SSMA-MATH CHALLENGES-(I)

DANIEL SITARU - ROMANIA

5477. Compute:

$$L = \lim_{n\to\infty}\left(\ln n + \lim_{x\to 0}\frac{1-\sqrt{1+x^2}\sqrt[3]{1+x^2}\cdot\ldots\cdot\sqrt[n]{1+x^2}}{x^2}\right).$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Ed Gray, Highland Beach, FL.
We rewrite the expression as:

$$1.\ \lim_{x\to 0}\frac{[1-(1+x^2)^{\frac{1}{2}+\frac{1}{3}+\frac{1}{n}+\ldots+\frac{1}{n}}]}{x^2}$$

$$2.\ \text{Let } N = \sum_{k=2}^{k=n}\frac{1}{k}, \text{ i.e., the harmonic series } -1$$

$$3.\ \text{Now consider } \lim_{x\to 0}\frac{[1-(1+x^2)^n]}{x^2}$$

We expand $(1+x^2)^N$ by the Binomial Theorem:

$$4.\ (1+x^2)^N = 1 + Nx^2 + \frac{N(N-1)}{2!}x^4 + \ldots$$

Then

$$5.\ \lim_{x\to 0}\frac{[1-(1+Nx^2+\frac{N(N-1)}{2}x^4+\ldots)]}{x^2}, \text{ or}$$

$$6.\ \lim_{x\to 0}\frac{[-Nx^2+\frac{-N(N-1)}{2}x^4+\ldots]}{x^2} = \frac{-Nx^2}{x^2} = -N$$

The original becomes

$$7.\ \lim_{n\to\infty}(\ln(n)-N) = \lim_{n\to\infty}\left(\ln(n)-\sum_{k=2}^{k=n}\frac{1}{k}\right) = \lim_{n\to\infty}(\ln(n)+1 \text{ - Harmonic series})$$

The Euler-Mascheroni Constant is defined as $\gamma = \lim_{n\to\infty}$ The Harmonic series - $\ln(n)$.
Therefore our expression in step 7 equals $1-\gamma$. □

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain.
Since

$$\lim_{x\to 0}\frac{1-(1+x^2)^{\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}}{x^2} = \lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\left[\frac{0}{0} = Indet.\right] \overset{\text{L'Hospital}}{=}$$

$$\lim_{x\to 0}\frac{0-(H_n-1)(1+x)^{H_n-2}2x}{2x} = (1-H_n)\lim_{x\to 0}(1+x^2)^{H_n-2} = 1-H_n,$$

$$L = \lim_{n\to\infty}(\ln n + 1 - H_n) = 1 - \lim_{n\to\infty}(H_N - \ln n) = 1-\gamma$$

where H_n is the n-th harmonic number and γ is the Euler Mascheroni constant. □

Solution 3 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.

$$\sqrt{1+x^2}\sqrt[3]{1+x^2}\dots\sqrt[n]{1+x^2} = (1+x^2)^{\frac{1}{2}+\frac{1}{3}+\frac{1}{n}} = (1+x^2)^{H_n-1}$$

We have

$$\lim_{n\to\infty}\left(\ln n + \lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)$$

Now

$$\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2} = -H_n + 1$$

thus

$$L = \lim_{n\to\infty}(\ln - \ln n - \gamma + o(1) + 1) = -\gamma + 1$$

□

Solution 4 by Julio Cesar Mohnsam and Mateus Gomes Lucas. both from IFSUL, Campus Pelots-RS, Brazil, and Ricardo Capiberibe Nunes of E.E. Amlio de Caravalho Bas, Campo Grande-MS, Brazil

$$L = \lim_{n\to\infty}\left(\ln n + \lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right) = \lim_{n\to\infty}(\ln n + \lim_{x\to 0}(1-H_n)(1+x^2)^{H_n-2})$$

because,

$$\lim_{x\to 0}\frac{[1-(1+x^2)^{H_n-1}]}{(x^2)} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{[1-(1+x^2)^{H_n-1}]'}{(x^2)'} =$$

$$= \lim_{x\to 0}(-H_n+1)(1+x^2)^{H_n-2} = -H_n + 1$$

Therefore:

$$L = \lim_{n\to\infty}(\ln - H_n + 1) = \lim_{n\to\infty}(\ln - H_n) + 1 = 1 + \lim_{n\to\infty}(\ln n - H_n) = 1 - \gamma$$

Note: γ is Euler-Mascheroni constant and $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$. □

5482. Prove that if n is a natural number then:

$$\frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n + (\tan 3^\circ)^n} + \frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n + (\tan 2^\circ)^n} + \frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n + (\tan 1^\circ)^n} \geq \frac{3}{2}$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Henry Ricardo, Westchester Are Math Circle, NY.
Since, for a fixed natural number $n, (\tan x)^n$ is an increasing positive function for $x \in [0, 90^\circ)$, we have

$$\frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n + (\tan 3^\circ)^n} \geq \frac{(\tan 5^\circ)^n}{(\tan 5^\circ)^n + (\tan 5^\circ)^n} = \frac{1}{2},$$

$$\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n + (\tan 2^\circ)^n} \geq \frac{(\tan 4^\circ)^n}{(\tan 4^\circ)^n + (\tan 4^\circ)^n} = \frac{1}{2},$$

$$\frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n + (\tan 1^\circ)^n} \geq \frac{(\tan 3^\circ)^n}{(\tan 3^\circ)^n + (\tan 3^\circ)^n} = \frac{1}{2},$$

so that adding these inequalities gives us the desired result. Equality holds if and only if $n = 0$ (assuming that 0 is considered a natural number). □

