SSMA-MATH CHALLENGES-(IV)

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5477. Compute:

$$L=\lim_{n\to\infty}\Big(\ln n+\lim_{x\to 0}\frac{1-\sqrt{1+x^2}\sqrt[3]{1+x^2}\cdot\ldots\cdot\sqrt[n]{1+x^2}}{x^2}\Big)$$

Proof. Let be $f_k(x)=\sqrt[k]{1+x^2}; k\in\overline{2,n}; n\in\mathbb{N}$

$$f_k'(x)=\frac{2x}{k\sqrt[k]{(1+x^2)^{k-1}}}$$

$$\lim_{x\to 0}\frac{1-\sqrt{1+x^2}\cdot\sqrt[3]{1+x^2}\cdot\ldots\cdot\sqrt[n]{1+x^2}}{x^2}=$$

$$=\lim_{x\to 0}\frac{1-f_2(x)f_3(x)\cdots f_n(x)}{x^2}\overbrace{=}^{(\frac{0}{0})}$$

$$=\lim_{x\to 0}\frac{\Big(1-f_2(x)f_3(x)\cdots f_n(x)\Big)'}{2x}=$$

$$=-\lim_{x\to 0}\frac{\Big(f_2(x)f_3(x)\cdots f_n(x)\Big)'}{2x}=$$

$$=-\frac{1}{2}\lim_{x\to 0}\Big(f_2(x)f_3(x)\cdots f_n(x)\Big)\cdot\sum_{k=2}^{n}\lim_{x\to 0}\Big(\frac{f_k'(x)}{x}\cdot\frac{1}{f_k(x)}\Big)=$$

$$=-\frac{1}{2}\sum_{k=2}^{n}\lim_{x\to 0}\frac{f_k'(x)}{x}=-\frac{1}{2}\sum_{k=2}^{n}\frac{2}{k\sqrt[k]{(1+x)^{k-1}}}=$$

$$=-\frac{1}{2}\sum_{k=2}^{n}\frac{2}{k}=-\sum_{k=2}^{n}\frac{1}{k}$$

$$L=\lim_{n\to\infty}\Big(\ln n-\frac{1}{2}-\frac{1}{3}-\cdots-\frac{1}{n}\Big)=$$

$$=\lim_{n\to\infty}\Big(1-\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\Big)\Big)=1-c$$

c - Euler's constant. □

5482. Prove that if $n\in\mathbb{N}$ then:

$$\frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n+(\tan 3^\circ)^n}+\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n+(\tan 2^\circ)^n}+\frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n+(\tan 1^\circ)^n}\geq\frac{3}{2}$$

Proof. Let be $f:\mathbb{R}\to\mathbb{R}; f(x)=\frac{a^x}{b^x+c^x}; a>b>c$.

$$f'(x)=\left(\frac{a^x}{b^x+c^x}\right)'=\frac{(a^x)'(b^x+c^x)-a^x(b^x+c^x)'}{(b^x+c^x)^2}=$$
$$=\frac{a^x\ln a(b^x+c^x)-a^x(b^x\ln b+c^x\ln c)}{(b^x+c^x)^2}=$$
$$=\frac{a^xb^x(\ln a-\ln b)+a^xc^x(\ln a-\ln c)}{(b^x+c^x)^2}>0$$

It follows f increasing on $R\Rightarrow f(x)\geq f(0),(\forall)x>0$
$\frac{a^x}{b^x+c^x}\geq\frac{1}{2};(\forall)x>0$
For $a=(\tan 5^\circ)^n; b=(\tan 4^\circ)^n; c=(\tan 3^\circ)^n$
$\frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n+(\tan 3^\circ)^n}\geq\frac{1}{2}$ because $\tan 5^\circ>\tan 4^\circ>\tan 3^\circ$
Analogous: $\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n+(\tan 2^\circ)^n}\geq\frac{1}{2}; \frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n+(\tan 1^\circ)^n}\geq\frac{1}{2}$
and by adding:

$$\frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n+(\tan 3^\circ)^n}+\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n+(\tan 2^\circ)^n}+\frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n+(\tan 1^\circ)^n}\geq\frac{3}{2}$$

