ABOUT A RMM INEQUALITY-XXIII

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1) In ΔABC the following relationship holds:

$$\frac{9R}{2}\le\sum_{cyc}\frac{m_b^2+m_c^2}{r_b+r_c}\le\frac{9R^2}{4r}$$

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Solution. **Lemma. 2) In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{m_b^2+m_c^2}{r_b+r_c}=\frac{s^4+s^2(40R^2-8Rr)-r(4R+r)^3}{8Rs^2}$$

Proof. Using $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ and $r_a=\frac{F}{s-a}$, it follows

$$\sum_{cyc}\frac{m_b^2+m_c^2}{r_b+r_c}=\sum_{cyc}\frac{\frac{2a^2+2c^2-b^2}{4}+\frac{2a^2+2b^2-c^2}{4}}{\frac{F}{s-b}+\frac{F}{s-c}}=$$

$$=\frac{1}{4F}\sum_{cyc}\frac{(s-b)(s-c)(4a^2+b^2+c^2)}{a}=\frac{1}{4F}\cdot\frac{\sum bc(s-b)(s-c)(4a^2+b^2+c^2)}{abc}$$

$$=\frac{1}{4rs}\cdot\frac{2r^2[s^4+s^2(40R^2-8Rr)-r(4R+r)^3]}{4Rrs}=$$

$$=\frac{1}{4sr}\cdot\frac{2r^2[s^4+s^2(40R^2-8Rr)-r(4R+r)^3]}{4Rrs}=$$

$$=\frac{s^4+s^2(40R^2-8Rr)-r(4R+r)^3}{8Rs^2},\text{ which follows from}$$

$$\sum bc(s-b)(s-c)(4a^2+b^2+c^2)=2r^2[s^4+s^2(40R^2-8Rr)-r(4R+r)^3]$$

Let's get back to the main problem. Using Lemma, inequality becomes:

$$\frac{9R}{2}\le\frac{s^4+s^2(40R^2-8Rr)-r(4R+r)^3}{8Rs^2}\le\frac{9R^2}{4r}$$

For RHS, we have:

$$\frac{s^4+s^2(40R^2-8Rr)-r(4R+r)^3}{8Rs^2}\le\frac{9R^2}{4r}\Leftrightarrow$$

$$s^4r+s^2r(40R^2-8Rr)-r^2(4R+r)^3\le 18R^3s^2\Leftrightarrow$$

$$s^2(18R^3-40R^2r+8Rr^2-rs^2)+r^2(4R+r)^3\ge 0$$

We distinguish the cases:

Case 1) If $18R^3-40R^2r+8Rr^2-rs^2\ge 0$, inequality is obviously true.

Case 2) If $18R^3-40R^2r+8Rr^2-rs^2<0$, inequality can be written as:

$r^2(4R+r)^3\ge s^2(rs^2-18R^3+40R^2r-8Rr^2)$, which follows from

$$16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}\ (Blundon-Gerretsen)$$

Remains to prove that:

$$r^2(4R+r)^3\ge\frac{R(4R+r)^2}{2(2R-r)}[r(16Rr-5r^2)-18R^3+40R^2r-8Rr^2]\Leftrightarrow$$

$$2r^2(2R-r)(4R+r)\ge R(-18R^3+40R^2r+8Rr^2-5r^3)\Leftrightarrow$$

$$18R^4-40R^3r+8R^2r^2+Rr^3-2r^4\ge 0\Leftrightarrow(R-2r)(18R^3-4R^2r+r^3)\ge 0,\text{ true from}$$

$R\ge 2r\ (Euler)$. Equality holds if and only if ΔABC equilateral. For LHS, we have:

$$\frac{9R}{2}\le\frac{s^4+s^2(40R^2-8Rr)-r(4R+r)^3}{8Rs^2}\Leftrightarrow$$

$$s^4+s^2(4R^2-8Rr)\ge r(4R+r)^3\Leftrightarrow$$

$s^2(s^2+4R^2-8Rr)\ge r(4R+r)^3$, which follows from

$$s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$$

Remains to prove that

$$\frac{r(4R+r)^2}{R+r}(16Rr-5r^2+4R^2-8Rr)\ge r(4R+r)^2\Leftrightarrow$$

$$(4R^2+8Rr-5r^2)\ge(R+r)(4R+r)\Leftrightarrow 4R^2+8Rr-5r^2\ge 4R^2+5Rr+r^2\Leftrightarrow$$

$$3Rr>6r^2\Leftrightarrow R\ge 2r\ (Euler).$$

Equality holds if and only if triangle is equilateral.