Solution 2 by Henry Ricardo, Westchester Are Math Circle, NY.
Since, for a fixed natural number $n, (\tan x)^n$ is an increasing positive function for $x \in [0, 90^\circ)$, we have

$$\frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n + (\tan 1^\circ)^n} \geq \frac{(\tan 3^\circ)^n}{(\tan 4^\circ)^n + (\tan 5^\circ)^n},$$

$$\frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n + (\tan 2^\circ)^n} \geq \frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n + (\tan 5^\circ)^n},$$

so that

$$\sum_{cyclic} \frac{(\tan 3^\circ)^n}{(\tan 2^\circ)^n + (\tan 1^\circ)^n} \geq$$

$$\geq \frac{(\tan 5^\circ)^n}{(\tan 4^\circ)^n + (\tan 3^\circ)^n} + \frac{(\tan 4^\circ)^n}{(\tan 3^\circ)^n + (\tan 5^\circ)^n} + \frac{(\tan 3^\circ)^n}{(\tan 4^\circ)^n + (\tan 5^\circ)^n}$$

Setting $a = (\tan 3^\circ)^n, b = (\tan 4^\circ)^n$, and $c = (\tan 5^\circ)^n$, we see that the right-hand side of the last inequality has the form

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b},$$

for $a, b, c > 0$, which is greater than or equal to $\frac{3}{2}$ by Nesbitt's inequality. Equality holds if and only if $n = 0$ (assuming that 0 is considered a natural number). □

Solution 3 by Ed Gray, Highland Beach, FL.
First we retrieve the required values:
1. $\tan 1^\circ = .017455065$
2. $\tan 2^\circ = .034920769$
3. $\tan 3^\circ = .052407779$
4. $\tan 4^\circ = .069926812$
5. $\tan 5^\circ = .087488664$

We rewrite the problem's equation as:

$$\frac{1}{\frac{\tan 4^\circ}{\tan 5^\circ} + \frac{\tan 3^\circ}{\tan 5^\circ}} + \frac{1}{\frac{\tan 3^\circ}{\tan 4^\circ} + \frac{\tan 2^\circ}{\tan 4^\circ}} + \frac{1}{\frac{\tan 2^\circ}{\tan 3^\circ} + \frac{\tan 1^\circ}{\tan 3^\circ}} \geq \frac{3}{2}$$

Substituting the values from steps 1-5 and performing the indicated divisions we define:

$$f(n) = \frac{1}{(.799267114)^n + (.599023652)^n} + \frac{1}{(.794551256)^n + (.499433116)^n} +$$

$$+ \frac{1}{(.66632797)^n + (.333062483)^n}$$

We note that $f(n)$ is an increasing function of n since the denominator clearly decrease as n increases.
Finally we note that

$$f(1) = .715158838 + 1.248899272 + 1.000609919 = 2.964668029 > \frac{3}{2}$$

Then the equality holds for all n since $f(n)$ is an increasing function. □

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

Lemma: For fixed positive reals a, b, c with $a < c, b < c$ let $f(x) = \frac{c^x}{b^x+a^x}$ for $x \geq 0$.
Then $f(x) \geq \frac{1}{2}$, for $x \geq 0$, with equality holding only for $x = 0$.
Proof. We calculate the derivative:

$$\begin{aligned} f'(x) &= \frac{(b^x + a^x)c^x \ln c - c^x(a^x \ln a + b^x \ln b)}{(b^x + a^x)^2} \\ &= c^x \frac{(b^x + a^x) \ln c - (a^x \ln a + b^x \ln b)}{(b^x + a^x)^2} \\ &= c^x \frac{b^x(\ln c - \ln b) + a^x(\ln c - \ln a)}{(b^x + a^x)^2}. \end{aligned}$$

The ln function is increasing, so $\ln c \geq \ln b$ and $\ln c > \ln a$; thus we see that the derivative is positive. Hence the function f is increasing, so $\frac{1}{2} = f(0) \leq f(x)$ for $x \geq 0$. Because the derivative is strictly positive, the function f actually grows: so $f(x) > \frac{1}{2}$ for $x > 0$.
To verify the inequality of the problem, we note that the tangent function is increasing, so in each summand the tangent term in the numerator is larger that each tangent term in the denominator. Hence we can apply the lemma to each of the three summands, forcing the sum $\geq \frac{3}{2}$. Note that equality holds if and only if $n = 0$.
Comment: We can apply the lemma to obtain some ugly inequalities which are clearly true:

$$\frac{3^n}{1^n + 2^n} + \frac{4^n}{2^n + 3^n} + \frac{5^n}{3^n + 4^n} + \ldots + \frac{(n+2)^n}{n^n + (n+1)^n} \geq \frac{n}{2}, \text{ and}$$

$$\frac{[(n+2)!]^n}{[n!]^n + [(n+1)!]^n} \geq \frac{1}{2}.$$

□

5488. Let a, and b be complex numbers. Solve the following equation:

$$x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3 = 0.$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Dionne Bailey, Elsie Campbell, Charles Diminnie. and Trey Smith, Angelo State University, San Angelo, TX
To begin, we note that

$$x^3 - 3ax^2 + 3(a^2 - b^2)x - a^3 + 3ab^2 - 2b^3$$

can be re-written as

$$(x^3 - 3ax^2 + 3a^2x - a^3) - 3b^2x + 3ab^2 - 2b^3$$

or

$$(x - a)^3 - 3b^2(x - a) - 2b^3.$$

Hence, if we substitute $y = x - a$, the given equation becomes

(1) $$y^3 - 2b^2y - 2b^3 = 0$$

Next, the left side of equation (1) can be re-grouped to obtain

$$\begin{aligned} y^3 - 3b^2y - 2b^3 &= (y^3 + b^3) - 3b^2(y + b) \\ &= (y + b)[(y^2 - by + b^2) - 3b^2] \\ &= (y + b)(y^2 - by - 2b^2) \\ &= (y + b)^2(y - 2b). \end{aligned}$$

