On a sequence in OEIS-A343010

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Today , I'll introduce my sequence $A343010$ in OEIS which is elementary and comes from very basic idea/question which we all know.

BACKGROUND

We all must have come across this very question :
"*Find three distinct Natural numbers so that their product is equal to their sum.*
The obvious solution to it is the numbers $1, 2, 3$ as

$$1 + 2 + 3 = 6 = 1 \cdot 2 \cdot 3$$

Okay , So after reading this far one might have been started asking questions like can we find four such numbers or in general such 'n' numbers.??

So let's try to Generalize this result up to 'n' variables.

GENERALIZATION

We need to find n numbers such that their sum is equal to their product. So we start with rewriting n as

$$n = \frac{n}{2} + \frac{n}{2} = \frac{n}{2} + \frac{(n-2)}{2} + 1$$

which implies

$$2n = n + 2 + (n-2) = n + 2 + \underbrace{1 + 1 + .. + .. + 1}_{(n-2)\text{times}}$$

So these are the n numbers which sum up to $2n$.
But also note that their product also equals $2n$ that is to say that

$$\underbrace{1 + 1 + .. + .. + 1}_{(n-2)\text{times}} + 2 + n = \underbrace{1 \cdot 1 \cdot .. \cdot .. \cdot 1}_{(n-2)\text{times}} \cdot 2 \cdot n = 2n$$

Hence we've found those n numbers but wait they aren't distinct because we've used 1 exactly $n-2$ times.
But let's (for now) leave this discussion by simply removing the distinct word

from the question that is

$\mathcal{Q}$: *Find n natural numbers such that their product is equal to their sum.*
$\mathbb{SOL}$: *$2, n$ and $(n-2)$ 1's are those such numbers which on addition as well as multiplication yields the same number and that same number is $2n$.*

GENERALIZATION IN A DIFFERENT SENSE

Now as we're done with the normal generalization let's try to see this from a different view.

For this we need to restrict ourselves up to 3 variables only.
And also change our question from the equality of product and summation from equality of product and some kconstant 'k' times the sum.
That is to say what are those three numbers a, b, c so that they satisfy the following relation :

$$a \cdot b \cdot c = k \cdot (a + b + c)$$

As a nice exercise ,one may try to find such natural numbers a, b, c when $k = 2$ that is $a \cdot b \cdot c = 2 \cdot (a + b + c)$

(Hint : try using inequalities and confining oneself up to certain integer bounds only)
Now , once done with 2 try plugging in 3 ,4 ,5...and so on.

One will ll observe that it's a bit difficult to find out a general solution but let's concentrate ourselves up to some general numbers that will always be a part of solution set.

There are multiple number of ways to do it but let's get back to our first question the solution was $1, 2, 3$. So another question :

$\mathcal{Q}$: *What do these numbers share in common except for they are consecutive natural numbers ???*
$\mathcal{SOL}$: *Note that these also are the Fibonacci numbers and to be more precise are consecutive Fibonacci numbers.*

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So , my sequence A343010 is the sequence of numbers k for which there exist three consecutive Fibonacci numbers a, b and c such that

$$k \cdot (a + b + c) = a \cdot b \cdot c..............(i)$$

FORMULA AND PROOF

Rewriting equation (i)

$$f_{n-1} \cdot f_n \cdot f_{n+1} = k \cdot (f_{n-1} + f_n + f_{n+1})$$

Using recurrence relation of Fibonacci numbers we get

$$f_{n-1} \cdot f_n \cdot f_{n+1} = 2k \cdot f_{n+1}$$

Simplifying gives

$$k = \frac{f_{n-1} \cdot f_n}{2}$$

Now since we're dealing with integers so the right hand side must contain a factor of 2.
Which is only possible when either f_{n-1} or f_n is even.
But f_d is even only when $d = 3e$ that is d is a multiple of 3. Hence in our case either

$$(n-1) = 3m; \text{ or } , n = 3m$$

$$\rightarrow n = 3m + 1, 3m$$

Hence plugging in the values we get ,

$$k = \frac{f_{3m} \cdot f_{3m+1}}{2}, \frac{f_{3m-1} \cdot f_{3m}}{2}$$

$$k = \frac{f_{3m} \cdot f_{3m\pm1}}{2}$$

for $m \in \mathbb{W}$
And the following result is due to Stefano Spezia (here $a(n)$ is the nth term.)

$$\sum_{n=1}^{\infty} a(n)x^n = \frac{x^2(1 + 3x + 3x^2 + x^3)}{(1 - 17x^2 - 17x^4 + x^6)}$$

FIRST FEW VALUES :

0, 1, 3, 20, 52, 357, 935, 6408, 16776, 114985, 301035, 2063324, 5401852, 37024845, 96932303, 664383888, 1739379600, 11921885137, 31211900499, 213929548580, 560074829380, 3838809989301, 10050135028343, 68884650258840, 180342355680792, 1236084894669817

References

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