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Number Theory Diophantine equation

1. Prove or Disprove the equation $9X^2 - 10XY + 9Y^2 = Z\uparrow^{2020}$ has infinitely many integer solution X, Y, Z such that $\gcd(X,Y,Z) = 1$, where,

$Z\uparrow^2 = Z^Z$, $Z\uparrow^3 = Z^{Z^Z}$, $Z\uparrow^4 = Z^{Z^{Z^Z}}$, defined analogusly for $Z\uparrow^{2020}$

Solution: We will prove that the equation has infinitely many integer solution.

We will first show that $9X^2 - 10XY + 9Y^2 = Z^Z$ has infinitely many integer solution X, Y, Z such that $\gcd(X,Y,Z) = 1$

The equation is equivalent as $2A^2 + 7B^2 = Z^Z$ where $A = X + Y$ and $B = X - Y$

Claim 1: $(2u^2 + 7v^2)^{2k+1} = 2m^2 + 7n^2$ for some positive integer m, n, and k be any natural number.

Proof: $(\sqrt{2}u + \sqrt{-7}v)^{2k+1} = \sum_{m=0}^{2k+1} \binom{2k+1}{m} (\sqrt{2}u)^m (\sqrt{-7}v)^{2k+1-m}$

$= \sum_{m\text{ odd}} \binom{2k+1}{m} (\sqrt{2}u)^m (\sqrt{-7}v)^{2k+1-m} + \sum_{m\text{ even}} \binom{2k+1}{m} (\sqrt{2}u)^m (\sqrt{-7}v)^{2k+1-m}$

$= \sqrt{2} \sum_{m\text{ odd}} \binom{2k+1}{m} 2^{\frac{m-1}{2}} (-7)^{\frac{2k+1-m}{2}} u^m v^{2k+1-m} +$

$\sqrt{-7} \sum_{m\text{ even}} \binom{2k+1}{m} 2^{\frac{m}{2}} (-7)^{\frac{2k+1-m-1}{2}} u^m v^{2k+1-m}$

Pick $m = \sum_{m\text{ odd}} \binom{2k+1}{m} 2^{\frac{m-1}{2}} (-7)^{\frac{2k+1-m}{2}} u^m v^{2k+1-m}$ and

$n = \sum_{m\text{ even}} \binom{2k+1}{m} 2^{\frac{m}{2}} (-7)^{\frac{2k+1-m-1}{2}} u^m v^{2k+1-m}$

Thus, $(\sqrt{2}u + \sqrt{-7}v)^{2k+1} = \sqrt{2}m + \sqrt{-7}n$ and $(\sqrt{2}u - \sqrt{-7}v)^{2k+1} = \sqrt{2}m - \sqrt{-7}n$

Thus, $(2u^2 + 7v^2)^{2k+1} = (\sqrt{2}m + \sqrt{-7}n)(\sqrt{2}m - \sqrt{-7}n) = 2m^2 + 7n^2$ ∎

Take $v = 1$, and let $A_{2k+1} = \sum_{m\text{ odd}} \binom{2k+1}{m} 2^{\frac{m-1}{2}} (-7)^{\frac{2k+1-m}{2}} u^m$ and set

$B_{2k+1} = \sum_{m\text{ even}} \binom{2k+1}{m} 2^{\frac{m}{2}} (-7)^{\frac{2k+1-m-1}{2}} u^m$

Claim 2: For every prime $p|2u^2+7$, we have $\frac{A_{2k+1}}{u} \equiv B_{2k+1} (\text{mod } p)$

Proof: $\frac{A_{2k+1}}{u} = \left(\binom{2k+1}{1} (-7)^k + \binom{2k+1}{3} 2(-7)^{k-1}u^2 + \cdots + \binom{2k+1}{2k+1} (2)^k u^{2k} \right)$

$$\frac{A_{2k+1}}{u} = \left(\binom{2k+1}{2k} (-7)^k + \binom{2k+1}{2k-2} 2(-7)^{k-1}u^2 + \cdots + \binom{2k+1}{0} (2)^k u^{2k} \right)$$

$$B_{2k+1} = \left(\binom{2k+1}{2k} 2^k u^{2k} + \binom{2k+1}{2k-2} 2^{k-1}(-7)u^{2k-2} + \cdots + \binom{2k+1}{0} (-7)^k \right)$$

