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SPECIAL TECHNIQUES FOR DEFINITE INTEGRALS

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Abstract: In this paper are presented a few general techniques to find definite integrals.

1. Introduction.

Theorem 1.

If $f: [a, b] \rightarrow \mathbb{R}$, $g: [a, b] \rightarrow \mathbb{R}^*$, $u: [a, b] \rightarrow \mathbb{R}$ continuous functions and $f(x) + f(s - x) = u(x)$, $g(x) = g(s - x)$, $\forall x \in [a, b]$, $s = a + b$, then:

$$\int_a^b \frac{f(x)}{g(x)} dx = \frac{1}{2} \int_a^b \frac{u(x)}{g(x)} dx$$

Proof. Using substitution $x = s - t$, we have:

$$\begin{aligned} \int_a^b \frac{f(x)}{g(x)} dx &= \int_a^b \frac{f(s-t)}{g(s-t)} (-dt) = \int_a^b \frac{u(t) - f(t)}{g(t)} dt = \int_a^b \frac{u(t)}{g(t)} dt - \int_a^b \frac{f(t)}{g(t)} dt \\ \int_a^b \frac{f(x)}{g(x)} dx &= \frac{1}{2} \int_a^b \frac{u(x)}{g(x)} dx \end{aligned}$$

Application 1. Find:

$$\Omega = \int_0^\pi \frac{(x+1) \sin x}{3 + \cos^2 x} dx$$

Solution.

$$\begin{aligned} \Omega &= \int_0^\pi \frac{(x+1) \sin x}{3 + \cos^2 x} dx = \int_0^\pi \frac{x \sin x}{3 + \cos^2 x} dx + \int_0^\pi \frac{\sin x}{3 + \cos^2 x} dx = I_1 + I_2 \\ I_1 &= \int_0^\pi \frac{x \sin x}{3 + \cos^2 x} dx = \int_0^\pi \frac{(\pi - x) \sin(\pi - x)}{3 + \cos^2 x} dx = \\ &= \pi \int_0^\pi \frac{\sin x}{3 + \cos^2 x} dx - \int_0^\pi \frac{x \sin x}{3 + \cos^2 x} dx \Rightarrow I_1 = \pi I_2 - I_1 \Rightarrow I_1 = \frac{\pi}{2} I_2 \end{aligned}$$



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$$I_2 = \int_0^\pi \frac{\sin x}{3 + \cos^2 x} dx = - \int_0^\pi \frac{(\cos x)'}{3 + \cos^2 x} dx = - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{\cos x}{\sqrt{3}} \right) \Big|_0^\pi = \frac{2\pi}{3\sqrt{3}}$$

Application 2. *Find:*

$$\Omega = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x \tan^{-1} x}{1 + e^{\tan x}} dx$$

Solution.

$$\begin{aligned} \Omega &= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x \tan^{-1} x}{1 + e^{\tan x}} dx = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{(-x) \tan^{-1}(-x)}{1 + e^{\tan(-x)}} dx = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x e^{\tan x} \cdot \tan^{-1} x}{1 + e^{\tan x}} dx \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{(1 + e^{\tan x} - 1)x \cdot \tan^{-1} x}{1 + e^{\tan x}} dx = \int_{-\sqrt{3}}^{\sqrt{3}} x \cdot \tan^{-1} x dx - I \\ 2I &= \int_{-\sqrt{3}}^{\sqrt{3}} x \cdot \tan^{-1} x dx = 2 \int_0^{\sqrt{3}} x \cdot \tan^{-1} x dx \\ \Omega &= \int_0^{\sqrt{3}} x \cdot \tan^{-1} x dx = \left(\frac{x^2}{2} - \tan^{-1} x \right) \Big|_0^{\sqrt{3}} - \frac{1}{2} \int_0^{\sqrt{3}} \frac{x^2}{1 + x^2} dx = \frac{\pi}{3} - \frac{\sqrt{3}}{2} \end{aligned}$$

Application 3. *Find:*

$$\Omega = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} x \cdot \log(1 + e^{x\sqrt{1-x^2}}) dx$$

Solution.

