SSMA-MATH CHALLENGES-(III)

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5579. Prove: If $a, b \in \mathbb{R}, a \leq b$, then

$$\log 5 \cdot \int_a^b 5^{x^2} dx + \log 5 \cdot \int_a^b 5x^4 dx \geq 5^b - 5^a$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Albert Stadler, Herrliberg, Switzerland.
It is enough to prove that

$$(*) \qquad 5^{x^2} + 5^{x^4} \geq 5^x$$

for all real values of x because then

$$\log 5 \int_a^b 5x^2 dx + \log 5 \int_a^b 5x^4 dx \geq \log 5 \int_a^b 5^x dx = 5^b - 5^a.$$

(*) holds true for $x \leq 0$, because $5^{x^2} + 5^{x^4} \geq 2 > 1 \geq 5^x$, if $x \leq 0$.
(*) holds true for $x \geq 1$, because $5^{x^2} + 5^{x^4} > 5^{x^2} \geq 5^x$, if $x \geq 1$.
Let $0 < x < 1$. Then, by the AM-GM inequality,

$$5^{x^2} + 5^{x^4} = \sum_{k=0}^{\infty} \frac{1}{k!}(x^{2k} + x^{4k}) \geq 2 \sum_{k=0}^{\infty} \frac{1}{k!} x^{3k} = 2 \cdot 5^{x^3}$$

and it remains to prove that

$$f(x) := \log 2 + x^3 \log 5 - x \log 5 \geq 0$$

for $0 < x < 1$, for then $2 \cdot 5^{x^3} \geq 5^x$.$f(x)$ assumes a local minimum at $x = \frac{1}{\sqrt{3}}$ and

$$f\left(\frac{1}{\sqrt{3}}\right) = \log(2) - \frac{2}{3\sqrt{3}} \log 5 > 0,$$

which concludes the proof. □

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA.

Because

$$\int_a^b 5^x dx = \frac{5^b - 5^a}{\log 5}$$

the desired inequality is equivalent to

$$\int_a^b (5^{x^2} + 5^{x^4} - 5^x) dx \geq 0.$$

If $a = b$, then

$$\int_a^b (5^{x^2} + 5^{x^4} - 5^x) dx = 0.$$

Next, suppose $a < b$, and consider the function

$$f(x) = \frac{1}{2}x^4 + \frac{1}{2}x^2 - x + \log_5 2.$$

Then $f'(x) = 2x^3 + x - 1$. An examination of the graphs of $y = 2x^3$ and $y = 1 - x$ reveals there exists a unique real number, say c for which $2c^3 = 1 - c$; that is, there exists a unique real number c for which $f'(c) = 0$. Moreover, for $x < c, f'(x) < 0$, and for $x > c, f'(x) > 0$, so f achieves an absolute minimum value at $x = c$. Now,

$$\begin{aligned} f(c) = \frac{1}{2}c^4 + \frac{1}{2}c^2 - c + \log_5 2 &= \frac{1}{4}(2c^4 + c^2 - c) + \frac{1}{4}c^2 - \frac{3}{4}c + \log_5 2 \\ &= \frac{1}{4}c(2c^3 + c - 1) + \frac{1}{4}\left(c - \frac{3}{2}\right)^2 - \frac{9}{16} + \log_5 2 \\ &= \frac{1}{4}\left(c - \frac{3}{2}\right)^2 - \frac{9}{16} + \log_5 2. \end{aligned}$$

With

$$f'\left(\frac{1}{2}\right) = -\frac{1}{4} < 0 \text{ and } f'\left(\frac{3}{5}\right) = \frac{4}{125} > 0,$$

It follows that $c < \frac{3}{5}$ and

$$f(c) > \frac{1}{4}\left(\frac{3}{5} - \frac{3}{2}\right)^2 - \frac{9}{16} + \log_5 2 = -\frac{9}{25} + \log_5 2 > 0.$$

Thus,

$$\frac{1}{2}x^4 + \frac{1}{2}x^2 - x + \log_5 2 > 0 \text{ or } x < \frac{1}{2}x^4 + \frac{1}{2}x^2 + \log_5 2$$

for all x. Because 5^x is an increasing function, it then follows that

$$5^x < 5^{\frac{1}{2}x^4 + \frac{1}{2}x^2 + \log_5 2} = 2 \cdot 5^{\frac{1}{2}x^4 + \frac{1}{2}x^2} = 2\sqrt{5^{x^4+x^2}} \le 5^{x^4} + 5^{x^2},$$

where the final inequality arises from the arithmetic mean – geometric mean inequality.

