A SIMPLE PROOF FOR POPOVICIU'S INEQUALITY INTEGRAL FORM

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Abstract. In this paper is given a simple proof for Popoviciu's inequality and an application.

Theorem

If $a, b, c > 0; f : (0, \infty) \to \mathbb{R}; f$ integrable and convexe function then:

$$\frac{1}{a}\int_0^a f(x)dx + \frac{1}{b}\int_0^b f(x)dx + \frac{1}{c}\int_0^c f(x)dx + \frac{9}{a+b+c}\int_0^{\frac{a+b+c}{3}} f(x)dx \geq$$

$$\geq \frac{4}{a+b}\int_0^{\frac{a+b}{2}} f(x)dx + \frac{4}{b+c}\int_0^{\frac{b+c}{2}} f(x)dx + \frac{4}{c+a}\int_0^{\frac{c+a}{2}} dx$$

Lemma

If $a > 0; f : (0, \infty) \to \mathbb{R}; f$ integrable and convexe then:

$$\int_0^1 f(ax)dx = \frac{1}{a}\int_0^a f(x)dx \tag{1}$$

Proof.

For $ax = y \Rightarrow x = \frac{1}{a}y; dx = \frac{1}{a}dy$

For $x = 0 \Rightarrow y = 0$. For $x = 1 \Rightarrow y = a$.

$$\int_0^1 f(ax)dx = \int_0^a f(y)\cdot\frac{1}{a}dy = \frac{1}{a}\int_0^a f(y)dy = \frac{1}{a}\int_0^a f(x)dx$$

Analogous with (1):

$$\int_0^1 f(bx)dx = \frac{1}{b}\int_0^b f(x)dx \tag{2}$$

$$\int_0^1 f(cx)dx = \frac{1}{c}\int_0^c f(x)dx \tag{3}$$

$$\int_0^1 f\Big(\frac{a+b+c}{3}\cdot x\Big) = \frac{3}{a+b+c}\int_0^{\frac{a+b+c}{3}} f(x)dx \tag{4}$$

$$\int_0^1 f\Big(\frac{a+b}{2}\cdot x\Big) = \frac{2}{a+b}\int_0^{\frac{a+b}{2}} f(x)dx \tag{5}$$

$$\int_0^1 f\Big(\frac{b+c}{2}\cdot x\Big) = \frac{2}{b+c}\int_0^{\frac{b+c}{2}} f(x)dx \tag{6}$$

$$\int_0^1 f\Big(\frac{c+a}{2}\cdot\Big) = \frac{2}{c+a}\int_0^{\frac{c+a}{2}} f(x)dx \tag{7}$$

By classical Popoviciu's inequality:

$$f(ax)+f(bx)+f(cx)+3f\Big(\frac{a+b+c}{3}\cdot x\Big) \geq$$

$$\geq 2f\Big(\frac{a+b}{2}\cdot x\Big)+2f\Big(\frac{b+c}{2}\cdot x\Big)+2f\Big(\frac{c+a}{2}\cdot x\Big) \tag{8}$$

Integrating (8):

$$\int_0^1 f(ax)dx+\int_0^1 f(bx)dx+\int_0^1 f(cx)dx+3\int_0^1 f\Big(\frac{a+b+c}{3}\cdot x\Big)dx \geq$$

$$\geq 2\int_0^1 f\Big(\frac{a+b}{2}\cdot\Big)dx+2\int_0^1 f\Big(\frac{b+c}{2}\cdot x\Big)dx+2\int_0^1 f\Big(\frac{c+a}{2}\cdot x\Big)dx$$

By $(1);(2);(3);(4);(5);(6);(7)$:

$$\frac{1}{a}\int_0^a f(x)dx+\frac{1}{b}\int_0^b f(x)dx+\frac{1}{c}\int_0^c f(x)dx+3\cdot\frac{3}{a+b+c}\int_0^{\frac{a+b+c}{3}} f(x)dx \geq$$

$$\geq 2\cdot\frac{2}{a+b}\int_0^{\frac{a+b}{2}} f(x)dx+2\cdot\frac{2}{b+c}\int_0^{\frac{b+c}{2}} f(x)dx+2\cdot\frac{2}{c+a}\int_0^{\frac{c+a}{2}} f(x)dx$$

$$\frac{1}{a}\int_0^a f(x)dx+\frac{1}{b}\int_0^b f(x)dx+\frac{1}{c}\int_0^c f(x)dx+\frac{9}{a+b+c}\int_0^{\frac{a+b+c}{3}} f(x)dx \geq$$

$$\geq \frac{4}{a+b}\int_0^{\frac{a+b}{2}} f(x)dx+\frac{4}{b+c}\int_0^{\frac{b+c}{2}} f(x)dx+\frac{4}{c+a}\int_0^{\frac{c+a}{2}} f(x)dx \tag{9}$$

If $a=b=c$:

$$LHS=\frac{3}{a}\int_0^a f(x)dx+\frac{9}{3a}\int_0^a f(x)dx=\frac{6}{a}\int_0^a f(x)dx$$

$$RHS=3\cdot\frac{4}{2a}\int_0^{\frac{a+a}{2}} f(x)dx=\frac{6}{a}\int_0^a f(x)dx$$

$$LHS=RHS$$

□

Application

If $n\in\mathbb{N}; n\geq 2; a,b,c>0$ then:

$$a^n+b^n+c^n+\frac{(a+b+c)^n}{3^{n-2}} \geq \frac{(a+b)^n}{2^{n-1}}+\frac{(b+c)^n}{2^{n-1}}+\frac{(c+a)^n}{2^{n-1}}$$

Proof.

We take in $(9): f(x)=x^n$

$$\frac{1}{a}\int_0^a x^n dx+\frac{1}{b}\int_0^b x^n dx+\frac{1}{c}\int_0^c x^n dx+\frac{9}{a+b+c}\int_0^{\frac{a+b+c}{3}} x^n dx \geq$$

$$\geq \frac{4}{a+b}\int_0^{\frac{a+b}{2}} x^n dx+\frac{4}{b+c}\int_0^{\frac{b+c}{2}} x^n dx+\frac{4}{c+a}\int_0^{\frac{c+a}{2}} x^n dx$$

$$\frac{1}{a}\cdot\frac{a^{n+1}}{n+1}+\frac{1}{b}\cdot\frac{b^{n+1}}{n+1}+\frac{1}{c}\cdot\frac{c^{n+1}}{n+1}+\frac{9}{a+b+c}\cdot\frac{(\frac{a+b+c}{3})^{n+1}}{n+1} \geq$$

$$\geq \frac{4}{a+b}\cdot\left(\frac{a+b}{2}\right)^{n+1}\cdot\frac{1}{n+1}+\frac{4}{b+c}\cdot\left(\frac{b+c}{2}\right)^{n+1}\cdot\frac{1}{n+1}+\frac{4}{c+a}\cdot\left(\frac{c+a}{2}\right)^{n+1}\cdot\frac{1}{n+1}$$

$$a^n+b^n+c^n+\frac{(a+b+c)^n}{3^{n-2}}\geq\frac{(a+b)^n}{2^{n-1}}+\frac{(b+c)^n}{2^{n-1}}+\frac{(c+a)^n}{2^{n-1}}$$

□

References

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