□

5488. Let be $a,b\in\mathbb{C}$. Solve the following equation:

$$x^3-3ax^2+3(a^2-b^2)x-a^3+3ab^2-2b^3=0$$

Proof. Let be $A=\begin{pmatrix}a&b&b\\b&a&b\\b&b&a\end{pmatrix};\quad TrA=3a;\quad TrA^*=3(a^2-b^2)$

$$\det A=a^3-3ab^2+2b^3$$
$$\det(A-xI_3)=\begin{vmatrix}a-x&b&b\\a&a-x&b\\b&b&a-x\end{vmatrix}=$$
$$=(a+2b-x)\begin{vmatrix}1&1&1\\b&a-x&b\\b&b&a-x\end{vmatrix}=$$
$$=(a+2b-x)\begin{vmatrix}1&0&0\\b&a-b-x&0\\b&0&a-b-x\end{vmatrix}=$$
$$=(a+2b-x)(a-b-x)^2$$
$$\det(A-xI_3)=-x^3+TrAx^2-TrA^*x+\det A=0$$
$$\Rightarrow x^3-TrAx^2+TrA^*x-\det A=0$$
$$x^3-3ax^2+3(a^2-b^2)x-a^3+3ab^2-2b^3=0$$
$$(a+2b-x)(a-b-x)^2=0$$
$$x_1=a+2b$$
$$x_2=x_3=a-b$$

□

5496. Let be $0 < a < b < c$. Prove that:

$$\sum(e^{a-b}+e^{b-a}) > 2a-2c+3+\sum\left(\frac{b}{a}\right)^{\sqrt{ab}}$$

Proof.

We use average logarithmic's ineqaulity:

$$\sqrt{xy} \leq \frac{x-y}{\ln x - \ln y} \leq \frac{x+y}{2}; x, y > 0$$

It follows:

$$\sqrt{ab} < \frac{a-b}{\ln a - \ln b} = \frac{b-a}{\ln b - \ln a} = \frac{b-a}{\ln(\frac{b}{a})}$$

$$\sqrt{ab}\ln\frac{b}{a} < b-a \Rightarrow e^{b-a} > \left(\frac{b}{a}\right)^{\sqrt{ab}}$$

Analogous:

$$e^{c-a} > \left(\frac{c}{a}\right)^{\sqrt{ac}}; e^{c-b} > \left(\frac{c}{b}\right)^{\sqrt{ac}}$$

$$\sum e^{b-a} > \sum\left(\frac{b}{a}\right)^{\sqrt{ab}} \tag{1}$$

In the known inequality: $e^x > x+1; x > 0$ we make $x = a-b$ then $x = b-c; x = a-c$
It follows:

$$e^{a-b} > a-b+1; e^{b-c} > b-c+1; e^{a-c} > a-c+1$$

$$\sum(e^{a-b}+e^{b-a}) \overbrace{>}^{(1)} \sum e^{b-a} + a-b+1+b-c+1+a-c+1$$

$$\sum(e^{a-b}+e^{b-a}) > 2a-2c+3+\sum\left(\frac{b}{a}\right)^{\sqrt{ab}}$$

□

5502. Prove that if $a, b, c > 0; a+b+c = e$ then:

$$e^{ac^e} \cdot e^{ba^e} \cdot e^{cb^e} > e^e \cdot a^{be^2} \cdot b^{ce^2} \cdot c^{ae^2}$$

Proof.