Finally,

$$5^{x^2} + 5^{x^4} - 5^x > 0$$

for all x, so

$$\int_a^b (5^{x^2} + 5^{x^4} - 5^x)dx > 0$$

whenever $a < b$. In summary,

$$\log 5 \cdot \int_a^b 5^{x^2} dx + \log 5 \cdot \int_a^b 5^{x^4} dx \ge 5^b - 5^a,$$

with equality holding if and only if $a = b$. □

Solution 3 by Seán M. Stewart, Bomaderry, NSW, Australia.

Consider the function $g(x) = 5^{x^2-x} + 5^{x^4-x} - 1$. Differentiating with respect to x we have

$$g'(x) = \log 5 \cdot 5^{-x}(5^{x^2}(2x - 1) + 5^{x^4}(4x^3 - 1)).$$

Stationary points occur when $g'(x) = 0$. Since $5^{-x} > 0$ for all x we have

$$5^{x^2}(2x - 1) + 5^{x^4}(4x^3 - 1) = 0.$$

Solving this equation numerically, we find a single stationary point occurs when $x = x^* = 0.578632089...$ At this stationary point we see that $g(x^*) = 0.147392262... > 0$. Differentiating again we find

$$g''(x) = 2\log 5 \cdot 5^{x^2-x} + 12\log 5 \cdot 5^{x^4-x}x^2 + \log^2 5 \cdot 5^{x^2-x}(2x-1)^2 + \log^2 5 \cdot 5^{x^4-x}(4x^3-1)^2,$$

and is clearly positive for all $x \in \mathbb{R}$. Since $g''(x) > 0, g$ is concave up with $x = x^*$ being a global minimum point. Since $g(x^*) > 0$, then $g(x) > 0$ for all $x \in \mathbb{R}$, or

$$5^{x^2} + 5^{x^4} > 5^x, \tag{1}$$

for all $x \in \mathbb{R}$ since $5^{-x} > 0$.
Now

$$\begin{aligned} \log 5 \cdot \int_a^b 5^{x^2} dx + \log 5 \cdot \int_a^b 5^{x^4} dx &= \log 5 \cdot \int_a^b (5^{x^2} + 5^{x^4}) dx \\ &> \int_a^b \log 5 \cdot 5^x dx \\ &= 5^x \Big|_a^b = 5^b - 5^a, \end{aligned}$$

where we have made use of the inequality given in (1). Noting that equality can only occur when $a = b$, we can write

$$\log 5 \cdot \int_a^b 5^{x^2} dx + \log 5 \cdot \int_a^b 5^{x^4} dx \geq 5^b - 5^a,$$

for $a, b \in \mathbb{R}, a \leq b$, as required to prove. □

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

Because $\int_a^b 5^x dx = \frac{5^b - 5^a}{\log 5}$, the given inequality is equivalent to

$$\log 5 \cdot \int_a^b 5^{x^2} dx + \log 5 \cdot \int_a^b 5^{x^4} dx \geq \log 5 \cdot \int_a^b 5^x dx$$

which is equivalent to $\int_a^b (5^{x^4} + 5^{x^2} - 5^x) dx \geq 0$.
This is true because
Claim: $5^{x^4} + 5^{x^2} - 5^x \geq 0$ for all real x.
In this modern electronic age, we could graph $f(x) = 5^{x^4} + 5^{x^2} - 5^x$ and readily accept the truth of this claim. In fact, the TI-83 graphing calculator shows that $f(x)$ achieves an absolute minimum of 0.360 at $x = .5307$.
We also present an analytic proof of the claim.
If $x \geq 1$, then we can consider a piece of $f(x) : f(x) \geq 5^{x^2} - 5^x \geq 0$, so $f(x) \geq 0$.
Consider the derivative:

$$f'(x) = 5^{x^4}\log 5 \cdot 4x^3 + 5^{x^2}\log 5 \cdot 2x - 5^x \log 5 = \log 5 \cdot 5^x\{4x^3 \cdot 5^{x^4-x} + 2x \cdot 5^{x^2-x} - 1\}.$$

