ABOUT NESBITT - IONESCU INEQUALITY

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If $a, b, c \in (0, \infty)$, then:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} \tag{N.I.}$$

Generalized: If $a, b, c, t, u \in \mathbb{R}_+^*$, then:

$$\frac{a}{tb+uc} + \frac{b}{tc+ua} + \frac{c}{ta+ub} \geq \frac{3}{t+u} \tag{1}$$

(2) Let be $n \in \mathbb{N}^* - \{1\}$ and $x_k \in \mathbb{R}_+^*, \forall k = \overline{1,n}, X_v = \sum_{k=1}^{n} x_k^v, \forall v \in \mathbb{R}_+^*$

Theorem.
If $n \in \mathbb{N} - \{1\}, a \in [0, \infty); b, c, d, m, t \in \mathbb{R}_+^*, \forall k = \overline{1,n}$ and $x_k \in \mathbb{R}_+^*, \forall k = \overline{1,n}$,

$$X_s = \sum_{k=1}^{n} x_k^s, \forall s \in \mathbb{R}_+^*, c \cdot X_t > d \cdot \max_{1 \leq k \leq n} x_k^t, \text{ then holds:}$$

$$\sum_{k=1}^{n} \frac{a \cdot X_m + b \cdot x_k^m}{c \cdot X_t - d \cdot x_k^t} \geq \frac{(an+b)n}{cn-d} \cdot \frac{X_m}{X_t} \tag{*}$$

Proof.
WLOG, suppose $x_1 \geq x_2 \geq \ldots \geq x_n$ and then:

$$\frac{1}{c \cdot X_t - d \cdot x_1^t} \geq \frac{1}{c \cdot X_t - dx_2^t} \geq \ldots \geq \frac{1}{c \cdot X_t - d \cdot x_n^t}$$

Applying Chebyshev's inequality for:

$$a \cdot X_m + b \cdot x_1^m \geq a \cdot X_m + b \cdot x_2^m \geq \ldots \geq a \cdot X_m + b \cdot x_n^m \tag{3}$$

$$\frac{1}{c \cdot X_t - d \cdot x_1^t} \geq \frac{1}{c \cdot X_t - dx_2^t} \geq \ldots \geq \frac{1}{c \cdot X_t - d \cdot x_n^t} \tag{4}$$

We get:

$$\sum_{k=1}^{n} \frac{a \cdot X_m + b \cdot x_k^m}{c \cdot X_t - d \cdot x_k^t} \geq \frac{1}{n} \left(\sum_{k=1}^{n} (a \cdot X_m + b \cdot x_k^m \right) \cdot \sum_{k=1}^{n} \frac{1}{c \cdot X_t - d \cdot x_k^t} =$$

$$= \frac{1}{n} \left(a \cdot n \cdot X_m + b \cdot \sum_{k=1}^{n} x_k^m \right) \cdot \sum_{k=1}^{n} \frac{1}{c \cdot X_t - d \cdot x_k^t} =$$

$$= \frac{1}{n} (a \cdot n \cdot X_m + b \cdot X_m) \cdot \sum_{k=1}^{n} \frac{1}{c \cdot X_t - d \cdot x_k^t} \overset{\text{Bergström}}{\geq}$$

$$\geq \frac{a \cdot n + b}{n} \cdot X_m \cdot \frac{n^2}{\sum_{k=1}^{n} (c \cdot X_t - d \cdot x_k^t)} =$$

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$$= (a \cdot n + b) \cdot \frac{n}{c \cdot n \cdot X_t - d \cdot X_t} \cdot X_m = \frac{(a \cdot n + b)n}{c \cdot n - d} \cdot \frac{X_m}{X_n}$$

If $m = t$, then inequality (*) becomes:

(**) $$\sum_{k=1}^{n} \frac{a \cdot X_m + b \cdot x_k^t}{c \cdot X_t - d \cdot x_k^t} \geq \frac{(a \cdot n + b)n}{c \cdot n - d}$$

If $a = 0, b = c = d = 1$, then we get:

(***) $$\sum_{k=1}^{n} \frac{x_k^m}{X_m - x_k^m} \geq \frac{n}{n-1}$$

If $m = 1$, then we get:

(N.I.) $$\sum_{k=1}^{n} \frac{x_k}{X - x_k} \geq \frac{n}{n-1}, \text{ where } X = X_1 = \sum_{k=1}^{n} x_k$$

For $n = 3$, we have:

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}; \forall x, y, z \in \mathbb{R}_+^*$$

If $n = 3$ and $a = 0, b = c = d = 1$, then (*) becomes as:

(****) $$\frac{x_a^m}{x_2^t + x_3^t} + \frac{x_2^m}{x_3^t + x_1^t} + \frac{x_3^m}{x_1^t + x_2^t} \geq \frac{3(x_1^m + x_2^m + x_3^m)}{2(x_1^t + x_2^t + x_3^t)}$$

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References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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