$$\begin{aligned} \Omega &= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} x \cdot \log(1 + e^{x\sqrt{1-x^2}}) dx \stackrel{x=-y}{=} - \int_{-\sqrt{2}/2}^{\sqrt{2}/2} y \cdot \log(1 + e^{-y\sqrt{1-y^2}}) dy = \\ &= - \int_{-\sqrt{2}/2}^{\sqrt{2}/2} y \cdot \log \left(\frac{1 + e^{y\sqrt{1-y^2}}}{e^{y\sqrt{1-y^2}}} \right) dy = -\Omega + \int_{-\sqrt{2}/2}^{\sqrt{2}/2} y \cdot \log(e^{y\sqrt{1-y^2}}) dy = \end{aligned}$$



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$$\begin{aligned}
 &= -\Omega + \int_{-\sqrt{2}/2}^{\sqrt{2}/2} y^2 \sqrt{1-y^2} dy \\
 2\Omega &= 2 \int_0^{\sqrt{2}/2} y^2 \sqrt{1-y^2} dy \stackrel{y=\sin t}{=} 2 \int_0^{\pi/4} \sin^2 t \sqrt{1-\sin^2 t} \cdot \cos t dt \\
 I &= \int_0^{\pi/4} \sin^2 t \cos^2 t dt = \frac{1}{4} \int_0^{\pi/4} \sin^2 2t dt = \frac{1}{4} \int_0^{\pi/4} \frac{1-\cos 4t}{2} dt = \frac{1}{8} \left(x - \frac{1}{4} \sin 4t \right)_0^{\pi/4} = \frac{\pi}{32}
 \end{aligned}$$

Application 4. *Find:*

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan x)}{2+\sin 2x + \cos 2x} dx$$

Solution.

$$\begin{aligned}
 I &= \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan x)}{2+\sin 2x + \cos 2x} dx = \int_0^{\frac{\pi}{4}} \frac{\log\left(1+\tan\left(\frac{\pi}{4}-y\right)\right)}{2+\sin\left(\frac{\pi}{2}-2y\right)+\cos\left(\frac{\pi}{2}-2y\right)} dy = \\
 &= \int_0^{\frac{\pi}{4}} \frac{\log 2 - \log(1+tgy)}{2+\sin 2y + \cos 2y} dy \\
 2I &= \log 2 \int_0^{\frac{\pi}{4}} \frac{1}{2+\frac{2\tan x}{1+\tan^2 x}+\frac{1-\tan^2 x}{1+\tan^2 x}} dx = \log 2 \int_0^1 \frac{1}{t^2+2t+3} dt \Rightarrow \\
 I &= \frac{\log 2}{2\sqrt{2}} \left(\tan^{-1}\sqrt{2} - \frac{\pi}{4} \right)
 \end{aligned}$$

Application 5. *Find:*

$$\Omega = \int_0^{2\pi} \frac{x+\tan(\sin x)}{2+\cos x} dx$$

Solution 1.

$$I = \int_0^{2\pi} \frac{x+\tan(\sin x)}{2+\cos x} dx = \int_0^{2\pi} \frac{2\pi-y-\tan(\sin x)}{2+\cos y} dy = 2\pi \int_0^{2\pi} \frac{1}{2+\cos y} dy - \Omega$$



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$$\begin{aligned}
 \frac{1}{\pi} I &= \int_0^{2\pi} \frac{1}{2 + \cos x} dx = \int_0^{\pi} \frac{1}{2 + \cos x} dx + \int_{\pi}^{2\pi} \frac{1}{2 + \cos x} dx = \\
 \int_0^{\pi} \frac{1}{2 + \cos x} dx + \int_0^{\pi} \frac{1}{2 - \cos x} dx &= \int_0^{\frac{\pi}{2}} \frac{1}{2 + \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{1}{2 - \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{1}{2 + \sin x} dx + \\
 \int_0^{\frac{\pi}{2}} \frac{1}{2 - \sin x} dx &= 2 \int_0^1 \frac{1}{t^2 + 3} dt + \frac{2}{3} \int_0^1 \frac{1}{t^2 + \frac{1}{3}} dt + \int_0^1 \frac{1}{(t+1)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dt + \\
 \int_0^1 \frac{1}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dt &= \frac{2\pi\sqrt{3}}{3} \Rightarrow I = \frac{2\pi^2\sqrt{3}}{3}
 \end{aligned}$$