Let be $f : (0, \infty) \to \mathbb{R}; f(x) = x^e - e\ln x$

$$f'(x) = ex^{e-1} - \frac{e}{x} = e\left(x^{e-1} - \frac{1}{x}\right) = \frac{e}{x}(x^e - 1)$$

$$f'(x) = 0 \Rightarrow x = 1$$

$$\lim_{\substack{x\to 0\\ x>0}} f(x) = \infty; \lim_{x\to\infty} f(x) = \infty$$

$$\min f(x) = f(1) = 1 \Rightarrow f(x) \geq 1; (\forall)x > 0$$

$$x^e - e\ln x \geq 1 \Rightarrow a^e - e\ln a \geq 1$$

we multiply with b:

$$ba^e - be\ln a \geq b$$

Analogous:

$$cb^e - ce\ln b \geq c$$

$$ac^e - ae\ln c \geq a$$

By adding:

$$ac^e + ba^e + cb^e \geq a + b + c + e(b\ln a + c\ln b + a\ln c)$$

$$ac^e + ba^e + cb^e \geq e\Big(1 + \ln(a^{be} \cdot b^{ce} \cdot c^{ae})\Big)$$

$$ac^e + ba^e + cb^e \geq e\ln e + \ln\Big(a^{be^2} \cdot b^{ce^2} \cdot c^{ae^2}\Big)$$

$$e^{ac^e + ba^e + cb^e} \geq e^e \cdot a^{be^2} \cdot b^{ce^2} \cdot c^{ae^2}$$

□

5506. Find:

$$\Omega = \det\left[\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}^{100}\right]$$

Proof.

$$\text{Let } X = \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}; Y = \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}$$

$$XY = \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 25 - 25 & -5 + 5 \\ 125 - 125 & -25 + 25 \end{pmatrix} = O_2$$

$$X^iY^i = O_2; (\forall) i, j \in \overline{1, 100}$$

$$(X + Y)^{100} = \sum_{k=0}^{100} \binom{100}{k} X^{100-k}Y^k = X^{100} + Y^{100}$$

$$\Omega = \det(X^{100} + Y^{100}) = \det\Big[(X + Y)^{100}\Big] =$$

$$= \Big[\det(X + Y)\Big]^{100} = \left(\det\left[\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} + \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}\right]\right)^{100} = \left(\begin{vmatrix} 26 & 0 \\ 0 & 26 \end{vmatrix}\right)^{100} = 26^{200}$$

□

5525. Solve for real numbers:

$$4\sin^2(x + y) = 1 + 4\cos^2 x + 4\cos^2 y$$

Proof.

$$4\cdot\frac{1-\cos(2x+2y)}{2}=1+4\cdot\frac{1+\cos 2x}{2}+4\cdot\frac{1+\cos 2y}{2}$$
$$2-2\cos(2x+2y)=1+2+2\cos 2x+2+2\cos 2y$$
$$2\cos 2x+2\cos(2x+2y)+2\cos 2y=-3$$
$$2(2\cos^2 x-1)+2\cdot 2\cos\frac{2x+2y+2y}{2}\cos\frac{2x+2y-2y}{2}=-3$$
$$4\cos^2 x-2+4\cos(x+2y)\cos x=-3$$
$$4\cos^2 x+4\cos(x+2y)\cos x=-1$$
$$4\cos^2 x+4\cos(x+2y)\cos x+\cos^2(x+2y)+1-\cos^2(x+2y)=0$$
$$[2\cos x+\cos(x+2y)]^2+\sin^2(x+2y)=0$$
$$\sin^2(x+2y)=0\Rightarrow x+2y=k\pi;k\in\mathbb{Z}\Rightarrow\cos(x+2y)=(-1)^k$$
$$2\cos x+(-1)^k=0\Rightarrow\cos x=\frac{(-1)^{k+1}}{2}$$
$$x\in\left\{\pm\arccos\frac{(-1)^{k+1}}{2}+2k\pi\middle|k\in\mathbb{Z}\right\}$$
$$y=\frac{1}{2}(k\pi-x)$$
$$y\in\frac{1}{2}\left\{k\pi\mp\arccos\frac{(-1)^{k+1}}{2}-2k\pi\middle|k\in\mathbb{Z}\right\}$$
$$y\in\left\{\frac{1}{2}\left(\mp\arccos\frac{(-1)^{k+1}}{2}-k\pi\right)\middle|k\in\mathbb{Z}\right\}$$

□

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