If $x \leq 0$, then each term inside the brackets is negative, so $f'(x) \leq 0$.
Thus $f(x)$ is decreasing to $f(0) = 1$, so $f(x) > 0$.
Now we complete our argument by showing $5^{x^4} + 5^{x^2} \geq 5^x$ on $[0, 1]$ by looking carefully at the behavior on small subintervals.
For convenience, let $g(x) = 5^{x^4} + 5^{x^2}$ and $h(x) = 5^x$.
We have $g'(x) = 5^{x^4}\log 5 \cdot 4x^3 + 5^{x^2}\log 5 \cdot 2x = 2x\log 5\{2x^2 \cdot 5^{x^4} + 2x \cdot 5^{x^2}\}$
Because $g'(x) < 0$ for $x < 0$ and $g'(x) > 0$ for $x > 0$, we see that $g(0) = 2$ is an absolute minimum.

We know that $g(x)$ and $h(x)$ are both increasing on $[0,1]$ and $h(0) = 1 < g(0) = 2$. We show that $h(x)$ lies below $g(x)$ on the entire interval by showing it true on six its left- hand height to (almost) the left-hand height of $g(x)$, thus lying below $g(x)$ throughout the subinterval. One can imagine stair-steps, with the graph of $g(x)$ lying (on or) above each step and the graph of $h(x)$ lying below the step.
On the subinterval $[0, .43], h(x) \leq h(.43) = 5.43 = 1.998 < 2 = g(0) \leq g(x)$.
On the subinterval $[.43, .54], h(x) \leq h(.54) = 5.54 = 2.385 < 2.403 = g(.43) \leq g(x)$.
On the subinterval $[.54, .627], h(x) \leq h(.627) = 5.627 = 2.743 < 2.7455 =$
$= g(.54) \leq g(x)$.
On the subinterval $[.627, .715], h(x) \leq h(.715) = 5.715 = 3.1606 < 3.165 =$
$= g(.627) \leq g(x)$
On the subinterval $[.715, .826], h(x) \leq h(.826) = 5.826 = 3.779 < 3.8 =$
$= g(.715) \leq g(x)$
On the subinterval $[.826, 1], h(x)(1) = 51 = 5 < 5.114 = g(.826) \leq g(x)$. □

5591. Solve for real numbers:

$$3^{\cos x+\cos y+\cos z} = 3^{\cos^2 x+\cos x} + 3^{\cos^2 y+\cos y} + 3^{\cos^2 z+\cos z}.$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA.

The given equation is equivalent to

$$3^{\cos^2 x-\cos y-\cos z} + 3^{\cos^2 y-\cos x-\cos z} + 3^{\cos^2 z-\cos x-\cos y} = 1.$$

By Jensen's inequality,

$$\begin{aligned} &3^{\cos^2 x-\cos y-\cos z} + 3^{\cos^2 y-\cos x-\cos z} + 3^{\cos^2 z-\cos x-\cos y} \\ &\geq 3 \cdot 3^{\frac{1}{3}(\cos^2 x-2\cos x+\cos^2 y-2\cos y+\cos^2 z-2\cos z)} \\ &= 3^{\frac{1}{3}(\cos x-1)^2+\frac{1}{3}(\cos y-1)^2+\frac{1}{3}(\cos z-1)^2} \\ &\geq 1. \end{aligned}$$

At the first inequality, equality holds if and only if

$$\cos^2 x + \cos x = \cos^2 y + \cos y = \cos^2 z + \cos z$$

at the second inequality, equality holds if and only if $\cos x = \cos y = \cos z = 1$. Thus,

$$3^{\cos^2 x-\cos y-\cos z} + 3^{\cos^2 y-\cos x-\cos z} + 3^{\cos^2 z-\cos x-\cos y} = 1$$

if and only if $\cos x = \cos y = \cos z = 1$. The solutions to

$$3^{\cos x+\cos y+\cos z} = 3^{\cos^2 x+\cos x} + 3^{\cos^2 y+\cos y} + 3^{\cos^2 z+\cos z}$$

are therefore the ordered triples $(x, y, z) = (2\pi j, 2\pi k, 2\pi l)$ for any integers j, k and l. □

Solution 2 by Albert Stadler, Herrliberg, Switzerland.
By the AM-GM inequality

$$3^{\cos^2 x+\cos x}+3^{\cos^2 y+\cos y}+3^{\cos^2 z+\cos z}\geq 3\cdot 3^{\frac{\cos^2 x+\cos x+\cos^2 y+\cos y+\cos^2 z+\cos z}{3}}.$$