Therefore, the solutions of (1) are $y = 2b$ and $y = -b$ (double solution).
Finally, since $y = x - a$, the solutions of the original equation are $x = a + 2b$ and $x = a - b$ (double solution). □

Solution 2 by Michel Bataille, Rouen, France.
Let $p(x)$ denote the polynomial on the left-hand side. Then, a short calculation gives

$$p(X + a) = X^3 - 3b^2X - 2b^3 = (X + b)^2(X - 2b)$$

which has $2b$ as a simple root and $-b$ as a double one. It immediately follows that the solutoin of the given equation are $a - b, a - b, a + 2b$. □

Solution 3 by Paul M. Harms, North Newton, KS.
The equation can be written as $(x - a)^3 - 3ab^2(x - a) - 2b^3 = 0$. If $b = 0$, the solution is $x = a$. If b is not zero, let $x - a = yb$. Then the equation become $b^3(y^3 - 3y - 2) = 0$.
We have $y^3 - 3y - 2 = (y - 2)(y + 1)^2 = 0$. The y solutions are $2, -1$ and -1. The solutions of the equation in the problem are $x = a + 2b$ and $x = a - b$ as a double root. □

Solution 4 by G. C. Greubel, Newport News, VA.

$$\begin{aligned} 0 &= x^3 - 3ax^2 + 3(a^2 - b^2)x - (a^3 - 3ab^2 + 2b^3) \\ &= x^3 - 3ax^2 + (a - b)(3a + 3b)x - ((a^2 - 2ab + b^2)(a + b)) \\ &= x^3 - (2(a - b) + (a + 2b))x^2 + (a - b)((a - b) + 2(a + 2b))x - (a - b)^2(a + 2b) \\ &= (x^2 - 2(a - b)x + (a - b)^2)(x - (a + 2b)) \\ &= (x - (a - b))^2(x - (a + 2b)). \end{aligned}$$

From this factorization the solutions of the cubic equation are

$$x \in \{a - b, a - b, a + 2b\}$$

Editor's comment: David Stone and John Hawkins made an instructive comment in their solution that merits being repeated. They wrote: "We confess – we did not immediately recognize the factorization. We originally used Cardan's Formula to find the solutions. However, there is a line of heuristic reasoning which would lead to the solution. If we consider $a = b$, the equation become $x^3 - 3ax^2 = 0$ which has $x = 0$ as a double root. Hence, the difference $a - b$ could be significant. Trying $x = a - b$ (via synthetic division) then proves to be productive." □

5496. Let a, b, c be real numbers such that $0 < a < b < c$. Prove that:

$$\sum_{cyclic} (e^{a-b} + e^{b-a}) \geq 2a - 2c + 3 + \sum_{cyclic} \left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY.
For $x > 0$ we apply the known inequality $e^x > x + 1$ to $x = a - b, b - c$ and $a - c$ to get

$$e^{a-b} > a - b + 1, \quad e^{b-c} > b - c + 1, \quad e^{a-c} > a - c + 1,$$

respectively. Adding these inequalities yields

$$e^{a-b} + e^{b-c} + e^{a-c} > 2a - 2c + 3 \tag{1}$$

For $x > y$, we see that

$$e^{x-y} > \left(\frac{x}{y}\right)^{\sqrt{xy}} \Leftrightarrow x - y > \sqrt{xy}\ln\left(\frac{x}{y}\right) \Leftrightarrow \sqrt{xy} < \frac{x - y}{\ln x - \ln y}$$

which is the left-hand member of the *logarithmic mean inequality*. Thus we have, since $0 < a < b < c$,

$$e^{b-a} > \left(\frac{b}{a}\right)^{\sqrt{ab}}, e^{c-b} > \left(\frac{c}{b}\right)^{\sqrt{bc}}, e^{c-a} > \left(\frac{c}{a}\right)^{\sqrt{ac}} > \left(\frac{a}{c}\right)^{\sqrt{ac}} \tag{2}$$

Adding (1) and (2), we find that

$$\sum_{cyclic}\left(e^{a-b} + e^{b-a}\right) > 2a - 2c + 3 + \sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

□

Solution 2 by Albert Stadler, Herrliberg, Switzerland.
We will prove the slightly stronger inequality

$$\sum_{cyclic}\left(e^{a-b} + e^{b-a}\right) \geq a - c + 3 + \sum_{cyclic}\left(\frac{b}{a}\right)^{\sqrt{ab}}.$$

