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UP.413 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\tan^{-1}k}{k}\cdot\tan^{-1}(n-k+1)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer, Solution 2 by Ravi Prakash-New Delhi-India, Solution 3 by Naren Bhandari-Bajura-Nepal

Solution 1 by proposer

$$0\le\left|\frac{\frac{\tan^{-1}1}{1}\cdot\tan^{-1}n+\frac{\tan^{-1}2}{2}\cdot\tan^{-1}(n-1)+\cdots+\frac{\tan^{-1}n}{n}\cdot\tan^{-1}1}{n}\right|=$$

$$=\frac{1}{n}\cdot\left|\frac{\tan^{-1}1}{1}\cdot\tan^{-1}n+\frac{\tan^{-1}2}{2}\cdot\tan^{-1}(n-1)+\cdots+\frac{\tan^{-1}n}{n}\cdot\tan^{-1}1\right|\le$$

$$\le\frac{\pi}{2}\cdot\frac{\left|\frac{\tan^{-1}1}{1}\right|+\left|\frac{\tan^{-1}2}{2}\right|+\cdots+\left|\frac{\tan^{-1}n}{n}\right|}{n}$$

Hence,

$$0 \le \lim_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\frac{\tan^{-1}k}{k}\cdot\tan^{-1}(n-k+1)\right|$$

$$\le \frac{\pi}{2}\lim_{n\to\infty}\frac{\left|\frac{\tan^{-1}1}{1}\right|+\left|\frac{\tan^{-1}2}{2}\right|+\cdots+\left|\frac{\tan^{-1}n}{n}\right|}{n}\overset{L.C-S}{=}$$

$$=\frac{\pi}{2}\cdot\lim_{n\to\infty}\left|\frac{\tan^{-1}(n+1)}{n+1}\right|=\frac{\pi}{2}\cdot\frac{\frac{\pi}{2}}{\infty}=\frac{\pi}{2}\cdot 0=0$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\tan^{-1}k}{k}\cdot\tan^{-1}(n-k+1)=0$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\because \tan^{-1}x<\frac{\pi}{2};\forall x>0\Rightarrow \tan^{-1}k\cdot\tan^{-1}(n-k+1)<\frac{\pi^2}{4};\forall k=\overline{1,n}$$

$$0<\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}\tan^{-1}k\tan^{-1}(n-k+1)<\frac{1}{n}\cdot\frac{\pi^2}{4}\cdot\sum_{k=1}^{n}\frac{1}{k}\overset{note}{=}a_n$$

As $\frac{1}{n}\to 0$as $n\to\infty$, then $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}\to 0$ as $n\to\infty$

Using Cauchy's first theorem, we have $a_n\to 0$ as $n\to\infty$.

By the Squeeze Theorem, it follows that:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\tan^{-1}k}{n}\tan^{-1}(n-k+1)=0$$

Solution 3 by Naren Bhandari-Bajura-Nepal

Let the sequence and function be defined as

$$q_n=\frac{\tan^{-1}k}{k}\cdot\tan^{-1}(n-k+1)\,,f(x)=\frac{\tan^{-1}x}{x};x\in(-0,\infty)$$

then it is easy to see that $q_n>0;\forall n>1$ and around $n\to\infty, q_n$ behaves like

$\frac{\pi}{2}\cdot\frac{\tan^{-1}k}{k}$ and on the other hand

$$0<\sum_{k=1}^{n}q_n<\frac{\pi}{2}\int_0^n f(x)\,dx=\frac{\pi}{2}\int_0^n\frac{\tan^{-1}x}{x}dx=\frac{\pi}{2}Ti_2(n)$$

where $Ti_2(x)$ is inverse tangent integral and which follows

$$0 < \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} q_n < \lim_{n\to\infty}\frac{\pi}{2n}\int_0^n f(x)\,dx = \lim_{n\to\infty}\frac{\pi}{2}\cdot\frac{Ti_2(n)}{n} = \frac{\pi^2}{2}\lim_{n\to\infty}\frac{\log n}{n} = 0$$

By squeeze theorem we have required limit 0. We use the asymptotic expansion of inverse tangent integral.