Hence,

$$3^{\cos x+\cos y+\cos z-1-\frac{\cos^2 x+\cos x+\cos^2 y+\cos y+\cos^2 z+\cos z}{3}}=$$
$$=3^{-\frac{(\cos x-1)^2+(\cos y-1)^2+(\cos z-1)^2}{3}}\geq 1$$

which implies $(\cos x-1)^2+(\cos y-1)^2+(\cos z-1)^2\leq 0, \cos x=\cos y=\cos z=1$, and finally, $x\equiv y\equiv z\equiv 0(\mod 2\pi)$. □

Solution 3 by Pratik Donga, India.
Let $\cos x=a, \cos y=b$ and $\cos z=c$. This implies:

$$3^{a+b+c}=3^{a^2+a}+3^{b^2+b}+3^{c^2+c}$$
$$\Rightarrow 3^a\cdot 3^b\cdot 3^c=3^{a^2}\cdot 3^{b^2}+3^b+3^{c^2}\cdot 3$$
$$(1)\qquad \Rightarrow 1=3^{a^2-b-c}+3^{b^2-a-c}+3^{c^2-a-b}$$

In Eq (1) if $a^2-b-c=b^2-a-c=c^2-a-b=-1$, then

$$1=3^{-1}+3^{-1}+3^{-1}=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{1}{3}(3)=1.$$

This implies, to make Eq (1) true, that the LHS=RHS, that is

$$(A)\qquad a^2-b-c=b^2-a-c=c^2-a-b=-1,$$

Now,

$$(2)\qquad a^2-b-c=b^2-a-c\Rightarrow a^2+a=b^2+b\Rightarrow a^2-b^2=b-a$$

Similarly,

$$(3)\qquad b^2-a-c=c^2-a-b\Rightarrow b^2+b=c^2+b=c\Rightarrow b^2-c^2=c-b$$

In Eq (2) $a^2-b^2=b-a\Rightarrow a+b=-1\Rightarrow c^2=-2$, and
in Eq (3) $b^2-c^2=c-b\Rightarrow b+c=-1\Rightarrow a^2=-2$, which are not possible for any real numbers.
Therefore, we cannot divide Eq (2) by $a-b$ nor Eq (3) by $b-c$.
So $a-b=b-c=0\Rightarrow a=b=c$ and also

$$(4)\qquad a^2+a=b^2+b=c^2+c\Rightarrow a=b=c$$

Put Eq (4) into Eq (A). So

$$a^2=b+c-1=a+a-1\Rightarrow (a-1)^2=0\Rightarrow a=1.$$

But since $a=b=c$ we have $a=b=c=1$ so $\cos x=\cos y=\cos z=1$ and since

$$x=\cos^{-1}1, y=\cos^{-1}1, z=\cos^{-1}1\rightarrow x=y=z=\{2k\pi|k\in\mathbb{Z}\}$$

$x=2k\pi, y=2l\pi$ and $z=2m\pi$, with k, l and m being integers. □

5597. If $x, y, z>0; xyz=1$ then:

$$\left(x+y-\frac{1}{\sqrt{z}}\right)^2+\left(y+z-\frac{1}{\sqrt{x}}\right)^2+\left(z+x-\frac{1}{\sqrt{y}}\right)^2\geq 3$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY.
First we note that Maclaurin's inequality gives us $\sqrt{\frac{xy+yz+zx}{3}} \geq \sqrt[3]{xyz} = 1$, or $xy + yz + zx \geq 3$. Then the AGM inequality yields

$$\sum_{cyclic} \left(x + y - \frac{1}{\sqrt{z}}\right)^2 = \sum_{cyclic} (x + y - \sqrt{xy})^2$$
$$\geq \sum_{cyclic} (\sqrt{xy})^2 = xy + yz + zx \geq 3.$$

□

Solution 2 by Henry Ricardo, Westchester Area Math Circle, NY.
First we note that the AGM inequality gives us $\frac{x+y+z}{3} \geq \sqrt[3]{xyz} = 1$, or $x+y+z \geq 3$. Then the AGM inequality and Radon's inequality yield

$$\sum_{cyclic} \left(x + y - \frac{1}{\sqrt{z}}\right)^2 = \sum_{cyclic} (x + y - \sqrt{xy})^2$$
$$\geq \sum_{cyclic} \left(\frac{x+y}{2}\right)^2 \geq \frac{[2(x+y+z)]^2}{12} \geq \frac{6^2}{12} = 3.$$