We will use the inequalities

$$e^x \geq 1 + x, x \text{ real} \tag{1}$$

$$1 \geq \left(\frac{y}{x}\right)^{\sqrt{xy}}, 0 \leq y \leq x \tag{2}$$

$$e^{y-x} \geq \left(\frac{y}{x}\right)^{\sqrt{xy}}, y \geq x \tag{3}$$

(1) and (2) are clear, while (3) is equivalent to each of the following lines:

$$y - x \geq \sqrt{xy}\log\left(\frac{y}{x}\right), \sqrt{\frac{y}{x}} - \sqrt{\frac{x}{y}} \geq \log\left(\frac{y}{x}\right),$$

$$x - \frac{1}{x} - \log x = \int_1^x \left(1 + \frac{1}{t^2} - \frac{1}{t}\right)dt \geq 0, x \geq 1 \text{ which holds true.}$$

Thus

$$\sum_{cyclic}\left(e^{a-b} + e^{b-a}\right) \geq 1 + a - b\left(\frac{b}{a}\right)^{\sqrt{ab}} + 1 + b - c + \left(\frac{b}{c}\right)^{\sqrt{bc}} + 1 + c - a + a^{a-c}$$

$$= 3 + \left(\frac{b}{a}\right)^{\sqrt{ab}} + \left(\frac{c}{b}\right)^{\sqrt{bc}} + e^{a-c} \geq 3 + \left(\frac{b}{a}\right)^{\sqrt{ab}} + \left(\frac{c}{b}\right)^{\sqrt{bc}} + 1 + a - c$$

$$\geq 3+\left(\frac{b}{a}\right)^{\sqrt{ab}}+\left(\frac{c}{b}\right)^{\sqrt{bc}}+\left(\frac{a}{c}\right)^{\sqrt{bc}}+a-c.$$

□

5502. Prove that if $a, b, c > 0$ and $a+b+c=e$ then:

$$e^{ac^e}\cdot e^{ba^e}\cdot e^{cb^e} > e^e\cdot a^{be^2}\cdot b^{ce^2}\cdot c^{ae^2}.$$

Here, $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Proposed by Daniel Sitaru - Romania

Solution by Paolo Perfetti. Department of Mathematics, Tor Vergata University, Rome, Italy

The inequality is equivalent to

$$ac^e+ba^e+cb^e > e+be^2\ln a+ce^2\ln b+ae^2\ln c$$

that is

$$a(c^e-e^2\ln c)+b(a^e-e^2\ln a)+c(b^e-e^2\ln b) > e$$

Let $f(x)=x^e-e^2\ln x$.

$$f''(x)=e(e-1)x^{e-2}+\frac{e^2}{x^2}>0$$

Thus by Jensen's inequality

$$e\sum_{cyc}\frac{a}{e}(c^e-e^2\ln c)\geq e\Big[\Big(\frac{a+b+c}{e}\Big)^e-a^2\ln\frac{a+b+c}{e}\Big]=e$$

□

Solution 2 by Moti Levy, Rehovot, Israel.

The function $\ln x$ is monotone increasing, then by applying log function on both sides of the inequality, we get

$$ac^e+ba^e+cb^e > e+be^2\ln a+ce^2\ln b+ae^2\ln c \tag{1}$$

or

$$\frac{a}{e}c^e+\frac{b}{e}a^e+\frac{c}{e}b^e > 1+e^2\Big(\frac{b}{e}\ln a+\frac{c}{e}\ln b+\frac{a}{e}\ln c\Big) \tag{2}$$

The function $\ln x$ is concave, hence

$$\ln\Big(\frac{ab+bc+ca}{e}\Big)\geq\frac{b}{e}\ln a+\frac{c}{e}\ln b+\frac{a}{e}\ln c \tag{3}$$

Thus we get for the right hand side of inequality (2):

$$1-e^2+e^2\ln(ab+bc+ca)\geq 1+e^2\Big(\frac{b}{e}\ln a+\frac{c}{e}\ln b+\frac{a}{e}\ln c\Big) \tag{4}$$

The function x^e is convex, hence we get for the left hand side of inequality (2):

$$\frac{a}{e}c^e+\frac{b}{e}a^e+\frac{c}{e}b^e\geq\Big(\frac{ab+bc+ca}{e}\Big)^e \tag{5}$$

By (4) and (5), to finish the solution, we have to show that

$$\Big(\frac{ab+bc+ca}{e}\Big)^e > 1-e^2+e^2\ln(ab+bc+ca) \tag{6}$$

Let us denote

$$x := (ab + bc + ca)^e \tag{7}$$

Since $ab + bc + ca \le \frac{e^2}{3}$, then

$$0 < x \le \left(\frac{e^2}{3}\right)^e \tag{8}$$

Setting (7) in (6), we need to show that

$$\frac{x}{e^e} > 1 - e^2 + e \ln x, \text{ for } 0 < x \le \left(\frac{e^2}{3}\right)^e,$$

or that

$$f(x) := x - e^{1+e} \ln x + e^e(e^2 - 1) > 0, \text{ for } 0 < x \le \left(\frac{e^2}{3}\right)^e \tag{9}$$

One can easily check that $f'(x) = 1 - \frac{e^{1+e}}{x} < 0$ for $0 < x \le \left(\frac{e^2}{3}\right)^e$. Hence, $f(x)$ is monotone decreasing function for $0 < x \le \left(\frac{e^2}{3}\right)^e$. Moreover, $\lim_{x\to 0} f(x) = +\infty$ and $f\left(\left(\frac{e^2}{3}\right)^e\right) = \left(\frac{e^2}{3}\right)^e - e^{1+e}\left(\ln\left(\frac{e^2}{3}\right)^e\right) + e(e^2 - 1) \cong 7.4789 > 0$. These and the monotonicity of $f(x)$ imply that $x - e^{1+e} \ln x + e^e(e^2 - 1) > 0$, for $0 < x \le \left(\frac{e^2}{3}\right)^e$. □

