

$$16\sin^4 A + (3\cos^2 A + \sin^2 A + \sqrt{3}\sin 2A)^2 \leq 18(\sin^4 A + 2\sin^2 A\cos^2 A + \cos^4 A)$$

$$16\sin^4 A + ((\sqrt{3}\cos A + \sin A)^2)^2 \leq 18(\sin^2 A + \cos^2 A)^2$$

$$16\sin^4 A + \left(2\sin\left(\frac{\pi}{3} + A\right)\right)^4 \leq 18 \cdot 1^2$$

$$16\sin^4 A + 16\sin^4\left(\frac{\pi}{3} + A\right) \leq 18$$

$$\sin^4 A + \sin^4\left(\frac{\pi}{3} + A\right) \leq \frac{18}{16} = \frac{9}{8} \tag{1}$$

Analogous:

$$\sin^4 B + \sin^4\left(\frac{\pi}{3} + B\right) \leq \frac{9}{8} \tag{2}$$

$$\sin^4 C + \sin^4\left(\frac{\pi}{3} + C\right) \le \frac{9}{8} \tag{3}$$

By adding (1); (2); (3):

$$\sin^4 A + \sin^4 B + \sin^4 C + \sin^4\left(\frac{\pi}{3} + A\right) + \sin^4\left(\frac{\pi}{3} + B\right) + \sin^4\left(\frac{\pi}{3} + C\right) \le \frac{27}{8}$$

Equality holds for an equilateral triangle:

$$A = B = C = \frac{\pi}{3}$$

□

5591. Solve for real numbers:

$$3^{\cos x + \cos y + \cos z} = 3^{\cos^2 x + \cos x} + 3^{\cos^2 y + \cos y} + 3^{\cos^2 z + \cos z}$$

Proof.

$$3^{\cos x + \cos y + \cos z} = 3^{\cos^2 x + \cos x} + 3^{\cos^2 y + \cos y} + 3^{\cos^2 z + \cos z} \ge$$

$$\overset{\text{AM-GM}}{\ge} 3\sqrt[3]{3^{\cos^2 x + \cos^2 y + \cos^2 z + \cos x + \cos y + \cos z}}$$

$$\left(3^{\sum_{cyc} \cos x}\right)^3 \ge 27 \cdot 3^{\sum_{cyc} \cos^2 x + \sum_{cyc} \cos x}$$

$$3\sum_{cyc} \cos x \ge 3 + \sum_{cyc} \cos^2 x + \sum_{cyc} \cos x$$

$$\sum_{cyc} \cos^2 x - 2\sum_{cyc} \cos x + 3 \le 0$$

$$(\cos x - 1)^2 + (\cos y - 1)^2 + (\cos z - 1)^2 \le 0$$

$$\Rightarrow \begin{cases} \cos x = 1 \\ \cos y = 1 \\ \cos z = 1 \end{cases} \Rightarrow \begin{cases} x = \pm\frac{\pi}{2} + 2m\pi; m \in \mathbb{Z} \\ y = \pm\frac{\pi}{2} + 2n\pi; n \in \mathbb{Z} \\ z = \pm\frac{\pi}{2} + 2p\pi; p \in \mathbb{Z} \end{cases}$$

□

5597. If $x, y, z > 0; xyz = 1$ then:

$$\left(x + y - \frac{1}{\sqrt{z}}\right)^2 + \left(y + z - \frac{1}{\sqrt{x}}\right)^2 + \left(z + x - \frac{1}{\sqrt{y}}\right)^2 \ge 3$$

Proof. First, we prove that:

$$2(x + y - \sqrt{xy})^2 \ge x^2 + y^2 \tag{1}$$

$$2\left(x^2 + y^2 + xy - 2x\sqrt{xy} - 2y\sqrt{xy} + 2xy\right) \ge x^2 + y^2$$

$$x^2 + y^2 + 6xy - 4x\sqrt{xy} - 4y\sqrt{xy} \ge 0$$

$$x^2 + 2xy + y^2 + 4xy - 4\sqrt{xy}(x + y) \ge 0$$

$$(x + y)^2 - 4\sqrt{xy}(x + y) + 4xy \ge 0$$

$$(x + y - 2\sqrt{xy})^2 \ge 0 \text{ (True)}$$

By (1):

(2) $$(x+y-\sqrt{xy})^2 \geq \frac{x^2+y^2}{2}$$

$$\sum_{cyc}\left(x+y-\frac{1}{\sqrt{z}}\right)^2 = \sum_{cyc}(x+y-\sqrt{xy})^2 \overset{(2)}{\geq}$$
$$\geq \sum_{cyc}\frac{x^2+y^2}{2} = x^2+y^2+z^2 \overset{\text{AM-GM}}{\geq}$$
$$\geq 3\sqrt[3]{(xyz)^2} = 3\cdot\sqrt[3]{1^2} = 3$$

Equality holds for $x=y=z=1$. □

5620. If $a,b\in[0,1]; a\leq b$ then:

$$4\sqrt{ab} \leq a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \leq 2(a+b)$$

Proof. Let be $f:[0,1]\to\mathbb{R}; f(x) = a\left(\frac{b}{a}\right)^x + b\left(\frac{a}{b}\right)^x$

$$f'(x) = a\left(\frac{b}{a}\right)^x \ln\frac{b}{a} + b\left(\frac{a}{b}\right)^x \ln\frac{a}{b} =$$
$$= \ln\frac{b}{a}\left[a\left(\frac{b}{a}\right)^x - b\left(\frac{a}{b}\right)^x\right]$$
$$f'(x) = 0 \Rightarrow a\left(\frac{b}{a}\right)^x = b\left(\frac{a}{b}\right)^x \Rightarrow a\left(\frac{b}{a}\right)^{2x} = b$$
$$\left(\frac{b}{a}\right)^{2x} = \left(\frac{b}{a}\right)^1 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$$

x	0		$\frac{1}{2}$		1
$f'(x)$	$- - - - - - - - - - - - - -$		0	$+ + + + + + + + + + + + + + +$	
$f(x)$	$a+b$	$\searrow$	$2\sqrt{ab}$	$\nearrow$	$a+b$

(1) $$\Rightarrow 2\sqrt{ab} \leq a\left(\frac{b}{a}\right)^x + b\left(\frac{a}{b}\right)^x \leq a+b$$

For $x=\sqrt{ab}\in[a,b]\subseteq[0,1]$ in (1): □

(2) $$2\sqrt{ab} \leq a\left(\frac{b}{a}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} \leq a+b$$

For $x=\frac{a+b}{2}\in[a,b]\subseteq[0,1]$ in (1):

(3) $$2\sqrt{ab} \leq a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}} \leq a+b$$

By adding (2);(3):

$$4\sqrt{ab} \leq a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \left(\frac{b}{a}\right)^{\frac{a+b}{2}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \left(\frac{a}{b}\right)^{\frac{a+b}{2}}\right) \leq 2(a+b)$$

$$4\sqrt{ab} \leq a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \leq 2(a+b)$$

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