Solution 2.

$$\begin{aligned}
 I &= \int_0^{2\pi} \frac{x + \tan(\sin x)}{2 + \cos x} dx = \int_{-\pi}^{\pi} \frac{\pi + y - \tan(\sin y)}{2 - \cos y} dy = \int_{-\pi}^{\pi} \frac{\pi}{2 - \cos y} dy + \\
 &\quad + \int_{-\pi}^{\pi} \frac{y - \tan(\sin y)}{2 - \cos y} dy = \int_0^{\pi} \frac{2\pi}{2 - \cos y} dy = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2 + \sin z} dz = \\
 &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2 + \frac{2\tan(\frac{z}{2})}{1 + \tan^2(\frac{z}{2})}} dz = 2\pi \int_{-1}^1 \frac{1}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dt = \frac{2\pi^2\sqrt{3}}{3}
 \end{aligned}$$

2. General result.

Theorem 2. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous function with property

$$\alpha f(x+b) + \beta f(c-x) = g(x+b), \forall x \in \mathbb{R}; \quad (1)$$

where $\alpha, \beta \in \mathbb{R}^*, \alpha + \beta \neq 0, b, c \in \mathbb{R}$, and $g: \mathbb{R} \rightarrow \mathbb{R}$ continuous function with primitives G . Then for all $m, n \in \mathbb{R}$, such that $m+n=b+c$,

$$I_{m,n} = \int_m^n f(x) dx = \frac{G(n) - G(m)}{\alpha + \beta}; \quad (2)$$

Proof. Replacing x with $x-b$, we get $\alpha f(x) + \beta f(b+c-x) = g(x)$, and hence,

$$f(x) = \frac{1}{\alpha} g(x) - \frac{\beta}{\alpha} f(b+c-x)$$

Integrating on the interval $[m, n]$, we get:



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$$\begin{aligned}
 I_{m,n} &= \int_m^n f(x) dx = \frac{1}{\alpha} \int_m^n g(x) dx - \frac{\beta}{\alpha} \int_m^n f(b+c-x) dx = \\
 &= \frac{1}{\alpha} [G(n) - G(m)] - \frac{\beta}{\alpha} \int_m^n f(b+c-x) dx \stackrel{b+c-x=t}{=} \\
 &= \frac{1}{\alpha} [G(n) - G(m)] + \frac{\beta}{\alpha} \int_n^m f(t) dt \\
 I_{m,n} \left(1 + \frac{\beta}{\alpha}\right) &= \frac{1}{\alpha} [G(n) - G(m)], \text{ then } I_{m,n} = \int_m^n f(x) dx = \frac{G(n) - G(m)}{\alpha + \beta}; (2)
 \end{aligned}$$

Application 6. If $h: \mathbb{R} \rightarrow \mathbb{R}$ is odd function, θ –periodic and with f'

continuous, then find:

$$\Omega = \int_0^{2\theta} \frac{x h'(x)}{1 + h^2(x)} dx$$

Solution. Let $f(x) = \frac{xh'(x)}{1+h^2(x)}$, then $f(x+2\theta) + f(-x) = 2\theta \cdot \frac{f(x)}{1+f^2(x)}$ and for $\alpha = \beta = 1$,
 $b = 2\theta, c = 0, g(x) = 2\theta \cdot \frac{h'(x)}{1+h^2(x)}$

Now, g has the primitives $G: \mathbb{R} \rightarrow \mathbb{R}, G(x) = 2\theta \cdot \tan^{-1} x$. For $m+n=2\theta$ and using

Theorem 2, we have:

$$I_{m,n} = \int_m^n f(x) dx = \theta [\tan^{-1} h(n) - \tan^{-1} h(m)]$$

Application 7. Find:

$$\Omega = \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$$

Solution. Let $(x) = \frac{x \sin x}{1 + \cos^2 x}$, then $f(x) + f(\pi - x) = \frac{\pi \sin x}{1 + \cos^2 x}$ and using **Theorem 2**, we get:

$$\begin{aligned}
 I_{m,n} &= \int_m^n f(x) dx = \frac{\pi}{2} [\tan^{-1}(\cos m) - \tan^{-1}(\cos n)]; m+n=\pi \\
 \text{For } m=0, n=\pi &\Rightarrow \Omega = \frac{\pi^2}{4}.
 \end{aligned}$$