Solution 3 by Kee – Wai Lau, Hong Kong, China.
For $0 < x < 1$, let $f(x)$ be the convex function $x^e - e^2 \ln x$. By taking logarithms, we see that the inequality of the problem is equivalent to

$$af(c) + bf(a) + cf(b) > e. \tag{1}$$

Let $\gamma_1 = \frac{a}{e}, \gamma_2 = \frac{b}{e}$ and $\gamma_3 = \frac{c}{e}$. By Jensen's inequality, the left side of (1) is greater than or equal to $ef(\gamma_1 c + \gamma_2 a + \gamma_3 b) = ef(\frac{ab+bc+ca}{e})$.
Since $f'(x) = \frac{e(x^e - e)}{x} < 0$ and

$$ab + bc + ca = \frac{2(a + b + c)^2 - (a - b)^2 - (b - c)^2 - (c - a)^2}{6} \le \frac{e^3}{3}, \text{ so}$$

$$f\left(\frac{ab + bc + ca}{e}\right) \ge f\left(\frac{e}{3}\right) = 1.49\ldots > 1.$$

Thus (1) holds and this completes the solution. □

Solution 4 by Michel Bataille, Rouen, France.
Taking logarithms and arranging, we see that the inequality is equivalent to

$$\frac{a}{e} \cdot c^e + \frac{b}{e} \cdot a^e + \frac{c}{e} \cdot b^e > 1 + e^2\left(\frac{b}{e} \cdot \ln a + \frac{c}{e} \cdot \ln b + \frac{a}{e} \cdot \ln c\right).$$

Since the functions $x \to x^e$ and $x \to \ln x$ are respectively convex and concave on $(0, \infty)$, Jensen's inequality yields

$$\frac{a}{e} \cdot c^e + \frac{b}{e} \cdot a^e + \frac{c}{e} \cdot b^e \ge \left(\frac{ab + bc + ca}{e}\right)^e$$

and

$$\frac{b}{e} \cdot \ln a + \frac{c}{e} \cdot \ln b + \frac{a}{e} \cdot \ln c \le \ln\left(\frac{ab + bc + ca}{e}\right)$$

Therefore, it is sufficient to prove that

$$U^e - e^2 \ln U - 1 > 0 \tag{1}$$

where $U = \frac{ab+bc+ca}{e}$.
Since $e^2 = (a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca)$, we have $U \leq \frac{e}{3}$, hence $U \in (0,1)$.
Now, let $f(x) = x^e - e^2 \ln x - 1$. The function f satisfies $f(1) = 0$ and $f'(x) = \frac{e(x^e - e)}{x}$.
It follows that f is strictly decreasing on the interval $(0,1]$ and so $f(U) > f(1)$, which is the desired inequality (1). □

5506. Find

$$\Omega = \det\left[\begin{pmatrix}1 & 5\\ 5 & 25\end{pmatrix}^{100} + \begin{pmatrix}25 & -5\\ -5 & 1\end{pmatrix}^{100}\right]$$

Solution 1 by Michel Bataille, Rouen, France.

$$\text{Let } A = \begin{pmatrix}1 & 5\\ 5 & 25\end{pmatrix}, B = \begin{pmatrix}25 & -5\\ -5 & 1\end{pmatrix}, O_2 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, I_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

It is readily checked that $AB = BA = O_2$ and $A + B = 26I_2$.
Since $AB = BA$, the binomial theorem gives

$$(A+B)^{100} = \sum_{k=0}^{100} \binom{100}{k} A^k B^{100-k} \tag{1}$$

Now, if $k \in \{1, 2, \ldots, 50\}$, then

$$A^k B^{100-k} = A^k B^k B^{100-2k} = (AB)^k B^{100-2k} = O_2 \cdot B^{100-2k} = O_2$$

(note that $A^k B^k = (AB)^k$ since $AB = BA$) and similarly, if $k \in \{51, 52, \ldots, 99\}$, then

$$A^k B^{100-k} = A^{2k-100}(AB)^{100-k} = O_2$$

As a result, (1) gives $(A+B)^{100} = A^{100} + B^{100}$, that is, $26^{100} I_2 = A^{100} + B^{100}$. We can conclude:

$$\Omega = \det(26^{100} I_2) = 26^{200}.$$

□

Solution 2 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND.
Observe

$$\begin{pmatrix}1 & 5\\ 5 & 25\end{pmatrix}^{100} = \left(\begin{bmatrix}1\\ 5\end{bmatrix}\begin{bmatrix}1 & 5\end{bmatrix}\right)^{100} = \begin{bmatrix}1\\ 5\end{bmatrix}\left(\begin{bmatrix}1 & 5\end{bmatrix}\begin{bmatrix}1\\ 5\end{bmatrix}\right)^{99}\begin{bmatrix}1 & 5\end{bmatrix} = 26^{99}\begin{pmatrix}1 & 5\\ 5 & 25\end{pmatrix}$$

and

$$\begin{bmatrix}25 & -5\\ -5 & 1\end{bmatrix}^{100} = \left(\begin{bmatrix}5\\ -1\end{bmatrix}\begin{bmatrix}5 & -1\end{bmatrix}\right)^{100} = \begin{bmatrix}5\\ -1\end{bmatrix}\left(\begin{bmatrix}5 & -1\end{bmatrix}\begin{bmatrix}5\\ -1\end{bmatrix}\right)^{99}\begin{bmatrix}5 & -1\end{bmatrix} = 2^{99}\begin{pmatrix}25 & -5\\ -5 & 1\end{pmatrix}$$