Equality holds if and only if $x = y = z = 1$. □

Solution 3 by Henry Ricardo, Westchester Area Math Circle, NY.
Using the AGM inequality twice, we see that

$$\sum_{cyclic} \left(x + y - \frac{1}{\sqrt{z}}\right)^2 = \sum_{cyclic} (x + y - \sqrt{xy})^2$$
$$\geq \sum_{cyclic} (\sqrt{xy})^2$$
$$\geq 3\sqrt[3]{(xyz)^2} = 3.$$

□

Solution 4 by Kee-Wai Lau, Hong, Kong, China.
By the AM-GM inequality, we have

$$2(x + y + z) - (\sqrt{xy} + \sqrt{yz} + \sqrt{zx})$$
$$= (x + y + z) + \frac{(\sqrt{x} - \sqrt{y})^2 + (\sqrt{y} - \sqrt{z})^2 + (\sqrt{z} - \sqrt{x})^2}{}$$
$$\geq x + y + z$$
$$\geq 3\sqrt[3]{xyz}$$
$$= 3.$$

Hence by the Cauchy-Schwarz inequality, we have

$$\left(x + y - \frac{1}{\sqrt{z}}\right)^2 + \left(y + z - \frac{1}{\sqrt{x}}\right)^2 + \left(z + x - \frac{1}{\sqrt{y}}\right)^2$$
$$\geq \frac{1}{3}\left(\left(x + y - \frac{1}{\sqrt{z}}\right) + \left(y + z - \frac{1}{\sqrt{x}}\right) + \left(z + x - \frac{1}{\sqrt{y}}\right)\right)^2$$

$$= \frac{1}{3}\Big((x+y-\sqrt{xy})+(y+z-\sqrt{yz})+(z+x-\sqrt{zx})\Big)^2$$
$$= \frac{1}{3}\Big(2(x+y+z)-(\sqrt{xy}+\sqrt{yz}+\sqrt{zx})\Big)^2$$
$$\geq 3, \text{ as requiered}$$

□

5620. Prove: If $a, b \in [0,1]; a \leq b$, then:

$$4\sqrt{ab} \leq a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \leq 2(a+b).$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Moti Levy, Rehovot, Israel.
Let

$$\alpha := \sqrt{ab} \leq 1, \quad \beta := \frac{a+b}{2} \leq 1, \quad r := \frac{b}{a},$$

Then the original inequality can be reformulated as

$$\sqrt{r} \leq \frac{r^\alpha + r^\beta + r^{1-\alpha} + r^{1-\beta}}{4} \leq \frac{1}{2} + r$$

Since $f(x) := r^x$ is convex function, then

$$\frac{r^\alpha + r^\beta + r^{1-\alpha} + r^{1-\beta}}{4} \geq r^{\frac{\alpha+\beta+(1-\alpha)+(1-\beta)}{4}} = \sqrt{r}.$$

The Bernoulli's inequality is

$$(1+x)^\alpha \leq 1 + \alpha x, \quad 0 \leq \alpha \leq 1, \quad x \geq -1.$$

Using the Bernoulli's inequality we get

$$r^\alpha \leq 1 + \alpha(r-1),$$
$$r^{1-\alpha} \leq 1 + (1-\alpha)(r-1)$$

hence

$$r^\alpha + r^{1-\alpha} \leq 1 + r.$$

It follows that

$$\frac{(r^\alpha + r^{1-\alpha}) + (r^\alpha + r^{1-\beta})}{4} \leq \frac{1}{2} + r.$$

Remark. The constraint $a \leq b$ is redundant. □

Solution 2 by Michel Bataille, Rouen, France.
We suppose $a, b \in (0,1]$ and do not use the hypothesis $a \leq b$.
Let

$$M = a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right).$$

Since $\sqrt{ab}$ and $\frac{a+b}{2}$ are in $(0,1]$, the functions $x \to x^{\sqrt{ab}}$ and $x \to x^{\frac{a+b}{2}}$ are concave on $(0,\infty)$.
It follows that

$$a\left(\frac{b}{a}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} \leq (a+b)\left(\frac{a}{a+b}\cdot\frac{b}{a} + \frac{b}{a+b}\cdot\frac{a}{b}\right)^{\sqrt{ab}} = a+b$$

and

$$a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}} \le (a+b)\left(\frac{a}{a+b}\cdot\frac{b}{a} + \frac{b}{a+b}\cdot\frac{a}{b}\right)^{\frac{a+b}{2}} = a+b.$$