Application 8. Find:

$$\Omega = \int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx$$

Solution. Let $f(x) = \log(1 + \tan x)$, then

$$f\left(\frac{\pi}{4} - x\right) = \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) = \log 2 - \log(1 + \tan x), \quad f(x) + f\left(\frac{\pi}{4} - x\right) = \log 2$$



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Using Theorem 2, we get:

$$I_{m,n} = \int_m^n \log(1 + \tan x) dx = \frac{n-m}{2} \log 2; m+n = \frac{\pi}{4}$$

For $m = 0, n = \frac{\pi}{4} \Rightarrow \Omega = \frac{\pi}{8} \log 2$

Application 9. If $h: [0, 1] \rightarrow \mathbb{R}$ is continuous function, then prove:

$$\int_0^\pi x \cdot h(\sin x) dx = \frac{\pi}{2} \int_0^\pi h(\sin x) dx$$

Solution. Let $f(x) = x \cdot f(\sin x) - \frac{\pi}{2} h(\sin x)$, then $f(x) + f(\pi - x) = 0$. Using **Theorem 2**, we get:

$$\int_m^n h(x) dx = 0; m+n = \pi$$

For $m = 0, n = \pi$, we get the problem.

Application 10. Prove that:

$$\int_0^1 \frac{dx}{\sqrt{x^4 - 4x^3 + 6x^2 - 4x + 2}} = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

Solution. Let $f: [0, 1] \rightarrow \mathbb{R}, f(x) = \frac{1}{\sqrt{x^4 - 4x^3 + 6x^2 - 4x + 2}} - \frac{1}{\sqrt{1+x^4}}$, then $f(x) + f(1-x) = 0$.

Using **Theorem 2**, we get:

$$\int_m^n f(x) dx = 0; m+n = 1$$

For $m = 0, n = 1$, we get the problem.

3. Extension result.

Theorem 3. If $f: [a - \theta, a + \theta] \rightarrow \mathbb{R}$ is continuous function with property

$\alpha f(a+x) + \beta f(a-x) = \gamma, \forall x \in [-\theta, \theta]; \alpha, \beta \in \mathbb{R}^*, \gamma \in \mathbb{R}$, then:

$$(i) \int_{a-\theta}^{a+\theta} f(x) dx = \frac{2\gamma}{\alpha + \beta} \cdot \theta; \alpha + \beta \neq 0$$

$$(ii) \int_{a-\theta}^{a+\theta} f(x) dx = \frac{\gamma}{\alpha} \cdot \theta + \frac{\alpha - \beta}{\alpha} \int_a^{a+\theta} f(x) dx$$

Proof. If $f: [a - \theta, a + \theta] \rightarrow \mathbb{R}$ is continuous function and $\varphi, \psi: [-\theta, \theta] \rightarrow [a - \theta, a + \theta]$,

$\psi(t) = a + t; \varphi(t) = a - t$, then

$$(i) \int_{a-\theta}^{a+\theta} f(x) dx = \int_{-\theta}^{\varphi(\theta)} f(\varphi(t)) dt = \int_{-\theta}^{\theta} f(\varphi(t)) \varphi'(t) dt = \int_{-\theta}^{\theta} f(a-t) dt =$$