It follows that

$$\Omega = \det\left[26^{99}\begin{pmatrix}1 & 5\\ 5 & 25\end{pmatrix} + 26^{99}\begin{pmatrix}25 & -5\\ -5 & 1\end{pmatrix}\right] = \det\left[\begin{pmatrix}26^{100} & 0\\ 0 & 26^{100}\end{pmatrix}\right] = 26^{200}.$$

□

Solution 3 by David A. Huckaby, Angelo State University, San Angelo, TX.
Let $A = \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}$ and $B = \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}$. Matrices A and B are each symmetric, hence orthogonally diagonalizable.
Solving the equation $\det(\lambda I - A) = 0$ yields $\lambda_1 = 0$ and $\lambda_2 = 26$ as the eigenvalues of A.
Solving the equation $(\lambda I - A)\overrightarrow{x} = \overrightarrow{0}$ successively for $\lambda = 0$ and $\lambda = 26$ yields $\overrightarrow{x_1} = \begin{pmatrix} -\frac{5}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} \end{pmatrix}$ and $\overrightarrow{x_2} = \begin{pmatrix} \frac{1}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} \end{pmatrix}$ as corresponding unit eigenvectors. So

$$A = \begin{pmatrix} -\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix} \begin{pmatrix} -\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix}$$

Similarly,

$$B = \begin{pmatrix} \frac{1}{\sqrt{26}} & -\frac{5}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \\ -\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix}.$$

Since for both A and B the matrix of eigenvectors is orthogonal, we have

$$A^{100} = \begin{pmatrix} -\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 26^{100} \end{pmatrix} \begin{pmatrix} -\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix} = \begin{pmatrix} 26^{99} & 5(26^{99}) \\ 5(26^{99}) & 25(26^{99}) \end{pmatrix},$$

and

$$B^{100} = \begin{pmatrix} \frac{1}{\sqrt{26}} & -\frac{5}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 26^{100} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \\ -\frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix} = \begin{pmatrix} 26(26^{99}) & -5(26^{99}) \\ -5(26^{99}) & 26^{99} \end{pmatrix}.$$

So $\Omega = \det[A^{100} + B^{100}] = \det \begin{pmatrix} 26^{100} & 0 \\ 0 & 26^{100} \end{pmatrix} = 26^{200}$. □

Solution 4 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece.

A way to calculate A^n for 2×2 matrix is to use the Hamilton-Cayley Theorem:

$$A^2 - Tr(A) \cdot A + \det A \cdot I_2 = 0.$$

For example, if we have a 2×2 matrix $A = \begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix}$ (or $A = \begin{pmatrix} a^2 & -a \\ -a & 1 \end{pmatrix}$) with $\det A = 0$ and $Tr(A) = a^2 + 1$, then the Hamilton-Cayley theorem becomes

$$A^2 = Tr(A) = (a^2+1)^2 A.$$
$$A^3 = (a^2+1)A^2 = (a^2+1)^2 A,$$
$$\cdots$$
$$A^n = (a^2+1)A^{n-1} = (a^2+1)^{n-1} A.$$

So we have:

$$\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} = (5^2+1)^{99} \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} = 26^{99} \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix},$$

$$\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}^{100} = (5^2+1)^{99} \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix} = 26^{99} \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ 5 & 1 \end{pmatrix}^{100} = 26^{99} \left(\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} + \begin{pmatrix} 25 & -5 \\ 5 & 1 \end{pmatrix} \right) = 26^{100} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and finally we have:

$$\Omega = \det\left(\begin{pmatrix}1 & 5\\ 5 & 25\end{pmatrix}^{100} + \begin{pmatrix}25 & -5\\ 5 & 1\end{pmatrix}^{100}\right) = \det\left(26^{100}\begin{pmatrix}1 & 5\\ 5 & 25\end{pmatrix}^{100} + \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = 26^{100}.$$

□

Solution 5 by Paolo Perfetti. Departament of Mathematics, Tor Vergata University, Rome, Italy

Let $c = \sqrt{26}$. We know that

$$\begin{pmatrix}1 & 5\\ 5 & 25\end{pmatrix} = \begin{pmatrix}-\frac{5}{c} & \frac{1}{c}\\ \frac{1}{c} & \frac{5}{c}\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 26\end{pmatrix}\begin{pmatrix}-\frac{5}{c} & \frac{1}{c}\\ \frac{1}{c} & \frac{5}{c}\end{pmatrix} = A \wedge A^{-1}$$

$$\begin{pmatrix}25 & -5\\ -5 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{c} & -\frac{5}{c}\\ \frac{5}{c} & \frac{1}{c}\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 26\end{pmatrix}\begin{pmatrix}\frac{1}{c} & \frac{5}{c}\\ -\frac{5}{c} & \frac{1}{c}\end{pmatrix} = B \wedge B^{-1}$$

$$\Omega = A \wedge^{100} A^{-1} + BA \wedge^{100} B^{-1}$$

$$A \wedge^{100} A^{-1} = \begin{pmatrix}26^{99} & 5\cdot 26^{99}\\ 5\cdot 26^{99} & 25\cdot 26^{99}\end{pmatrix}$$

$$B \wedge^{100} B^{-1} = \begin{pmatrix}25\cdot 26^{99} & -5\cdot 26^{99}\\ -5\cdot 26^{99} & 26^{99}\end{pmatrix}$$