By addition, $M \le 2(a+b)$.
If m is a positive real number, the function $x \to m^x$ is convex on $\mathbb{R}$. Taking successively $m = \frac{b}{a}$ and $m = \frac{a}{b}$ and setting $k = \frac{1}{2}(\sqrt{ab} + \frac{a+b}{2})$, it follows that

$$\left(\frac{b}{a}\right)^{\sqrt{ab}} + \left(\frac{b}{a}\right)^{\frac{a+b}{2}} \ge 2\left(\frac{b}{a}\right)^k$$

and

$$\left(\frac{a}{b}\right)^{\sqrt{ab}} + \left(\frac{a}{b}\right)^{\frac{a+b}{2}} \ge 2\left(\frac{a}{b}\right)^k.$$

Using $x + y \ge 2\sqrt{xy}$ for positive x, y, we deduce that

$$M \ge 2\left(a\left(\frac{b}{a}\right)^k + b\left(\frac{a}{b}\right)^k\right) \ge 2\cdot 2\left(a\left(\frac{b}{a}\right)^k \cdot b\left(\frac{a}{b}\right)^k\right)^{\frac{1}{2}}$$

and $M \ge 4\sqrt{ab}$ follows. □

Solution 3 by Arkady Alt, San Jose, California.
Applying inequality $x + y \ge 2\sqrt{xy}, x, y > 0$ to $(x, y) = \left(a(\frac{b}{a})^{\sqrt{ab}}, b(\frac{a}{b})^{\sqrt{ab}}\right)$ and to $(x, y) = \left(a(\frac{b}{a})^{\frac{a+b}{2}}, b(\frac{a}{b})^{\frac{a+b}{2}}\right)$ we obtain

$$a\left(\frac{b}{a}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} \ge 2\sqrt{a\left(\frac{b}{a}\right)^{\sqrt{ab}} \cdot b\left(\frac{a}{b}\right)^{\sqrt{ab}}} = 2\sqrt{ab} \text{ and}$$

$$a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}} \ge 2\sqrt{a(ba)^{\frac{a+b}{2}} \cdot b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}} = 2\sqrt{ab}.$$

Thus,

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \ge 4\sqrt{ab}.$$

For function $f(t) = t^p$ which for $p \in [0, 1]$ is concave down on $(0, \infty)$, holds inequality $\frac{ax^p + by^p}{a+b} \le (\frac{ax+by}{a+b})^p$ for any $x, y > 0$.
Since $\sqrt{ab}, \frac{a+b}{2} \in [0, 1]$ then applying this inequality to $(x, y, p) = (\frac{b}{a}, \frac{a}{b}, \sqrt{ab})$ and $(x, y, p) = (\frac{b}{a}, \frac{a}{b}, \frac{a+b}{2})$ we obtain

$$\frac{a(\frac{b}{a})^{\sqrt{ab}} + b(\frac{a}{b})^{\sqrt{ab}}}{a+b} \le \left(\frac{a\cdot\frac{b}{a} + b\cdot\frac{a}{b}}{a+b}\right)^{\sqrt{ab}} = 1 \Leftrightarrow$$

$$a\left(\frac{b}{a}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} \le a + b \text{ and}$$

$$\frac{a(\frac{b}{a})^{\frac{a+b}{2}} + b(\frac{a}{b})^{\frac{a+b}{2}}}{a+b} \le \left(\frac{a\cdot\frac{b}{a} + b\cdot\frac{a}{b}}{a+b}\right)^{\frac{a+b}{2}} = 1 \Leftrightarrow$$

$$a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}} \le a + b.$$

Therefore,

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\right)\leq 2(a+b).$$

□

Solution 4 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.
In the following proof, we use Heinz's inequality:

$$\sqrt{ab}\leq\frac{a^{1-\alpha}b^{\alpha}+a^{\alpha}b^{1-\alpha}}{2}\leq\frac{a+b}{2}\text{ for }a,b>0,\alpha\in[0,1],$$

first with $\alpha=\sqrt{ab}$ and then with $\alpha=\frac{a+b}{2}$.
First we rearrange the central term in the proposed inequality:

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)$$

$$=a\left(\frac{b}{a}\right)^{\sqrt{ab}}+a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}+b\left(\frac{a}{b}\right)^{\sqrt{ab}}+b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}$$

$$=\left(a\cdot\frac{b^{\sqrt{ab}}}{a^{\sqrt{ab}}}+b\cdot\frac{a^{\sqrt{ab}}}{b^{\sqrt{ab}}}\right)+\left(a\cdot\frac{b^{\frac{a+b}{2}}}{a^{\frac{a+b}{2}}}+b\cdot\frac{a^{\frac{a+b}{2}}}{b^{\frac{a+b}{2}}}\right)$$

$$=2\left(\frac{a^{1-\sqrt{ab}}b^{\sqrt{ab}}+a^{\sqrt{ab}}b^{1-\sqrt{ab}}}{2}\right)+2\left(\frac{a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}}+a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}}{2}\right)$$

Now the Heinz inequality yields

$$4\sqrt{ab}\leq 2\left(\frac{a^{1-\sqrt{ab}}b^{\sqrt{ab}}+a^{\sqrt{ab}}b^{1-\sqrt{ab}}}{2}\right)+2\left(\frac{a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}}+a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}}{2}\right)$$

$$\leq 2\left(\frac{a+b}{2}\right)+2\left(\frac{a+b}{2}\right)=2(a+b).$$

□

Solution 5 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC.

First we prove that

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\geq 4\sqrt{ab}\tag{1}$$

To prove this, we will use the well-known inequality that for all $p>0$ and any real number r

$$p^r+\frac{1}{p^r}\geq 2.\tag{2}$$

For all $a>0$ and $b>0$, using (2) in the above step, we can write

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)$$

$$=\sqrt{ab}\left[\sqrt{\frac{a}{b}}\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+\sqrt{\frac{b}{a}}\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\right]$$

$$= \sqrt{ab}\left[\left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{b}{a}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)+\left(\left(\frac{a}{b}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)\right]$$

$$\sqrt{ab}\left[\left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)+\left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)\right]$$

$$\geq \sqrt{ab}(2+2) = 4\sqrt{ab}.$$

Now, we prove that

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\leq 2(a+b). \tag{3}$$

To prove (3), we notice that, for $a \in (0,1]$ and $b \in (0,1]$.
(4)

With $a \leq b$, we have $\begin{cases} b^{\sqrt{ab}} - a^{\sqrt{ab}} \geq 0 \\ b^{1-\sqrt{ab}} - a^{1-\sqrt{ab}} \geq 0, \end{cases}$ and $\begin{cases} b^{\frac{a+b}{2}} - a^{\frac{a+b}{2}} \geq 0 \\ b^{1-\frac{a+b}{2}} - a^{1-\frac{a+b}{2}} \geq 0 \end{cases}$

Also,

$$-2a-2b = -a^{\sqrt{ab}}a^{1-\sqrt{ab}} - a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}} - b^{\sqrt{ab}}b^{1-\sqrt{ab}} - b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}} \tag{5}$$

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)-2a-2b$$

$$= a^{1-\sqrt{ab}}b^{\sqrt{ab}} + a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}} + a^{\sqrt{ab}}b^{1-\sqrt{ab}} + a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}$$

$$-a^{\sqrt{ab}}a^{1-\sqrt{ab}} - a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}} - b^{\sqrt{ab}}b^{1-\sqrt{ab}} - b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}$$

$$= (a^{1-\sqrt{ab}}b^{\sqrt{ab}} - a^{\sqrt{ab}}a^{1-\sqrt{ab}}) + (a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}} - a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}})$$

$$+(a^{\sqrt{ab}}b^{1-\sqrt{ab}} - b^{\sqrt{ab}}b^{1-\sqrt{ab}}) + (a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}} - b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}})$$

$$= a^{1-\sqrt{ab}}(b^{\sqrt{ab}} - a^{\sqrt{ab}}) + a^{1-\frac{a+b}{2}}(b^{\frac{a+b}{2}} - a^{\frac{a+b}{2}})$$

$$-b^{1-\sqrt{ab}}(b^{\sqrt{ab}} - a^{\sqrt{ab}}) - b^{1-\frac{a+b}{2}}(b^{\frac{a+b}{2}} - a^{\frac{a+b}{2}})$$

$$= -\left[(b^{\sqrt{ab}} - a^{\sqrt{ab}})(b^{1-\sqrt{ab}} - a^{1-\sqrt{ab}}) + (b^{\frac{a+b}{2}} - a^{\frac{a+b}{2}})(b^{1-\frac{a+b}{2}} - a^{1-\frac{a+b}{2}})\right] \leq 0.$$