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$$\begin{aligned}
 &= \int_{-\theta}^{\theta} \left(\frac{\gamma}{\alpha} - \frac{\beta}{\alpha} f(a-t) \right) dt = \frac{2\gamma}{\alpha} \cdot \theta - \frac{\beta}{\alpha} \int_{-\theta}^{\theta} f(a-t) dt = \\
 &= \frac{2\gamma}{\alpha} \cdot \theta + \frac{\beta}{\alpha} \int_{-\theta}^{\theta} f(\psi(t)) \psi'(t) dt = \frac{2\gamma}{\alpha} \cdot \theta + \frac{\beta}{\alpha} \int_{a-\theta}^{a+\theta} f(x) dx \\
 \int_{a-\theta}^{a+\theta} f(x) dx + \frac{\beta}{\alpha} \int_{a-\theta}^{a+\theta} f(x) dx &= \frac{2\gamma}{\beta} \cdot \theta \Rightarrow \int_{a-\theta}^{a+\theta} f(x) dx = \frac{2\gamma}{\alpha + \beta} \cdot \theta; \alpha + \beta \neq 0 \\
 (\text{ii}) \int_{a-\theta}^{a+\theta} f(x) dx &= \int_{a-\theta}^a f(x) dx + \int_a^{a+\theta} f(x) dx \\
 \int_{a-\theta}^a f(x) dx &= \int_{\varphi(-\theta)}^{\varphi(0)} f(x) dx = \int_{-\theta}^0 f(\varphi(t)) \varphi'(t) dt = \int_{-\theta}^0 f(a+t) dt = \\
 &= \int_{-\theta}^0 \left(\frac{\gamma}{\alpha} - \frac{\beta}{\alpha} f(a-t) \right) dt = \frac{\gamma}{\alpha} \cdot \theta - \frac{\beta}{\alpha} \int_{-\theta}^0 f(a-t) dt = \\
 &= \frac{\gamma}{\alpha} \cdot \theta + \frac{\beta}{\alpha} \int_{\psi(-\theta)}^{\psi(0)} f(\psi(t)) \psi'(t) dt = \frac{\gamma}{\alpha} \cdot \theta + \frac{\beta}{\alpha} \int_{a+\theta}^a f(x) dx
 \end{aligned}$$

So, we have:

$$\int_{a-\theta}^{a+\theta} f(x) dx = \frac{\gamma}{\alpha} \cdot \theta + \frac{\beta}{\alpha} \int_{a+\theta}^a f(x) dx + \int_a^{a+\theta} f(x) dx = \frac{\gamma}{\alpha} \cdot \theta + \frac{\alpha - \beta}{\alpha} \int_a^{a+\theta} f(x) dx$$

Definition. Function $f: [a - \theta, a + \theta] \rightarrow \mathbb{R}$ is **a –even function**,

(**a –odd function**) if $f(a+x) = f(a-x); \forall x \leq |\theta|$,

$$(f(a+x) = -f(a-x); \forall |x| \leq \theta).$$

Application 11. Find:

$$\Omega_n = \int_0^1 \frac{4x^3 - 6x^2 + 8x - 3}{(x^2 - x + 1)^n} dx; n \in \mathbb{N}$$

Solution. $g(x) = x^2 - x + 1$ is $\frac{1}{2}$ – even and $h(x) = 4x^3 - 6x^2 + 8x - 3$ is $\frac{1}{2}$ – odd, so

$$f(x) = \frac{h(x)}{g^n(x)}$$
 is $\frac{1}{2}$ – odd and using **Theorem 3**, we get:



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$$\Omega_n = \int_0^1 \frac{4x^3 - 6x^2 + 8x - 3}{(x^2 - x + 1)^n} dx = 0.$$

Application 12. Find:

$$\Omega_n = \int_0^1 (2x - 1)^{2n+1} e^{x-x^2} dx; n \in \mathbb{N}$$

Solution. $g(x) = (2x - 1)^{2n+1}$ is $\frac{1}{2}$ – odd function and $h(x) = e^{x-x^2}$ is $\frac{1}{2}$ – even function, then $f(x) = g(x) \cdot h(x)$ is $\frac{1}{2}$ – odd function. Using **Theorem 3**, we get:

$$\Omega_n = \int_0^1 (2x - 1)^{2n+1} e^{x-x^2} dx = 0.$$

Corollary 1. For any function $f: [a - \theta, a + \theta] \rightarrow \mathbb{R}$ exist an function $f_1, a - \text{even}$ and $f_2, a - \text{odd}$ such that $f(x) = f_1(x) + f_2(x); \forall x \in [a - \theta, a + \theta]$.