Thus

$$\Omega = \det\begin{pmatrix}26^{99}\cdot 26 & 0\\ 0 & 26^{99}\cdot 26\end{pmatrix} = 26^{200}.$$

□

5525. Find real values for x and y such that:

$$4\sin^2(x+y) = 1 + 4\cos^2 x + 4\cos^2 y$$

Proposed by Daniel Sitaru - Romania

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

Put $u = e^{2ix}, v = e^{2iy}$. Then the given equation reads as

$$0 = (e^{2ix+2iy} + e^{-2ix-2iy} - 2) + 1 + (e^{2ix} + e^{-2ix} + 2) + (e^{2iy} + e^{-2iy} + 2) =$$

$$= u\frac{1}{uv} + u + \frac{1}{u} + v + \frac{1}{v} + 3 = \frac{(uv+u+1)(uv+v+1)}{uv}.$$

So either $v = \frac{1}{u} - 1$ or $\frac{1}{v} = -u - 1$. If x and y run through the real numbers v and $\frac{1}{v}$ represent circles in the complex plane with radius 1 and center 0, while $-u-1$ and $\frac{-1}{u} - 1$ represent circles with radius 1 and center -1. Therefore $(u,v) \in \{(e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}}), (e^{-\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}})\}$ which translates to $x \equiv y \equiv \pm\frac{\pi}{3}(\mod \pi)$.

□

Solution 2 by Michael C. Faleski, University Center, MI.

Let's rewrite the statement of the problem using several trigonometric indentities. This leads to

$$4(\sin x\cos y + \sin x\cos y)^2 = 1 + 4\cos^2 x + 4\cos^2 y$$

$$4(\sin^2 x\cos^2 y + \sin^2 y\cos^2 x + 2\sin x\sin y\cos x\cos y) = 1 + 4\cos^2 x + 4\cos^2 y$$

$$4((1-\cos^2 x)\cos^2 y + \cos^2 x(1-\cos^2 y) + 2\sin x\sin y\cos x\cos y) = 1 + 4\cos^2 x + 4\cos^2 y$$

$$-8\cos^2 x\cos^2 y + 8\sin x\sin y\cos x\cos y = 1$$
$$-8\Big(\frac{1}{2}+\frac{1}{2}\cos(2x)\Big)\Big(\frac{1}{2}+\frac{1}{2}\cos(2y)\Big) + 2\sin 2x\sin 2y = 1$$
$$-2(1+\cos 2x+\cos 2y+\cos 2x\cos 2y) + 2\sin 2x\sin 2y = 1$$
$$-2-2\cos 2x-2\cos 2y-2\cos 2x\cos 2y+2\sin 2x\sin 2y = 1$$
$$-2\cos 2x-2\cos 2y-2(\cos 2x\cos 2y-\sin 2x\sin 2y) = 3$$
$$\cos 2x+\cos 2y+\cos(2x+2y) = -\frac{3}{2}.$$

And now we use $\cos a = \cos b = 2\cos(\frac{1}{2}(a+b))\cos(\frac{1}{2}(a-b))$ to produce
$2\cos(x+y)\cos(x-y) + (2\cos^2(x+y)-1) = -\frac{3}{2}$, so we have
$2\cos^2(x+y) + 2\cos(x-y)\cos(x+y) + \frac{1}{2} = 0$, or
$\cos^2(x+y) + \cos(x-y)\cos(x+y) + \frac{1}{4} = 0$.
We will now use the quadratic formula to solve for $\cos(x+y)$.

$$\cos(x+y) = \frac{-\cos(x-y) \pm \sqrt{\cos^2(x-y)-1}}{2}.$$

As we are required to have real solutions, this means that
$\cos^2(x-y)-1 \geq 0 \to \cos^2(x-y) \geq 1$. This condition is only true for
$\cos^2(x-y) = 1 \to \cos(x-y) = 1$.
Letting $y = x-a$, we find $\cos a = 1 \to a = 2n\pi, \forall n\in\mathbb{Z}$.

$$\cos(x+y) = -\frac{\cos(x-y)}{2} = -\frac{1}{2}.$$

Since $y = \pm 2n\pi$, then for $0 \leq x \leq 2\pi, x = y$. Hence, $\cos 2x = -\frac{1}{2}$, which leads to

$2x = \frac{2}{3}\pi, \frac{4}{3}\pi \to x = \Big(\frac{1}{3}\pi, \frac{2}{3}\pi\Big)$. So for $0 \leq x, y \leq 2\pi, (x,y) = \Big(\frac{1}{3}\pi, \frac{1}{3}\pi\Big), \Big(\frac{2}{3}\pi, \frac{2}{3}\pi\Big)$.

□

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain.

