NEW REFINEMENT FOR RADON'S INEQUALITY

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Theorem. (Radon's Inequality)
If $x_k, y_k \in (0,\infty), \forall k = \overline{1,n}, n \geq 2$ and $t \geq 0$, then:

$$\text{(R)} \qquad \frac{x_1^{t+1}}{y_1^t} + \frac{x_2^{t+1}}{y_2^t} + \ldots + \frac{x_n^{t+1}}{y_n^t} \geq \frac{(x_1 + x_2 + \ldots + x_n)^{t+1}}{(y_1 + y_2 + \ldots + y_n)^t}$$

Equality holds if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \ldots = \frac{x_n}{y_n}$.
Theorem. (Bersgström's Inequality)
If $x_k, y_k \in (0,\infty), \forall k = \overline{1,n}, n \geq 2$, then:

$$\text{(B)} \qquad \sum_{k=1}^{n} \frac{x_k^2}{y_k} \geq \frac{(x_1 + x_2 + \ldots + x_n)^2}{y_1 + y_2 + \ldots + y_n}$$

Equality holds if and only if $\exists u \in \mathbb{R}_+^*$ such that $|x_k| = u \cdot y_k; \forall k = \overline{1,n}$.
We observe that inequality (B) cannot be a consequence of inequality (R) because inequaliyt (R) is not possible for $x_k \in \mathbb{R} - \mathbb{R}_+^*; \forall k = \overline{1,n}$.
Theorem.
If $u \geq 0, v > 0$ and $t \geq 0, x_k, y_k \in \mathbb{R}_+^*, \forall k = \overline{1,n}, x_{n+1} = x_1, y_{n+1} = y_1$, then:
(*)

$$\sum_{k=1}^{n} \frac{x_k^{t+1}}{y_k^t} \geq \frac{1}{2u+v}\left(u \cdot \sum_{k=1}^{n} \frac{(x_k + x_{k+1})^{t+1}}{(y_k + y_{k+1})^t} + v \cdot \sum_{k=1}^{n} \frac{x_k^{t+1}}{y_k^{t+1}}\right) \geq \frac{(x_1 + x_2 + \ldots + x_n)^{t+1}}{(y_1 + y_2 + \ldots + y_n)^t}$$

Proof.
We have:

$$(2u+v) \cdot \sum_{k=1}^{n} \frac{x_k^{t+1}}{y_k^t} = 2u \cdot \sum_{k=1}^{n} \frac{x_k^{t+1}}{y_k^t} + v \cdot \sum_{k=1}^{n} \frac{x_k^{t+1}}{y_k^t} =$$

$$= u \cdot \sum_{k=1}^{n} \left(\frac{x_k^{t+1}}{y_k^t} + \frac{x_{k+1}^{t+1}}{y_k^t}\right) + v \cdot \sum_{k=1}^{n} \frac{x_k^{t+1}}{y_k^t} \overset{(R)}{\geq} u \cdot \sum_{k=1}^{n} \frac{(x_k + x_{k+1})^{t+1}}{(y_k + y_{k+1})^t} + v \cdot \sum_{k=1}^{n} \frac{x_k^{t+1}}{y_k^t} \overset{(R)}{\geq}$$

$$\geq u \cdot \frac{(\sum_{k=1}^{n}(x_k + x_{k+1}))^{t+1}}{(\sum_{k=1}^{n}(y_k + y_{k+1}))^t} + v \cdot \frac{(\sum_{k=1}^{n} x_k)^{t+1}}{(\sum_{k=1}^{n} y_k)^t} \geq$$

$$\geq u \cdot \frac{(2\sum_{k=1}^{n} x_k)^{t+1}}{(2\sum_{k=1}^{n} y_k)^t} + v \cdot \frac{(\sum_{k=1}^{n} x_k)^{t+1}}{(\sum_{k=1}^{n} y_k)^t} =$$

$$= \frac{2^{t+1} \cdot u}{2^t} \cdot \frac{(\sum_{k=1}^{n} x_k)^{t+1}}{(\sum_{k=1}^{n} y_k)^t} + v \cdot \frac{(\sum_{k=1}^{n} x_k)^{t+1}}{(\sum_{k=1}^{n} y_k)^t} = (2u+v) \cdot \frac{(\sum_{k=1}^{n} x_k)^{t+1}}{(\sum_{k=1}^{n} y_k)^t} \Leftrightarrow$$

$$\sum_{k=1}^{n} \frac{x_k^{t+1}}{y_k^t} \geq \frac{1}{2u+v}\left(u \cdot \sum_{k=1}^{n} \frac{(x_k + x_{k+1})^{t+1}}{(y_k + y_{k+1})^t} + v \cdot \sum_{k=1}^{n} \frac{x_k^{t+1}}{y_k^{t+1}}\right) \geq \frac{(x_1 + x_2 + \ldots + x_n)^{t+1}}{(y_1 + y_2 + \ldots y_n)^t}$$

If $u = 0$, inequality (*) becomes as:

$$\text{(R)} \qquad \frac{x_1^{t+1}}{y_1^t} + \frac{x_2^{t+1}}{y_2^t} + \ldots + \frac{x_n^{t+1}}{y_n^t} \geq \frac{(x_1 + x_2 + \ldots + x_n)^{t+1}}{(y_1 + y_2 + \ldots + y_n)^t}$$

If $u = v$, inequality (*) becomes as:

$$\sum_{k=1}^{n} \frac{x_k^{t+1}}{y_k^t} \geq \frac{1}{3}\left(\sum_{k=1}^{n} \frac{(x_k + x_{k+1})^{t+1}}{(y_k + y_{k+1})^t} + \sum_{k=1}^{n} \frac{x_k^{t+1}}{y_k^t}\right) \geq \frac{(x_1 + x_2 + \ldots + x_n)^{t+1}}{(y_1 + y_2 + \ldots + y_n)^t}$$

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References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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