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Let $a,b,c>0$. Prove that:

$$\sum\left(\frac{a}{b^2}+\frac{a^2}{b}\right)\ge\sum\left(\sqrt{a^2-ab+b^2}+\frac{1}{\sqrt{a^2-ab+b^2}}\right)\ge 6$$

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We have:

$$\sqrt{a^2-ab+b^2}=\sqrt{(a+b)^2-3ab}\ge\sqrt{(a+b)^2-\frac{3}{4}(a+b)^2}=\frac{a+b}{2};$$

Using AM-GM Inequality, we have:

$$\frac{a}{b^2}+\frac{1}{a}\ge\frac{2}{b},\qquad\frac{b}{c^2}+\frac{1}{b}\ge\frac{2}{c},\qquad\frac{c}{a^2}+\frac{1}{c}\ge\frac{2}{a}$$

Thus,

$$\sum\frac{a}{b^2}+\sum\frac{1}{a}\ge 2\sum\frac{1}{a}$$

$$\rightarrow\sum\frac{a}{b^2}\ge\sum\frac{1}{a}=\frac{1}{2}\sum\left(\frac{1}{a}+\frac{1}{b}\right)\ge\sum\frac{2}{a+b}\ge\sum\frac{1}{\sqrt{a^2-ab+b^2}};\ (1)$$

We have:

$$\sum\frac{a^2}{b}=\sum\frac{a^2-ab+b^2}{b}=\sum\left(\frac{a^2-ab+b^2}{b}+b\right)-\sum a$$
$$\geq\sum 2\sqrt{a^2-ab+b^2}-\sum a;$$

$$\to\sum\frac{a^2}{b}\geq\sum 2\sqrt{a^2-ab+b^2}-\sum a$$
$$\geq\sum\sqrt{a^2-ab+b^2}+\sum\frac{a+b}{2}-\sum a=\sum\sqrt{a^2-ab+b^2};\ (2)$$

From (1) & (2) we have:

$$\sum\left(\frac{a}{b^2}+\frac{a^2}{b}\right)\geq\sum\left(\sqrt{a^2-ab+b^2}+\frac{1}{\sqrt{a^2-ab+b^2}}\right)$$

Again, using AM-GM Inequality we have:

$$\sum\left(\sqrt{a^2-ab+b^2}+\frac{1}{\sqrt{a^2-ab+b^2}}\right)\geq 2+2+2=6.$$

Hence,

$$\sum\left(\frac{a}{b^2}+\frac{a^2}{b}\right)\geq\sum\left(\sqrt{a^2-ab+b^2}+\frac{1}{\sqrt{a^2-ab+b^2}}\right)\geq 6.$$

Equality if and only if $a=b=c=1$.