This completes the proof of (3).
Now, combining the inequalities from (1) and (3), we conclude that

$$4\sqrt{ab} \leq a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\leq 2(a+b)$$

□

5628. Prove that if $x, y, z, u, v, w \in (0, \infty)$, then:

$$\frac{x^2}{u}e^{\frac{u}{x}} + \frac{y^2}{v}e^{\frac{y^2}{y}} + \frac{z^2}{w}e^{\frac{w}{z}} \geq \frac{(x+y+z)^2}{u+v+w}e^{\frac{u+v+w}{x+y+z}}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
The solution follows by Jensen's inequality, considering function $f(t) = \frac{1}{t}e^t$, which is convex because $f''(t) = \frac{e^t(t^2-2t+2)}{t^3} = \frac{e^t((t-1)^2+1)}{t^3} > 0$ for $t > 0$. By doing $a = \frac{u}{x}, b = \frac{v}{y}, c = \frac{w}{z}$, the given inequality may be written as

$$xf(a) + tf(b) + zf(c) \geq (x+y+z)f\Big(\frac{xa+yb+zc}{x+y+z}\Big)$$

Because $xa + yb + zc = u + v + w$. □

Solution 2 by Henry Ricardo, Westchester Area Math Circle, NY.
The function $f(x) = \frac{1}{x}e^x$ is convex on $(0,\infty) : f''(x) = e^x\frac{(x-1)^2+1}{x^3} > 0$.
Therefore we can apply the weighted for of Jensen's inequality

$$\frac{w_1f(a) + w_2f(b) + w_3f(c)}{w_1 + w_2 + w_3} \geq f\Big(\frac{w_1a + w_2b + w_3c}{w_1 + w_2 + w_3}\Big)$$

with weghts x, y, z and variables $\frac{u}{x}, \frac{v}{y}, \frac{w}{z}$:

$$x\Big(\frac{x}{u}e^{\frac{u}{x}}\Big) + y\cdot\Big(\frac{y}{v}e^{\frac{v}{y}}\Big) + z\cdot\Big(\frac{z}{w}e^{\frac{w}{z}}\Big) \geq (x+y+z)\frac{x+y+z}{y+v+w}e^{\frac{u+v+w}{x+y+z}}$$

$$= \frac{(x+y+z)^2}{u+v+w}e^{\frac{u+v+x}{x+y+z}}.$$

□

Solution 3 by Kee-Wai Lau, Hong Kong, China.
Since $e^t = +\sum_{k=12}^{\infty}\frac{t^k}{k!}$ we need only to prove that

$$(1) \qquad \frac{x^2}{u} + \frac{y^2}{v} + \frac{z^2}{w} \geq \frac{(x+y+z)^2}{u+v+w}$$

and

$$(2) \qquad x\Big(\frac{u}{x}\Big)^k + y\Big(\frac{v}{y}\Big)^k + z\Big(\frac{w}{z}\Big)^k \geq (x+y+z)\Big(\frac{u+v+w}{x+y+z}\Big)^k$$

for any nonnegative integer k.
By the inequality of Cauchy - Schwarz, we have

$$(x+y+z)^2 = \Big(\frac{x}{\sqrt{u}}\sqrt{u} + \frac{u}{\sqrt{v}}\sqrt{v} + \frac{x}{\sqrt{w}}\sqrt{w}\Big)^2 \leq \Big(\frac{x^2}{u} + \frac{y^2}{v} + \frac{z^2}{w}\Big)(u+v+w)$$

and (1) holds. For $k = 0, 1, (2)$ is trivial. For $k \geq 2$ and $t > 0$, let $f(t) = t^k$, which is convex.
Hence by Jensen's inequality, we have

$$\frac{x}{x+y+z}\Big(\frac{u}{x}\Big)^k + \frac{k}{x+y+z}\Big(\frac{v}{y}\Big)^k + \frac{z}{x+y+z}\Big(\frac{w}{z}\Big)^k$$

□

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