Corollary 2. If $f, g: [a - \theta, a + \theta] \rightarrow \mathbb{R}$ integrable functions and f is $a - \text{odd}$, then

$$\int_{a-\theta}^{a+\theta} f(x)g(x) dx = \int_a^{a+\theta} f(x)(g(x) + g(2a - x)) dx$$

Application 13. Let $f: [-1, 1] \rightarrow \mathbb{R}$ continuous with property $f(x) +$

$f(-x) = \pi; \forall x \in [-1, 1]$. Find:

$$\Omega_n = \int_0^{(2n+1)\pi} f(\cos x) dx; \forall n \in \mathbb{N}$$

Solution. We have:

$$\Omega_n = \Omega_{n-1} + \int_{(2n-1)\pi}^{(2n+1)\pi} f(\cos x) dx, g(x) = f(\cos x) \text{ is } 2n\pi - \text{odd}, \text{ then:}$$

$$I = \int_{(2n-1)\pi}^{(2n+1)\pi} f(\cos x) dx = 2 \int_{2n\pi}^{2n\pi+\pi} f(\cos x) dx = 2 \int_0^\pi f(\cos(t + 2n\pi)) dt =$$

$$= 2 \int_0^\pi f(\cos t) dt = -2 \int_1^{-1} \frac{f(u)}{\sqrt{1-u^2}} du = 2 \int_{-1}^1 \frac{f(u)}{\sqrt{1-u^2}} du$$

$$g(u) = \frac{1}{\sqrt{1-u^2}} \text{ is } 0 - \text{even} \Rightarrow$$

$$I = 2 \int_{-1}^1 \frac{f(u)}{\sqrt{1-u^2}} du = 2 \int_0^1 \frac{f(u) + f(-u)}{\sqrt{1-u^2}} du = 2\pi \int_0^1 \frac{du}{\sqrt{1-u^2}} = \pi^2$$



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$$\text{So, } \Omega_n = \Omega_{n-1} + \pi^2 \Rightarrow \Omega_n = (n+1)\pi^2$$

Application 14. Find:

$$\Omega_n = \int_0^{\frac{\pi}{4}} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

Solution. $f(x) = \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}$ is $\frac{\pi}{2}$ – even, then using **Theorem 3**, we get:

$$\begin{aligned} \Omega_n &= \int_0^{\frac{\pi}{4}} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx = \int_0^{\pi} x f(x) dx = \int_{\frac{\pi}{2}}^{\pi} f(x)(x + \pi - x) dx = \\ &\left(\because g(x) = 1 \text{ is } \frac{3\pi}{4} \text{ – even and } f\left(\frac{3\pi}{2} - x\right) = \frac{\cos^{2n} x}{\sin^{2n} x + \cos^{2n} x} \right) \\ &= \pi \int_{\frac{\pi}{2}}^{\pi} f(x) dx = \pi \int_{\frac{3\pi}{4}}^{\pi} \left(f(x) + f\left(\frac{3\pi}{2} - x\right) \right) dx = \pi \int_{\frac{3\pi}{4}}^{\pi} dx = \frac{\pi^2}{4} \end{aligned}$$

Application 15. Find:

$$\Omega = \int_0^{\frac{\pi}{4}} \frac{\log(1 + \tan x)}{\sin 2x + \cos 2x} dx$$

Solution. $f(x) = \frac{1}{\sin 2x + \cos 2x} = \frac{1}{\sqrt{2} \cos\left(2x - \frac{\pi}{4}\right)}$ ⇒ f is $\frac{\pi}{8}$ – even.

$$\begin{aligned} \Omega &= \int_0^{\frac{\pi}{4}} \frac{\log(1 + \tan x)}{\sin 2x + \cos 2x} dx = \int_0^{\frac{\pi}{4}} f(x) \log(1 + \tan x) dx = \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} f(x) \left(\log(1 + \tan x) + \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) \right) dx = \\ &= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} f(x) \left(\log(1 + \tan x) + \log\left(\frac{2}{1 + \tan x}\right) \right) dx = \frac{\log 2}{\sqrt{2}} \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{dx}{\cos\left(2x - \frac{\pi}{4}\right)} \end{aligned}$$

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