PROPOSED PROBLEM

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4265. Prove that if $a, b, c \in (0,1); a+b+c=1$ then:

$$\frac{4}{\pi}(\arctan a + \arctan b + \arctan c) > \frac{1}{2-(ab+bc+ca)}$$

Proof. Let be $f:[0,1] \to \mathbb{R}; f(x) = \arctan x - \frac{\pi x^2}{2(x^2+1)}$

$$f'(x) = \frac{1}{1+x^2} - \frac{\pi}{2} \cdot \frac{(x^2)'(x^2+1) - x^2(x^2+1)'}{(x^2+1)^2}$$

$$f'(x) = \frac{2(x^2-\pi x+1)}{2(x^2+1)^2}; f'(x) = 0 \Rightarrow x_1 = \frac{\pi - \sqrt{\pi^2-4}}{2} \in [0,1]$$

$$f(0) = f(1) = 0; f(x_1) > 0 \Rightarrow \min_{x \in [0,1]} f(x) = 0 \Rightarrow$$

$$\Rightarrow f(x) > 0; (\forall) x \in (0,1)$$

$$\arctan x - \frac{\pi x^2}{2(x^2+1)} > 0 \tag{0.1}$$

Let be $x = a; x = b$, respectively $x = c$ in 0.1

By adding:

$$\sum \arctan a - \frac{\pi}{2} \sum \frac{a^2}{a^2+1} > 0$$

$$\frac{2}{\pi} \sum \arctan a > \sum \frac{a^2}{a^2+1} \geq \overbrace{\geq}^{\text{Cauchy-Schwarz}}$$

$$\geq \frac{(a+b+c)^2}{a^2+b^2+c^2+3} = \frac{1}{(a+b+c)^2 - 2(ab+bc+ca)+3} =$$

$$\geq \frac{1}{4-2(ab+bc+ca)}$$

By multiplying with 2:

$$\frac{4}{\pi}(\arctan a + \arctan b + \arctan c) > \frac{1}{2-(ab+bc+ca)}$$

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