Remark. Let's replace r_a with h_a.

3) In ΔABC the following relationship holds:

$$9r\le\sum_{cyc}\frac{m_b^2+m_c^2}{h_b+h_c}\le\frac{9R^2}{4r}$$

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Solution. **Lemma. 4) In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{m_b^2+m_c^2}{h_b+h_c}=\frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)}$$

Proof. Using identities $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ and $h_a=\frac{2F}{a}$, it follows

$$\sum_{cyc}\frac{m_b^2+m_c^2}{h_b+h_c}=\sum_{cyc}\frac{\frac{2a^2+2c^2-b^2}{4}+\frac{2a^2+2b^2-c^2}{4}}{\frac{2F}{b}+\frac{2F}{c}}=$$

$$=\frac{1}{8F}\sum_{cyc}\frac{bc(a+b)(a+c)(4a^2+b^2+c^2)}{b+c}=\frac{1}{8F}\cdot\frac{\sum bc(a+b)(a+c)(4a^2+b^2+c^2)}{\prod(b+c)}=$$

$$=\frac{1}{8rs}\cdot\frac{2[s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3]}{2s(s^2+r^2+2Rr)}=$$

$$=\frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)}$$, which follows from

$$\sum bc(a+b)(a+c)(4a^2+b^2+c^2)=$$

$$=2[s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3]$$

Let's get back to the main problem. Using Lemma, inequality becomes as:

$$9r\le\frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)}\le\frac{9R^2}{4r}$$

For RHS, we have:

$$s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3$$
$$\le 18R^2s^2(s^2+r^2+2Rr)\Leftrightarrow$$
$$s^4(18R^2-36Rr-r^2-s^2)+s^2r(36R^3+90R^2r+40Rr^2+r^3)+r^3(4R+r)^3\ge 0$$

We distinguish the following cases:

Case 1) If $(18R^2-36Rr-r^2-s^2)\ge 0$, inequality is obviously true.

Case 2) If $(18R^2-36Rr-r^2-s^2)<0$, inequality becomes:

$$s^2r(36R^3+90R^2r+40Rr^2+r^3)+r^3(4R+r)^3\ge s^4(s^2+r^2+36Rr-18R^2)$$

which follows from

$$\frac{r(4R+r)^2}{R+r}\le 16Rr-5r^2\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2(Blundon-Gerretsen)$$

Remains to prove that:

$$\frac{r(4R+r)^2}{R+r}.r(36R^3+90R^2r+40Rr^2+r^3)+r^3(4R+r)^3\ge$$
$$\ge\frac{R(4R+r)^2}{2(2R-r)}\cdot(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+r^2+36Rr-18R^2)\Leftrightarrow$$
$$2r^2(2R-r)(36R^3+90R^2r+40Rr^2+r^3)+2(2R-r)r^3(4R+r)(R+r)\ge$$
$$\ge R(4R^2+4Rr+3r^2)(-14R^2+40Rr+4r^2)(R+r)\Leftrightarrow$$
$$r^2(2R-r)(36R^3+90R^2r+40Rr^2+r^3)+(2R-r)r^3(4R+r)(R+r)\ge$$
$$\ge R(4R^2+4Rr+3r^2)(-7R^2+20Rr+2r^2)(R+r)\Leftrightarrow$$
$$r^2(72R^4+144R^3r-10R^2r^2-38Rr^3-r^4)+r^3(8R^3+10R^2r+2Rr^2-4R^2r-5Rr^2-r^3)\ge$$
$$\ge(R^2+Rr)(-28R^4+80R^3r+8R^2r^2-28R^3r+80R^2r^2+8Rr^3-21R^2r^2+60Rr^3+6r^4)$$
$$\Leftrightarrow r^2(72R^4+152R^3r-4R^2r^2-41Rr^3-2r^4)\ge$$
$$\ge(R^2+Rr)(-28R^4+52R^3r+67R^2r^2+68Rr^3+6r^4)\Leftrightarrow$$
$$28R^6-24R^5r-47R^4r^2+17R^3r^3-78R^2r^4-47Rr^5-2r^6\ge 0\Leftrightarrow$$

