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In any Δ ABC holds

$$94,5+\sum\frac{\sin^7 A}{\sin^7 B+\sin^7 C}\le 96\left(\frac{R}{2r}\right)^6$$

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$$:\left(\sum a^3\right)\left(\sum a^4\right)=\sum a^7+2s\left(\sum a^3b^3\right)-abc\left(\sum a^2b^2\right)$$

$$\Rightarrow -\sum a^7=-\left(\sum a^3\right)\left(2\sum a^2b^2-16s^2r^2\right)-abc\left(\sum a^2b^2\right)$$
$$+2s\left[\left(s^2+4Rr+r^2\right)^3-3.4Rrs.2s\left(s^2+2Rr+r^2\right)\right]$$
$$=32r^2s^3\left(s^2-6Rr-3r^2\right)-\left[\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right]\left(4Rrs+4s\left(s^2-6Rr-3r^2\right)\right)$$
$$+2s\left(s^2+4Rr+r^2\right)^3-48Rrs^3\left(s^2+2Rr+r^2\right)$$
$$\Rightarrow\sum a^7=48Rrs^3\left(s^2+2Rr+r^2\right)$$
$$+\left[\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right]\left(4Rrs+4s\left(s^2-6Rr-3r^2\right)\right)$$
$$-32r^2s^3\left(s^2-6Rr-3r^2\right)-2s\left(s^2+4Rr+r^2\right)^3$$

$$\Rightarrow \sum a^7 \overset{(i)}{=} 2s\left(s^6 - (14Rr + 21r^2)s^4 + r^2s^2(112R^2 + 140Rr + 35r^2) - r^3(224R^3 + 224R^2r + 70Rr^2 + 7r^3)\right)$$

$$\text{Again}, \sum \frac{a^3}{b+c} = \sum \frac{\left(2s-(b+c)\right)^3}{b+c} = \sum \frac{8s^3 - (b+c)^3 - 6s(b+c)a}{b+c}$$

$$= \frac{8s^3}{2s(s^2+2Rr+r^2)} \sum (c+a)(a+b) - \sum (b^2+c^2+2bc) - 6s(2s)$$

$$= \frac{4s^2(5s^2+4Rr+r^2)}{s^2+2Rr+r^2} - 4s^2 - 2(s^2-4Rr-r^2) - 12s^2$$

$$\Rightarrow \sum \frac{a^3}{b+c} \overset{(ii)}{=} \frac{2s^4 - (12Rr+12r^2)s^2 + r^2(16R^2+12Rr+2r^2)}{s^2+2Rr+r^2}$$

$$\text{Now}, 94,5 + \sum \frac{\sin^7 A}{\sin^7 B + \sin^7 C} = 94,5 + \sum \frac{a^7}{b^7+c^7}$$

$$= 94,5 + \left(\sum a^7\right)\sum \frac{1}{b^7+c^7} - 3 \overset{\text{Chebyshev}}{\leq} 91,5$$

$$+ 2\left(\sum a^7\right)\sum \frac{1}{(b^6+c^6)(b+c)} \overset{A-G}{\leq} 91,5 + \left(\sum a^7\right)\sum \frac{a^3}{a^3b^3c^3(b+c)}$$

$$= 91,5 + \left(\sum a^7\right).\frac{1}{64R^3r^3s^3}.\sum \frac{a^3}{b+c}$$

$$\overset{\text{via (i),(ii)}}{=} \frac{183}{2} + \frac{2s\left(s^6 - (14Rr+21r^2)s^4 + r^2s^2(112R^2+140Rr+35r^2) - r^3(224R^3+224R^2r+70Rr^2+7r^3)\right)}{64R^3r^3s^3}.\frac{2s^4 - (12Rr+12r^2)s^2 + r^2(16R^2+12Rr+2r^2)}{s^2+2Rr+r^2}$$

$$= \frac{\left(s^6 - (14Rr+21r^2)s^4 + r^2s^2(112R^2+140Rr+35r^2) - r^3(224R^3+224R^2r+70Rr^2+7r^3)\right)\left(2s^4 - (12Rr+12r^2)s^2 + r^2(16R^2+12Rr+2r^2)\right) + 2928R^3r^3s^2(s^2+2Rr+r^2)}{32R^3r^3s^2(s^2+2Rr+r^2)}$$

$$\overset{?}{\leq} 96\left(\frac{R}{2r}\right)^6 \Leftrightarrow 2r^3s^{10} - r^4s^8(40R+54r) + r^5s^6(408R^2+712Rr+324r^2)$$

$$- s^4(48R^9 - 912R^3r^6 + 3976R^2r^7 + 2520Rr^8 + 476r^9)$$

$$- rs^2(96R^{10} + 48R^9r - 10336R^4r^6 - 11888R^3r^7 - 5992R^2r^8 - 1624Rr^9 - 154r^{10})$$

$$- r^8(3584R^5 + 6272R^4r + 4256R^3r^2 + 1400R^2r^3 + 224Rr^4 + 14r^5) \underset{(\bullet)}{\overset{?}{\leq}} 0$$

$$\text{Now, Rouche} \Rightarrow s^2 - (m-n) \geq 0 \text{ and } s^2 - (m+n) \leq 0, \text{ where } m = 2R^2 + 10Rr - r^2 \text{ and } n$$

$$= 2(R-2r)\sqrt{R^2-2Rr}$$

$$\therefore \left(s^2 - (m+n)\right)\left(s^2 - (m-n)\right) \leq 0 \Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0$$

$$\Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3 \leq 0$$

$$\Rightarrow 2r^3s^{10} - 2r^3(4R^2 + 20Rr - 2r^2)s^8 + 2r^4(4R+r)^3 \leq 0$$

and $\therefore$ **in order to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet)$**

$$\leq 2r^3s^{10} - 2r^3(4R^2 + 20Rr - 2r^2)s^8 + 2r^4(4R+r)^3s^6$$

$$\Leftrightarrow r^3(8R^2 - 32r^2)s^8 - 26r^5s^8 - 2r^4(4R+r)^3s^6 + r^5s^6(408R^2 + 712Rr + 324r^2)$$

$$- s^4(48R^9 - 912R^3r^6 + 3976R^2r^7 + 2520Rr^8 + 476r^9)$$

$$-rs^2(96R^{10}+48R^9r-10336R^4r^6-11888R^3r^7-5992R^2r^8-1624Rr^9-154r^{10})$$
$$-r^8(3584R^5+6272R^4r+4256R^3r^2+1400R^2r^3+224Rr^4+14r^5)\overset{(\bullet\bullet)}{\lesseqgtr}0$$

Via Gerretsen, LHS of $(\bullet\bullet)$

$$\le r^3(8R^2-32r^2)(4R^2+4Rr+3r^2)s^6-26r^5(16Rr-5r^2)s^6$$
$$-2r^4(4R+r)^3s^6+r^5s^6(408R^2+712Rr+324r^2)$$
$$-s^4(48R^9-912R^3r^6+3976R^2r^7+2520Rr^8+476r^9)$$
$$-rs^2(96R^{10}+48R^9r-10336R^4r^6-11888R^3r^7-5992R^2r^8-1624Rr^9-154r^{10})$$
$$-r^8(3584R^5+6272R^4r+4256R^3r^2+1400R^2r^3+224Rr^4+14r^5)$$
$$=r^3(32R^4-96R^3r+208R^2r^2+144Rr^3+356r^4)s^6$$
$$-s^4(48R^9-912R^3r^6+3976R^2r^7+2520Rr^8+476r^9)$$
$$-rs^2(96R^{10}+48R^9r-10336R^4r^6-11888R^3r^7-5992R^2r^8-1624Rr^9-154r^{10})$$
$$-r^8(3584R^5+6272R^4r+4256R^3r^2+1400R^2r^3+224Rr^4+14r^5)\overset{\text{Gerretsen}}{\lesseqgtr}r^3(32R^4-96R^3r+208R^2r^2+144Rr^3+356r^4)(4R^2+4Rr+3r^2)s^4$$
$$-s^4(48R^9-912R^3r^6+3976R^2r^7+2520Rr^8+476r^9)$$
$$-rs^2(96R^{10}+48R^9r-10336R^4r^6-11888R^3r^7-5992R^2r^8-1624Rr^9-154r^{10})$$
$$-r^8(3584R^5+6272R^4r+4256R^3r^2+1400R^2r^3+224Rr^4+14r^5)\overset{?}{\lesseqgtr}0$$

$$\left(\because 32R^4-96R^3r+208R^2r^2+144Rr^3+356r^4=(R-2r)(16R^3+16R^2(R-2r)+144Rr^2+432r^3)+1220r^3\overset{\text{Euler}}{\gtrsim}0\right)$$

$$\Leftrightarrow(48R^9-128R^6r^3+256R^5r^4-544R^4r^5-2032R^3r^6+1352R^2r^7+664Rr^8-592r^9)s^4$$
$$+rs^2(96R^{10}+48R^9r-10336R^4r^6-11888R^3r^7-5992R^2r^8-1624Rr^9-154r^{10})$$
$$+r^8(3584R^5+6272R^4r+4256R^3r^2+1400R^2r^3+224Rr^4+14r^5)\underset{(\bullet\bullet\bullet)}{\overset{?}{\gtrsim}}0$$

$$\because 48R^9-128R^6r^3+256R^5r^4-544R^4r^5-2032R^3r^6+1352R^2r^7+664Rr^8-592r^9$$
$$=(R-2r)(48R^8+96R^7r+192R^6r^2+256R^5r^3+768R^4r^4+992R^3r^5-48R^2r^6+1256Rr^7+3176r^8)+5760r^9\overset{\text{Euler}}{\gtrsim}0\therefore\text{LHS of }(\bullet\bullet\bullet)\overset{\text{Gerretsen}}{\gtrsim}$$
$$(48R^9-128R^6r^3+256R^5r^4-544R^4r^5-2032R^3r^6+1352R^2r^7+664Rr^8-592r^9)(16Rr-5r^2)s^2$$

$$+rs^2(96R^{10}+48R^9r-10336R^4r^6-11888R^3r^7-5992R^2r^8-1624Rr^9-154r^{10})$$
$$+r^8(3584R^5+6272R^4r+4256R^3r^2+1400R^2r^3+224Rr^4+14r^5)$$
$$\Leftrightarrow (864R^{10}-192R^9r-2048R^7r^3+4736R^6r^4-9984R^5r^5-40128R^4r^6+19904R^3r^7$$
$$-2128R^2r^8-14416Rr^9+2806r^{10})s^2$$
$$+r^8(3584R^5+6272R^4r+4256R^3r^2+1400R^2r^3+224Rr^4+14r^5)\underset{(\bullet\bullet\bullet\bullet)}{\overset{?}{\gtrless}} 0$$

Case 1 $864R^{10}-192R^9r-2048R^7r^3+4736R^6r^4-9984R^5r^5-40128R^4r^6$
$+19904R^3r^7-2128R^2r^8-14416Rr^9+2806r^{10}$
≥ 0 *and* then, LHS of $(\bullet\bullet\bullet\bullet)>0$

Case 2 $864R^{10}-192R^9r-2048R^7r^3+4736R^6r^4-9984R^5r^5-40128R^4r^6$
$+19904R^3r^7-2128R^2r^8-14416Rr^9+2806r^{10}$
<0 *and* then, LHS of $(\bullet\bullet\bullet\bullet)$

$$=-\Big(-(864R^{10}-192R^9r-2048R^7r^3+4736R^6r^4-9984R^5r^5-40128R^4r^6$$
$$+19904R^3r^7-2128R^2r^8-14416Rr^9+2806r^{10})\Big)s^2$$
$$+r^8(3584R^5+6272R^4r+4256R^3r^2+1400R^2r^3+224Rr^4+14r^5)\overset{\text{Gerretsen}}{\geq}$$
$$=-\Big(-(864R^{10}-192R^9r-2048R^7r^3+4736R^6r^4-9984R^5r^5-40128R^4r^6$$
$$+19904R^3r^7-2128R^2r^8-14416Rr^9+2806r^{10})\Big)(4R^2+4Rr+3r^2)$$
$$+r^8(3584R^5+6272R^4r+4256R^3r^2+1400R^2r^3+224Rr^4+14r^5)\overset{?}{\gtrless}0$$
$$\Leftrightarrow 3456t^{12}+2688t^{11}+1824t^{10}-8768t^9+10752t^8-27136t^7-186240t^6-107264t^5$$
$$-43008t^4-2208t^3-51424t^2-31800t+8432\overset{?}{\gtrless}0\ \left(\text{where } t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\Big(3456t^{11}+9600t^{10}+21024t^9+33280t^8+77312t^7+127488t^6+68736t^5$$
$$+30208t^4+17408t^3+32608t^2+11684t+2108(t-2)\Big)\overset{?}{\gtrless}0\to \text{true}$$
$$\because t\overset{\text{Euler}}{\gtrless}2$$

$\therefore$ LHS of $(\bullet\bullet\bullet\bullet)\geq 0$ $\therefore$ combining *cases* 1 *and* 2, in *all* triangles ABC, LHS of $(\bullet\bullet\bullet\bullet)$
≥ 0 holds true $\Rightarrow(\bullet\bullet\bullet\bullet)\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)$ is true in *all* triangles ABC

$$\therefore \text{ in } any\ \Delta\,ABC,\ 94,5+\sum\frac{\sin^7A}{\sin^7B+\sin^7C}\leq 96\left(\frac{R}{2r}\right)^6\ \text{(QED)}$$