$2u^2 \equiv -7 (\text{mod } p) \Rightarrow (-7)^k \equiv 2^k u^{2k} (\text{mod } p), 2(-7)^{k-1}u^2 \equiv 2^{k-1}(-7)u^{2k-2} (\text{mod }$

and so on for each binomial cofficient thus $\frac{A_{2k+1}}{u} \equiv B_{2k+1} (\text{mod } p)$ ∎

Now we are near to end, choose 7 not divides u, Clearly A_{2k+1} and B_{2k+1} is odd with

u odd since $2k+1 | \binom{2k+1}{i}$ for $1 \le i \le k-1$,

$2u^2+7 | \binom{2u^2+7}{i}$, and $2u^2+7 \nmid A_{2u^2+7}$ thus by claim 1 $2u^2+7 \nmid B_{2u^2+7}$

thus $2A_{2u^2+7}^2 + 7B_{2u^2+7}^2 = (2u^2+7)^{2u^2+7}$, as by Claim 2 for any prime $p|2u^2+7$

$p \nmid A_{2u^2+7}$ thus $\gcd(A_{2u^2+7}, B_{2u^2+7}, 2u^2+7) = 1$,

Observe that one can replace $2u^2+7$ with $(2u^2+7)k$ in exponent where k odd

and we have $2A_{(2u^2+7)k}^2 + 7B_{(2u^2+7)k}^2 = (2u^2+7)^{(2u^2+7)k}$

with $\gcd(A_{(2u^2+7)k}, B_{(2u^2+7)k}, 2u^2+7) = 1$, Since $(2u^2+7) \uparrow^n = (2u^2+7)k$

Now $X_{(2u^2+7)\uparrow^n} = \frac{A_{(2u^2+7)\uparrow^n} + B_{(2u^2+7)\uparrow^n}}{2}$, $Y_{(2u^2+7)\uparrow^n} = \frac{A_{(2u^2+7)\uparrow^n} - B_{(2u^2+7)\uparrow^n}}{2}$

As $A_{(2u^2+7)\uparrow^n}$ and $B_{(2u^2+7)\uparrow^n}$ odd both $X_{(2u^2+7)\uparrow^n}$ and $Y_{(2u^2+7)\uparrow^n}$ are integers

$\gcd(\frac{A_{(2u^2+7)\uparrow^n} + B_{(2u^2+7)\uparrow^n}}{2}, \frac{A_{(2u^2+7)\uparrow^n} - B_{(2u^2+7)\uparrow^n}}{2})$

$= \gcd(A_{(2u^2+7)\uparrow^n}, \frac{A_{(2u^2+7)\uparrow^n} - B_{(2u^2+7)\uparrow^n}}{2}) = A(\text{say})$, clearly A must be odd , thus

let p be a prime such that $p|A_{(2u^2+7)\uparrow^n}$ if $p|A$ then $p| \frac{A_{(2u^2+7)\uparrow^n} - B_{(2u^2+7)\uparrow^n}}{2}$ then $p|B_{(2u^2+7)\uparrow^n}$

but $\gcd(A_{(2u^2+7)\uparrow n}, B_{(2u^2+7)\uparrow n})$

$= 1$ so contadiction thus $\gcd\left(\frac{A_{(2u^2+7)\uparrow n} + B_{(2u^2+7)\uparrow n}}{2}, \frac{A_{(2u^2+7)\uparrow n} - B_{(2u^2+7)\uparrow n}}{2}\right)$

$= 1$

Hence $\gcd\left(X_{(2u^2+7)\uparrow n}, Y_{(2u^2+7)\uparrow n}, (2u^2 + 7) \uparrow n\right) = 1$ completing the proof.

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