$4\sin^2(x+y) = 1+4\cos^2 x+4\cos^2 y \Leftrightarrow 4(1-\cos^2(x+y)) = 1+2\cos(2x)+2+2\cos(2y)$

$$\Leftrightarrow 4-4\cos^2(x+y) = 5+4\cos\Big(\frac{2x+2y}{2}\Big)\cos\Big(\frac{2x-2y}{2}\Big)$$
$$\Leftrightarrow 0 = 4-4\cos^2(x+y)+4\cos(x+y)\cos(x-y)+1$$
$$\Leftrightarrow 0 = (2\cos(x+y)+\cos(x-y))^2+\sin^2(x-y)$$
$$\Leftrightarrow 2\cos(x+y)+\cos(x-y) = 0 = \sin(x-y) \Leftrightarrow x-y = k\pi, k\in\mathbb{Z}$$
$$\cos(x+y)+\cos(k\pi) = 0 \Leftrightarrow x-y = k\pi; \cos(x+y) = \frac{(-1)^{k+1}}{2}, k\in\mathbb{Z}$$
$$\Leftrightarrow x-y = k\pi; x+y = \arccos\frac{(-1)^{k+1}}{2} \in \mathbb{Z}$$
$$\Leftrightarrow x = \frac{1}{2}\Big(\arccos\frac{(-1)^{k+1}}{2}+k\pi\Big), y = \frac{1}{2}\Big(\arccos\frac{(-1)^{k+1}}{2}-k\pi\Big), k\in\mathbb{Z}$$

□

Solution 4 by Kee-Wai Lau, Hong Kong, China.
Since $\sin(x+y) = \sin x\cos y + \cos x \sin y$, so the given equation is equivalent to $1 - 8\sin x\cos x\sin y\cos y + 8\cos^2 x\cos^2 y = 0$. Clearly $\cos x \neq 0$ and $\cos y \neq 0$. So dividing both sides of the last equation by $\cos^2 x\cos^2 y$, we obtain

$$\sec^2 x\sec^2 y - 8\tan x\tan y + 8 = 0 \text{ or } (1+\tan^2 x)(1+\tan^2 y) - 8\tan x\tan y + 8 = 0,$$

or

$$(\tan x - \tan y)^2 + (\tan x\tan y - 3)^2 = 0.$$

Thus $\tan x = \tan y$ and $\tan x\tan y = 3$, so that $\tan x = \tan y = \sqrt{3}$ or $\tan x = \tan y = -\sqrt{3}$
It follows that

$$(x,y) = \Big(\frac{\pi}{3} + m\pi, \frac{\pi}{3} + n\pi\Big), \Big(\frac{2\pi}{3} + m\pi, \frac{2\pi}{3} + n\pi\Big),$$

where m and n are arbitrary integers. □

Solution 5 by Ulrich Abel, Technische Hochshule Mittelhessen, Germany.
Using $\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$ we see that the equation

$$4\sin^2(x+y) = 1 + 4\cos^2(x) + 4\cos^2(y)$$

is equivalent to

$$0 = 3 + 2\cos(2x+2y) + 2\cos(2x) + 2\cos(2y) =: f(x,y).$$

Using $\sin(2a) + \sin(2b) = 2\sin(a+b)\cos(a-b)$ we obtain

$$\begin{aligned} \mathit{gradf}(x,y) &= -4\cdot(\sin(2x+2y)+\sin(2x), \sin(2x+2y)+\sin(2y)) \\ &= -8\cdot(\sin(2x+y)\cos y, \sin(x+2y)\cos x). \end{aligned}$$

Therefore, $\mathit{gradf}(x,y) = (0,0)$ happens if $2x = \pi(\mod 2\pi)$ and $2y = \pi(\mod 2\pi)$. The critical points $(\frac{2n+1}{2}\pi, \frac{2m+1}{2}\pi)$ with integers n, m satisfy

$$f\Big(\frac{2n+1}{2}\pi, \frac{2m+1}{2}\pi\Big) = 3 + 2\cdot 1 + 2(-1)^{n+1} + 2(-1)^{m+1} > 0.$$

$2x = \pi(\mod 2\pi)$ and $2x + y = 0(\mod \pi)$. The critical points $(\frac{2n+1}{2}\pi, m\pi - (2n+1)\pi)$ with integers n, m satisfy

$$f\Big(\frac{2n+1}{2}\pi, m\pi - (2n+1)\pi\Big) = 3 + 2\cdot(-1) + 2(-1)^{n+1} + 2\cdot 1 > 0.$$

$2y = \pi(\mod 2\pi)$ and $x + 2y = 0(\mod \pi)$ is symmetrical to the preceding case. $2x + y = 0(\mod \pi)$ and $x + 2y = 0(\mod \pi)$. This implies $3x + 3y = (n+m)\pi$ and $x - y = (n-m)\pi$ with integers n, m. We infer that $(x,y) = \frac{\pi}{3}(2n-m, 2m-n)$ are the remaining critical points of f.

$$\begin{aligned} &f\Big(\frac{2n-m}{3}\pi, \frac{2m-n}{3}\pi\Big) \\ &= 3 + 2\cos\frac{2(n+m)\pi}{3} + 2\cos\frac{(4n-2m)\pi}{3} + 2\cos\frac{(4m-2n)\pi}{3} \\ &= 3 + 2\Big(2\cos^2\frac{(n+m)\pi}{3} - 1\Big) + 4\cos\frac{(n+m)\pi}{3}\cos(n-m)\pi \\ &= 1 + 4\cos^2\frac{N\pi}{3} + 4(-1)^N\cos\frac{N\pi}{3} = \Big(1 + 2(-1)^n\cos\frac{N\pi}{3}\Big)^2 \geq 0 \end{aligned}$$

with $N := n + m$. Consequently, the function value is equal to zero iff N is not a multiple of 3.

In total, we have $f(x, y) \geq 0$ on R^2 and $f(x, y) = 0$ if and only if $(x, y) = (2n - m, 2m - n)\frac{\pi}{3}$, for all integers n, m satisfying $n + m \neq 0(\mod 3)$. The solutions of the above trigonometric identity are exactly the zeros of f.

□

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