$(R-2r)(28R^5+32R^4r+17R^3r^2+51R^2r^3+24Rr^4+r^5)\ge 0$, true from $R\ge 2r$ $(Euler)$.

Equality holds if and only if triangle is equilateral.

For LHS, we have:

$$\frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)}\ge 9r\Leftrightarrow$$

$$s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3\ge$$
$$\ge 72r^2s^2(s^2+r^2+2Rr)\Leftrightarrow$$
$$s^6+s^4(36Rr-71r^2)-s^2r^2(72R^2+184Rr+73r^2)\ge r^3(4R+r)^3\Leftrightarrow$$
$$s^2[s^4+s^2(36Rr-71r^2)-r^2(72R^2+184Rr+73r^2)]\ge r^3(4R+r)^3\Leftrightarrow$$
$s^2[s^2(s^2+36Rr-71r^2)-r^2(72r^2+184Rr+73r^2)]\ge r^3(4R+r)^3$, which follows from $s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$. Remains to prove:
$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(16Rr-5r^2+36Rr-71r^2)-r^2(72R^2+184Rr+73r^2)]\ge$$
$$\ge r^3(4R+r)^3\Leftrightarrow$$
$$(16R-5r)(52R-76r)-(72R^2+184Rr+73r^2)\ge(R+r)(4R+r)\Leftrightarrow$$
$756R^2-1665Rr+306r^2\ge 0\Leftrightarrow(R-2r)(756R-143r)\ge 0$, true from $R\ge 2r(Euler)$.

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_b^2+m_c^2}{h_b+h_c}\ge\frac{2r}{R}\sum_{cyc}\frac{m_b^2+m_c^2}{r_b+r_c}$$

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Solution. Using Lemmas, we have:
$$\sum_{cyc}\frac{m_b^2+m_c^2}{h_b+h_c}=\frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)}$$
$$\sum_{cyc}\frac{m_b^2+m_c^2}{r_b+r_c}=\frac{s^4+s^2(40R^2-8Rr)-r(4R+r)^3}{8Rs^2}$$

Inequality becomes
$$\frac{s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3}{8rs^2(s^2+r^2+2Rr)}\ge$$
$$\ge\frac{2r}{R}\cdot\frac{s^4+s^2(40R^2-8Rr)-r(4R+r)^3}{8Rs^2}\Leftrightarrow$$
$$R^2[s^6+s^4(36Rr+r^2)-s^2r^2(72R^2+40Rr+r^2)-r^3(4R+r)^3]\ge$$
$$\ge 2r^2(s^2+r^2+2Rr)[s^4+s^2(40R^2-8Rr)-r(4R+r)^3]\Leftrightarrow$$
$$s^2[s^4(R^2-2r^2)+s^2(36R^3r-79R^2r^2+120Rr^3-2r^4)-$$
$$-r^2(72R^4+8R^3r+49R^2r^2+40Rr^3+2r^4)]+$$
$+2r^3(-64R^5+80R^4r+148R^3r^2+71R^2r^3+14Rr^4+r^5)\ge 0$, which follows from $s^2\ge 16Rr-5r^2(Gerretsen)$ and $R\ge 2r\ (Euler)$.

Equality holds if and only if triangle is equilateral.

REFERENCE:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro