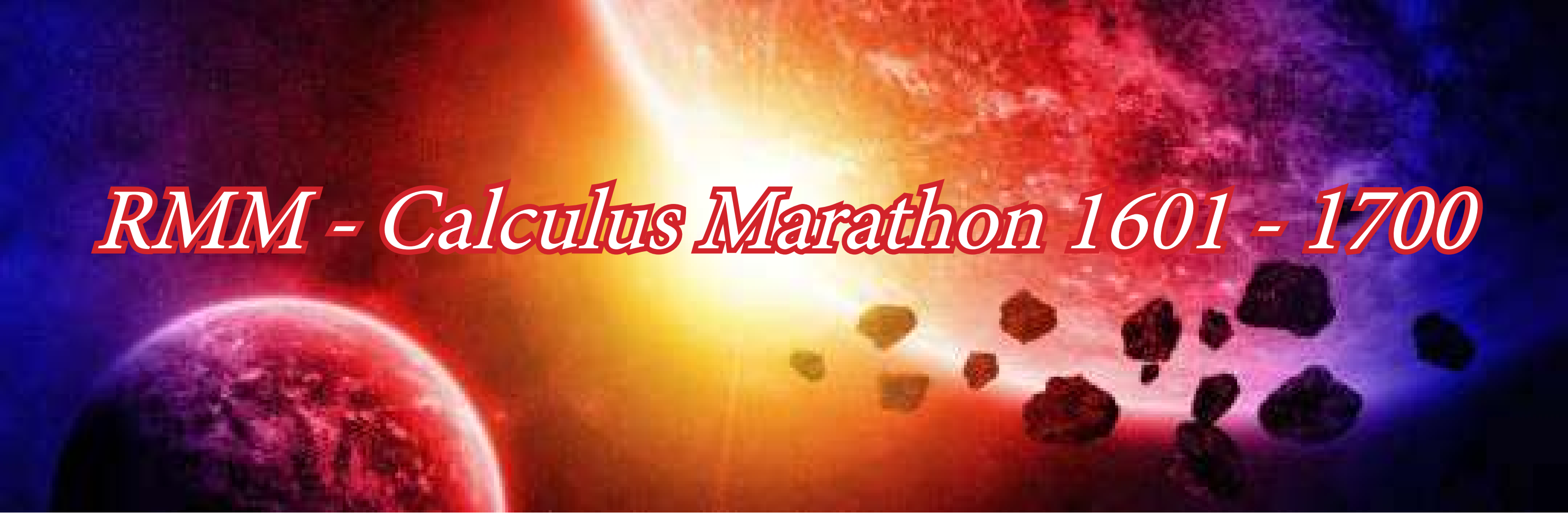

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1601. For any complex number q with $Re(q) \le 1$, let

$$R(q) = \int_0^q \frac{e^x - 1}{x} \log^2\left(\frac{q}{x}\right) dx$$

Then prove the following identity:

$$\int_0^\infty e^{-q}\left(1 + \frac{1}{q}\right) R(q)\, dq = 2\big(\zeta(3) + \zeta(4)\big)$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Rana Ranino-Setif-Algerie

$$R(q) = \int_0^q \frac{e^x - 1}{x} \log^2\left(\frac{q}{x}\right) dx \overset{x=qt}{=} \int_0^1 \frac{e^{qt} - 1}{t} \log^2 t\, dt \overset{IBP}{=}$$

$$= \left[\frac{1}{3}(e^{qt} - 1)\log^3 t\right]_0^1 - \frac{1}{3}\int_0^1 q e^{qt} \log^3 t\, dt = -\frac{1}{3}\int_0^1 q e^{qt}\log^3 t\, dt$$

$$\Omega = \int_0^\infty e^{-q}\left(1 + \frac{1}{q}\right) R(q)\, dq = -\frac{1}{3}\int_0^1 \log^3 t \int_0^\infty (1+q)e^{-q(1-t)} dq dt =$$

$$= -\frac{1}{3}\int_0^1 \left(\frac{1}{(1-t)^2} + \frac{1}{1-t}\right)\log^3 t\, dt = -\frac{1}{3}\sum_{n=0}^\infty (n+2)\int_0^1 t^n \log^3 t\, dt$$

$$\int_0^1 t^n \log^3 t\, dt \overset{t=e^{-y}}{=} -\int_0^\infty y^3 e^{-(n+1)y} dy = -\frac{\Gamma(4)}{(n+1)^4} = -\frac{6}{(1+n)^4}$$

Hence,

$$\Omega = 2\sum_{n=0}^\infty \frac{n+2}{(n+1)^4} = 2\sum_{n=0}^\infty \left(\frac{1}{(n+1)^3} + \frac{1}{(n+1)^4}\right) = 2\big(\zeta(3) + \zeta(4)\big)$$

Solution 2 by Rana Ranino-Setif-Algerie

$$R(q) = \int_0^q \frac{e^x - 1}{x}\log^2\left(\frac{q}{x}\right) dx \overset{x=qt}{=} \int_0^1 \frac{e^{qt} - 1}{t}\log^2 t\, dt =$$

$$= \sum_{k=1}^\infty \frac{q^k}{k!}\int_0^1 t^{k-1}\log^2 t\, dt = 2\sum_{k=1}^\infty \frac{q^k}{k^3 k!}$$

$$\Omega = \int_0^\infty e^{-q}\left(1 + \frac{1}{q}\right) R(q)\, dq = 2\sum_{k=1}^\infty \frac{1}{k^3 k!}\int_0^\infty (q^k + q^{k-1})e^{-q}\, dq =$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{k^3k!}(k!+(k-1)!)$$

Therefore,

$$\int_0^{\infty} e^{-q}\left(1+\frac{1}{q}\right)R(q)\,dq = 2\sum_{k=1}^{\infty}\frac{1}{k^3}+2\sum_{k=1}^{\infty}\frac{1}{k^4} = 2\big(\zeta(3)+\zeta(4)\big)$$

1602. For $x>0$ let $u_1 = 4x^2+2(4k+1)x+4k^2+2k$

$$u_2 = x^4+4kx^3+(6k^2+1)x^2+2k(2k^2+1)x+k^4+k^2$$

$$u_3(x) = x^4+4(k+1)x^3+(6k^2+12k+7)x^2+2(2k^3+6k^2+7k+3)x+k^4+4k^3+7k^2+6k+2$$

$$a_k(x)=\sqrt{u_1+\sqrt{u_1+\sqrt{u_1+\cdots}}};\ b_k(x)=\sqrt{u_2+\sqrt{u_2+\sqrt{u_2+\cdots}}};$$

$$c_k(x)=\sqrt{u_3+\sqrt{u_3+\sqrt{u_3+\cdots}}};\ \Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\int_2^3\frac{a_k(x)dx}{b_k(x)\cdot c_k(x)}\right)$$

Prove that the roots of the equation $\sin\Omega = \frac{1}{\sqrt{170}}(8u^3-60u^2+142u-104)$ are in arithmetic progression.

Proposed by Costel Florea-Romania

Solution by Adrian Popa-Romania

$$a_k(x)=\sqrt{u_1+\sqrt{u_1+\sqrt{u_1+\cdots}}}\Rightarrow a_k^2(x)-a_k(x)-u_1=0$$

$$a_k(x)=\frac{1+\sqrt{1+4u_1}}{2}\text{ and } 1+4u_1=\big(4x+(4k+1)\big)^2\Rightarrow$$

$$a_k(x)=\frac{1+4x+4k+1}{2}=2x+2k+1$$

$$b_k(x)=\sqrt{u_2+\sqrt{u_2+\sqrt{u_2+\cdots}}};\ b_k(x)=\sqrt{u_2+b_k(1)}\Rightarrow$$

$$\Rightarrow b_k^2-b_k(x)-u_2=0\Rightarrow b_k(x)=\frac{1+\sqrt{1+4u_2}}{2}\Rightarrow b_k(x)=x^2+2kx+k^2+1$$

$$c_k(x)=\sqrt{u_3+\sqrt{u_3+\sqrt{u_3+\cdots}}}=\sqrt{u_3+c_k(x)}\Rightarrow c_k^2(x)-c_k(x)-u_3=0$$

$$c_k(x)=x^2+2(k+1)x+k^2+2k+2$$

$$\frac{a_k(x)}{b_k(x)\cdot c_k(x)}=\frac{2x+2k+1}{(x^2+2kx+k^2+1)(x^2+2(k+1)x+(k+1)^2+1)}=$$

$$=\frac{1}{x^2+2kx+k^2+1}-\frac{1}{x^2+2(k+1)x+(k+1)^2+1}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\int_2^3\left(\frac{1}{x^2+2kx+k^2+1}-\frac{1}{x^2+2(k+1)x+(k+1)^2+1}\right)dx=$$

$$=\lim_{n\to\infty}\int_2^3\sum_{k=1}^{n}\left(\frac{1}{x^2+2kx+k^2+1}-\frac{1}{x^2+2(k+1)x+(k+1)^2+1}\right)dx=$$

$$=\lim_{n\to\infty}\int_2^3\left(\frac{1}{x^2+2x+2}-\underbrace{\frac{1}{x^2+2(n+1)x+(n+1)^2+1}}_{\to 0}\right)dx=$$

$$=\int_2^3\frac{1}{(x+1)^2+1}dx=[\tan^{-1}(x+1)]_2^3=\tan^{-1}\frac{1}{13}\Rightarrow\Omega=\tan^{-1}\left(\frac{1}{13}\right)$$

Now, we have: $\tan x=\frac{\sin x}{\sqrt{1-\sin^2 x}}\Rightarrow\tan^2 x=\frac{\sin^2 x}{1-\sin^2 x}\Rightarrow\tan^2 x-\tan^2 x\cdot\sin^2 x=\sin^2 x$

$$\sin x=\frac{\tan x}{\sqrt{1+\tan^2 x}}\Rightarrow\sin(\tan^{-1}x)=\frac{\tan(\tan^{-1}x)}{\sqrt{1+\tan^2(\tan^{-1}x)}}\Rightarrow\sin\Omega=\frac{1}{\sqrt{170}}$$

But: $\sin\Omega=\frac{1}{\sqrt{170}}(8u^3-60u^2+142u-104)$, then

$$8u^3-60u^2+142u-105=0\Leftrightarrow\left(u-\frac{5}{2}\right)(8u^2-40u+42)=0$$

$u_1=\frac{3}{2},u_2=\frac{5}{2},u_3=\frac{7}{2}$. Then $u_2=\frac{u_1+u_3}{2}$

1603. For $k\geq 2$, prove the following identity:

$$\frac{\partial}{\partial k}\int_0^{\frac{\pi}{2}}\sin x\log\left(\frac{\sin^2 x}{k-\sqrt{k^2-\sin^2 x}}\right)dx=\coth^{-1}k$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Asmat Qatea-Afghanistan

$$\because \int \frac{1}{\sqrt{a^2+x^2}}dx = \log\left(x+\sqrt{a^2+x^2}\right)+C$$

We have:

$$\frac{\partial}{\partial k}\int_0^{\frac{\pi}{2}} \sin x \log\left(\frac{\sin^2 x}{k-\sqrt{k^2-\sin^2 x}}\right)dx =$$

$$= \int_0^{\frac{\pi}{2}} \sin^3 x \cdot \frac{\dfrac{1-\dfrac{2k}{2\sqrt{k^2-\sin^2 x}}}{\left(k-\sqrt{k^2-\sin^2 x}\right)^2}}{\dfrac{\sin^2 x}{k-\sqrt{k^2-\sin^2 x}}}dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{k^2-\sin^2 x}}dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{k^2-1+\cos^2 x}}dx \overset{\cos x = u}{=} \int_0^1 \frac{du}{\sqrt{k^2-1+u^2}} = \log\left(u+\sqrt{u^2+k^2-1}\right|_0^1 =$$

$$= \log(1+k) - \frac{1}{2}\log(k^2-1) =$$

$$= \log(k+1) - \frac{1}{2}\log(k+1) - \frac{1}{2}\log(k-1) = \frac{1}{2}\log\left(\frac{k+1}{k-1}\right) = \coth^{-1} k$$

1604. Find:

$$\Omega = \int_0^{\frac{\pi}{2}} x^4\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$$

Proposed by Sujeethan Balendran-SriLanka

Solution by Cornel Ioan Vălean-Romania

Using well-known result:

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{4^n}\binom{2n}{n}\cos(2nx) = \frac{\cos\frac{x}{2}}{\sqrt{2\cos x}};\ |x| < \frac{\pi}{2}$$

which by simple rearrangements may be brought to the following useful form:

$$2\sqrt{2}\sum_{n=0}^{\infty} \frac{1}{4^n}\binom{2n}{n}\cos(4nx) = \sqrt{\tan x}+\sqrt{\cot x}$$

and this is exactly what we need in our proof.

So, we write and obtain that:

$$\Omega=\int_0^{\frac{\pi}{2}}x^4\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx=\int_0^{\frac{\pi}{2}}x^4\left(2\sqrt{2}+2\sqrt{2}\sum_{n=1}^{\infty}\frac{1}{4^n}\binom{2n}{n}\cos(4nx)\right)dx=$$

$$=\frac{\pi^5}{40\sqrt{2}}+\frac{\pi^3}{2^{\frac{7}{2}}}\sum_{n=1}^{\infty}\frac{1}{n^2 4^n}\binom{2n}{n}-\frac{3\pi}{2^{\frac{9}{2}}}\sum_{n=1}^{\infty}\frac{1}{n^4 4^n}\binom{2n}{n}=$$

$$=\frac{3\log 2}{4\sqrt{2}}\pi\zeta(2)+\frac{\log^4 2}{8\sqrt{2}}\pi+\frac{79}{1920\sqrt{2}}\pi^5-\frac{5\log^2 2}{16\sqrt{2}}\pi^3,$$

where the binoharmonic series are both well-known in the mathematical literature, and straightforward (one reduces to trivial primitives, and the other one reduces to Beta function limits). So, we have that:

$$\sum_{n=1}^{\infty}\frac{1}{n^2 4^n}\binom{2n}{n}=\frac{\pi^2}{6}-2\log^2(2)$$

and for the other one we get

$$\sum_{n=1}^{\infty}\frac{1}{n^4 4^n}\binom{2n}{n}=\frac{\pi^4}{40}-4\log(2)\zeta(3)+\frac{1}{3}\log^2(2)\pi^2-\frac{2}{3}\log^4(2)$$

1605. Prove that:

$$\int_0^{\infty}\left(\frac{\cos^2(\pi x)\,\Gamma\left(\frac{1}{2}+x\right)}{(1+x^2)\Gamma(1+x)}+\frac{\sin(2\pi x)\,\Gamma(x)}{2(1+x^2)\Gamma\left(\frac{1}{2}+x\right)}\right)dx=Re\left(\frac{\pi(1+e^{-2\pi})\Gamma\left(\frac{1}{2}-i\right)}{2\Gamma(1-i)}\right)$$

Proposed by Sujeethan Balendran-SriLanka

Solution by Felix Marin-Venezuela

$$J=\int_0^{\infty}\left(\frac{\cos^2(\pi x)\,\Gamma\left(\frac{1}{2}+x\right)}{(1+x^2)\Gamma(1+x)}+\frac{\sin(2\pi x)\,\Gamma(x)}{2(1+x^2)\Gamma\left(\frac{1}{2}+x\right)}\right)dx=$$

$$=\sqrt{\pi}Re\left(\int_{-\infty}^{\infty}e^{i\pi x}\binom{-\frac{1}{2}}{x}\frac{dx}{1+x^2}\right)=$$

$$=\sqrt{\pi}Re\left(\int_{-\infty}^{\infty}e^{i\pi x}\left(\int_{-\pi}^{\pi}\frac{\left(1+e^{i\phi}\right)^{-\frac{1}{2}}}{e^{ix\phi}}\frac{d\phi}{2\pi}\right)\right)\frac{dx}{1+x^2}=$$

$$=\frac{1}{2\sqrt{\pi}}e^{-\pi}Im\left(\oint_{|z|=1}z^{-1-i}(1+z)^{-\frac{1}{2}}dz\right)\overset{\epsilon\to 0^+}{\sim}$$

$$\sim Im\left[-\int_{-1}^{-\epsilon}\frac{(-x)^{-1-i}e^{i\pi(-1-i)}}{\sqrt{1+x}}dx-\int_{\pi}^{-\pi}\epsilon^{-1-i}e^{i(-1-i)}\epsilon e^{i\theta}i\,d\theta\right]$$

$$-Im\left(\int_{-\epsilon}^{-1}\frac{(-x)^{-1-i}e^{-i\pi(-1-i)}}{\sqrt{1+x}}dx\right)=$$

$$=\frac{\sqrt{\pi}}{2}e^{-\pi}\cdot Im\left(e^{\pi}\int_{\epsilon}^{1}x^{-1-i}(1-x)^{-\frac{1}{2}}dx+2i\sinh(\pi)\,\epsilon^{-i}-e^{-\pi}\int_{\epsilon}^{1}x^{-1-i}(1-x)^{-\frac{1}{2}}dx\right)$$

$$=$$

$$=\frac{\sqrt{\pi}}{2}e^{-\pi}\cdot Im\left(2\sinh(\pi)\int_{\epsilon}^{1}x^{-1-i}(1-x)^{-\frac{1}{2}}dx+2i\sinh(\pi)\,\epsilon^{-i}\right)=$$

$$=\frac{\pi(1-e^{-2\pi})}{2}Im\left(\frac{\Gamma(-i)}{\Gamma\left(\frac{1}{2}-i\right)}\right)\cong 1.2473$$

1606. Find:

$$\Omega=\int_0^{\infty}\sinh^{-1}(\operatorname{csch}x)\,dx$$

Proposed by Abdul Mukhtar-Nigeria

Solution 1 by Asmat Qatea-Afghanistan

$\because \sinh^{-1}x=\log\left(x+\sqrt{1+x^2}\right)$ and $\operatorname{csch}x=\frac{2}{e^x-e^{-x}}$, we get:

$$\Omega=\int_0^{\infty}\log\left(\frac{2}{e^x-e^{-x}}+\sqrt{1+\left(\frac{2}{e^x-e^{-x}}\right)^2}\right)dx=\int_0^{\infty}\log\left(\frac{2+e^x+e^{-x}}{e^x-e^{-x}}\right)dx=$$

$$=\int_0^{\infty}\log\left(\frac{2e^{-x}+e^{-2x}+1}{1-e^{-2x}}\right)dx\overset{e^{-x}=u}{=}\int_0^1\frac{1}{u}\log\left(\frac{u^2+2u+1}{1-u^2}\right)du=$$

$$=\int_0^1\frac{1}{u}\log\left(\frac{1+u}{1-u}\right)du=\int_0^1\frac{1}{u}\log(1+u)\,du-\int_0^1\frac{1}{u}\log(1-u)\,du=$$

$$= -\int_0^1 \frac{1}{u}\sum_{k=1}^{\infty}\frac{(-1)^k u^k}{k}du - \int_0^1 \frac{1}{u}\sum_{k=1}^{\infty}\frac{-u^k}{k}du =$$

$$= -\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^1 u^{k-1}du + \sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 u^{k-1}du = -\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} + \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{12} + \frac{\pi^2}{6}$$

Therefore,

$$\Omega = \int_0^{\infty} \sinh^{-1}(\operatorname{csch} x)\,dx = \frac{\pi^2}{4}$$

Solution 2 by Ose Favour-Nigeria

$$\because \operatorname{csch} x = \frac{2}{e^x - e^{-x}} \text{ and } \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\Omega = \int_0^{\infty} \log(scsh\, x + \coth x)\,dx = \int_0^{\infty} \log(e^x + e^{-x} + 2)\,dx - \int_0^{\infty} \log(e^x - e^{-x})\,dx$$

$$= \Phi - \Psi$$

$$\Phi \overset{u=e^{-x}}{=} \int_0^1 \frac{1}{u}\log\left(\frac{1}{u} + u + 2\right)du = \int_0^1 \frac{1}{u}\log\left(\frac{(u+1)^2}{u}\right)du =$$

$$= 2\int_0^1 \frac{\log(1+u)}{u}du - \int_0^1 \frac{\log u}{u}du = \frac{\pi^2}{6} - li(1)$$

$$\Psi = \int_0^{\infty} \log(e^x(1 - e^{-2x}))dx \overset{u=e^{-x}}{=} = \int_0^1 \frac{1}{u}\log\left(\frac{1-u^2}{u}\right)du =$$

$$= \int_0^1 \frac{\log(1-u^2)}{u}du - \int_0^1 \frac{\log u}{u}du = -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 u^{2n-1}du - li(1) =$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} - li(1) = -\frac{1}{2}\zeta(2) - li(1) = -\frac{\pi^2}{12} - li(1)$$

Therefore,

$$\Omega = \int_0^{\infty} \sinh^{-1}(\operatorname{csch} x)\,dx = \frac{\pi^2}{4}$$

Solution 3 by proposer

$$\Omega = \int_0^{\infty} \sinh^{-1}(\operatorname{csch} x)\,dx = (x\cdot\sinh^{-1}x\cdot\operatorname{csch} x)_0^{\infty} + \int_0^{\infty}\frac{x\cdot\cosh x}{\sinh^2 x}\cdot\frac{dx}{\sqrt{1+\operatorname{csch}^2 x}} =$$

$$=\int_0^\infty \frac{x}{\sqrt{\cosh^2 x}}\cdot\frac{\cosh x}{\sinh x}dx=\int_0^\infty \frac{x}{\sinh x}dx=\int_0^\infty \frac{2xdx}{e^x-e^{-x}}=\int_0^\infty \frac{2xdx}{e^x(1-e^{-2x})}=$$

$$=\int_0^\infty \frac{2x\cdot e^{-2x}}{1-e^{-2x}}dx=\int_0^\infty 2x\cdot e^{-x}\cdot\sum_{n=0}^\infty (e^{-2x})^n=\sum_{n=0}^\infty\int_0^\infty 2x\cdot e^{-(2n+1)x}dx \overset{t=(2n+1)x}{=}$$

$$=2\sum_{n=0}^\infty \frac{1}{(2n+1)^2}\int_0^\infty e^{-t}\,dt=2\sum_{n=0}^\infty \Gamma(2)=2\sum_{n=0}^\infty \frac{1}{(2n+1)^2}=$$

$$=2\left\{\sum_{n=0}^\infty \frac{1}{n^2}-\sum_{n=1}^\infty\frac{1}{(2n)^2}\right\}=\frac{\pi^2}{4}$$

1607. Prove that:

$$\int_0^\infty \frac{xe^{\pi x}}{(1+4x^2)^2(e^{2\pi x}-1)}dx=\frac{1}{16}(2G-1)$$

Where $G=\sum_{n=1}^\infty \frac{(-1)^{n-1}}{(2n-1)^2}$ – Catalan's constant.

Proposed by Ngulmun George Baite-India

Solution 1 by Felix Marin-Venezuela

$$\int_0^\infty \frac{xe^{\pi x}}{(1+4x^2)^2(e^{2\pi x}-1)}dx=\int_0^\infty \frac{x}{(1+4x^2)(2\sinh(\pi x))}dx=$$

$$=\frac{1}{8}\left\{i\int_0^\infty \frac{[1+2(ix)]^{-2}-[1+2(-ix)]^2}{2\sinh(\pi x)}\right\}$$

Where $\{*\}'$s enclosed expression can be evaluated with the *Abel-Plana Formula.*

Namely,

$$\int_0^\infty \frac{xe^{\pi x}}{(1+4x^2)^2(e^{2\pi x}-1)}dx=\frac{1}{8}\left\{\sum_{n=0}^\infty (-1)^n(1+2n)^{-2}-\left[\frac{1}{2}(1+2x)^{-2}\right]_{x=0}\right\}=$$

$$=\frac{1}{8}\left\{\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2}-\frac{1}{2}\right\}=\frac{1}{16}(2G-1)\cong 0.0520$$

Solution 2 by Ajetunmobi Abdulqoyyum-Nigeria

$$\because \sinh(x)=\frac{e^x-e^{-x}}{2}\Rightarrow \frac{1}{\sinh(x)}=\frac{2}{e^x-e^{-x}}\Rightarrow \operatorname{csch}(x)=\frac{2e^x}{e^{2x}-1}$$

$$\Rightarrow \operatorname{csch}(\pi x)=\frac{2e^{\pi x}}{e^{2\pi x}-1}\Rightarrow \frac{e^{\pi x}}{e^{2\pi x}-1}=\frac{1}{2}\operatorname{csch}(\pi x)$$

Thus,

$$\Omega=\int_0^{\infty}\frac{xe^{\pi x}}{(1+4x^2)^2(e^{2\pi x}-1)}dx=\frac{1}{2}\int_0^{\infty}\frac{x\cdot\operatorname{csch}(\pi x)}{(1+4x^2)^2}dx\overset{(*)}{=}$$

$$\left(\because \operatorname{csch}(x)=\frac{1}{x}+2x\sum_{n\ge1}\frac{(-1)^n}{n^2\pi^2+x^2}\Rightarrow \operatorname{csch}(\pi x)=\frac{1}{\pi x}+\frac{2x}{\pi}\sum_{n\ge1}\frac{(-1)^n}{n^2+x^2}\right)$$

$$\overset{(*)}{=}\frac{1}{2}\left(\int_0^{\infty}\frac{x}{(1+4x^2)^2}\left(\frac{1}{\pi x}+\frac{2x}{\pi}\sum_{n\ge1}\frac{(-1)^n}{n^2+x^2}\right)\right)dx=$$

$$=\frac{1}{2}\left(\frac{1}{\pi}\int_0^{\infty}\frac{dx}{(1+4x^2)^2}+\frac{2}{\pi}\sum_{n\ge1}(-1)^n\int_0^{\infty}\frac{x^2}{(n^2+x^2)(1+4x^2)^2}dx\right)$$

$$A=\int_0^{\infty}\frac{dx}{(1+4x^2)^2}=\frac{1}{2}\int_0^{\infty}\frac{dx}{(1+x^2)^2}=\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\pi}{8}$$

$$B=\int_0^{\infty}\frac{x^2}{(n^2+x^2)(1+4x^2)^2}dx=$$

$$=\int_0^{\infty}\left(\frac{4n^2}{(4n^2-1)^2(4x^2+1)}-\frac{1}{(4x^2+1)^2(4n^2-1)}-\frac{n^2}{(4n^2-1)^2(n^2+x^2)}\right)dx=$$

$$=\frac{4n^2}{4n^2-1}\int_0^{\infty}\frac{dx}{4x^2+1}-\frac{1}{4n^2-1}\int_0^{\infty}\frac{dx}{(1+4x^2)^2}-\frac{n^2}{(4n^2-1)^2}\int_0^{\infty}\frac{dx}{n^2+x^2}=$$

$$=\frac{n^2\pi}{(4n^2-1)^2}-\frac{\pi}{8(4n^2-1)}-\frac{n\pi}{2(4n^2-1)^2}$$

$$\Omega=\frac{1}{2}\left(\frac{1}{8}+2\left(\sum_{n\ge1}\frac{n^2(-1)^n}{(4n^2-1)^2}-\frac{1}{8}\sum_{n\ge1}\frac{(-1)^n}{4n^2-1}-\frac{1}{2}\sum_{n\ge1}\frac{(-1)^n n}{(4n^2-1)^2}\right)\right)$$

$$C=\sum_{n\ge1}\frac{n^2(-1)^n}{(4n^2-1)^2}=\frac{1}{4}\left(\sum_{n\ge1}\frac{(-1)^n}{4n^2-1}+\sum_{n\ge1}\frac{(-1)^n}{(4n^2-1)^2}\right)=$$

$$=\frac{1}{4}\left(\frac{1}{2}\sum_{n\ge1}(-1)^n\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)+\sum_{n\ge1}\frac{(-1)^n}{(2n-1)^2(2n+1)^2}\right)=$$

$$=\frac{1}{4}\left(\frac{1}{2}\left(-\frac{\pi}{4}-\left(\frac{\pi}{4}-1\right)\right)+\sum_{n\geq 1}\frac{(-1)^n}{4}\left(\frac{1}{2n+1}-\frac{1}{2n-1}+\frac{1}{(2n+1)^2}+\frac{1}{(2n-1)^2}\right)\right.$$

$$=-\frac{\pi}{32}$$

$$D=\sum_{n\geq 1}\frac{(-1)^n}{4n^2-1}=\sum_{n\geq 1}\frac{(-1)^n}{(2n-1)(2n+1)}=\frac{1}{2}-\frac{\pi}{4}$$

$$E=\sum_{n\geq 1}\frac{(-1)^n n}{(4n^2-1)^2}=\sum_{n\geq 1}(-1)^n\left(\frac{n}{(2n-1)^2(2n+1)^2}\right)=$$

$$=\frac{1}{8}\sum_{n\geq 1}(-1)^n\left(\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\right)=\frac{1}{8}-\frac{G}{4}$$

Hence,

$$\Omega=\frac{1}{2}\left(\frac{1}{8}+2\left(-\frac{\pi}{32}-\frac{1}{8}\left(\frac{1}{2}-\frac{\pi}{4}\right)-\frac{1}{16}+\frac{G}{8}\right)\right)=\frac{1}{16}(2G-1)$$

Therefore,

$$\int_0^\infty \frac{xe^{\pi x}}{(1+4x^2)^2(e^{2\pi x}-1)}dx=\frac{1}{16}(2G-1)$$

1608. Prove that:

$$\Omega=\int_0^{\frac{\pi}{2}}\log(1+\tan^4 x)\cos^2 x\,dx=\frac{\pi\log(6+4\sqrt{2})}{4}-\frac{\pi}{2}$$

Proposed by Ajetunmobi Abdulqoyyum-Nigeria

Solution 1 by Kartick Chandra Betal-India

$$\Omega=\int_0^{\frac{\pi}{2}}\log(1+\tan^4 x)\cos^2 x\,dx=\int_0^\infty \frac{\log(1+x^4)}{(1+x^2)^2}dx=$$

$$=\int_0^\infty \frac{x^2\{\log(1+x^4)-4\log x\}}{(1+x^2)^2}dx$$

$$2\Omega=\int_0^\infty \frac{\log(1+x^4)}{1+x^2}dx-4\int_0^\infty \frac{x^2\log x}{(1+x^2)^2}dx$$

$$\Omega=\frac{1}{2}\int_0^{\frac{\pi}{2}}\log(1+\tan^4 x)\,dx+2\int_0^\infty \frac{\log x}{(1+x^2)^2}dx=$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\log(1-2\sin^2 x\cos^2 x)\,dx-2\int_0^{\frac{\pi}{2}}\log(\cos x)\,dx+\int_0^{\infty}\frac{1-x^2}{(x^2+1)^2}\log x\,dx=$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\log\left(1-\frac{\sin^2 2x}{2}\right)dx+\pi\log 2-\int_0^{\infty}\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2}\log x\,dx=$$

$$=\frac{1}{4}\int_0^{\pi}\log\left(1-\frac{\sin^2 x}{2}\right)dx+\pi\log 2-\frac{\log x}{x+\frac{1}{x}}\Bigg|_0^{\infty}-\int_0^{\infty}\frac{dx}{x\left(x+\frac{1}{x}\right)}=$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\log\left\{\left(\frac{1}{\sqrt{2}}\right)^2\sin^2 x+\cos^2 x\right\}dx+\pi\log 2-\int_0^{\infty}\frac{dx}{1+x^2}=$$

$$=\frac{\pi}{2}\log\left(\frac{\frac{1}{\sqrt{2}}+1}{2}\right)+\pi\log 2-\frac{\pi}{2}=\frac{\pi}{4}\log\left(\frac{3+2\sqrt{2}}{8}\cdot 16\right)-\frac{\pi}{2}=\frac{\pi}{4}\log(6+4\sqrt{2})-\frac{\pi}{2}$$

Solution 2 by Jack Desire-Nigeria

$$\Omega=\int_0^{\frac{\pi}{2}}\log(1+\tan^4 x)\cos^2 x\,dx=$$

$$=\int_0^{\frac{\pi}{2}}\cos^2 x\log(\cos^4 x+\sin^4 x)\,dx-4\int_0^{\frac{\pi}{2}}\cos^2 x\log(\cos x)\,dx=$$

$$=\int_0^{\frac{\pi}{2}}\cos^2 x\log((\cos^2 x+\sin^2 x)^2-2\sin^2 x\cos^2 x)\,dx-4\int_0^{\frac{\pi}{2}}\cos^2 x\log(\cos x)\,dx$$

$$=\int_0^{\frac{\pi}{2}}\cos^2 x\log(1-2\sin^2 x\cos^2 x)\,dx-4\int_0^{\frac{\pi}{2}}\cos^2 x\log(\cos x)\,dx$$

$$\Omega_1=\int_0^{\frac{\pi}{2}}\cos^2 x\log(1-2\sin^2 x\cos^2 x)\,dx=-\int_0^{\frac{\pi}{2}}\cos^2 x\sum_{k=1}^{\infty}\frac{2^k\sin^{2k}x\cos^{2k}x}{k}dx=$$

$$=-\sum_{k=1}^{\infty}\frac{2^k}{k}\int_0^{\frac{\pi}{2}}\sin^{2k}x\cos^{2k+2}x\,dx=\sum_{k=1}^{\infty}\frac{2^k}{k}\left(\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}{2\Gamma(2k+2)}\right)=$$

$$=-\sum_{k=1}^{\infty}\frac{2^{k-1}}{k}\left(\frac{\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{(2k+1)\Gamma(2k+1)}\right)=-\sum_{k=1}^{\infty}\frac{2^{k-2}}{k}\left(\frac{\Gamma^2\left(k+\frac{1}{2}\right)}{\Gamma(2k+1)}\right)$$

$$\because \frac{\Gamma(k)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(2k)}=\frac{\sqrt{\pi}}{2^{2k-1}}\Rightarrow\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma(k+1)}{\Gamma(2k+1)}=\frac{\sqrt{\pi}}{2^{2k}};\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(2k+1)}=\sqrt{\pi}/2^{2k}\Gamma(k+1)$$

$$\Omega_1=-\sum_{k=1}^{\infty}\frac{2^{k-2}}{k}\left(\frac{\sqrt{\pi}\Gamma\left(k+\frac{1}{2}\right)}{2^{2k}\Gamma(k+1)}\right)=-\sqrt{\pi}\sum_{k=1}^{\infty}\frac{2^{-k-2}}{k}\left(\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)}\right)=$$

$$=-\pi\sum_{k=1}^{\infty}\frac{2^{-k-2}}{k}\binom{-\frac{1}{2}}{k}(-1)^k=-\frac{\pi}{4}\sum_{k=1}^{\infty}\frac{2^{-k}}{k}\binom{-\frac{1}{2}}{k}(-1)^k=$$

$$=-\frac{\pi}{4}\int_0^{\frac{1}{2}}x^{-1}\sum_{k=1}^{\infty}x^k(-1)^k\binom{-\frac{1}{2}}{k}dx=-\frac{\pi}{4}\int_0^{\frac{1}{2}}\frac{1}{x}\left(\frac{1}{\sqrt{1-x}}-1\right)dx\overset{\sqrt{1-x}=u}{=}$$

$$=-\frac{\pi}{2}\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{1-u^2}\left(\frac{1}{u}-1\right)u\,du=-\frac{\pi}{2}\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1-u}{1-u^2}du=-\frac{\pi}{2}\log(1+u)\Big|_{\frac{1}{\sqrt{2}}}^{1}=$$

$$=-\frac{\pi}{2}\log\left(\frac{2\sqrt{2}}{1+\sqrt{2}}\right)=-\frac{\pi}{2}\log(4-2\sqrt{2})$$

Now,

$$\Omega_2=4\int_0^{\frac{\pi}{2}}\cos^2 x\log(\cos x)\,dx=4\frac{\partial}{\partial a}\Big|_{a=2}\int_0^{\frac{\pi}{2}}\cos^a x\,dx=4\sqrt{\pi}\frac{\partial}{\partial a}\left(\frac{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}{2\Gamma\left(\frac{a}{2}+1\right)}\right)=$$

$$=2\sqrt{\pi}\left(\frac{\frac{1}{2}\psi\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+1\right)}-\frac{\frac{1}{2}\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\psi\left(\frac{a}{2}+1\right)}{\Gamma\left(\frac{a}{2}+1\right)}\right)=$$

$$=\sqrt{\pi}\left(\frac{\psi\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}-\frac{\Gamma\left(\frac{3}{2}\right)\psi(2)}{\Gamma(2)}\right)=\frac{\pi}{2}\left(\psi\left(\frac{3}{2}\right)-\psi(2)\right)=\frac{\pi}{2}(2-2\log 2)$$

Hence,

$$\Omega=\Omega_1-\Omega_2=-\frac{\pi}{2}\log(4-2\sqrt{2})-\frac{\pi}{2}(1-2\log 2)=-\frac{\pi}{2}+\frac{\pi}{2}\log(2+\sqrt{2})=$$

$$=-\frac{\pi}{2}+\frac{\pi}{4}\log(6+4\sqrt{2})$$

1609. If $x\in\mathbb{R}_+$ and $F(x)=\int_0^x\frac{e^t-1}{t}\log\left(\frac{x}{t}\right)dt$ then prove:

$$\int_0^{\infty}e^{-x}F(x)\,dx=\zeta(2)$$

Proposed by Angad Singh-India

Solution 1 by Togrul Ehmedov-Azerbaijan

$$\int_0^\infty e^{-x}\int_0^x \frac{e^t-1}{t}\log\left(\frac{x}{t}\right)dt\,dx \overset{\frac{x}{t}=z}{\underset{t=\frac{x}{z}}{=}} \int_0^\infty e^{-x}\int_1^\infty \frac{e^{\frac{x}{z}}-1}{x}\cdot\log z\cdot\frac{xdz}{z^2}dx =$$

$$=\int_0^\infty e^{-x}\int_1^\infty \left(e^{\frac{x}{z}}-1\right)\cdot\frac{\log z}{z}dzdx=\int_1^\infty\int_0^\infty\left(e^{-x\left(1-\frac{1}{z}\right)}-e^{-x}\right)\cdot\frac{\log z}{z}dxdz=$$

$$=\int_1^\infty\int_0^\infty\frac{\log z}{z}\int_0^\infty\left(e^{-\frac{x(z-1)}{z}}-e^{-x}\right)dx\,dz=$$

$$=\int_1^\infty\frac{\log z}{z}\left[\frac{z}{1-z}e^{-\frac{x(z-1)}{z}}+e^{-x}\right]_0^\infty dz=\int_1^\infty\frac{\log z}{z}\left(\frac{z}{z-1}-1\right)dz=$$

$$=\int_1^\infty\frac{\log z}{z(z-1)}dz=-\int_0^1\frac{\log z}{1-z}dz=-\sum_{k=1}^\infty z^{k-1}\log z\,dz=\sum_{k=1}^\infty\frac{1}{k^2}=\zeta(2)$$

Solution 2 by Ajenikoko Gbolahan-Nigeria

$$F(x)=\int_0^x\frac{e^t-1}{t}\log\left(\frac{x}{t}\right)dt\overset{u=\frac{x}{t}}{=}\int_1^\infty\frac{e^{\frac{x}{u}}-1}{\frac{x}{u}}\log u\frac{xdu}{u^2}=\int_1^\infty\frac{e^{\frac{x}{u}}-1}{u}\log u\,du$$

Then,

$$\Psi=\int_0^\infty e^{-x}F(x)\,dx=\int_0^\infty e^{-x}\int_1^\infty\frac{e^{\frac{x}{u}}-1}{u}\log u\,du\,dx=\int_0^\infty\int_1^\infty\frac{e^{-x+\frac{x}{u}}-e^{-x}}{u}\log u\,du\,dx$$

Using Fubini's theorem to switch order of integration.

$$\Psi=\int_1^\infty\frac{\log u}{u}\int_0^\infty\left(e^{-x+\frac{x}{u}}-e^{-x}\right)dx\,du=\int_1^\infty\frac{\log u}{u}\left(\frac{u}{u-1}-1\right)du=$$

$$=\int_1^\infty\left(\frac{\log u}{u-1}-\frac{\log u}{u}-\frac{\log u}{u}\right)du$$

$$\because\int_1^\infty\frac{\log u}{u}du\text{ does not converges}$$

$$\Psi=\int_1^\infty\frac{\log u}{u-1}du=-\sum_{n=0}^\infty\frac{\partial}{\partial s}\Big|_{s=0}\int_1^\infty x^{n+s}\,ds=-\sum_{n=0}^\infty\frac{1}{-(n+1)^2}=\sum_{n=0}^\infty\frac{1}{(n+1)^2}=\zeta(2)$$

Solution 3 by Rana Ranino-Setif-Algerie

$$F(x)=\int_0^x\frac{e^t-1}{t}\log\left(\frac{x}{t}\right)dt\overset{t=xy}{=}\int_0^1\frac{1-e^{xy}}{y}\log y\,dt$$

$$\Omega=\int_0^\infty F(x)e^{-x}\,dx=\int_0^\infty\int_0^1\frac{1-e^{xy}}{y}\log y\,e^{-x}\,dt\,dx=$$

$$=\int_0^1\frac{\log y}{y}\int_0^\infty(1-e^{xy})e^{-x}\,dxdy=\int_0^1\frac{\log y}{y}\left(1-\frac{1}{1-y}\right)dy=-\int_0^1\frac{\log y}{1-y}dy=$$

$$=-\int_0^1\frac{\log(1-y)}{y}dy=Li_2(1)=\zeta(2)$$

Solution 4 by Kartick Chandra Betal-India

$$F(x)=\int_0^x\frac{e^t-1}{t}\log\left(\frac{x}{t}\right)dt=-\int_0^1\frac{e^{tx}-1}{t}\log t\,dt=\int_0^1\left(\log t\int_0^x e^{ty}\,dy\right)dt$$

$$\int_0^\infty e^{-x}F(x)\,dx=-\int_0^\infty e^{-x}\left(\int_0^1\log t\int_0^x e^{ty}\,dy\right)dt=$$

$$=-\int_0^1\log t\int_0^\infty e^{ty}\int_y^\infty e^{-x}\,dx\,dy\,dz=-\int_0^1\log t\int_0^\infty e^{ty}\cdot e^{-y}\,dy\,dt=$$

$$=-\int_0^1\log t\frac{dt}{1-t}=-\int_0^1\frac{\log(1-t)}{t}dt=\zeta(2)$$

Solution 5 by Jack Desire-Nigeria

$$F(x)=\int_0^x\frac{e^t-1}{t}\log\left(\frac{x}{t}\right)dt=\int_0^x\sum_{k=1}^\infty\frac{t^{k-1}}{k!}\log\left(\frac{x}{t}\right)dx=$$

$$=\sum_{k=1}^\infty\frac{1}{k!}\int_0^x t^{k-1}\{\log x-\log t\}\,dt=\sum_{k=1}^\infty\frac{1}{k!}\left\{\frac{x^k\log x}{k}-\left(\frac{x^k\log x}{k}-\frac{x^k}{k^2}\right)\right\}=$$

$$=\sum_{k=1}^\infty\frac{1}{k!}\left(\frac{x^k}{k^2}\right)$$

$$\int_0^\infty e^{-x}F(x)\,dx=\sum_{k=1}^\infty\frac{1}{k!\,k^2}\int_0^\infty e^{-x}x^k\,dx=\sum_{k=1}^\infty\frac{1}{k^2k!}\cdot k!=\sum_{k=1}^\infty\frac{1}{k^2}=\zeta(2)$$

Solution 6 by proposer

$$F(x)=\int_0^x\frac{e^t-1}{t}\log\left(\frac{x}{t}\right)dt=\int_0^x\int_t^x\frac{e^t-1}{ty}dy\,dt=\int_0^x\int_0^y\frac{e^t-1}{ty}dt\,dy=$$

$$=\int_0^x\frac{1}{y}\int_0^y\sum_{k=1}^\infty\frac{t^{k-1}}{k!}dt\,dy=\int_0^x\frac{1}{y}\sum_{k=1}^\infty\frac{y^k}{k\cdot k!}dy=\sum_{k=1}^\infty\frac{x^k}{k^2\cdot k!}$$

Hence,

$$\int_0^\infty e^{-x}F(x)\,dx=\sum_{k=1}^\infty\frac{1}{k^2\cdot k!}\int_0^\infty e^{-x}x^k\,dx=\sum_{k=1}^\infty\frac{1}{k^2}=\zeta(2)$$

1610. For $|x|\le 1$, let $T(x)=\sum_{n=1}^\infty\frac{\left((-1)^{\frac{1}{2}n(n-1)}+1\right)x^n}{n^2+n}$ then prove that:

$$\int_0^1\left(x+\frac{1}{x}\right)\left(T(-x)+T(x)\right)dx=\frac{\pi^2}{24}-1+\log 2$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$T(x)=\sum_{n=1}^\infty\frac{\left((-1)^{\frac{1}{2}n(n-1)}+1\right)x^n}{n^2+n};\ T(-x)=\sum_{n=1}^\infty\frac{\left((-1)^{\frac{1}{2}n(n+1)}+(-1)^n\right)x^n}{n^2+n}$$

$$T(x)+T(-x)=\sum_{n=1}^\infty\left(\frac{1}{n}-\frac{1}{n+1}\right)\left((-1)^{\frac{1}{2}n(n-1)}+(-1)^{\frac{1}{2}n(n+1)}+(-1)^n+1\right)x^n$$

$$(-1)^{\frac{1}{2}n(n-1)}+(-1)^{\frac{1}{2}n(n+1)}+(-1)^n+1=\begin{cases}4, if\ n=4k\\0, otherwise\end{cases}$$

$$T(x)+T(-x)=4\sum_{k=1}^\infty\left(\frac{1}{4k}-\frac{1}{4k+1}\right)x^{4k}$$

$$\Omega=\int_0^1\left(x+\frac{1}{x}\right)\left(T(-x)+T(x)\right)dx=4\sum_{k=1}^\infty\left(\frac{1}{4k}-\frac{1}{4k+1}\right)\int_0^1\left(x^{4k+1}+x^{4k-1}\right)dx=$$

$$=4\sum_{k=1}^\infty\left(\frac{1}{4k}-\frac{1}{4k+1}\right)\left(\frac{1}{4k}+\frac{1}{4k+1}\right)=\sum_{k=1}^\infty\left(\frac{1}{4k^2}-\frac{1}{2k}+\frac{1}{2k+1}\right)=$$

$$=\frac{1}{4}\zeta(2)-1+\underbrace{\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)}_{(\log 2)}$$

Therefore,

$$\int_0^1\left(x+\frac{1}{x}\right)\left(T(-x)+T(x)\right)dx=\frac{\pi^2}{24}-1+\log 2$$

1611. Find:

$$\Omega(n)=\int\frac{x^{2n-1}(1-x^2)}{e^{nx^2}}dx,n\in\mathbb{N},n\geq 1$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega(n)=\int\frac{x^{2n-1}(1-x^2)}{e^{nx^2}}dx\overset{t=x^2}{=}\frac{1}{2}\int t^{n-1}e^{-nt}\,dt-\frac{1}{2}\int t^ne^{-nt}\,dt$$

$$\frac{1}{2}\int t^{n-1}e^{-nt}\,dt\overset{IBP}{=}\frac{1}{2n}t^ne^{-nt}+\frac{1}{2}\int t^ne^{-nt}\,dt$$

$$\Omega(n)=\frac{1}{2n}t^ne^{-nt}+C=\frac{x^{2n}}{2ne^{nx^2}}+C$$

Solution 2 by Adrian Popa-Romania

$$\Omega(n)=\int\frac{x^{2n-1}(1-x^2)}{e^{nx^2}}dx=\int x^{2n-1}e^{-nx^2}\,dx-\int x^{2n+1}e^{-nx^2}\,dx=I_1-I_2$$

$$I_1=\int x^{2n-1}e^{-nx^2}\,dx\overset{u=x^{2n-2}e^{-nx^2}}{=}$$

$$=\frac{x^{2n}e^{-nx^2}}{2}-(n-1)\int x^{2n-1}e^{-nx^2}\,dx+n\int x^{2n+1}e^{-nx^2}\,dx=$$

$$=\frac{x^{2n}e^{-nx^2}}{2}-(n-1)I_1+nI_2$$

$$n(I_1-I_2)=\frac{x^{2n}e^{-nx^2}}{2}\Rightarrow I_1-I_2=\Omega(n)=\frac{x^{2n}e^{-nx^2}}{2}+C$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\Omega(n)=\int\frac{x^{2n-1}(1-x^2)}{e^{nx^2}}dx=\int\frac{x^{2n-2}(1-x^2)}{e^{nx^2}}xdx\overset{x^2=t}{=}$$

$$=\frac{1}{2}\int e^{-nt}t^{n-1}(1-t)\,dt=$$

$$=\frac{1}{2n}e^{-nt}t^n-\frac{1}{2}\int e^{-nt}t^{n-1}\,dt+\frac{1}{2}\int e^{-nt}t^{n-1}dt=$$

$$=\frac{1}{2n}e^{-nt}t^n+C=\frac{1}{2n}e^{-nx^2}x^{2n}+C$$

Solution 4 by Almas Babirov-Azerbaijan

$$\Omega(n)=\int\frac{x^{2n-1}(1-x^2)}{e^{nx^2}}dx=\int x^{2n-1}e^{-nx^2}dx-\int x^{2n+1}e^{-nx^2}dx=$$

$$=\frac{1}{2n}\int e^{-nx^2}d(x^2)+\frac{1}{2n}\int x^{2n}d\left(e^{-nx^2}\right)=$$

$$=\frac{1}{2n}\int e^{-nx}d(x^2)+\frac{1}{2n}x^{2n}e^{-nx^2}-\int e^{-nx^2}d(x^2)=$$

$$=\frac{1}{2n}x^{2n}e^{-nx^2}+C$$

Solution 5 by Gilmer Lopez-Cajamarca-Peru

$$\Omega(n)=\int\frac{x^{2n-1}(1-x^2)}{e^{nx^2}}dx=\int x^{2n-1}e^{nx^2}dx-\int\frac{x^{2n}\cdot x}{e^{nx^2}}dx\overset{u=x^{2n}}{=}$$

$$=\int\frac{x^{2n-1}}{e^{nx^2}}dx-\left(-\frac{x^{2n}}{2ne^{nx^2}}+\int\frac{2n\cdot x^{2n-1}}{2ne^{nx^2}}dx\right)=$$

$$=\int\frac{x^{2n-1}}{e^{nx^2}}dx+\frac{x^{2n}}{2ne^{nx^2}}-\int\frac{x^{2n-1}}{e^{nx^2}}dx=\frac{x^{2n}}{2ne^{nx^2}}+C$$

1612. Find:

$$\Omega=\lim_{\varepsilon\to 0^+}\int_\varepsilon^1\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\lim_{\varepsilon\to 0^+}\int_\varepsilon^1\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx=\int_0^1\frac{x\sqrt{x}(1-x^2)\log x}{1-x^6}dx=$$

$$=\int_0^1\frac{\left(x^{\frac{3}{2}}-x^{\frac{7}{2}}\right)\log x}{1-x^6}dx=\frac{1}{36}\int_0^1\frac{\left(x^{\frac{5}{12}-1}-x^{\frac{3}{4}-1}\right)\log x}{1-x}dx=$$

$$=\frac{1}{36}\left\{\psi^{(1)}\left(\frac{3}{4}\right)-\psi^{(1)}\left(\frac{5}{12}\right)\right\}=\frac{1}{36}\left\{\pi^2-8G-\psi^{(1)}\left(\frac{5}{12}\right)\right\}$$

Therefore,

$$\Omega=\lim_{\varepsilon\to 0^+}\int_\varepsilon^1\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx=\frac{1}{36}\left\{\pi^2-8G-\psi^{(1)}\left(\frac{5}{12}\right)\right\}$$

Solution 2 by Mohammad Rostami-Afghanistan

$$\Omega=\lim_{\varepsilon\to 0^+}\int_\varepsilon^1\frac{x\sqrt{x}\log x}{x^4+x^2+1}dx=\int_0^1\frac{x\sqrt{x}(1-x^2)\log x}{1-x^6}dx=$$

$$=\int_0^1\frac{x\sqrt{x}(1-x^2)\log x}{(1-x^2)(1+x^2+x^4)}dx=\int_0^1\frac{x\sqrt{x}(1-x^2)\log x}{1-x^6}dx=$$

$$=\int_0^1\frac{x\sqrt{x}\log x}{1-x^6}dx-\int_0^1\frac{x^3\sqrt{x}\log x}{1-x^6}dx=I_1-I_2$$

$$I_1=\int_0^1\frac{x\sqrt{x}\log x}{1-x^6}dx=\int_0^1 x\sqrt{x}\sum_{k=0}^\infty x^{6k}\frac{\partial}{\partial a}\bigg|_{a=0}x^a dx=$$

$$=\sum_{k=0}^\infty\frac{\partial}{\partial a}\bigg|_{a=0}\int_0^1 x^{6k+a+\frac{3}{2}}dx=\sum_{k=0}^\infty\left[\frac{1}{6k+a+\frac{5}{2}}\right]'_{a=0}=$$

$$=\sum_{k=0}^\infty\frac{-1}{\left(6k+\frac{5}{2}\right)^2}=-\frac{1}{36}\sum_{k=0}^\infty\frac{1}{\left(k+\frac{5}{12}\right)^2}=-\frac{1}{36}\psi^{(1)}\left(\frac{5}{12}\right)$$

$$I_2=\int_0^1\frac{x^3\sqrt{x}\log x}{1-x^6}dx=\int_0^1 x^3\sqrt{x}\sum_{n=0}^\infty x^{6n}\frac{\partial}{\partial b}\bigg|_{b=0}x^b dx=$$

$$=\sum_{n=0}^\infty\frac{\partial}{\partial b}\bigg|_{b=0}\int_0^1 x^{6n+b+\frac{7}{2}}dx=\sum_{n=0}^\infty\left[\frac{1}{6n+b+\frac{9}{2}}\right]'_{b=0}=\sum_{k=0}^\infty\frac{-1}{\left(6n+\frac{9}{2}\right)^2}=$$

$$=-\frac{1}{36}\sum_{n=0}^\infty\frac{1}{\left(n+\frac{9}{12}\right)^2}=-\frac{1}{36}\psi^{(1)}\left(\frac{9}{12}\right)$$

Therefore,

$$\Omega=I_1-I_2=\frac{1}{36}\left[\psi^{(1)}\left(\frac{3}{4}\right)-\psi^{(1)}\left(\frac{5}{12}\right)\right]$$

Solution 3 by Abdul Mukhtar-Nigeria

$$\because\ x^4+x^2+1=\frac{1-x^6}{1-x^2}$$

$$\Omega=\int_0^1\frac{x^{\frac{3}{2}}\log x-x^{\frac{7}{2}}\log x}{1-x^6}dx=\int_0^1\left(x^{\frac{3}{2}}\log x-x^{\frac{7}{2}}\log x\right)\sum_{k=0}^{\infty}x^{6k}\,dx=$$

$$=\sum_{k=0}^{\infty}\left(\int_0^1 x^{6k+\frac{3}{2}}\log x\,dx-\int_0^1 x^{6k+\frac{7}{2}}\log x\,dx\right)$$

$$H_1=\int_0^1 x^{6k+\frac{3}{2}}\log x\,dx\,;H_2=\int_0^1 x^{6k+\frac{7}{2}}\log x\,dx\Rightarrow\Omega=\sum_{k=0}^{\infty}(H_1-H_2)$$

$$I_a=\int_0^1 x^a\log x\,dx\overset{IBP}{=}\int_0^1\frac{\log x}{a+1}d(x^{a+1})=\frac{\log x}{a+1}x^{a+1}\Big|_0^1-\frac{1}{a+1}\int_0^1 x^a\,dx=$$

$$=-\frac{1}{a+1}\int_0^1 x^a\,dx=-\frac{1}{(a+1)^2};a>-1$$

$$H_1=-\frac{1}{\left(6k+\frac{5}{2}\right)^2}=-\frac{1}{36}\cdot\frac{1}{\left(k+\frac{5}{2}\right)^2}$$

$$H_2=-\frac{1}{36}\cdot\frac{1}{\left(k+\frac{9}{2}\right)^2}=-\frac{1}{36}\cdot\frac{1}{\left(k+\frac{3}{4}\right)^2}$$

Therefore,

$$\Omega=\sum_{k=0}^{\infty}(H_1-H_2)=\frac{1}{36}\left[\psi^{(1)}\left(\frac{3}{4}\right)-\psi^{(1)}\left(\frac{5}{12}\right)\right]$$

1613. If $0<a\le b$ then:

$$3\int_a^b\sqrt{x^4+x^2+1}\,dx\ge(b-a)\sqrt{(a^2+ab+b^2)^2+3(a^2+ab+b^2)+9}$$

Proposed by Daniel Sitaru-Romania

Solution by Kamel Gandoulli Rezgui-Tunisia

Let $f(t)=\sqrt{t^4+t^2+1}>0;a,b>0$ and let $c=\frac{\sqrt{a^2+ab+b^2}}{\sqrt{3}}\le\frac{\sqrt{3}}{\sqrt{3}}b$. From M.V.T., we get:

$$\exists\alpha\in[a,b]:f(\alpha)=\frac{1}{b-a}\int_a^b f(t)\,dt$$

$$f\nearrow\Rightarrow \alpha\ge\frac{a+b}{2};\frac{\alpha}{\frac{a+b}{2}}\ne 0$$

If $a\to 0, c\to\frac{b}{\sqrt{3}}$ and $\frac{a+b}{2}\to\frac{b}{2}$ then, $\frac{\alpha}{\frac{a+b}{2}}=\frac{2}{\sqrt{3}}=c$ impossible!

$$c<\alpha\Rightarrow f(\alpha)\ge f(c)\Rightarrow$$

$$\frac{1}{b-a}\int_a^b f(t)\,dt\ge f\left(\frac{\sqrt{a^2+ab+b^2}}{\sqrt{3}}\right)=\frac{\sqrt{(a^2+ab+b^2)^2+3(a^2+ab+b^2)+9}}{3}$$

Therefore,

$$3\int_a^b\sqrt{x^4+x^2+1}\,dx\ge(b-a)\sqrt{(a^2+ab+b^2)^2+3(a^2+ab+b^2)+9}$$

1614. If $0<a\le b$ then:

$$\int_a^b\sinh x\left(e^{\sinh^2 x}+e^{\cosh^2 x}\right)dx\ge\frac{289}{105}\left(\sqrt[17]{\cosh^{18} b}-\sqrt[17]{\cosh^{18} a}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Kamel Gandouli Rezgui-Tunisia

Let $y=\cosh x\Rightarrow x=\cosh^{-1}y, dx=\frac{1}{\sqrt{y^2-1}}dy$

$$\sinh(\cosh^{-1}y)=\sqrt{y^2-1}$$

$$\Omega=\int_a^b\sinh x\left(e^{\sinh^2 x}+e^{\cosh^2 x}\right)dx=$$

$$=\int_{\cosh a}^{\cosh b}\sqrt{y^2-1}\left(e^{y^2}+e^{y^2-1}\right)\frac{dy}{\sqrt{y^2-1}}=\int_{\cosh a}^{\cosh b}\left(e^{y^2}+e^{y^2-1}\right)dy$$

Let $f(y)=\left(e^{y^2}+e^{y^2-1}\right)-\frac{51}{35}y^{\frac{1}{17}}\Rightarrow f'(y)=2y\left(e^{y^2}+e^{y^2-1}\right)-\frac{3}{35}y^{-\frac{33}{17}}=$

$$=y\left(2\left(e^{y^2}+e^{y^2-1}\right)-\frac{3}{35}y^{-\frac{33}{17}}\right)=0\Rightarrow 2\left(1+\frac{1}{e}\right)=\frac{3}{35}y^{-\frac{33}{17}}e^{-y^2}$$

$$\log 2+\log(1+e)-1=\log\left(\frac{3}{35}\right)-\frac{33}{17}\log y-y^2:E$$

$$\Rightarrow y=\alpha_0\cong 0.15.\text{ Let }\min f(\alpha)\cong 0.2\Rightarrow e^{y^2}+e^{y^2-1}-\frac{51}{35}y^{\frac{1}{17}}\ge 0;\forall y\ge 0$$

$$\int_{\cosh a}^{\cosh b}\left(e^{y^2}+e^{y^2-1}\right)dy\ge\int_{\cosh a}^{\cosh b}\frac{51}{35}y^{\frac{1}{17}}dy$$

$$\Rightarrow \int_a^b \sinh x\left(e^{\cosh^2 x}+e^{\sinh^2 x}\right)dx \ge \frac{51}{35}\frac{y^{\frac{18}{17}}}{\frac{1}{17}+1}\Bigg|_{\cosh a}^{\cosh b} =$$

$$=\frac{51}{35}\cdot\frac{17}{18}\left(\sqrt[17]{\cosh^{18} b}-\sqrt[17]{\cosh^{18} a}\right)=\frac{289}{105}\left(\sqrt[17]{\cosh^{18} b}-\sqrt[17]{\cosh^{18} a}\right)$$

Equality holds for $a=b$.

1615.

$$\Omega(n)=\sum_{k=2}^{n-1}\int_0^{\frac{\pi}{4}}\frac{\sin^{2k+1}x\cos^k x+\sin^k x\cos^{2k+1}x}{\sin^{3k+3}x+\cos^{3k+3}x}dx$$

Find:

$$\omega=\lim_{n\to\infty}\frac{\Omega(n)}{n}$$

Proposed by Costel Florea-Romania

Solution by Adrian Popa-Romania

$$I=\int_0^{\frac{\pi}{4}}\frac{\sin^{2k+1}x\cos^k x+\sin^k x\cos^{2k+1}x}{\sin^{3k+3}x+\cos^{3k+3}x}dx=$$

$$=\int_0^{\frac{\pi}{4}}\frac{\sin^k x\cos^k x\left(\sin^{k+1}x+\cos^{k+1}x\right)}{(\sin^{k+1}x+\cos^{k+1}x)(\sin^{2k+2}x-\sin^{k+1}x\cos^{k+1}x+\cos^{2k+2}x)}dx=$$

$$=\int_0^{\frac{\pi}{4}}\frac{\sin^k x\cos^k x}{\cos^{2k+2}x\,(\tan^{2k+2}x-\tan^{k+1}x+1)}dx=$$

$$=\int_0^{\frac{\pi}{4}}\frac{\tan^k x}{\cos^2 x\,(\tan^{2k+2}x-\tan^{k+1}x+1)}dx \overset{\tan^{k+1}x=t}{=}$$

$$=\frac{1}{k+1}\int_0^1\frac{dt}{t^2-t+1}=\frac{1}{k+1}\int_0^1\frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{1}{k+1}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\Bigg|_0^1=$$

$$=\frac{2\pi}{3\sqrt{3}(k+1)}$$

$$\Omega(n)=\sum_{k=1}^{n}\frac{2\pi}{3\sqrt{3}(k+1)}=\frac{\pi}{3\sqrt{3}}\left(\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n-1}\right)$$

$$\omega=\lim_{n\to\infty}\frac{\Omega(n)}{n}=\lim_{n\to\infty}\frac{\frac{\pi}{3\sqrt{3}}\left(\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n-1}\right)}{n}\overset{C-S}{=}\frac{2\pi}{3\sqrt{3}}\lim_{n\to\infty}\frac{\frac{1}{n}}{n+1-n}=0$$

1616. Find a closed form:

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n n}{(4n+3)^2}$$

Proposed by Ajetunmobi Abdulqoyyum-Nigeria

Solution 1 by Kartick Chandra Betal-India

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n n}{(4n+3)^2}=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{4n+3}-\frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)^2}=$$

$$=\frac{1}{4}\int_0^1\sum_{n=0}^{\infty}(-x^4)^n x^2dx-\frac{3}{4}\int_0^1\sum_{n=0}^{\infty}(-x^4)^n x^2\log x\,dx=$$

$$=\frac{1}{4}\int_0^1\frac{x^2}{1+x^4}dx-\frac{3}{4}\int_0^1\frac{x^2\log x}{1+x^4}dx=\frac{1}{4}\int_1^{\infty}\frac{dx}{1+x^4}-\frac{3}{64}\int_0^1\frac{x^{\frac{3}{4}-1}\log x}{1+x}dx=$$

$$=\frac{1}{8}\int_0^{\infty}\frac{dx}{1+x^4}-\frac{3}{64}\cdot\frac{1}{2}\left[\frac{\partial}{\partial s}\left\{\psi\left(\frac{s+1}{2}\right)-\psi\left(\frac{s}{2}\right)\right\}\right]_{s=\frac{3}{4}}=$$

$$=\frac{1}{32}\int_1^{\infty}\frac{x^{\frac{1}{4}-1}}{1+x}dx-\frac{3}{256}\left[\psi'\left(\frac{7}{8}\right)-\psi'\left(\frac{3}{8}\right)\right]=$$

$$=\frac{\pi\sqrt{2}}{32}-\frac{3}{256}\left\{\psi'\left(\frac{7}{8}\right)-\psi'\left(\frac{3}{8}\right)\right\}$$

Solution 2 by Syed Sahahabudeen-India

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n n}{(4n+3)^2}=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{4n+3}-\frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)^2}=$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{8n+3}-\frac{1}{8n+7}-\frac{3}{(8n+3)^2}+\frac{3}{(8n+7)^2}\right)$$

We have:

$$\sum_{n=0}^{\infty}\left(\frac{1}{8n+3}-\frac{1}{8n+7}\right)=\frac{1}{8}\left(\psi\left(\frac{7}{8}\right)-\psi\left(\frac{3}{8}\right)\right)$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{(8n+7)^2}-\frac{1}{(8n+3)^2}\right)=\frac{1}{64}\left(\psi^{(1)}\left(\frac{7}{8}\right)-\psi^{(1)}\left(\frac{3}{8}\right)\right)$$

Therefore,

$$\Omega=\frac{1}{32}\left(\psi\left(\frac{7}{8}\right)-\psi\left(\frac{3}{8}\right)\right)+\frac{3}{256}\left(\psi^{(1)}\left(\frac{7}{8}\right)-\psi^{(1)}\left(\frac{3}{8}\right)\right)$$

Solution 3 by Jack Desire-Nigeria

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n n}{(4n+3)^2}=\frac{1}{16}\sum_{n=0}^{\infty}\frac{(-1)^n n}{\left(n+\frac{3}{4}\right)^2}=\frac{1}{16}\sum_{n=0}^{\infty}(-1)^n n\int_0^1 x^{n-\frac{1}{4}}(-\log x)\,dx=$$

$$=-\frac{1}{16}\int_0^1 x^{-\frac{1}{4}}\log x\,dx\sum_{n=0}^{\infty}(-x)^n n;\left(\sum_{n=0}^{\infty}(-x)^n=-\frac{x}{(1+x)^2}\right)$$

$$\Omega=-\frac{1}{16}\int_0^1 x^{-\frac{1}{4}}\log x\left(-\frac{x}{(1+x)^2}\right)dx=\frac{1}{16}\int_0^1\frac{x^{\frac{3}{4}}\log x}{(1+x)^2}dx$$

Applying IBP, we have:

$$\Omega=\frac{1}{16}\cdot\left.\frac{-x^{\frac{3}{4}}\log x}{1+x}\right|_0^1+\int_0^1\left(\frac{x^{-\frac{1}{4}}}{1+x}+\frac{\frac{3}{4}x^{-\frac{1}{4}}\log x}{1+x}\right)dx$$

We know that:

$$\int_0^1\frac{x^{\alpha}}{1+x}dx=\frac{1}{2}\left(\psi\left(\frac{\alpha}{2}+1\right)-\psi\left(\frac{\alpha}{2}+\frac{1}{2}\right)\right)$$

$$\int_0^1\frac{x^{\alpha}\log x}{1+x}dx=\frac{1}{4}\left[\psi'\left(\frac{\alpha}{2}+1\right)-\psi'\left(\frac{\alpha}{2}+\frac{1}{2}\right)\right]$$

$$\int_0^1\frac{x^{-\frac{1}{4}}}{1+x}dx=\frac{1}{2}\left[\psi\left(\frac{7}{8}\right)-\psi\left(\frac{3}{8}\right)\right];\frac{3}{4}\int_0^1\frac{x^{-\frac{1}{4}}\log x}{1+x}dx=\frac{3}{16}\left[\psi'\left(\frac{7}{8}\right)-\psi'\left(\frac{3}{8}\right)\right]$$

$$\Omega=\frac{1}{16}\left\{\frac{1}{2}\left[\psi\left(\frac{7}{8}\right)-\psi\left(\frac{3}{8}\right)\right]+\frac{3}{16}\left[\psi'\left(\frac{7}{8}\right)-\psi'\left(\frac{3}{8}\right)\right]\right\}$$

$$\psi\left(\frac{7}{8}\right)-\psi\left(\frac{3}{8}\right)=\int_0^1\frac{x^{-\frac{5}{8}}-x^{-\frac{1}{8}}}{1-x}dx\overset{x=u^8}{=}\int_0^1\frac{u^{-5}-u^{-1}}{1-u^8}8u^7du=$$

$$=8\int_0^1\frac{u^2-u^6}{1-u^8}du=8\int_0^1\frac{u^2}{1+u^4}du=8\int_0^1\frac{1}{\frac{1}{u^2}+u^2}du=$$

$$=4\left(\int_0^1\frac{1-\frac{1}{u^2}}{\left(u+\frac{1}{u}\right)^2-2}du+\int_0^1\frac{1+\frac{1}{u^2}}{\left(u-\frac{1}{u}\right)^2+2}du\right)=$$

$$=4\left(\frac{1}{2\sqrt{2}}\log\left(\frac{u+\frac{1}{u}-\sqrt{2}}{u+\frac{1}{u}+\sqrt{2}}\right)\Bigg|_0^1+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u^2-1}{u\sqrt{2}}\right)\Bigg|_0^1\right)=$$

$$=\sqrt{2}\log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)+\pi\sqrt{2}=\pi\sqrt{2}+\sqrt{2}\log(3-2\sqrt{2})$$

Therefore,

$$\Omega=\frac{1}{32}\left(\pi\sqrt{2}+\sqrt{2}\log(3-2\sqrt{2})\right)+\frac{3}{256}\left[\psi'\left(\frac{7}{8}\right)-\psi'\left(\frac{3}{8}\right)\right]$$

Solution 4 by Ajenikoko Gbolahan-Nigeria

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n n}{(4n+3)^2}=\frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{4n+3}-\frac{3(-1)^n}{4n+3}\right]$$

Using that:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+k}=\sum_{n=0}^{\infty}\frac{1}{2n+k}-\frac{1}{2n+3+k}$$

$$\Omega=\frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{1}{8n+3}-\frac{1}{8n+7}\right]-\frac{3}{4}\sum_{n=0}^{\infty}\left[\frac{1}{(8n+3)^2}-\frac{1}{(8n+7)^2}\right]=$$

$$=\frac{1}{32}\left[\Phi\left(1,1,\frac{3}{8}\right)-\Phi\left(1,1,\frac{7}{8}\right)\right]-\frac{3}{32}\left[\Phi\left(1,2,\frac{3}{8}\right)-\Phi\left(1,2,\frac{7}{8}\right)\right]$$

where $\Phi(a,b,c)$ –Lerch transcendent function.

Solution 5 by Rana Ranino-Setif-Algerie

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n n}{(4n+3)^2}=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(4n+3-3)}{(4n+3)^2}=\underbrace{\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{4n+3}}_{A}-\underbrace{\frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)^2}}_{B}$$

$$A=\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{4n+2}\,dx=\int_0^1\frac{x^2}{1+x^4}dx$$

$$=\frac{1}{4\sqrt{2}}\int_0^1\left(\frac{2x}{x^2-x\sqrt{2}+1}-\frac{2x}{x^2+x\sqrt{2}+1}\right)dx$$

$$=\frac{1}{4\sqrt{2}}\int_0^1\left(\frac{2x-\sqrt{2}}{x^2-x\sqrt{2}+1}-\frac{2x+\sqrt{2}}{x^2+x\sqrt{2}+1}\right)dx+\frac{1}{4}\int_0^1\frac{dx}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}+$$

$$+\frac{1}{4}\int_0^1\frac{dx}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}=$$

$$=\frac{1}{4\sqrt{2}}\log\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}-1)+\frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}+1)=$$

$$=\frac{1}{2\sqrt{2}}\log(\sqrt{2}-1)+\frac{\pi}{4\sqrt{2}}$$

$$B=\frac{1}{64}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{3}{8}\right)^2}-\frac{1}{\left(n+\frac{7}{8}\right)^2}=\frac{1}{64}\left\{\psi^{(1)}\left(\frac{3}{8}\right)-\psi^{(1)}\left(\frac{7}{8}\right)\right\}$$

Therefore,

$$\Omega=\sum_{n=0}^{\infty}\frac{(-1)^n n}{(4n+3)^2}=\frac{1}{8\sqrt{2}}\log(\sqrt{2}-1)+\frac{\pi}{16\sqrt{2}}+\frac{3}{256}\left\{\psi^{(1)}\left(\frac{7}{8}\right)-\psi^{(1)}\left(\frac{3}{8}\right)\right\}$$

1617. Let be the function $f:[0,\infty)\to[0,\infty)$ continuous such that:

$(f(f(x))=(2^{x+1}+x-1)f(x),\forall x\geq 0$. Prove that: f invertible and find

$$\Omega=\lim_{x\to 0}\frac{f^{-1}(x)}{x}$$

Proposed by Florică Anastase-Romania

Solution 1 by Kamel Gandouli Rezgui-Tunisia

If $x,x'\geq 0$ such that $f(x)=f'(x)\Rightarrow f(f(x))=f(f(x'))$

$$\begin{cases}f(f(x))=(2^{x+1}+x-1)f(x)\\ f(f(x'))=(2^{x'+1}+x'-1)f(x')\end{cases}$$

$h(x) = 2^{x+1} + x - 1 = e^{(x+1)\log 2} + x - 1$ injective because $h \nearrow$ and continuous, then

$x = x' \Rightarrow f$ −injective.

If $y \in \mathbb{R}_+ \Rightarrow \exists t \in \mathbb{R}_+$ such that $(2^{t+1} + t - 1)f(t) = y$

Because $\alpha(2^{t+1} + t - 1)$ − bijective $\forall \alpha \geq 0 \Rightarrow f$ −surjective. So, f −invertible.

$$f\left(f\left(f^{-1}(x)\right)\right) = (2^{x+1} + x - 1)x \Rightarrow f(x) = (2^{x+1} + x - 1)x$$

$$\lim_{x\to 0}\frac{f(x)}{x} = 1 \Rightarrow f(0) = 0 \Rightarrow f^{-1}(0) = 0$$

$$\lim_{x\to 0}\frac{f(x)}{x} = \lim_{t\to 0}\frac{f\left(f^{-1}(t)\right)}{f^{-1}(t)} = \lim_{t\to 0}\frac{t}{f^{-1}(t)} = 1$$

Therefore,

$$\Omega = \lim_{x\to 0}\frac{f^{-1}(x)}{x} = 1$$

Solution 2 by Surjeet Singhania-India

First of all we will show f is bijective function. Here we go.

Put $x = 0$ in functional equation we will get $(f \circ f)_{(0)} = f(0)$

Claim $f(0) = 0$, if not then $f(0) = m \in \mathbb{R}_+$ implies $f(m) = m$.

Put $x = m$ in equation $f(m) = (2^{m+1} + x - 1)f(m) \Rightarrow$

$2^{m+1} + m - 1 = 1$ only $m = 0$ satisfy equation. Hence, $f(m)$ is only solution for $m = 0$.

Now, $\forall x \in \mathbb{R}_+, \frac{(f\circ f)_{(x)}}{f(x)} = 2^{x+1} + x - 1 = g(x)$, where g is continuous and increasing function, so g −bijective.

Suppose f is not injective then for some $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2) \Rightarrow$

$\frac{(f\circ f)_{(x_1)}}{f(x_1)} = \frac{(f\circ f)_{(x_2)}}{f(x_2)} \Rightarrow g(x_1) = g(x_2)$. Contradiction, then g −injective, hence f −injective function.

If f −is not surjective, then $\exists y \in \mathbb{R}_+$ such that no $\exists x \in \mathbb{R}^+, f(x) = y$ since we prove f − is injective function $(f \circ f)_{(x)} \neq f(y) \Rightarrow g(y) \neq \frac{f(y)}{f(x)}$. Contradiction, then g −is surjective, hence f −is surjective, then f −bijective. So, f −invertible.

Since $f(0)=0 \Rightarrow \lim_{x\to 0^+} f(x)=0 \Rightarrow \lim_{x\to 0^+} f^{-1}(x)=0$

$$\Omega = \lim_{x\to 0}\frac{f^{-1}(x)}{x} = \lim_{x\to 0}\frac{f(x)}{(f\circ f)_{(x)}} = \lim_{x\to 0}\frac{1}{2^{x+1}+x-1} = 1$$

Solution 3 by proposer

a) Let $f(0)=a, a\in[0,\infty)$

For $x=0 \Rightarrow (f(f(0))=f(0) \Rightarrow f(a)=a \Rightarrow a$ fix point of f.

For $x=a \Rightarrow (f(f(a))=(2^{a+1}+a-1)f(a) \Rightarrow a=(2^{a+1}+a-1)a \Rightarrow$

$a=0$ unique fix point of f.

Let $g:(0,\infty)\to R, g(x)=f(x)-x$ which isn't zero and have same sign on $(0,\infty)$.

Let $b\in(0,\infty)$ with $f(b)\neq 0 \Rightarrow$

$$g(f(b)) = f(f(b)) - f(b) = (2^{b+1}+b-2)f(b) > 0 \Rightarrow$$

$f(x)>0, \forall x\in[0,\infty)$ and how $\lim_{x\to\infty} f(x)=\infty \Rightarrow f([0,\infty))=[0,\infty) \Rightarrow f$ surjective.

Let $x_1,x_2\in[0,\infty)$ with $f(x_1)=f(x_2)$

Then $f(x_1)=f(x_2) \Leftrightarrow f(f(x_1))=f(f(x_2)) \Leftrightarrow$

$(2^{1+x_1}+x_1-1)f(x_1)=(2^{1+x_2}+x_2-1)f(x_2) \Leftrightarrow x_1=x_2 \Leftrightarrow f$ injective.

So, f bijective and exist $f^{-1}:[0,\infty)\to[0,\infty)$ inverse.

$$\Omega = \lim_{x\to 0}\frac{f^{-1}(x)}{x} = \lim_{y\to 0}\frac{y}{f(y)} =$$

$$= \lim_{x\to 0}\frac{f(x)}{f(f(x))} = \lim_{x\to 0}\frac{f(x)}{(2^{x+1}+x-1)f(x)} = \lim_{x\to 0}\frac{1}{2^{x+1}+x-1} = 1$$

1618. Find:

$$\Omega = \lim_{p\to\infty}\left[\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)\right]^{H_p},$$

where H_p –harmonic number.

Proposed by Florică Anastase-Romania

Solution 1 by Daniela Ruxandra Tonilă-Romania

$$\Omega=\lim_{p\to\infty}\left[\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)\right]^{H_p}$$

$$\log\Omega=\lim_{p\to\infty}H_p\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)=\lim_{p\to\infty}H_p\left(\lim_{n\to\infty}\sum_{k=1}^{n}\log\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)=$$

$$=\lim_{p\to\infty}H_p\left(\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\log\left(1+\frac{(k+1)^p}{n^{p+1}}\right)}{\frac{(k+1)^p}{n^{p+1}}}\cdot\frac{(k+1)^p}{n^{p+1}}\right)=$$

$$=\lim_{p\to\infty}H_p\left(\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+1)^p}{n^{p+1}}\right)=\lim_{p\to\infty}H_p\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k+1}{n}\right)^p\right)=$$

$$=\lim_{p\to\infty}H_p\int_0^1 x^p\,dx=\lim_{p\to\infty}\frac{H_p}{p+1}\overset{C-S}{=}\lim_{p\to\infty}\frac{H_{p+1}-H_p}{p+1-p}=0$$

Therefore,

$$\Omega=\lim_{p\to\infty}\left[\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)\right]^{H_p}=1$$

Solution 2 by proposer

From well-known inequality: $\frac{x}{x+1}\le\ln(1+x)\le x,\forall x\in(-1,\infty)$ we get:

$$\frac{(k+1)^p}{(k+1)^p+n^{p+1}}\le\ln\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\le\frac{(k+1)^p}{n^{p+1}}$$

Thus,

$$\sum_{k=1}^{n}\frac{(k+1)^p}{(n+1)^p+n^{p+1}}\le\sum_{k=1}^{n}\frac{(k+1)^p}{(k+1)^p+n^{p+1}}\le\ln\left[\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right]\le\sum_{k=1}^{n}\frac{(k+1)^p}{n^{p+1}}$$

We have:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+1)^p}{(n+1)^p+n^{p+1}}\overset{C-S}{=}\lim_{n\to\infty}\frac{(n+2)^p}{(n+2)^p-(n+1)^p+(n+1)^{p+1}-n^{p+1}}$$

$$=\lim_{n\to\infty}\frac{(n+2)^p}{(n+2)^p-(n+1)^p+C_{p+1}^0n^{p+1}+C_{p+1}^1n^p+C_{p+1}^2n^{p-1}+\cdots+C_{p+1}^{p+1}-n^{p+1}}=$$

$$=\frac{1}{p+1}$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+1)^p}{n^{p+1}}=\lim_{n\to\infty}\frac{(n+2)^p}{(n+1)^{p+1}-n^{p+1}}=$$

$$=\lim_{n\to\infty}\frac{(n+2)^p}{C_{p+1}^0 n^{p+1}+C_{p+1}^1 n^p+C_{p+1}^2 n^{p-1}+\cdots+C_{p+1}^{p+1}-n^{p+1}}=\frac{1}{p+1}$$

So, we get:

$$\lim_{n\to\infty}\ln\left[\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right]=\frac{1}{p+1}\Rightarrow\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)=e^{\frac{1}{p+1}}$$

Therefore,

$$\Omega=\lim_{p\to\infty}\left[\lim_{n\to\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{(k+1)^p}{n^{p+1}}\right)\right)\right]^{H_p}=\lim_{p\to\infty}\left(e^{\frac{1}{p+1}}\right)^{H_p}=e^{\lim_{n\to\infty}\frac{H_p}{p+1}}\overset{LC-S}{=}=1$$

1619. Prove that:

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(1-4n)}=\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

Proposed by Ajenikoko Gbolahan-Nigeria

Solution 1 by Dawid Bialek-Poland

Recall:

$$-\sin^{-1}x=\sum_{n=0}^{\infty}\frac{-\binom{2n}{n}}{4^n(2n+1)}x^{2n+1};\ |x|\le 1$$

and derivative both sides w.r.t. x

$$-\frac{1}{\sqrt{1-x^2}}=-\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n}x^{2n};\ (x^2\to x^4)\Rightarrow-\frac{1}{\sqrt{1-x^4}}=-\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n}x^{4n};\ (1)$$

Multiply both sides (1) by x^{-2}, we get:

$$\frac{-1}{x^2\sqrt{1-x^4}}=-\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n}x^{4n-2}$$

and integrate them:

$$-\int_0^1 \frac{1}{x^2\sqrt{1-x^4}}dx = -\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n}\int_0^1 x^{4n-2}\,dx$$

$$-\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(1-4n)} = -\int_0^1 \frac{1}{x^2\sqrt{1-x^4}}dx;\ (x^4\to t)\Rightarrow$$

$$-\frac{1}{4}\int_0^1 t^{-\frac{5}{4}}\cdot(1-t)^{-\frac{1}{2}}dt \overset{(beta\ f-n)}{=} -\frac{1}{4}\cdot B\left(-\frac{1}{4},\frac{1}{2}\right) = -\frac{1}{4}\cdot\frac{\Gamma\left(-\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)} = -\frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

$$\because \Gamma(n+1) = n\Gamma(n) \Rightarrow \Gamma\left(-\frac{1}{4}+1\right) = -\frac{1}{4}\Gamma\left(-\frac{1}{4}\right)$$

Therefore,

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(1-4n)} = \frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

Solution 2 by Syed Shahabudeen-Kerala-India

$$\Omega = \sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n(1-4n)} = -\int_0^1 \frac{1}{x^2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}x^{4n}dx =;\left(\because \frac{(2n)!}{4^n n!} = \left(\frac{1}{2}\right)_n\right)$$

$$= -\int_0^1 \frac{1}{x^2}\ {}_1F_0\left(\frac{1}{2};;x^4\right)dx = -\int_0^1 \frac{1}{x^2\sqrt{1-x^4}}dx =;\left(\because\ {}_1F(a;;z) = \frac{1}{(1-z)^a}\right)$$

$$-\frac{1}{4}\int_0^1 \frac{1}{t^{\frac{5}{4}}\sqrt{1-t}}dt = \frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

1620. Prove that:

$$\sum_{n=1}^{\infty}\left(\frac{H_{6n-1}}{6n} - \frac{H_{6n-3}}{6n-2} - \frac{H_{6n-4}}{6n-3} + \frac{H_{6n-6}}{6n-5}\right) = -\frac{\pi^2}{18}$$

Proposed by Asmat Qatea-Afghanistan

Solution by Kartick Chandra Betal-India

$$\sum_{n=1}^{\infty}\left(\frac{H_{6n-1}}{6n} - \frac{H_{6n-3}}{6n-2} - \frac{H_{6n-4}}{6n-3} + \frac{H_{6n-6}}{6n-5}\right) = \sum_{n=1}^{\infty}\left(\frac{H_{6n}}{6n} - \frac{H_{6n-2}}{6n-2} - \frac{H_{6n-3}}{6n-3} + \frac{H_{6n-5}}{6n-5}\right) -$$

$$-\sum_{n=1}^{\infty}\left(\frac{1}{(6n)^2}-\frac{1}{(6n-2)^2}-\frac{1}{(6n-3)^2}+\frac{1}{(6n-5)^2}\right)=$$

$$=-\int_0^1\left(1-x^2-x^3+x^5\right)\log(1-x)\sum_{n=1}^{\infty}(x^6)^{n-1}dx+$$

$$+\int_0^1\left(1-x^2-x^3+x^5\right)\log(x)\sum_{n=1}^{\infty}(x^6)^{n-1}dx=$$

$$=\int_0^1\frac{(1-x^2)(1-x^3)}{1-x^6}\{\log x-\log(1-x)\}\,dx=$$

$$=\int_0^1\frac{1-x^2}{1+x^3}\{\log x-\log(1-x)\}=\int_0^1\frac{[1-(1-x)^2][\log(1-x)-\log x]}{[1+(1-x)][1-(1-x)+(1-x)^2]}dx=$$

$$=\int_0^1\frac{(2-x)x[\log(1-x)-\log x]}{(2-x)(1-x+x^2)}dx=\int_0^1\frac{x}{1-x+x^2}[\log(1-x)-\log x]\,dx=$$

$$=\int_0^1\frac{x\log\left(\frac{1}{x}-1\right)}{1-x+x^2}dx=\int_1^{\infty}\frac{\log(x-1)}{x(1-x+x^2)}dx=\int_0^{\infty}\frac{\log x}{(x+1)(1+x+x^2)}dx=$$

$$=\int_0^1\left(\frac{2}{1+x}-\frac{2x+1}{1+x+x^2}\right)\log x\,dx=$$

$$=2\int_0^1\frac{\log x}{1+x}dx-\log x\log(1+x+x^2)\Big|_0^1+\int_0^1\frac{\log(1-x^3)-\log(1-x)}{x}dx=$$

$$=-2\cdot\eta(2)+\frac{1}{3}\int_0^1\frac{\log(1-x)}{x}dx-\int_0^1\frac{\log(1-x)}{x}dx=$$

$$=-\zeta(2)+\frac{2}{3}\zeta(2)=-\frac{\zeta(2)}{3}=-\frac{\pi^2}{18}$$

Therefore,

$$\sum_{n=1}^{\infty}\left(\frac{H_{6n-1}}{6n}-\frac{H_{6n-3}}{6n-2}-\frac{H_{6n-4}}{6n-3}+\frac{H_{6n-6}}{6n-5}\right)=-\frac{\pi^2}{18}$$

1621. Find:

$$\Omega=\lim_{x\to 0}\frac{\left(x-3\tan\left(\frac{\pi}{x+4}\right)+3\right)\left(e^{\cos x}-e\right)}{T_5(x)-5x}$$

T_5 –Cebyshev's polynome first kind.

Proposed by Mohammad Hamed Nasery-Afghanistan

Solution by Kamel Gandouli Rezgui-Tunisia

$$\Omega = \lim_{x\to 0}\frac{\left(x-3\tan\left(\frac{\pi}{x+4}\right)+3\right)\left(e^{\cos x}-e\right)}{T_5(x)-5x} =$$

$$= \lim_{x\to 0}\frac{\frac{x-3\tan\left(\frac{\pi}{x+4}\right)-(-3)}{x}\cdot\frac{e^{\cos x}-e}{x^2}}{\frac{T_5(x)-5}{x^3}}$$

$$\left(x-3\tan\left(\frac{\pi}{x+4}\right)\right)'(0) = 1-3\left(1+\tan^2\left(\frac{\pi}{x+4}\right)\right)\left(-\frac{\pi}{(x+4)^2}\right)(0) =$$

$$= 1+\frac{3\pi}{8}$$

$$\frac{e^{\cos x}-e}{x^2} \cong -\frac{e}{2}+o(x^3) \to -\frac{e}{2}$$

$$\frac{T_5(x)}{x^3} \to -20$$

Hence,

$$\Omega = \lim_{x\to 0}\frac{\frac{x-3\tan\left(\frac{\pi}{x+4}\right)-(-3)}{x}\cdot\frac{e^{\cos x}-e}{x^2}}{\frac{T_5(x)-5}{x^3}} = \frac{(8+3\pi)e}{320}$$

1622.

$$x_1 = 1, x_2 = 2, (n-1)x_n + nx_{n-1} = n(n-1)\log\left(\frac{x_{n-1}^2+(n-1)^2}{x_{n-2}^2+(n-2)^2}\right),$$

$$n \geq 3. \text{ Find:}$$

$$\Omega = \lim_{n\to\infty}\frac{x_n\left(\sqrt[n]{\gamma}-1\right)^2}{n}$$

Proposed by Ruxandra Daniela Tonilă-Romania

Solution by Kamel Gandouli Rezgui-Tunisia

$$(n-1)x_n + nx_{n-1} = n(n-1)\log\left(\frac{x_{n-1}^2+(n-1)^2}{x_{n-2}^2+(n-2)^2}\right), n \geq 3$$

$$w_n = \frac{x_n}{n}, x_1 = 1, x_2 = 2 \Rightarrow w_1 = w_2 = 1$$

$$\Rightarrow w_n + w_{n-1} = \log\left(\frac{(n-1)^2}{(n-2)^2}\cdot\frac{w_{n-1}^2+1}{w_{n-2}^2+1}\right) = \log\left(\frac{n-1}{n-2}\right)^2 + \log\left(\frac{w_{n-1}^2+1}{w_{n-2}^2+1}\right) =$$

$$= 2\log\left(\frac{n-1}{n-2}\right) + \log\left(\frac{w_{n-1}^2+1}{w_{n-2}^2+1}\right)$$

$$\log\left(\frac{n-1}{n-2}\right) \xrightarrow{n\to\infty} 0;\ \log\left(\frac{w_{n-1}^2+1}{w_{n-2}^2+1}\right) \le \log\left(\frac{w_{n-1}^2+1}{1}\right) \le 2w_{n-1} \to \infty$$

$$w_n \le w_{n-1} \Rightarrow \log\left(\frac{w_{n-1}^2+1}{w_{n-2}^2+1}\right) \le 0 \Rightarrow w_n \le 2\log\left(\frac{n-1}{n-2}\right) \to 0 \Rightarrow w_n \to 0$$

$$\frac{x_n}{n} \to 0 \Rightarrow \Omega = \lim_{n\to\infty}\frac{x_n(\sqrt[n]{\gamma}-1)^2}{n} = 0$$

1623. For $n \ge 1, \varphi$ —Golden ration and let

$$\int_0^\infty \frac{e^{-\pi x}\cos\left(\frac{\pi x}{n} - n\pi x\right)}{\sqrt{x}}dx = f(n)\int_0^\infty \frac{e^{-\pi x}\cos\left(\frac{\pi x}{n} + n\pi x\right)}{\sqrt{x}}dx$$

then prove the following identity:

$$\frac{f(\sqrt{\varphi})}{f(\varphi)} = \frac{\sqrt{\dfrac{\left(\sqrt{\sqrt{5}+1}+2\right)\left(\sqrt{5}+2\sqrt{3(\sqrt{5}+3)}+1\right)}{2\sqrt{2}+\sqrt{5}+1}}}{\sqrt{3}\cdot\sqrt[4]{\sqrt{5}+1}}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$I(a) = \int_0^\infty \frac{e^{-\pi x}\cos(a\pi x)}{\sqrt{x}}dx = Re\left(\int_0^\infty x^{\frac{1}{2}-1}e^{-\pi x(1-ia)}dx\right) = Re\left(\frac{1}{\sqrt{1-ia}}\right) =$$

$$= \frac{\cos\left(\frac{1}{2}\tan^{-1}a\right)}{\sqrt[4]{1+a^2}}$$

$$\because \cos\left(\frac{1}{2}\tan^{-1}a\right) = \sqrt{\frac{1+\cos(\tan^{-1}a)}{2}} = \sqrt{\frac{1+\sqrt{1+a^2}}{2\sqrt{1+a^2}}} \Rightarrow I(a) = \sqrt{\frac{1+\sqrt{1+a^2}}{2(1+a^2)}}$$

$$f(\varphi)=\frac{I\left(\varphi-\frac{1}{\varphi}\right)}{I\left(\varphi+\frac{1}{\varphi}\right)}=\frac{I(1)}{I(\sqrt{5})}=\sqrt{1+\sqrt{2}}\cdot\sqrt{\frac{3}{1+\sqrt{6}}}=\sqrt{\frac{3(1+\sqrt{2})}{1+\sqrt{6}}}$$

$$f(\sqrt{\varphi})=\frac{I\left(\sqrt{\varphi}-\frac{1}{\sqrt{\varphi}}\right)}{I\left(\sqrt{\varphi}+\frac{1}{\sqrt{\varphi}}\right)}=\frac{I\left(\sqrt{\sqrt{5}-2}\right)}{I\left(\sqrt{\sqrt{5}+2}\right)}=\sqrt{\frac{\left(1+\sqrt{\sqrt{5}-1}\right)(\sqrt{5}+3)}{(\sqrt{5}-1)\left(1+\sqrt{\sqrt{5}+3}\right)}}$$

$$\Rightarrow\frac{f(\sqrt{\varphi})}{f(\varphi)}=\sqrt{\frac{\left(1+\sqrt{\sqrt{5}-1}\right)(\sqrt{5}+3)(1+\sqrt{6})}{3(\sqrt{5}-1)\left(1+\sqrt{\sqrt{5}+3}\right)(1+\sqrt{2})}}=$$

$$=\sqrt{\frac{\left(1+\sqrt{\sqrt{5}-1}\right)(\sqrt{5}+3)(1+\sqrt{6})(\sqrt{2}-1)(\sqrt{5}+1)\left(\sqrt{\sqrt{5}+3}-1\right)(\sqrt{5}-2)}{12}}=$$

$$=\sqrt{\frac{\left(2+\sqrt{\sqrt{5}+1}\right)(1+\sqrt{6})(\sqrt{2}-1)\left(\sqrt{\sqrt{5}+3}-1\right)}{3\sqrt{\sqrt{5}+1}}}=$$

$$=\frac{\sqrt{\left(2+\sqrt{\sqrt{5}+1}\right)(1+\sqrt{6})(\sqrt{2}-1)\left(\sqrt{\sqrt{5}+3}-1\right)}}{\sqrt{3}\cdot\sqrt[4]{\sqrt{5}+1}}=$$

$$=\frac{\sqrt{\left(2+\sqrt{\sqrt{5}+1}\right)(1+\sqrt{6})(\sqrt{2}-1)\left(\sqrt{\sqrt{5}+3}-1\right)(1+2\sqrt{2}+\sqrt{5})}}{\sqrt{3}\cdot\sqrt[4]{\sqrt{5}+1}\cdot\left(\sqrt{1+2\sqrt{2}+\sqrt{5}}\right)}=$$

$$=\frac{\sqrt{\frac{\left(2+\sqrt{\sqrt{5}+1}\right)(1+\sqrt{6})(1+\sqrt{5})}{1+2\sqrt{2}+\sqrt{5}}}}{\sqrt{3}\cdot\sqrt[4]{\sqrt{5}+1}}=\frac{\sqrt{\frac{\left(2+\sqrt{\sqrt{5}+1}\right)\left(1+\sqrt{5}+\sqrt{(\sqrt{6}+\sqrt{30})^2}\right)}{1+2\sqrt{2}+\sqrt{5}}}}{\sqrt{3}\cdot\sqrt[4]{\sqrt{5}+1}}$$

$$=\frac{\sqrt{\frac{\left(\sqrt{\sqrt{5}+1}+2\right)\left(\sqrt{5}+2\sqrt{3(\sqrt{5}+3)}+1\right)}{2\sqrt{2}+\sqrt{5}+1}}}{\sqrt{3}\cdot\sqrt[4]{\sqrt{5}+1}}$$

1624. Prove that:

$$\frac{1}{3}\left\{7-8\sqrt{7}\cos\left[\frac{1}{3}\left(arcsin\left(\frac{3\sqrt{21}}{14}\right)+4\pi\right)\right]\right\}+8\cos\left(\frac{4\pi}{7}\right)=1$$

Proposed by Carlos Paiva-Fortaleza-Brazil

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \cos\left(4.\frac{4\pi}{7}\right)=\cos\left(\frac{16\pi}{7}\right)=\cos\left(\frac{2\pi}{7}\right)\ and\ \cos\left(3.\frac{4\pi}{7}\right)=\cos\left(\frac{12\pi}{7}\right)$$

$$=\cos\left(\frac{2\pi}{7}\right)\ \to\ \cos\left(4.\frac{4\pi}{7}\right)=\cos\left(3.\frac{4\pi}{7}\right)$$

$$Similarly, we\ have:\ \cos\left(4.\frac{2\pi}{7}\right)=\cos\left(3.\frac{2\pi}{7}\right)\ and\ \cos\left(4.\frac{6\pi}{7}\right)=\cos\left(3.\frac{6\pi}{7}\right)$$

$$\to Since\ \cos(4x)=8\cos^4 x-8\cos^2 x+1\ and\ \cos(3x)=4\cos^3 x-3\cos x$$

$$\to x_2=\cos\left(\frac{2\pi}{7}\right), x_4=\cos\left(\frac{4\pi}{7}\right), x_6$$

$$=\cos\left(\frac{6\pi}{7}\right)\ are\ the\ solutions\ of\ the\ equation:\ 8x^4-8x^2+1$$

$$=4x^3-3x$$

$$\leftrightarrow (x-1)(8x^3+4x^2-4x-1)=0 \leftrightarrow 8x^3+4x^2-4x-1=0\ (\because x_2,x_4,x_6\neq 1)$$

$$Let\ x=y-\frac{1}{6}\to 8\left(y^3-\frac{1}{2}y^2+\frac{1}{12}y-\frac{1}{216}\right)+4\left(y^2-\frac{1}{3}y+\frac{1}{36}\right)-4y+\frac{2}{3}-1=0$$

$$\leftrightarrow 8y^3-\frac{14}{3}y=\frac{7}{27}$$

$$Let\ y=\frac{\sqrt{7}}{3}z\ \to 4z^3-3z=\frac{1}{2\sqrt{7}}=\frac{\sqrt{7}}{14},$$

$$let\ z=cos\theta\ \to\ 4\cos^3\theta-3\cos\theta=\cos 3\theta=\frac{\sqrt{7}}{14}$$

$$\to z=cos\theta\ =\cos\left[\frac{1}{3}\left(arccos\left(\frac{\sqrt{7}}{14}\right)+2k\pi\right)\right], k=0,1\ or\ 2.$$

$$\rightarrow y=\frac{\sqrt{7}}{3}\cos\left[\frac{1}{3}\left(arccos\left(\frac{\sqrt{7}}{14}\right)+2k\pi\right)\right]\rightarrow x$$
$$=\frac{\sqrt{7}}{3}\cos\left[\frac{1}{3}\left(arccos\left(\frac{\sqrt{7}}{14}\right)+2k\pi\right)\right]-\frac{1}{6}, k=0,1 \text{ or } 2.$$

$$It's\ clear\ that: x_2>0>x_4>x_6 \text{ and } 0<\frac{\sqrt{7}}{14}<\frac{1}{2}\rightarrow arccos\left(\frac{\sqrt{7}}{14}\right)\in\left(\frac{\pi}{3},\frac{\pi}{2}\right)$$
$$\rightarrow \frac{1}{3}arccos\left(\frac{\sqrt{7}}{14}\right)\in\left(\frac{\pi}{9},\frac{\pi}{6}\right)$$
$$\rightarrow x_2=\frac{1}{3}arccos\left(\frac{\sqrt{7}}{14}\right) \text{ and } \frac{1}{3}\left(arccos\left(\frac{\sqrt{7}}{14}\right)+2\pi\right)$$
$$\in\left(\frac{7\pi}{9},\frac{5\pi}{6}\right) \text{ and } \frac{1}{3}\left(arccos\left(\frac{\sqrt{7}}{14}\right)+4\pi\right)\in\left(\frac{13\pi}{9},\frac{3\pi}{2}\right)$$
$$\rightarrow \cos\left[\frac{1}{3}\left(arccos\left(\frac{\sqrt{7}}{14}\right)+4\pi\right)\right]>\cos\left[\frac{1}{3}\left(arccos\left(\frac{\sqrt{7}}{14}\right)+2\pi\right)\right]\rightarrow x_4$$
$$=\frac{\sqrt{7}}{3}\cos\left[\frac{1}{3}\left(arccos\left(\frac{\sqrt{7}}{14}\right)+4\pi\right)\right]-\frac{1}{6}$$
$$Since\ \sin^2\left(arccos\left(\frac{\sqrt{7}}{14}\right)\right)=1-\frac{7}{14^2}=\frac{189}{14^2}\rightarrow \sin\left(arccos\left(\frac{\sqrt{7}}{14}\right)\right)=\frac{3\sqrt{21}}{14}$$
$$\rightarrow arccos\left(\frac{\sqrt{7}}{14}\right)=arcsin\left(\frac{3\sqrt{21}}{14}\right).$$
$$\rightarrow x_4=\cos\left(\frac{4\pi}{7}\right)=\frac{\sqrt{7}}{3}\cos\left[\frac{1}{3}\left(arcsin\left(\frac{3\sqrt{21}}{14}\right)+4\pi\right)\right]-\frac{1}{6}$$

Therefore,

$$\frac{1}{3}\left\{7-8\sqrt{7}\cos\left[\frac{1}{3}\left(arcsin\left(\frac{3\sqrt{21}}{14}\right)+4\pi\right)\right]\right\}+8\cos\left(\frac{4\pi}{7}\right)$$
$$=\frac{7}{3}-8\left(\cos\left(\frac{4\pi}{7}\right)+\frac{1}{6}\right)+8\cos\left(\frac{4\pi}{7}\right)=1.$$

1625. Prove that:

$$\sqrt[3]{\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}\cos\frac{2\pi}{9}}=-\sqrt[3]{\frac{3(\sqrt[3]{9}-1)}{4}}$$

Proposed by Carlos Paiva-Brazil

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{Since}: \cos\left(3.\frac{2\pi}{9}\right)=\cos\left(3.\frac{4\pi}{9}\right)=\cos\left(3.\frac{8\pi}{9}\right)=-\frac{1}{2}\ \textit{and}\ \cos 3x=4\cos^3 x-3\cos x$$

$$\to a=\cos\frac{2\pi}{9}, b=\cos\frac{4\pi}{9}, c=\cos\frac{8\pi}{9}\ \textit{are the solutions of equation}\ -\frac{1}{2}=4x^3-3x\ \textit{or}\ 8x^3-6x+1=0$$

$$\to \textit{From Vieta's formulas, we have}: \sum_{a,b,c} a=0, \sum_{a,b,c} ab=\frac{-6}{8}=-\frac{3}{4}\ \textit{and}\ abc=-\frac{1}{8}.$$

$$\textit{Let}\ x=\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}\ \textit{and}\ y=\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}$$

$$\to x^3=\left(\sum_{a,b,c}\sqrt[3]{ab}\right)^3=\sum_{a,b,c} ab+3\sqrt[3]{abc}\left(\sum_{a,b,c}\sqrt[3]{ab}\right)\left(\sum_{a,b,c}\sqrt[3]{a}\right)-3\sqrt[3]{(abc)^2}=-\frac{3}{4}-\frac{3}{2}xy-\frac{3}{4}=-\frac{3}{2}-\frac{3}{2}xy\ (1)$$

$$y^3=\left(\sum_{a,b,c}\sqrt[3]{a}\right)^3=\sum_{a,b,c} a+3\left(\sum_{a,b,c}\sqrt[3]{ab}\right)\left(\sum_{a,b,c}\sqrt[3]{a}\right)-3\sqrt[3]{abc}=3xy+\frac{3}{2}\ (2)$$

$$\to (1)\times(2)\to (xy)^3=-\frac{3}{2}(xy+1).\frac{3}{2}(2xy+1)\overset{z=xy}{\Leftrightarrow} 4z^3+9(2z+1)(z+1)=0$$

$$\leftrightarrow 4z^3+18z^2+27z+9=0$$

$$\overset{z=t-\frac{3}{2}}{\Leftrightarrow} 4\left(t^3-\frac{9t^2}{2}+\frac{27t}{4}-\frac{27}{8}\right)+18\left(t^2-3t+\frac{9}{4}\right)+27t-\frac{81}{2}+9=0\leftrightarrow 4t^3=\frac{9}{2}\leftrightarrow t=\frac{\sqrt[3]{9}}{2}\to z=\frac{\sqrt[3]{9}-3}{2}$$

$$(1)\to x^3=-\frac{3}{2}-\frac{3}{2}z=-\frac{3(\sqrt[3]{9}-1)}{4}\ \to \sum_{a,b,c}\sqrt[3]{ab}=-\sqrt[3]{\frac{3(\sqrt[3]{9}-1)}{4}}.$$

Therefore,

$$\sqrt[3]{\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}\cos\frac{2\pi}{9}}=-\sqrt[3]{\frac{3(\sqrt[3]{9}-1)}{4}}.$$

Solution 2 by proposer

We know that $\cos 3\cdot 40^\circ=\cos 120^\circ=-\frac{1}{2}$

$$\cos 3\cdot 80^\circ=\cos 240^\circ=-\frac{1}{2},\cos 3\cdot 160^\circ=\cos 480^\circ=-\frac{1}{2}$$

Then, $\cos 3x=-\frac{1}{2}, 8\cos^3 x-6\cos x+1=0$

Let $a^3=\cos 40^\circ, b^3=\cos 80^\circ, c^3=\cos 160^\circ$ and a^3,b^3,c^3 –the roots at the cubic

$8\cos^3 x-6\cos x+1=0$. By Vieta's formulae

$$\begin{cases} a^3+b^3+c^3=0\\ a^3b^3+b^3c^3+c^3a^3=-\frac{3}{4}\\ a^3b^3c^3=-\frac{1}{8};\left(\because abc=-\frac{1}{2}\right)\end{cases}$$

We also know that:

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

Let $a+b+c=\theta_1, ab+bc+ca=\theta_2\Rightarrow \theta_1(\theta_1^2-3\theta_2)=\frac{3}{2}$

$$a^3b^3+b^3c^3+c^3a^3=[(ab+bc+ca)^2-3abc(a+b+c)](ab+bc+ca)$$

$$-\frac{3}{4}-3\cdot\frac{1}{4}=\theta_2\left(\theta_2^2+\frac{3\theta_1}{2}\right)$$

$$\Rightarrow\begin{cases}2\theta_1(\theta_1^2-3\theta_2)=3\\ \theta_2(2\theta_2^2+3\theta_1)=-3\end{cases}\Rightarrow \theta_1=\frac{-2\theta_2^3-3}{3\theta_2}$$

$$16\theta_2^9-36\theta_2^6+27\theta_2^2+54=0\Leftrightarrow(4\theta_2^3-3)^3=-343=-3^5\Leftrightarrow 4\theta_2^3-3-3\sqrt[3]{9}$$

$$\theta_2=-\sqrt[3]{\frac{3(\sqrt[3]{9}-1)}{4}}$$

Therefore,

$$\sqrt[3]{\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}\cos\frac{2\pi}{9}}=-\sqrt[3]{\frac{3(\sqrt[3]{9}-1)}{4}}$$

Solution 3 by Samir Cabiyev-Azerbaijan

$$\sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=A$$

$$\sqrt[3]{\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}\cos\frac{2\pi}{9}}=B$$

$$\because (x+y+z)^3=x^3+y^3+z^3+3(x+y+z)(xy+yz+zx)-3xyz$$

$$\Rightarrow A^3=\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}+3AB-3\sqrt[3]{\cos\frac{2\pi}{9}\cdot\cos\frac{4\pi}{9}\cdot\cos\frac{8\pi}{9}}$$

$$-3\sqrt[3]{\left(\cos\frac{2\pi}{9}\cdot\cos\frac{4\pi}{9}\cdot\cos\frac{8\pi}{9}\right)^2}$$

$$x_1=\cos\frac{2\pi}{9},x_2=\cos\frac{4\pi}{9},x_3=\cos\frac{8\pi}{9}$$

$$4\cos^3\varphi-3\cos\varphi+\frac{1}{2}=0$$

$$\begin{cases} x_1+x_2+x_3=0 \\ x_1x_2+x_2x_3+x_3x_1=-\dfrac{3}{4} \\ x_1x_2x_3=-\dfrac{1}{8}\end{cases}\Rightarrow\begin{cases} A^3-3AB=\dfrac{3}{2} \\ B^3+\dfrac{3}{2}AB=-\dfrac{3}{2}\end{cases}\Rightarrow A^3+2B^3=-\frac{3}{2}$$

$$A^3=-2B^3-\frac{3}{2}\Rightarrow A=-\sqrt[3]{2B^3+\frac{3}{2}}\Rightarrow -2B^3-\frac{3}{2}+3B\sqrt[3]{2B^3+\frac{3}{2}}=\frac{3}{2}$$

$$-(2B^3+3)^3=-27\cdot B^3\left(2B^3+\frac{3}{2}\right).$$

Let $B^3=t\Rightarrow(2t+3)^3+27\left(2t^2+\frac{3}{2}t\right)\Rightarrow\left(2t-\frac{6}{4}\right)^3=-\frac{6^3\cdot 9}{4^3}\Rightarrow$

$$t=\frac{3-3\sqrt[3]{9}}{4},B^3=t\Rightarrow B=-\sqrt[3]{\frac{3(\sqrt[3]{9}-1)}{4}}$$

Therefore,

$$\sqrt[3]{\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}\cos\frac{2\pi}{9}}=-\sqrt[3]{\frac{3(\sqrt[3]{9}-1)}{4}}$$

1626. Prove that:

$$\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}+(-1)^{n+1}H_n}{n+1}=\frac{1}{4}(\pi+2\log^2 2-\log 4-4)$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Amrit Awasthi-India

The sum can be written as,

$$S=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n+1}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n+1}=\Psi+\Omega,\text{ where}$$

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n+1}=\sum_{n=1}^{\infty}\frac{H_n}{n+1}\cos\big(x(x+1)\big),\text{ at } x=\pi$$

$$\sum_{n=1}^{\infty}H_n y^{n-1}=\frac{1}{y}\left(\sum_{n=1}^{\infty}\frac{y^n}{n}\right)\left(\sum_{n=0}^{\infty}y^n\right)=-\frac{\log(1-y)}{y(1-y)};|y|<1$$

Multiply with z and integrate between 0 to z, it follows that

$$\sum_{n=1}^{\infty}\frac{H_n z^{n+1}}{n+1}=\frac{1}{2}\log^2(1-z)$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{H_n\cos\big(n(n+1)\big)}{n+1}=Re\left(\sum_{n=1}^{\infty}\frac{H_n e^{i(n+1)n}}{n+1}\right)=Re\left(\frac{1}{2}\log^2(1-z)\right)=$$

$$=Re\left[\frac{1}{2}(\log(1-e^{ix}))^2\right]=Re\left[\frac{1}{2}\big(\log((1-\cos x)-i\sin x)\big)^2\right]=$$

$$=Re\left(\frac{1}{2}(\log(1-\cos x-i\sin x))^2\right)=Re\left(\frac{1}{2}\left(\log\left(2\sin\frac{x}{2}+i\arg(1-e^{ix})\right)^2\right)\right)=$$

$$=\frac{1}{2}\left(\log^2\left(2\sin\frac{x}{2}\right)-\left(\tan^{-1}\left(-\frac{\sin x}{1-\cos x}\right)\right)^2\right)=$$

$$=\frac{1}{2}\left(\log^2\left(2\sin\frac{x}{2}\right)-\left(\frac{\pi}{2}-\cot^{-1}\left(\cot\frac{x}{2}\right)^2\right)\right)\Rightarrow\Omega=\frac{1}{2}\left(\log^2\left(2\sin\frac{x}{2}\right)-\left(\frac{\pi-x}{2}\right)^2\right)$$

$$\text{For } x=\pi\Rightarrow\Omega\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}H_n=\frac{1}{2}\log^2 2=\frac{1}{4}(2\log^2 2);(I)$$

$$\Psi=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n+1}=-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots=$$

$$=\left(-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)-\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)=$$

$$=\left(\frac{\pi}{4}-1\right)-\frac{1}{2}\log 2=\frac{1}{4}(\pi-\log 4-4);(II)$$

From$(I),(II)$ we get,

$$S=\frac{1}{4}(\pi+2\log^2 2-\log 4-4)$$

Solution 2 by Jack Desire-Nigeria

$$\text{Let: }\Omega=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}+(-1)^{n+1}H_n}{n+1}$$

$$(-1)^{\frac{n(n+1)}{2}} is+ve \text{ for } n=3,7,11,\dots,(4n-1)$$

$$(-1)^{\frac{n(n+1)}{2}}\ is+ve \text{ for } n=4,8,12,\dots,(4n)$$

$$(-1)^{\frac{n(n+1)}{2}}\ is-ve \text{ for } n=1,5,9,\dots,(4n-3)$$

$$(-1)^{\frac{n(n+1)}{2}} is-ve \text{ for } n=2,6,10,\dots,(4n-2)$$

$$\text{Let: }\Omega_1=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n+1},\Omega_2=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}H_n}{n+1}$$

We have:

$$\Omega_1=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n+1}=\sum_{n=1}^{\infty}\left(\frac{1}{4n}+\frac{1}{4n+1}-\frac{1}{4n-2}-\frac{1}{4n-1}\right)=$$

$$=\frac{1}{4}\left[\psi(n)+\psi\left(n+\frac{1}{4}\right)-\psi\left(n-\frac{1}{2}\right)-\psi\left(n-\frac{1}{4}\right)\right]_1^{\infty}=$$

$$=\frac{1}{4}\left(-\psi(1)-\psi\left(\frac{5}{4}\right)+\psi\left(\frac{1}{2}\right)+\psi\left(\frac{3}{4}\right)\right)=\frac{1}{4}\left(\psi\left(\frac{1}{2}\right)-\psi(1)+\psi\left(\frac{3}{4}\right)-\psi\left(\frac{5}{4}\right)\right)$$

$$\text{We know: }\int_0^1\frac{1}{1+x}dx=\log 2=\frac{1}{2}\left(\psi(1)-\psi\left(\frac{1}{2}\right)\right)$$

$$\because\frac{1}{4}\left(\psi\left(\frac{1}{2}\right)-\psi(1)\right)=-\frac{1}{2}\log 2\text{, when }\alpha=\frac{1}{2}$$

$$\int_0^1\frac{\sqrt{x}}{1+x}dx=\int_0^1\frac{2u^2}{1+u^2}du=2\int_0^1\left(1-\frac{1}{u^2+1}\right)du=2\left(1-\frac{\pi}{4}\right)$$

$$=\frac{1}{2}\left(\psi\left(\frac{5}{4}\right)-\psi\left(\frac{3}{4}\right)\right)$$

$$\frac{1}{4}\left(\psi\left(\frac{3}{4}\right)-\psi\left(\frac{5}{4}\right)\right)=-\frac{1}{2}\left(2-\frac{\pi}{2}\right)=\frac{\pi}{4}-1$$

$$\Omega_1=\frac{1}{4}\left(\psi\left(\frac{1}{2}\right)-\psi(1)+\psi\left(\frac{3}{4}\right)-\psi\left(\frac{5}{4}\right)\right)=-\frac{1}{2}\log 2+\frac{\pi}{4}-1$$

$$\Omega_2=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n+1}$$

$$\text{We know: }\sum_{n=1}^{\infty}(-1)^{n+1}x^nH_n=\frac{\log(1+x)}{1+x}$$

Integrating both sides w.r.t. x from 0 to 1, we get

$$\Omega_2=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n+1}=\int_0^1\frac{\log(1+x)}{1+x}dx=\frac{1}{2}\log^2 2$$

$$\Omega=\Omega_1+\Omega_2=\frac{1}{4}(\pi+2\log^2 2-\log 4-4)$$

Solution 3 by Ajenikoko Gbolahan-Nigeria

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}+(-1)^{n+1}H_n}{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n+1}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n+1}$$

$$\Phi=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n+1}=-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\cdots=$$

$$=\sum_{n=1}^{\infty}\left[\frac{1}{4n}+\frac{1}{4n+1}-\frac{1}{4n-1}-\frac{1}{4n-2}\right]=\sum_{n=1}^{\infty}\int_0^1(x^{4n-1}+x^{4n}-x^{4n-2}-x^{4n-3})dx=$$

$$=\int_0^1\sum_{n=1}^{\infty}\left(\frac{x^{4n}}{x}+x^{4n}-\frac{x^{4n}}{x^2}-\frac{x^{4n}}{x^3}\right)dx=$$

$$=\int_0^1\left(\frac{1}{x}\cdot\frac{x^4}{1-x^4}+\frac{x^4}{1-x^4}-\frac{1}{x^2}\cdot\frac{x^4}{1-x^4}-\frac{1}{x^3}\cdot\frac{x^4}{1-x^4}\right)dx=$$

$$=\int_0^1\left(\frac{x^3}{1-x^4}+\frac{x^4}{1-x^4}-\frac{x^2}{1-x^4}-\frac{x}{1-x^4}\right)dx=\int_0^1\left(\frac{1}{x^2+1}-\frac{x}{x^2+1}-1\right)dx=$$

$$=\frac{\pi}{4}-\frac{\log 2}{2}-1$$

$$\Psi=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n+1}$$

We know that: $\sum_{n=1}^{\infty}H_nx^n=\frac{\log(1-x)}{x-1}$

Subtitute $x\to -x$ and integrate from 0 to 1 w.r.t. x then,

$$\Psi=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n+1}=\int_0^1\frac{\log(1+x)}{1+x}dx=\int_1^2\frac{\log u}{u}du=\frac{\log^2 2}{2}$$

Thus, $\Omega=\Phi+\Psi=\frac{\pi}{4}-\frac{\log 2}{2}-1+\frac{\log^2 2}{2}=\frac{1}{4}(\pi+2\log^2 2-\log 4-4)$

Solution 4 by Syed Shahabudeen-India

Let: $\Omega=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}+(-1)^{n+1}H_n}{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n+1}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n+1}=A+B$

$$A=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n+1};\ B=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n+1}$$

$$A=\sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{n+1}=Re\left(\sum_{n=1}^{\infty}\frac{e^{i\frac{n(n+1)\pi}{2}}}{n+1}\right)=\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n(n+1)\pi}{2}\right)}{n+1}=$$

$$=\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi n^2}{2}\right)\cos\left(\frac{\pi n}{2}\right)-\sin\left(\frac{\pi n^2}{2}\right)\sin\left(\frac{n\pi}{n}\right)}{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}$$

$$=\frac{\pi}{4}-1-\frac{\log 2}{2}$$

$$B=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n+1}=-\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_{n+1}-\frac{1}{n+1}\right)}{n+1}=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^2}+\sum_{m=2}^{\infty}\frac{(-1)^mH_m}{m};\left(\text{here: }\sum_{m=2}^{\infty}\frac{(-1)^mH_m}{m}=1-\frac{\pi^2}{12}+\frac{\log^2 2}{2}\right)$$

$$=\left(\frac{\pi^2}{12}-1\right)+1-\frac{\pi^2}{12}+\frac{\log^2 2}{2}=\frac{\log^2 2}{2}$$

$$\Omega=\frac{\pi}{4}-1-\frac{\log 2}{2}+\frac{\log^2 2}{2}=\frac{1}{4}(\pi+2\log^2 2-\log 4-4)$$

1627. Prove that:

$$\int_0^{\frac{\pi}{4}}\int_0^1\int_0^1\frac{a^2+b^2\sin^2 x}{b^2+a^2\cos^2 x}\tan x\,da\,db\,dx=$$

$$=\frac{1}{12}-\frac{5\pi}{12}+\frac{\log 3}{6}+\frac{5\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}{6\sqrt{2}}+\frac{17\tan^{-1}(\sqrt{2})}{12\sqrt{2}}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Almas Babirov-Azerbaijan

$$\Omega=\int_0^{\frac{\pi}{4}}\int_0^1\int_0^1\frac{a^2+b^2\sin^2 x}{b^2+a^2\cos^2 x}\tan x\,da\,db\,dx$$

$$\text{Let }\Omega_1=\int_0^1\frac{a^2+b^2\sin^2 x}{b^2+a^2\cos^2 x}\tan x\,dx=$$

$$=\int_0^1\frac{a^2\cos^2 x+b^2+(b\sin x\cos x)^2-b^2}{b^2+a^2\cos^2 x}\cdot\frac{\tan x}{\cos^2 x}da=$$

$$= \int_0^1 \frac{\tan x}{\cos^2 x}\left(1 + \frac{b^2(\sin^2 x\cos^2 x - 1)}{b^2\left(1 + \frac{a^2\cos^2 x}{b^2}\right)}\right) da =$$

$$= \frac{\tan x}{\cos^2 x}\int_0^1 da + \frac{\tan x}{\cos^2 x}\int_0^1 \frac{\sin^2 x\cos^2 x - 1}{1 + \frac{a^2\cos^2 x}{b^2}} da =$$

$$= \frac{\tan x}{\cos^2 x}\cdot a\Big|_0^1 + \underbrace{\frac{\tan x(\sin^2 x\cos^2 x - 1)}{\cos^3 x}}_{f(x)}\int_0^1 \frac{b}{1 + \frac{a^2\cos^2 x}{b^2}} d\left(\frac{a\cos x}{b}\right) =$$

$$= \underbrace{\frac{\sin x}{\cos^3 x}}_{g(x)} + f(x)\cdot b\left(\tan^{-1}\left(\frac{a\cos x}{b}\right)\right|_0^1 = g(x) + f(x)\cdot b\cdot\tan^{-1}\left(\frac{\cos x}{b}\right)$$

$$\Omega_2 = \int_0^1 \left(g(x) + f(x)b\cdot\tan^{-1}\left(\frac{\cos x}{b}\right)\right) db =$$

$$= \int_0^1 g(x)\,db + f(x)\int_0^1 b\cdot\tan^{-1}\left(\frac{\cos x}{b}\right) db =$$

$$= g(x)b|_0^1 + \frac{f(x)}{2}\int_0^1 \tan^{-1}\left(\frac{\cos x}{b}\right) db^2 =$$

$$= g(x) + \frac{f(x)}{2}\left(b^2\tan^{-1}\left(\frac{\cos x}{b}\right)\right)\Big|_0^1 - \frac{g(x)}{2}\int_0^1 b^2 d\left(\tan^{-1}\left(\frac{\cos x}{b}\right)\right) =$$

$$= g(x) + \frac{f(x)}{2}tan^{-1}(\cos x) + \frac{f(x)}{2}\cos x\int_0^1 \frac{b^1\cdot\frac{\cos x}{b^2}}{1 + \frac{\cos^2 x}{b^2}} db =$$

$$= g(x) + \frac{f(x)}{2}\tan^{-1}(\cos x) + \frac{1}{2}f(x)\cos x\int_0^1 \frac{1}{1 + \frac{\cos^2 x}{b^2}} db =$$

$$= g(x) + \frac{1}{2}f(x)\tan^{-1}(\cos x) + \frac{1}{2}f(x)\cos x - \frac{1}{2}f(x)\cos^2 x\tan^{-1}\left(\frac{1}{\cos x}\right)$$

$$\Omega = \int_0^{\frac{\pi}{4}}\left(g(x) + \frac{1}{2}f(x)\tan^{-1}(\cos x) + \frac{1}{2}f(x)\cos x - \frac{1}{2}f(x)\cos^2 x\tan^{-1}\left(\frac{1}{\cos x}\right)\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan x}{\cos^2 x} dx + \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^4 x}(\sin^2 x\cos^2 x - 1)\tan^{-1}(\cos x)\, dx +$$

$$+\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^3 x}(\sin^2 x\cos^2 x-1)\,dx-\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^2 x}\left(\sin^2 x\cos^2 x\tan^{-1}\left(\frac{1}{\cos x}\right)\right)dx=$$

$$=A_1+\frac{1}{2}(A_2+A_3-A_4)$$

$$A_1=\int_0^{\frac{\pi}{4}}\frac{\tan x}{\cos^2 x}dx=\frac{1}{2}\tan^2 x\Big|_0^{\frac{\pi}{4}}=\frac{1}{2}$$

$$A_2=\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^4 x}(\sin^2 x\cos^2 x-1)\tan^{-1}(\cos x)\,dx=$$

$$=\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^2 x}\tan^{-1}(\cos x)\,dx-\int_0^{\frac{\pi}{2}}\sin x\tan^{-1}(\cos x)\,dx-\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^4 x}\tan^{-1}(\cos x)\,dx$$

$$=\int_0^{\frac{\pi}{4}}\tan^{-1}x\,d\left(\frac{1}{\cos x}\right)+\int_0^{\frac{\pi}{4}}\tan^{-1}(\cos x)\,d(\cos x)-\frac{1}{3}\int_0^{\frac{\pi}{4}}\tan^{-1}(\cos x)\,d\left(\frac{1}{\cos^3 x}\right)=$$

$$=\left(\frac{\tan^{-1}(\cos x)}{\cos x}+\cos x\tan^{-1}(\cos x)-\frac{1}{3}\cdot\frac{\tan^{-1}(\cos x)}{\cos^3 x}\right)\Bigg|_0^{\frac{\pi}{4}}+$$

$$+\int_0^{\frac{\pi}{4}}\frac{\sin x\,dx}{\cos x\,(1+\cos^2 x)}+\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{1+\cos^2 x}dx-\frac{1}{3}\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^3 x\,(1+\cos^2 x)}dx=$$

$$=\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)+\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\frac{2\sqrt{2}}{3}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\frac{\pi}{4}-\frac{\pi}{4}+\frac{\pi}{12}+$$

$$+\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos x}dx-\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{1+\cos^2 x}dx+\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{1+\cos^2 x}dx-\frac{1}{3}\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^3 x}dx-$$

$$-\frac{1}{3}\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos x}dx=-\frac{5\pi}{12}+\frac{5\sqrt{2}}{6}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\frac{4}{3}\log\left(\frac{1}{\sqrt{2}}\right)-\frac{1}{6}\frac{1}{\cos^2 x}\Big|_0^{\frac{\pi}{4}}+$$

$$+\frac{1}{6}\log(1+\cos^2 x)\Big|_0^{\frac{\pi}{4}}=-\frac{1}{6}-\frac{5\pi}{12}+\frac{5\sqrt{2}}{6}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{6}\log 3+\frac{1}{3}\log 2$$

$$A_3=\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^3 x}(\sin^2 x\cos^2 x-1)\,dx=$$

$$=\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos x}dx-\int_0^{\frac{\pi}{4}}\sin x\cos x\,dx-\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos x}dx=$$

$$=\left(-\log(\cos x)+\frac{1}{4}\cos 2x-\frac{1}{2\cos^2 x}\right)\Big|_0^{\frac{\pi}{4}}=\frac{1}{2}\log 2-\frac{3}{4}$$

$$A_4=\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^2 x}\left(\sin^2 x\cos^2 x\tan^{-1}\left(\frac{1}{\cos x}\right)\right)dx=$$

$$=\int_0^{\frac{\pi}{4}}\sin x\tan^{-1}\left(\frac{1}{\cos x}\right)dx-\int_0^{\frac{\pi}{4}}\sin x\cos^2 x\tan^{-1}\left(\frac{1}{\cos x}\right)dx-$$

$$-\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos^2 x}\tan^{-1}\left(\frac{1}{\cos x}\right)dx=$$

$$=-\int_0^{\frac{\pi}{4}}\tan^{-1}\left(\frac{1}{\cos x}\right)d(\cos x)+\frac{1}{3}\int_0^{\frac{\pi}{4}}\tan^{-1}\left(\frac{1}{\cos x}\right)d(\cos^3 x)-$$

$$-\int_0^{\frac{\pi}{4}}\tan^{-1}\left(\frac{1}{\cos x}\right)d\left(\frac{1}{\cos x}\right)=$$

$$=\left(-\cos x\tan^{-1}\left(\frac{1}{\cos x}\right)+\frac{1}{3}\cos^3 x\tan^{-1}\left(\frac{1}{\cos x}\right)-\frac{1}{\cos x}\tan^{-1}\left(\frac{1}{\cos x}\right)\right)\Big|_0^{\frac{\pi}{4}}+$$

$$+\int_0^{\frac{\pi}{4}}\frac{\cos x\cdot\frac{\sin x}{\cos^2 x}}{1+\frac{1}{\cos^2 x}}dx-\frac{1}{3}\int_0^{\frac{\pi}{4}}\frac{\cos^3 x\cdot\frac{\sin x}{\cos^2 x}}{1+\frac{1}{\cos^2 x}}dx+\int_0^{\frac{\pi}{4}}\frac{1}{\cos x}\cdot\frac{\frac{\sin x}{\cos^2 x}\cdot\cos^2 x}{1+\cos^2 x}dx=$$

$$=\frac{5\pi}{12}-\frac{17\sqrt{2}}{12}\tan^{-1}(\sqrt{2})+\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{1+\cos^2 x}dx+\frac{1}{6}\int_0^{\frac{\pi}{4}}\frac{\cos^2 x\, d(\cos^2 x)}{1+\cos^2 x}+\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos x}dx-$$

$$-\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{1+\cos^2 x}dx=$$

$$=\frac{5\pi}{12}-\frac{17\sqrt{2}}{12}\tan^{-1}(\sqrt{2})+\frac{1}{6}\cos^2 x\Big|_0^{\frac{\pi}{4}}-\frac{1}{6}\log(1+\cos^2 x)\Big|_0^{\frac{\pi}{4}}-\log\left(\frac{1}{\sqrt{2}}\right)=$$

$$=-\frac{1}{12}+\frac{5\pi}{12}-\frac{17\sqrt{2}}{12}\tan^{-1}(\sqrt{2})-\frac{1}{6}\log 3+\frac{5}{6}\log 2$$

Hence,

$$\Omega=\frac{1}{2}(2A_2+A_2+A_3+A_4)=$$

$$=\frac{1}{12}-\frac{5\pi}{12}+\frac{\log 3}{6}+\frac{5\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}{6\sqrt{2}}+\frac{17\tan^{-1}(\sqrt{2})}{12\sqrt{2}}$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\frac{\pi}{4}}\int_0^1\int_0^1\frac{a^2+b^2\sin^2 x}{b^2+a^2\cos^2 x}\tan x\,da\,db\,dx$$

$$\text{Let }\Omega_1=\int_0^1\frac{a^2+b^2\sin^2 x}{b^2+a^2\cos^2 x}\tan x\,dx\overset{\boxed{t=\tan x}}{=}\int_0^1\frac{a^2+(a^2+b^2)t^2}{b^2+a^2+b^2t^2}\left(\frac{t}{1+t^2}\right)dt=$$

$$=\frac{1}{2}\int_0^1\frac{a^2+(a^2+b^2)t}{(a^2+b^2+b^2t)(1+t)}dt=$$

$$=\frac{a^4+b^4+a^2b^2}{2a^2b^2}\int_0^1\frac{b^2}{a^2+b^2+b^2t}dt-\frac{b^2}{2a^2}\int_0^1\frac{1}{1+t}dt=$$

$$=\frac{1}{2}\left(\left(\frac{b}{a}\right)^2+\left(\frac{a}{b}\right)^2+1\right)\log\left(1+\frac{\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2}\right)-\frac{\log 2}{2}\left(\frac{b}{a}\right)^2$$

$$\Omega\overset{\boxed{t=\frac{b}{a}}}{=}\frac{1}{2}\int_0^1 a\left(\int_0^{\frac{1}{a}}\left(\frac{1}{t^2}+t^2+1\right)\log\left(1+\frac{t^2}{1+t^2}\right)-\log 2\,t^2\right)dt\,da=$$

$$=\int_0^{\frac{1}{a}}\left(\frac{1}{t^2}+t^2+1\right)\log\left(1+\frac{t^2}{1+t^2}\right)dt\overset{\boxed{IBP}}{=}\left(t-\frac{1}{t}+\frac{t^3}{3}\right)\log\left(1+\frac{t^2}{1+t^2}\right)\Bigg|_0^{\frac{1}{a}}-$$

$$-\int_0^{\frac{1}{a}}\left(t-\frac{1}{t}+\frac{t^3}{3}\right)\left(\frac{4t}{1+2t^2}-\frac{2t}{1+t^2}\right)dt$$

$$\Psi=\left(\frac{1}{a}-a+\frac{1}{3a^3}\right)\log\left(1+\frac{1}{1+a^2}\right)-\int_0^{\frac{1}{a}}\left(\frac{1}{3}+\frac{10}{3(1+t^2)}-\frac{17}{3(1+2t^2)}\right)dt=$$

$$=\left(\frac{1}{a}-a+\frac{1}{3a^3}\right)\log\left(1+\frac{1}{1+a^2}\right)-\frac{2t+20\tan^{-1}t-17\sqrt{2}\tan^{-1}(\sqrt{2}t)}{6}\Bigg|_0^{\frac{1}{a}}=$$

$$=\left(\frac{1}{a}-a+\frac{1}{3a^3}\right)\log\left(1+\frac{1}{1+a^2}\right)-\frac{1}{3a}-\frac{10}{3}\tan^{-1}\left(\frac{1}{a}\right)+\frac{17\sqrt{2}}{6}\tan^{-1}\left(\frac{\sqrt{2}}{a}\right)$$

$$\Omega=\frac{1}{6}\int_0^1\left(\left(3-3a^2+\frac{1}{a^2}\right)\log\left(1+\frac{1}{1+a^2}\right)-\frac{\log 2}{a^2}\right)da-$$

$$-\frac{1}{6}\int_0^1\left(1+10a\tan^{-1}\left(\frac{1}{a}\right)-\frac{17}{\sqrt{2}}a\tan^{-1}\left(\frac{\sqrt{2}}{a}\right)da\right)=\frac{1}{6}(A-B)$$

$$A\overset{\boxed{IBP}}{=}\left(\left(3a-a^3-\frac{1}{a}\right)\log\left(1+\frac{1}{1+a^2}\right)+\frac{\log 2}{a}\right)\Bigg|_0^1-$$

$$-\int_0^1\left(3a-a^3-\frac{1}{a}\right)\left(\frac{2a}{2+a^2}-\frac{2a}{1+a^2}\right)da=$$

$$=\log 3+\int_0^1\left(\frac{22}{2+a^2}-\frac{10}{1+a^2}-2\right)da=\log 3-2-\frac{5\pi}{2}+\frac{22}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\int_0^1 a\tan^{-1}\left(\frac{1}{a}\right)da=\frac{a^2}{2}\tan^{-1}\left(\frac{1}{a}\right)\Bigg|_0^1+\frac{1}{2}\int_0^1\frac{x^2}{1+x^2}dx=$$

$$=\left(\frac{a^2}{2}\tan^{-1}\left(\frac{1}{a}\right)+\frac{a}{2}-\frac{1}{2}\tan^{-1}a\right)\Bigg|_0^1=\frac{1}{2}$$

$$\int_0^1 a\tan^{-1}\left(\frac{\sqrt{2}}{a}\right)da=\left(\frac{a^2}{2}\tan^{-1}\left(\frac{\sqrt{2}}{a}\right)+\frac{a}{\sqrt{2}}-2\tan^{-1}\left(\frac{a}{\sqrt{2}}\right)\right)\Bigg|_0^1=$$

$$=\frac{1}{2}\tan^{-1}(\sqrt{2})+\frac{1}{\sqrt{2}}-\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$B=-\frac{5}{2}-\frac{17}{2\sqrt{2}}\tan^{-1}(\sqrt{2})+\frac{17}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\Omega=\frac{1}{6}(A-B)=\frac{1}{6}\left(\log 3+\frac{1}{2}-\frac{5\pi}{12}+\frac{5}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)+\frac{17}{2\sqrt{2}}\tan^{-1}(\sqrt{2})\right)=$$

$$=\frac{1}{12}-\frac{5\pi}{12}+\frac{\log 3}{6}+\frac{5\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}{6\sqrt{2}}+\frac{17\tan^{-1}(\sqrt{2})}{12\sqrt{2}}$$

Therefore,

$$\int_0^{\frac{\pi}{4}}\int_0^1\int_0^1\frac{a^2+b^2\sin^2 x}{b^2+a^2\cos^2 x}\tan x\,da\,db\,dx=$$

$$=\frac{1}{12}-\frac{5\pi}{12}+\frac{\log 3}{6}+\frac{5\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)}{6\sqrt{2}}+\frac{17\tan^{-1}(\sqrt{2})}{12\sqrt{2}}$$

1628. Prove that:

$$\int_0^{\frac{\pi}{2}}\left(\sqrt[3]{\cot x}+\frac{1}{\sqrt[3]{\cot x}}\right)^2\sin^3 x\,dx=\frac{4}{3}+\frac{2^{\frac{8}{3}}\sqrt{3}\Gamma\left(\frac{4}{3}\right)^3}{\pi}-\frac{2^{\frac{4}{3}}\pi^2}{\Gamma\left(-\frac{2}{3}\right)^3}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\frac{\pi}{2}}\left(\sqrt[3]{\cot x}+\frac{1}{\sqrt[3]{\cot x}}\right)^2\sin^3 x\,dx=$$

$$=\int_0^{\frac{\pi}{2}}\left(2\sin^3 x+(\sin x)^{\frac{11}{3}}(\cos x)^{-\frac{2}{3}}+(\sin x)^{\frac{7}{3}}(\cos x)^{\frac{2}{3}}\right)dx=$$

$$=B\left(2,\frac{1}{2}\right)+\frac{1}{2}B\left(\frac{7}{3},\frac{1}{6}\right)+\frac{1}{2}B\left(\frac{5}{3},\frac{5}{6}\right)=\frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}+\frac{\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{1}{6}\right)}{2\Gamma\left(\frac{5}{2}\right)}+\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{5}{6}\right)}{2\Gamma\left(\frac{5}{2}\right)}$$

$$\because \Gamma(z+1)=z\Gamma(z)\Rightarrow\Gamma\left(\frac{5}{2}\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3}{4}\Gamma\left(\frac{1}{2}\right)=\frac{3\sqrt{\pi}}{4}$$

$$\Omega=\frac{4}{3}+\frac{2\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{1}{6}\right)}{3\sqrt{\pi}}+\frac{2\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{5}{6}\right)}{3\sqrt{\pi}}=$$

$$=\frac{4}{3}+\frac{8\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{6}\right)}{9\sqrt{\pi}}+\frac{4\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}{9\sqrt{\pi}}=\frac{4}{3}+A+B,\text{ where}$$

$$\because\Gamma\left(\frac{1}{6}\right)=\frac{2^{\frac{2}{3}}\sqrt{\pi}\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}=\frac{2^{\frac{2}{3}}\sqrt{\pi}\Gamma^2\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}=\frac{2^{-\frac{1}{3}}\sqrt{3}\Gamma^2\left(\frac{1}{3}\right)}{\sqrt{\pi}}=\frac{2^{-\frac{1}{3}}9\sqrt{\pi}\Gamma^2\left(\frac{4}{3}\right)}{\sqrt{\pi}}$$

$$\because\Gamma(z)\Gamma(1-z)=\pi\csc(\pi z)$$

$$A=\frac{8\Gamma\left(\frac{4}{3}\right)2^{-\frac{1}{3}}9\sqrt{3}\Gamma^2\left(\frac{4}{3}\right)}{9\sqrt{\pi}\cdot\sqrt{\pi}}=\frac{2^{\frac{8}{3}}\sqrt{3}\Gamma^3\left(\frac{4}{3}\right)}{\pi}$$

$$\Gamma\left(\frac{1}{3}\right)=\frac{2^{\frac{1}{3}}\sqrt{\pi}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}$$

$$B=\frac{2^{\frac{7}{3}}\Gamma^2\left(\frac{2}{3}\right)}{9\Gamma\left(\frac{1}{3}\right)}=\frac{2^{\frac{7}{3}}\Gamma^2\left(\frac{2}{3}\right)\Gamma^2\left(\frac{1}{3}\right)}{9\Gamma^3\left(\frac{1}{3}\right)}=\frac{2^{\frac{13}{2}}\pi^2}{27\Gamma^3\left(\frac{1}{3}\right)}=-\frac{2^{\frac{4}{3}}\pi^2}{\Gamma^3\left(-\frac{2}{3}\right)}$$

Therefore,

$$\int_0^{\frac{\pi}{2}}\left(\sqrt[3]{\cot x}+\frac{1}{\sqrt[3]{\cot x}}\right)^2\sin^3 x\,dx=\frac{4}{3}+\frac{2^{\frac{8}{3}}\sqrt{3}\Gamma\left(\frac{4}{3}\right)^3}{\pi}-\frac{2^{\frac{4}{3}}\pi^2}{\Gamma\left(-\frac{2}{3}\right)^3}$$

1629. Let $f: R^+\to R$ satisfies the functional equation

$$(*): f(ab)=e^{ab-a-b}\left[e^b.f(a)+e^a.f(b)\right],\forall a,b>0$$

$If\ f(1)=0$ and $f'(1)=e$.Prove that: $f(a)=e^a.ln(a),\forall a>0.$

Proposed by Hikmat Hammadov-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*): f(ab)=e^{ab-a-b}\left[e^b.f(a)+e^a.f(b)\right],\forall a,b>0$$

$$(*)\leftrightarrow\frac{f(ab)}{e^{ab}}=\frac{f(a)}{e^a}+\frac{f(b)}{e^b}\leftrightarrow g(ab)=g(a)+g(b),(\forall a,b>0),$$

$$where\ g(a)=\frac{f(a)}{e^a},\forall a>0.$$

$$Since\ f(1)=0\ and\ f'(1)=e\ \to g(1)=0\ and\ \lim_{a\to 1}\frac{g(a)-g(1)}{a-1}=\lim_{a\to 1}\frac{f(a)}{(a-1)e^a}$$

$$=\frac{f'(1)}{e}=1\ \to\ g'(1)=1.$$

$$Also, we\ have:\ \lim_{a\to a_0}\frac{g(a)-g(a_0)}{a-a_0}\overset{b\to\frac{a}{a_0}}{\cong}\lim_{b\to 1}\frac{g(b.a_0)-g(a_0)}{b.a_0-a_0}=\lim_{b\to 1}\frac{g(b)}{a_0(b-1)}$$

$$=\frac{g'(1)}{a_0}=\frac{1}{a_0},\forall a_0>0$$

$$\to\ g\ is\ differentiable\ on\ R^+\ with\ g'(a)=\frac{1}{a},\forall a\in R^+\ \to\ g(a)=ln(a),\forall a\in R^+$$

$$\to\ f(a)=e^a.ln(a),\forall a\in R^+.$$

$$Therefore,\qquad f(a)=e^a.ln(a),\forall a\in R^+.$$

1630. If $0\le a\le\frac{\pi}{12}$ then:

$$\int_0^a\sin x.\cos(6x).\cos^6(4x).\cos^{15}(2x)\,dx\le\frac{1}{193}\left(1-\cos^{193}a\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ x \in [0, a], since\ a \le \frac{\pi}{12} \to \cos x, \cos(2x), \cos(4x), \cos(6x), \sin x \ge 0$$

$$We\ have: \cos(2x) = 2\cos^2 x - 1 \overset{AM-GM}{\le} (\cos^4 x + 1) - 1 = \cos^4 x \to \cos(2x) \le \cos^4 x\ (1)$$

$$\to \cos(4x) \overset{(1)}{\le} \cos^4(2x) \overset{(1)}{\le} (\cos^4 x)^4 = \cos^{16} x \to \cos(4x) \le \cos^{16} x\ (2)$$

$$\cos(3x) = \cos x.(4\cos^2 x - 3) \overset{AM-GM}{\le} \cos x.[(\cos^8 x + 1 + 1 + 1) - 3] = \cos^9 x \to \cos(3x) \le \cos^9 x\ (i)$$

$$\to \cos(6x) \overset{(1)}{\le} \cos^4(3x) \overset{(i)}{\le} (\cos^9 x)^4 = \cos^{36} x \to \cos(6x) \le \cos^{36} x\ (3)$$

$$(1),(2),(3) \to \cos(6x).\cos^6(4x).\cos^{15}(2x) \le \cos^{36} x.(\cos^{16} x)^6.(\cos^4 x)^{15} = \cos^{192} x$$

$$\to \int_0^a \sin x.\cos(6x).\cos^6(4x).\cos^{15}(2x)\,dx \le \int_0^a \sin x.\cos^{192} x\,dx = \left[\frac{-1}{193}\cos^{193} x\right]_0^a = \frac{1}{193}(1 - \cos^{193} a).$$

1631. **Find:**

$$\Omega = \lim_{x \to 1^+}\left(2\log|1 - x| - 4\log\left|1 - \sqrt{1 + \sqrt{1 - x}}\right|\right),$$

where $|\cdot|$ – denotes absolute sign.

Proposed by Naren Bhandari-Bajura-Nepal

Solution 1 by Surjeet Singhania-India

Choose $\frac{1}{10^{10}} > \delta > 0$ such that $0 < |z - 1| < \delta$, we know that

$$\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{16} + O(x^5), \forall\{x \in \mathbb{C}||x| < 1\}$$

Hence, $\sqrt{1 + \sqrt{1 - z}} = 1 + 2^{-1}\sqrt{1 - z} - 8^{-1}\left(\sqrt{1 - z}\right)^2 + 16^{-1}\left(\sqrt{1 - z}\right)^3 + O\left(\left(\sqrt{1 - z}\right)^5\right)$

$$4\log\left|1-\sqrt{1+\sqrt{1-z}}\right| = 4\log\left|2^{-1}\sqrt{1-z}-8^{-1}\left(\sqrt{1-z}\right)^2+\cdots\right|$$

$$4\log\left|1-\sqrt{1+\sqrt{1-z}}\right| = 2\log|1-z|+4\log\left|\frac{1}{2}+\frac{\sqrt{1-z}}{8}+\cdots\right|$$

Hence,

$$\Omega=\lim_{x\to 1^+}\left(2\log|1-x|-4\log\left|1-\sqrt{1+\sqrt{1-x}}\right)=-4\log\left(\frac{1}{2}\right)=4\log 2$$

Solution 2 by Muhammad Afzal-Pakistan

$$\Omega=\lim_{x\to 1^+}\left(2\log|1-x|-4\log\left|1-\sqrt{1+\sqrt{1-x}}\right)\overset{x\to 1-x}{=}$$

$$=\lim_{x\to 0^+}\left(2\log|x|-4\log\left|1-\sqrt{1+\sqrt{x}}\right|\right)\overset{\sqrt{1+\sqrt{x}}\sim 1+\frac{1}{2}\sqrt{x}}{=}$$

$$=\lim_{x\to 0^+}\left(2\log|x|-4\log\left|\frac{\sqrt{x}}{2}\right|\right)=\lim_{x\to 0^+}(2\log|x|-2\log|x|+4\log 2)=4\log 2$$

1632.

$$\text{If } S_n=\sum_{k=0}^{n}\sum_{m=0}^{k}\frac{(-1)^{m+k}}{m!}\text{ then prove:}$$

$$\lim_{n\to\infty}(S_{n-1}+S_n)=e$$

Proposed by Angad Singh-India

Solution by Amrit Awasthi-India

$$S_n=\sum_{k=0}^{n}\sum_{m=0}^{k}\frac{(-1)^{m+k}}{m!}=\sum_{k=0}^{n}\sum_{m=0}^{k}\frac{(-1)^m}{m!}\cdot(-1)^k$$

Changing the order of summation we get as $0\le m\le k\le n$,

$$S_n=\sum_{m=0}^{n}\frac{(-1)^m}{m!}\sum_{k=m}^{n}(-1)^k=\sum_{m=0}^{n}\frac{(-1)^m}{m!}\cdot\Omega_n;$$

Now, if n is odd then $\Omega_n=(-1)^m$ and in that case $\Omega_{n+1}=0$. Hence,

$$\lim_{n\to\infty}(S_{n-1}+S_n)=\lim_{n\to\infty}\sum_{m=0}^{n}\frac{(-1)^m}{m!}\cdot(-1)^m=\lim_{n\to\infty}\sum_{m=0}^{n}\frac{1}{m!}=e$$

Similarly, if we assume that n is even, then $S_n = 0$ and S_{n+1} contributes to the given limits.

1633. $\Omega(n) = (1+2^2)(1+2^4)(1+2^8)\cdot \ldots \cdot \left(1+2^{2^{n-1}}\right), n \ge 1$

Find:

$$\Omega = \lim_{n\to\infty}\left(1+\frac{1}{3\Omega(n)}\right)^{2^{2^n}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Florentin Vișescu-Romania

$$\Omega(n)\cdot(1-2^2) = (1-2^2)(1+2^2)(1+2^4)(1+2^8)\cdot \ldots \cdot \left(1+2^{2^{n-1}}\right)$$

$$-3\cdot\Omega(n) = \left(1-2^{2^3}\right)\left(1+2^{2^3}\right)\cdot \ldots \cdot \left(1-2^{2^n}\right)$$

$$-3\cdot\Omega(n) = \left(1-2^{2^n}\right)$$

$$\Omega = \lim_{n\to\infty}\left(1+\frac{1}{3\Omega(n)}\right)^{2^{2^n}} = \lim_{n\to\infty}\left(1+\frac{1}{2^{2^n}-1}\right)^{\left(2^{2^n}-1\right)\cdot\frac{2^{2^n}}{2^{2^n}-1}} = \exp\left\{\lim_{n\to\infty}\frac{2^{2^n}}{2^{2^n}-1}\right\} = e$$

Solution 2 by Ty Halpen-USA

$$\Omega(n) = (1+2^2)(1+2^4)(1+2^8)\cdot \ldots \cdot \left(1+2^{2^{n-1}}\right) =$$

$$= \frac{1-2^2}{1-2^2}\cdot(1+2^2)(1+2^4)(1+2^8)\cdot \ldots \cdot \left(1+2^{2^{n-1}}\right) =$$

$$= -\frac{1}{3}(1-2^4)(1+2^4)(1+2^8)\cdot \ldots \cdot \left(1+2^{2^{n-1}}\right) = -\frac{1}{3}\left(1-2^{2^n}\right)$$

$$\Omega = \lim_{n\to\infty}\left(1+\frac{1}{3\Omega(n)}\right)^{2^{2^n}} = \lim_{n\to\infty}\left(1+\frac{1}{2^{2^n}-1}\right)^{\left(2^{2^n}-1\right)\cdot\frac{2^{2^n}}{2^{2^n}-1}} = \exp\left\{\lim_{n\to\infty}\frac{2^{2^n}}{2^{2^n}-1}\right\} = e$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\Omega(n)\cdot(1-2^2) = (1-2^2)(1+2^2)(1+2^4)(1+2^8)\cdot \ldots \cdot \left(1+2^{2^{n-1}}\right)$$

$$= \left(1-2^{2^3}\right)\left(1+2^{2^3}\right)\cdot \ldots \cdot \left(1-2^{2^n}\right) = \left(1-2^{2^n}\right) \Rightarrow 3\Omega(n) = 2^{2^n}-1$$

$$\Omega = \lim_{n\to\infty}\left(1+\frac{1}{3\Omega(n)}\right)^{2^{2^n}} = \lim_{n\to\infty}\left(1+\frac{1}{2^{2^n}-1}\right)^{\left(2^{2^n}-1\right)}\left(1+\frac{1}{2^{2^n}-1}\right) =$$

$$= \exp\left\{\lim_{n\to\infty}\frac{2^{2^n}}{2^{2^n}-1}\right\} = e$$

1634. **Find:** $\Omega =$

$$\lim_{y\to\infty}\left(\frac{2}{y^2}\left(\lim_{z\to\infty}\frac{1}{z^4}\left(\lim_{x\to 0}\frac{[(y^2+y+1)x^p+1]^{z^2+z+1}-[(z^2+z+1)x^p+1]^{y^2+y+1}}{x^{2p}}\right)\right)\right)^{\phi y},$$

where $y,z\in\mathbb{N}, p\in\mathbb{N}^*$ and ϕ –Golden Ratio.

Proposed by Costel Florea-Romania

Solution by Kamel Gandouli Rezgui-Tunisia

Let $a = y^2+y+1\in\mathbb{N}, b = z^2+z+1\in\mathbb{N}$

$$\Omega = \lim_{y\to\infty}\left(\frac{2}{y^2}\left(\lim_{z\to\infty}\frac{1}{z^4}\left(\lim_{x\to 0}\frac{(ax^p+1)^b-(bx^p+1)^a}{x^{2p}}\right)\right)\right)^{\phi y}$$

$$\lim_{x\to 0}\frac{(ax^p+1)^b-(bx^p+1)^a}{x^{2p}} = \lim_{x\to 0}\frac{1}{x^{2p}}\left(\sum_{k=1}^{b}\binom{b}{k}(ax^p)^k - \sum_{k=1}^{a}\binom{a}{k}(ax^p)^k\right) =$$

$$= \lim_{x\to 0}\frac{1}{x^{2p}}\left(\binom{b}{1}ax^p - \binom{a}{1}bx^p + \binom{b}{2}a^2x^{2p} - \binom{a}{2}b^2x^{2p} + \sum_{k=3}^{b}\binom{b}{k}(ax^p)^k - \sum_{k=1}^{a}\binom{a}{k}(ax^p)^k\right)$$

$$= \binom{b}{1}a^2 - \binom{a}{1}b^2$$

$$\binom{b}{2} = \frac{b(b-1)}{2!} = \frac{(z^2+z+1)(z^2+z)}{2}$$

$$z\to\infty \Rightarrow \frac{\binom{b}{2}}{z^4}\to\frac{1}{2} \text{ and } \frac{b^2}{z^4}\to 1$$

Hence,

$$\lim_{z\to\infty}\frac{1}{z^4}\left(\lim_{x\to 0}\frac{[(y^2+y+1)x^p+1]^{z^2+z+1}-[(z^2+z+1)x^p+1]^{y^2+y+1}}{x^{2p}}\right) =$$

$$= \lim_{z\to\infty}\frac{1}{z^4}\left(\lim_{x\to 0}\frac{(ax^p+1)^b-(bx^p+1)^a}{x^{2p}}\right) = \frac{a^2}{2} - \binom{a}{2} = \frac{a}{2} = \frac{y^2+y+1}{2}$$

Thus,

$$\Omega = \lim_{y\to\infty}\left(\frac{2}{y^2}\left(\lim_{z\to\infty}\frac{1}{z^4}\left(\lim_{x\to 0}\frac{(ax^p+1)^b-(bx^p+1)^a}{x^{2p}}\right)\right)\right)^{\Phi y} =$$

$$= \lim_{y\to\infty}\left(\frac{y^2+y+1}{y^2}\right)^{\phi y} = \lim_{y\to\infty}\exp\left(\phi y\cdot\frac{y\log\left(\frac{y^2+y+1}{y^2}\right)}{\frac{y^2+y+1}{y^2}}\cdot\frac{y+1}{y^2}\right) = e^{\phi}$$

1635. Prove that:

$$\sum_{n=0}^{\infty}\frac{(-1)^{0+1+2+\cdots+n}}{(2n+1)^2} = \frac{\sqrt{2}\pi^2}{16}$$

Proposed by Amrit Awasthi-India

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega = \sum_{n=0}^{\infty}\frac{(-1)^{0+1+2+\cdots+n}}{(2n+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{(2n+1)^2};$$

$$(-1)^{\frac{n(n+1)}{2}} = \begin{cases} 1 \text{ if } & n = 4k, 4k+3 \\ -1 \text{ if } & n = 4k+1, 4k+2 \end{cases}$$

$$\Omega = \sum_{n=0}^{\infty}\left(\frac{1}{(8n+1)^2}+\frac{1}{(8n+7)^2}\right) - \sum_{n=0}^{\infty}\left(\frac{1}{(8n+3)^2}+\frac{1}{(8n+5)^2}\right) =$$

$$= \frac{1}{64}\left\{\psi^{(1)}\left(\frac{1}{8}\right)+\psi^{(1)}\left(\frac{7}{8}\right)\right\} - \frac{1}{64}\left\{\psi^{(1)}\left(\frac{3}{8}\right)+\psi^{(1)}\left(\frac{5}{8}\right)\right\}$$

$$\because \psi^{(1)}(z)+\psi^{(1)}(1-z) = \frac{\pi^2}{\sin^2(\pi z)}$$

$$\Omega = \frac{\pi^2}{64}\left(\frac{1}{\sin^2\left(\frac{\pi}{8}\right)} - \frac{1}{\sin^2\left(\frac{3\pi}{8}\right)}\right) = \frac{\pi^2}{64}\left(\frac{1}{\sin^2\left(\frac{\pi}{8}\right)} - \frac{1}{\cos^2\left(\frac{\pi}{8}\right)}\right) = \frac{\pi^2}{64}\cdot\frac{4\cos\left(\frac{\pi}{4}\right)}{\sin^2\left(\frac{\pi}{4}\right)}$$

$$= \frac{\pi^2\sqrt{2}}{16}$$

Solution 2 by Syed Shahabudeen-Kerala-India

$$\Omega = \sum_{n=0}^{\infty}\frac{(-1)^{0+1+2+\cdots+n}}{(2n+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{(2n+1)^2} = Re\sum_{n=0}^{\infty}\frac{e^{i\cdot\frac{n(n+1)\pi}{2}}}{(2n+1)^2} =$$

$$= \sum_{n=0}^{\infty}\frac{\cos\frac{\pi n^2}{2}\cos\frac{n\pi}{2} - \sin\frac{\pi n^2}{2}\sin\frac{\pi n}{2}}{(2n+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+1)^2} - \sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)^2}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+1)^2}=\sum_{n=0}^{\infty}\frac{1}{(8n+1)^2}-\sum_{n=0}^{\infty}\frac{1}{(8n+5)^2}=\frac{1}{64}\left(\psi^{(1)}\left(\frac{1}{8}\right)-\psi^{(1)}\left(\frac{5}{8}\right)\right)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+3)^2}=\sum_{n=0}^{\infty}\frac{1}{(8n+3)^2}-\sum_{n=0}^{\infty}\frac{1}{(8n+7)^2}=\frac{1}{64}\left(\psi^{(1)}\left(\frac{3}{8}\right)-\psi^{(1)}\left(\frac{7}{8}\right)\right)$$

$$\Rightarrow \Omega=\frac{1}{64}\left(\psi^{(1)}\left(\frac{1}{8}\right)+\psi^{(1)}\left(\frac{7}{8}\right)-\psi^{(1)}\left(\frac{5}{8}\right)-\psi^{(1)}\left(\frac{5}{8}\right)\right)=$$

$$=\frac{1}{64}\left(\frac{\pi^2}{\sin^2\left(\frac{\pi}{8}\right)}-\frac{\pi^2}{\sin^2\left(\frac{3\pi}{8}\right)}\right)=\frac{\pi^2}{64}\cdot 4\sqrt{2}=\frac{\pi^2\sqrt{2}}{16}$$

$$\because \frac{\pi^2}{\sin^2\left(\frac{\pi}{8}\right)}=4+2\sqrt{2};\ \frac{\pi^2}{\sin^2\left(\frac{3\pi}{8}\right)}=4-2\sqrt{2}$$

1636. Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}i(i+2)(i+4)\right]^{-1}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Ravi Prakash-New Delhi-India

$$\sum_{i=1}^{k}i(i+2)(i+4)=\sum_{i=3}^{k+2}(i-2)i(i+2)=\sum_{i=3}^{k+2}(i^3-4i)=\sum_{i=1}^{k+2}(i^3-4i)+3=$$

$$=\left(\frac{(k+2)(k+3)}{2}\right)^2-\frac{4}{2}(k+2)(k+3)+3=$$

$$=\frac{1}{4}[(k^2+5k+6)^2-8(k^2+5k+6)+12]=$$

$$=\frac{1}{4}(k^4+10k^3+29k^2+20k)=\frac{1}{4}k(k+1)(k+4)(k+5)$$

$$\Rightarrow\left[\sum_{i=1}^{k}i(i+2)(i+4)\right]^{-1}=\frac{4}{k(k+1)(k+4)(k+5)}=$$

$$=\frac{1}{5}\cdot\frac{1}{k}-\frac{1}{3}\cdot\frac{1}{k+1}+\frac{1}{3}\cdot\frac{1}{k+4}-\frac{1}{5}\cdot\frac{1}{k+4}\Rightarrow\sum_{k=1}^{n}\left[\sum_{i=1}^{k}i(i+2)(i+4)\right]^{-1}=$$

$$=\frac{1}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}-\frac{1}{n+5}\right)=$$

$$-\frac{1}{3}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}\right)$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}i(i+2)(i+4)\right]^{-1}=$$

$$=\frac{1}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)-\frac{1}{3}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=\frac{43}{450}$$

1637. Find a closed form:

$$\Omega=\sum_{n=1}^{\infty}\frac{n(n+1)^2(2n+1)}{4^n\cdot n!}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{For } n\ge 1:\ \frac{n(n+1)^2(2n+1)}{n!}=\frac{(n+1)^2(2n+1)}{(n-1)!}$$

Write: $(n+1)^2(2n+1)$

$$=A(n-1)(n-2)(n-3)+B(n-1)(n-2)+C(n-1)+D$$

Equaling coefficient of n^3, we get: $A=2$. Put $n=1\Rightarrow\ D=12;\ n=2\Rightarrow C=33$,

$n=3\Rightarrow B=17$. Thus, for $n\ge 4$ we have:

$$\frac{n(n+1)^2(2n+1)}{n!}=\frac{2}{(n-4)!}+\frac{17}{(n-3)!}+\frac{33}{(n-2)!}+\frac{12}{(n-1)!}$$

$$\text{Now, }\Omega=\sum_{n=1}^{\infty}\frac{n(n+1)^2(2n+1)}{4^n\cdot n!}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{(n+1)^2(2n+1)}{4^{n-1}(n-1)!}=$$

$$=\frac{1}{4}\cdot\frac{12}{0!}+\frac{1}{4^2}\cdot\frac{12}{1!}+\frac{1}{4^2}\cdot\frac{33}{0!}+\frac{1}{4^3}\cdot\frac{12}{2!}+\frac{1}{4^3}\cdot\frac{33}{1!}+\frac{1}{4^3}\cdot\frac{17}{0!}+\frac{1}{4^4}\cdot\frac{12}{3!}+\frac{1}{4^4}\cdot\frac{33}{2!}+\frac{1}{4^4}\cdot\frac{17}{1!}$$
$$+\frac{1}{4^4}\cdot\frac{2}{0!}+\cdots$$

Adding, we get:

$$\Omega=\sum_{n=1}^{\infty}\frac{n(n+1)^2(2n+1)}{4^n\cdot n!}=\frac{12}{4}e^{\frac{1}{4}}+\frac{33}{16}e^{\frac{1}{4}}+\frac{17}{64}e^{\frac{1}{4}}+\frac{12}{256}e^{\frac{1}{4}}=\frac{683\sqrt[4]{e}}{128}$$

Solution 2 by Serlea Kabay-Liberia

$$\text{Since } e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Rightarrow x^2e^x=\sum_{n=1}^{\infty}\frac{x^{n+1}}{(n-1)!}\text{ and differentiating,}$$

$$e^x(x^2+2x)=\sum_{n=1}^{\infty}\frac{(n+1)x^n}{(n-1)!}\Rightarrow e^x(x^3+2x^2)$$

$$=\sum_{n=1}^{\infty}\frac{(n+1)x^{n+1}}{(n-1)!},\text{differentiating again,}$$

$$xe^x(x^2+5x+4)=\sum_{n=1}^{\infty}\frac{(n+1)^2x^n}{(n-1)!},$$

$$\Rightarrow x^3e^{x^2}(x^4+5x^2+4)=\sum_{n=1}^{\infty}\frac{(n+1)^2x^{2n+1}}{(n-1)!},\text{differentiating and let } x=\frac{1}{2},\text{we get:}$$

$$\frac{683\sqrt[4]{e}}{128}=\sum_{n=1}^{\infty}\frac{(n+1)^2(2n+1)}{4^n\cdot(n-1)!}$$

Therefore,

$$\Omega=\sum_{n=1}^{\infty}\frac{n(n+1)^2(2n+1)}{4^n\cdot n!}=\frac{683\sqrt[4]{e}}{128}$$

Solution 3 by Hasan Bostanlik-Turkiye

$$f(x)=e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Rightarrow f'(x)=e^x=\sum_{n=0}^{\infty}\frac{nx^{n-1}}{n!},$$

$$f''(x)=e^x=\sum_{n=0}^{\infty}\frac{n(n-1)x^{n-2}}{n!},f'''(x)=e^x=\sum_{n=0}^{\infty}\frac{n(n-1)(n-2)x^{n-3}}{n!}$$

$$f^{(iv)}(x)=e^x=\sum_{n=0}^{\infty}\frac{n(n-1)(n-2)(n-3)x^{n-4}}{n!}$$

$$x=1\Rightarrow e=\sum_{n=0}^{\infty}\frac{n}{n!}\Rightarrow\sum_{n=0}^{\infty}\frac{n}{4^n\cdot n!}=\frac{\sqrt[4]{e}}{4}$$

$$e=\sum_{n=0}^{\infty}\frac{n^2-n}{n!}=\sum_{n=0}^{\infty}\frac{n^2}{n!}-e\Rightarrow 2e=\sum_{n=0}^{\infty}\frac{n^2}{n!}\Rightarrow\sum_{n=0}^{\infty}\frac{n^2}{4^n\cdot n!}=\frac{5}{16}\cdot\sqrt[4]{e}$$

$$e=\sum_{n=0}^{\infty}\left(\frac{n^3}{n!}-\frac{3n}{n!}+\frac{2}{n!}\right)=\sum_{n=0}^{\infty}\frac{n^3}{n!}-3\cdot 2e+2e\Rightarrow 5e=\sum_{n=0}^{\infty}\frac{n^3}{n!}$$

$$\sum_{n=0}^{\infty}\frac{n^3}{4^n\cdot n!}=\frac{29\cdot\sqrt[4]{e}}{64}$$

$$e=\sum_{n=0}^{\infty}\frac{n^4}{n!}-6\cdot 5e+16\cdot 2e-6e\Rightarrow 15e=\sum_{n=0}^{\infty}\frac{n^4}{n!}\Rightarrow\sum_{n=0}^{\infty}\frac{n^4}{4^n\cdot n!}=\frac{201}{256}\cdot\sqrt[4]{e}$$

Therefore,

$$\Omega=\sum_{n=1}^{\infty}\frac{n(n+1)^2(2n+1)}{4^n\cdot n!}=\frac{683\sqrt[4]{e}}{128}$$

Solution 4 by George Moses-Benin-Nigeria

$$\Omega=\sum_{n=1}^{\infty}\frac{n(n+1)^2(2n+1)}{4^n\cdot n!}=\sum_{n=0}^{\infty}\frac{(n+1)(n+2)^2(2n+3)}{4^{n+1}\cdot(n+1)!}=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(n+2)^2(2n+3)}{4^n\cdot n!}$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}\frac{2n^3+11n^2+20n+12}{4^n\cdot n!}=\frac{1}{4}\sum_{n=0}^{\infty}\frac{2n^3+11n^2+20n}{4^n\cdot n!}+\frac{12}{4}\sum_{n=0}^{\infty}\frac{1}{4^n\cdot n!}$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}\frac{2(n+1)^3+11(n+1)^2+20(n+1)}{4^{n+1}\cdot(n+1)!}+3\sqrt[4]{e}=$$

$$=\frac{1}{16}\sum_{n=0}^{\infty}\frac{2(n+1)^2+11(n+1)+20}{4^n\cdot n!}+3\sqrt[4]{e}$$

$$f(x)=xe^x\sum_{k=0}^{\infty}\frac{x^{k+1}}{k!}\Rightarrow f'(x)=(x+1)e^x=\sum_{k=0}^{\infty}\frac{(k+1)x^k}{k!}=(x^2+x)e^x=$$

$$=\sum_{k=0}^{\infty}\frac{(k+1)x^{k+1}}{k!}$$

$$f''(x)=(x^2+3x+1)e^x=\sum_{k=0}^{\infty}\frac{(k+1)^2x^k}{k!}$$

$$\Omega=\frac{1}{8}\sum_{n=0}^{\infty}\frac{(n+1)^2}{4^n\cdot n!}+\frac{11}{16}\sum_{n=0}^{\infty}\frac{n+1}{4^n\cdot n!}+\frac{5}{4}\sum_{n=0}^{\infty}\frac{1}{4^n\cdot n!}+3\sqrt[4]{e}=$$

$$=\frac{1}{8}\left(\left(\left(\frac{1}{4}\right)^2+3\left(\frac{1}{4}\right)+1\right)\right)e^{\frac{1}{4}}+\frac{11}{16}\left(\left(\frac{1}{4}+1\right)e^{\frac{1}{4}}\right)+\frac{5}{4}e^{\frac{1}{4}}+3e^{\frac{1}{4}}=\frac{683}{128}e^{\frac{1}{4}}$$

Solution 5 by Syed Shahabudeen-Kerala-India

$$\Omega=\sum_{n=1}^{\infty}\frac{n(n+1)^2(2n+1)}{4^n\cdot n!}=2\sum_{n=1}^{\infty}\frac{n^4}{4^n\cdot n!}+5\sum_{n=1}^{\infty}\frac{n^3}{4^n\cdot n!}+4\sum_{n=1}^{\infty}\frac{n^2}{4^n\cdot n!}+\sum_{n=1}^{\infty}\frac{n}{4^n\cdot n!}$$

$$\because e^xB_k(x)=\sum_{n=1}^{\infty}\frac{n^kx^n}{n!},\text{ where } B_k(x)-\text{Bell polynomials.}$$

$$\Omega=2e^xB_4\left(\frac{1}{4}\right)+5e^xB_3\left(\frac{1}{4}\right)+4e^xB_2\left(\frac{1}{4}\right)+e^xB_1\left(\frac{1}{4}\right)$$

$$B_4(x)=x^4+6x^3+7x^2+x,B_4\left(\frac{1}{4}\right)=\frac{201}{256}$$

$$B_3(x)=x^3+3x^2+x,B_3\left(\frac{1}{4}\right)=\frac{29}{64}$$

$$B_2(x)=x^2+x,B_2\left(\frac{1}{4}\right)=\frac{5}{16}$$

$$B_1(x)=x,B_1\left(\frac{1}{4}\right)=\frac{1}{4}$$

Therefore,

$$\Omega=\sum_{n=1}^{\infty}\frac{n(n+1)^2(2n+1)}{4^n\cdot n!}=\frac{201}{128}e^{\frac{1}{4}}+\frac{145}{64}e^{\frac{1}{4}}+\frac{20}{16}e^{\frac{1}{4}}+\frac{1}{4}e^{\frac{1}{4}}=\frac{683}{128}\sqrt[4]{e}$$

Solution 6 by Hikmat Mammadov-Azerbaijan

$$\sum_{n=1}^{\infty}\frac{x^n}{n!}=e^x-1$$

$$\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!}=e^x,\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!}=x^2e^x\Rightarrow\frac{d}{dx}\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n}=(2x+x^2)e^x\Rightarrow$$

$$\sum_{n=1}^{\infty}\frac{n(n+1)x^n}{n!}=(2x+x^2)e^x$$

$$\sum_{n=1}^{\infty}\frac{n(n+1)x^{n+1}}{n!}=(2x^2+x^3)e^x\Rightarrow\frac{d}{dx}\sum_{n=1}^{\infty}\frac{n(n+1)x^{n+1}}{n!}=(4x+5x^2+x^3)e^x$$

$$\sum_{n=1}^{\infty}\frac{n(n+1)^2x^n}{n!}=(4x+5x^2+x^3)e^x \text{ replacing } x=x^2\Rightarrow$$

$$\sum_{n=1}^{\infty}\frac{n(n+1)x^{2n}}{n!}=(4x^2+5x^4+x^6)e^{x^2}\Rightarrow\sum_{n=1}^{\infty}\frac{n(n+1)^2x^{2n+1}}{n!}=\left(4x^3+5x^5+x^7\right)e^{x^2}$$

$$\sum_{n=1}^{\infty}\frac{n(n+1)^2(2n+1)x^{2n}}{n!}=(12x^2+33x^4+17x^6+2x^8)e^x \text{ and putting } x=\frac{1}{2}$$

$$\Omega=\sum_{n=1}^{\infty}\frac{n(n+1)^2(2n+1)}{4^n\cdot n!}=\frac{683}{128}\sqrt[4]{e}$$

Solution 7 by Yen Tung Chung-Taichung-Taiwan

$$\Omega=\sum_{n=1}^{\infty}\frac{n(n+1)^2(2n+1)}{4^n\cdot n!}=\sum_{n=1}^{\infty}\frac{(n+1)^2(2n+1)}{4^n\cdot(n-1)!}\overset{m=n-1}{=}$$

$$=\sum_{m=0}^{\infty}\frac{(m+2)^2(2m+3)}{4^{m+1}\cdot m!}=\sum_{m=0}^{\infty}\frac{2m(m-1)(m-2)+17m(m-1)+33m+12}{4^{m+1}\cdot m!}=$$

$$=\underbrace{\sum_{m=3}^{\infty}\frac{2}{4^{m+1}(m-3)!}}_{\text{let } k=m-3}+\underbrace{\sum_{m=2}^{\infty}\frac{17}{4^{m+1}(m-2)!}}_{\text{let } k=m-2}+\underbrace{\sum_{m=1}^{\infty}\frac{33}{4^{m+1}(m-1)!}}_{\text{let } k=m-1}+\underbrace{\sum_{m=0}^{\infty}\frac{12}{4^{m+1}m!}}_{\text{let } k=m}=$$

$$=\sum_{k=0}^{\infty}\frac{2}{4^{k+4}\cdot k!}+\sum_{k=0}^{\infty}\frac{17}{4^{k+3}\cdot k!}+\sum_{k=0}^{\infty}\frac{33}{4^{k+2}\cdot k!}+\sum_{k=0}^{\infty}\frac{12}{4^{k+1}\cdot k!}=$$

$$=\frac{1}{128}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}\right)^k}{k!}+\frac{17}{64}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}\right)^k}{k!}+\frac{33}{16}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}\right)^k}{k!}+3\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}\right)^k}{k!}=$$

$$=\left(\frac{1}{128}+\frac{17}{64}+\frac{33}{16}+3\right)e^{\frac{1}{4}}=\frac{683}{128}e^{\frac{1}{4}}$$

1638. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^4}\cdot\sum_{1\le i<j\le n}\sin^2\frac{(j-i)\pi}{n}\cdot\sum_{1\le i<j\le n}\cos^2\frac{(j-i)\pi}{n}$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Adrian Popa-Romania

$$\Omega=\lim_{n\to\infty}\frac{1}{n^4}\cdot\sum_{1\le i<j\le n}\sin^2\frac{(j-i)\pi}{n}\cdot\sum_{1\le i<j\le n}\cos^2\frac{(j-i)\pi}{n}=$$

$$=\Omega=\lim_{n\to\infty}\left(\frac{1}{n^2}\cdot\sum_{1\le i<j\le n}\sin^2\frac{(j-i)\pi}{n}\right)\cdot\left(\frac{1}{n^2}\cdot\sum_{1\le i<j\le n}\cos^2\frac{(j-i)\pi}{n}\right)=\Omega_1\cdot\Omega_1$$

$$\Omega_1=\lim_{n\to\infty}\left(\frac{1}{n^2}\cdot\sum_{1\le i<j\le n}\sin^2\frac{(j-i)\pi}{n}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{1}{2n^2}\sum_{1\le i<j\le n}\frac{1}{2}-\frac{1}{2n^2}\sum_{1\le i<j\le n}\cos\left(\frac{2j\pi}{n}-\frac{2i\pi}{n}\right)\right)=$$

$$=\lim_{n\to\infty}\left(\frac{n(n+1)}{2n^2\cdot 2}-\frac{1}{2n^2}\sum_{1\le i<j\le n}\left(\cos\frac{2i\pi}{n}\cos\frac{2j\pi}{n}+\sin\frac{2i\pi}{n}\sin\frac{2j\pi}{n}\right)\right)$$

$$\left(\sum_{i=1}^{n}\cos\frac{2i\pi}{n}\right)^2=\sum_{i=1}^{n}\cos^2\frac{2i\pi}{n}+2\sum_{1\le i<j\le n}\cos\frac{2i\pi}{n}\cos\frac{2j\pi}{n}\Rightarrow$$

$$\sum_{1\le i<j\le n}\cos\frac{2i\pi}{n}\cos\frac{2j\pi}{n}=\frac{1}{2}\left(\sum_{i=1}^{n}\cos\frac{2i\pi}{n}\right)^2-\frac{1}{2}\sum_{i=1}^{n}\cos^2\frac{2i\pi}{n};(*)$$

$$\left(\sum_{i=1}^{n}\sin\frac{2i\pi}{n}\right)^2=\sum_{i=1}^{n}\sin^2\frac{2i\pi}{n}+2\sum_{1\le i<j\le n}\sin\frac{2i\pi}{n}\sin\frac{2j\pi}{n}\Rightarrow$$

$$\sum_{1\le i<j\le n}\sin\frac{2i\pi}{n}\sin\frac{2j\pi}{n}=\frac{1}{2}\left(\left(\sum_{i=1}^{n}\sin\frac{2i\pi}{n}\right)^2-\sum_{i=1}^{n}\sin^2\frac{2i\pi}{n}\right)$$

Hence,

$$\Omega_1=\frac{1}{4}-\lim_{n\to\infty}\frac{1}{4n^2}\left(\left(\sum_{i=1}^{n}\cos\frac{2i\pi}{n}\right)^2+\left(\sum_{i=1}^{n}\sin\frac{2i\pi}{n}\right)^2-\sum_{i=1}^{n}\left(\cos^2\frac{2i\pi}{n}+\sin^2\frac{2i\pi}{n}\right)\right)=$$

$$=\frac{1}{4}-\frac{1}{4}\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\cos\frac{2i\pi}{n}\right)^2-\frac{1}{4}\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\sin\frac{2i\pi}{n}\right)^2+\lim_{n\to\infty}\frac{1}{4n^2}\sum_{i=1}^{n}1=$$

$$=\frac{1}{4}-\frac{1}{4}\left(\int_0^1\cos 2\pi x\,dx\right)^2-\frac{1}{4}\left(\int_0^1\sin 2\pi x\,dx\right)^2+\lim_{n\to\infty}\frac{1}{4n}=$$

$$=\frac{1}{4}-\frac{1}{4}\left(\frac{1}{2\pi}\sin 2\pi x\Big|_0^1\right)^2-\frac{1}{4}\left(-\frac{1}{2\pi}\cos 2\pi x\Big|_0^1\right)^2=\frac{1}{4}$$

$$\Omega_2=\lim_{n\to\infty}\frac{1}{2n^2}\sum_{1\le i<j\le n}1+\lim_{n\to\infty}\frac{1}{2n^2}\sum_{1\le i<j\le n}\left(\cos\frac{2\pi i}{n}\cos\frac{2\pi j}{n}+\sin\frac{2\pi i}{n}\sin\frac{2\pi j}{n}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{2n^2}\cdot\frac{n(n-1)}{2}=\frac{1}{4}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{n^4}\cdot\sum_{1\le i<j\le n}\sin^2\frac{(j-i)\pi}{n}\cdot\sum_{1\le i<j\le n}\cos^2\frac{(j-i)\pi}{n}=\frac{1}{16}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{Let: } S_1=\sum_{1\le i,j\le n}\sin^2\left[\left(\frac{j-i}{n}\right)\pi\right]=$$

$$=\sin^2\frac{\pi}{n}+\left(\sin^2\frac{\pi}{n}+\sin^2\frac{2\pi}{n}\right)+\left(\sin^2\frac{\pi}{n}+\sin^2\frac{2\pi}{n}+\sin^2\frac{3\pi}{n}\right)+\cdots+$$

$$+\left(\sin^2\frac{\pi}{n}+\sin^2\frac{2\pi}{n}+\cdots+\sin^2\left(\frac{(n-1)\pi}{n}\right)\right)+\cdots+(n-k)\sin^2\left(\frac{k\pi}{n}\right)+\cdots$$

$$\ldots+1\cdot\sin^2\left(\frac{(n-1)\pi}{n}\right)=\sum_{k=1}^{n}(n-k)\sin^2\left(\frac{k\pi}{n}\right)\Rightarrow$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i,j\le n}\sin^2\left[\left(\frac{j-i}{n}\right)\pi\right]=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)\sin^2\left(\frac{k\pi}{n}\right)=$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)\left(1-\cos\left(\frac{2k\pi}{n}\right)\right)=\frac{1}{2}\int_0^1(1-x)(1-\cos(2k\pi))\,dx=$$

$$=\frac{1}{2}(1-x)\left(x-\frac{1}{2\pi}\sin(2\pi x)\right)\Big|_0^1+\frac{1}{2}\int_0^1\left(x-\frac{1}{2\pi}\sin(2\pi x)\right)dx=$$

$$=\frac{1}{2}\left[\frac{1}{2}x^2+\frac{1}{4\pi^2}\cos(2\pi x)\right]\Big|_0^1=\frac{1}{4}$$

$$\text{Let: } S_2=\sum_{1\le i<j\le n}\cos^2\left(\frac{(j-i)\pi}{n}\right)=\sum_{k=1}^{n}(n-k)\cos^2\left(\frac{k\pi}{n}\right)\overset{as\ in\ S_1}{\Longrightarrow}$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\cos^2\left(\frac{(j-i)\pi}{n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)\cos^2\left(\frac{k\pi}{n}\right)=\int_0^1(1-x)\cos^2 x\,dx=$$

$$=\frac{1}{4}\int_0^1(1-x)[1+\cos(2\pi x)]\,dx=$$

$$=\frac{1}{2}(1-x)\left[x+\frac{1}{2\pi}\sin(2\pi x)\right]\Big|_0^1+\frac{1}{2}\int_0^1\left(x+\frac{1}{2\pi}\sin(2\pi x)\right)dx=\frac{1}{4}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{n^4}\cdot\sum_{1\le i<j\le n}\sin^2\frac{(j-i)\pi}{n}\cdot\sum_{1\le i<j\le n}\cos^2\frac{(j-i)\pi}{n}=\frac{1}{16}$$

1639. Prove that:

$$\int_0^\infty\frac{1}{\sqrt{1+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}=\frac{2}{\pi}\log(3+2\sqrt{2})$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Asmat Qatea-Afghanistan

$$\int_0^\infty\frac{1}{\sqrt{1+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}\overset{x\to x^2}{=}\int_0^\infty\frac{2}{\sqrt{1+e^{\frac{\pi}{x}}}}\frac{dx}{x^2}\overset{\frac{\pi}{x}\to x}{=}\frac{2}{\pi}\int_0^\infty\frac{dx}{\sqrt{1+e^x}}\overset{(*)}{=};\quad(*)\begin{cases}1+e^x=u^2\\ dx=\dfrac{2udu}{u^2-1}\end{cases}$$

$$=\frac{2}{\pi}\int_{\sqrt{2}}^\infty\frac{2du}{u^2-1}=\frac{2}{\pi}\int_{\sqrt{2}}^\infty\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du=\frac{2}{\pi}\log\left(\frac{u-1}{u+1}\right)\Big|_{\sqrt{2}}^\infty=$$

$$=\frac{2}{\pi}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)=\frac{2}{\pi}\log(3+2\sqrt{2})$$

Therefore,

$$\int_0^\infty \frac{1}{\sqrt{1+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}=\frac{2}{\pi}\log(3+2\sqrt{2})$$

Solution 2 by Adrian Popa-Romania

$$\frac{\pi}{\sqrt{x}}=t\Rightarrow \pi x^{-\frac{1}{2}}=t\Rightarrow \frac{dx}{\sqrt{x^3}}=-\frac{2}{\pi}dt\Rightarrow$$

$$\int_0^\infty \frac{1}{\sqrt{1+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}=\frac{2}{\pi}\int_0^\infty \frac{dt}{\sqrt{1+e^t}}\overset{(*)}{=};\quad (*)\begin{cases}\sqrt{1+e^t}=u\\ t=\log(u^2-1),dt=\dfrac{2u}{u^2-1}du\end{cases}$$

$$=\frac{2}{\pi}\int_{\sqrt{2}}^\infty \frac{2du}{u^2-1}=\frac{2}{\pi}\int_{\sqrt{2}}^\infty\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du=\frac{2}{\pi}\log\left(\frac{u-1}{u+1}\right)\Big|_{\sqrt{2}}^\infty=$$

$$=\frac{2}{\pi}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)=\frac{2}{\pi}\log(3+2\sqrt{2})$$

Therefore,

$$\int_0^\infty \frac{1}{\sqrt{1+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}=\frac{2}{\pi}\log(3+2\sqrt{2})$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\int_0^\infty \frac{1}{\sqrt{1+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}\overset{(*)}{=};\quad (*)\begin{cases}\dfrac{\pi}{\sqrt{x}}=2t\\ -\dfrac{1}{2}\cdot\dfrac{\pi}{x^{\frac{3}{2}}}dx=dt\end{cases}$$

$$=\frac{1}{\pi}\int_\infty^0-\frac{4dt}{\sqrt{1+e^{2t}}}=\frac{4}{\pi}\int_0^\infty\frac{e^{-t}dt}{\sqrt{1-e^{-2t}}}=-\frac{4}{\pi}\log\left(e^{-t}+\sqrt{1+e^{-2t}}\right)\Big|_0^\infty=$$

$$=\frac{2}{\pi}\log(1+\sqrt{2})^2=\frac{2}{\pi}\log(3+2\sqrt{2})$$

Therefore,

$$\int_0^\infty \frac{1}{\sqrt{1+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}=\frac{2}{\pi}\log(3+2\sqrt{2})$$

1640. Let $a>0$. Find:

$$\Omega=\int\frac{dx}{x^4\sqrt{x^2-a^2}};x\in(a,\infty)$$

Proposed by Marin Chirciu-Romania

Solution 1 by Kartick Chandra Betal-India

$$\Omega=\int\frac{dx}{x^4\sqrt{x^2-a^2}}\overset{(*)}{=};\quad (*):\begin{cases}x=a\sec y\\dx=a\sec y\tan y\,dy\end{cases}$$

$$=\int\frac{a\sec y\tan y}{a^4\sec^4 y\cdot a\tan y}dy=\frac{1}{a^4}\int\cos^3 y\,dy=\frac{1}{4a^4}\int(\cos 3y+3\cos y)\,dy=$$

$$=\frac{1}{4a^4}\left(\frac{\sin 3y}{3}+\sin y\right)=\frac{1}{4a^4}\left(\frac{3\sin y-4\sin^3 y}{3}+\sin y\right)=\frac{1}{4a^4}\left(2\sin y-\frac{4}{3}\sin^3 y\right)$$

$$=\frac{1}{4a^4}(2\sqrt{1-\left(\frac{a}{x}\right)^2}-\frac{4}{3}\left(1-\frac{a^2}{x^2}\right)\sqrt{1-\frac{a^2}{x^2}}=$$

$$=\frac{1}{4a^4}\sqrt{1-\frac{a^2}{x^2}}\left(2-\frac{4(x^2-a^2)}{3x^2}\right)=\frac{\sqrt{x^2-a^2}}{2a^4x}\cdot\frac{3x^2-2x^2+2a^2}{3x^2}=$$

$$=\frac{x^2+2a^2}{6x^3x^4}\sqrt{x^2-a^2}+C$$

Solution 2 by Ose Favour-Nigeria

$$\Omega=\int\frac{dx}{x^4\sqrt{x^2-a^2}}\overset{(*)}{=};\quad (*):\begin{cases}x=a\sec\theta\\dx=a\sec\theta\tan\theta\,d\theta\end{cases}$$

$$=\frac{1}{a^4}\int\cos^3\theta\,d\theta=\frac{1}{a^4}\int(1-\sin^2\theta)\cos\theta\,d\theta\overset{u=\sin\theta}{=}$$

$$=\frac{1}{a^4}\int(1-u^2)\,du=\frac{u}{a^4}-\frac{u^3}{3a^4}+C=\frac{\sin\theta}{a^4}-\frac{\sin^3\theta}{3a^4}+C$$

$$\sec\theta=\frac{x}{a},\cos\theta=\frac{a}{x},\sin\theta=\sqrt{1-\frac{a^2}{x^2}}=\frac{1}{x}\sqrt{x^2-a^2}$$

$$\Omega=\frac{\sqrt{x^2-a^2}}{xa^4}-\frac{\sqrt{(x^2-a^2)^3}}{3x^3a^4}+C$$

Solution 3 by Yen Tung Chung-Taichung-Taiwan

$$\Omega=\int\frac{dx}{x^4\sqrt{x^2-a^2}}\overset{(*)}{=};\quad (*):\begin{cases} y=\sqrt{1-\frac{a^2}{x^2}} \\ -\frac{2a^2}{x^3}dx=-2y\,dy\end{cases}\Rightarrow\frac{a^2}{x^2}=1-y^2$$

$$\overset{(*)}{=}\int\frac{\frac{1}{x^2}}{\sqrt{1-\frac{a^2}{x^2}}}\cdot\frac{1}{x^3}dx=\int\frac{\frac{1}{a^2}(1-y^2)}{y}\cdot\frac{1}{a^2}y\,dy=\frac{1}{a^4}\int(1-y^2)\,dy=$$

$$=\frac{1}{a^4}\left(y-\frac{1}{3}y^3\right)+C=\frac{1}{a^4}\left(\sqrt{1-\frac{a^2}{x^2}}-\frac{1}{3}\left(\sqrt{1-\frac{a^2}{x^2}}\right)^3\right)+C=$$

$$=\frac{(2x^2+a^2)\sqrt{x^2-a^2}}{3a^4x^3}+C$$

Solution 4 by Ravi Prakash-New Delhi-India

$$\Omega=\int\frac{dx}{x^4\sqrt{x^2-a^2}}\overset{(*)}{=};\quad (*):\begin{cases}\frac{a}{x}=t \\ -\frac{a}{x^2}dx=dt\end{cases}$$

$$\overset{(*)}{=}-\int\frac{dx}{ax^3\sqrt{1-\left(\frac{a}{x}\right)^2}}\left(-\frac{a}{x}\right)dx=-\frac{1}{a}\int\frac{dt}{\left(\frac{a}{t}\right)^3\sqrt{1-t^2}}=$$

$$=-\frac{1}{a^4}\int\frac{t^3}{\sqrt{1-t^2}}dt=-\frac{1}{a^4}\int\frac{(t^3-t)+t}{\sqrt{1-t^2}}dt=\frac{1}{a^4}\left[\int t\sqrt{1-t^2}\,dt-\int\frac{t}{\sqrt{1-t^2}}dt\right]=$$

$$=\frac{1}{a^4}\left[-\frac{1}{3}(1-t^2)^{\frac{3}{2}}+\sqrt{1-t^2}\right]+C=$$

$$=\frac{1}{a^4}\left[\sqrt{1-\left(\frac{a}{x}\right)^2}-\frac{1}{3}\left(1-\frac{a^2}{x^2}\right)^{\frac{3}{2}}\right]=\frac{1}{a^4}\left[\frac{\sqrt{x^2-a^2}}{x}-\frac{1}{3}\cdot\frac{(x^2-a^2)^{\frac{3}{2}}}{x^3}\right]+C$$

Solution 5 by Hikmat Mammadov-Azerbaijan

$$\Omega=\int\frac{dx}{x^4\sqrt{x^2-a^2}}=\int\frac{dx}{x^5\sqrt{1-ax^{-2}}}\overset{(*)}{=};\quad (*):\begin{cases}2ax^{-3}=2t\,dt \\ \frac{1}{xa^2}=1-t^2\end{cases}\Rightarrow\begin{cases}x^{-3}dx=\frac{t}{a}dt \\ \frac{1}{x^2}=a(1-t^2)\end{cases}$$

$$\overset{(*)}{=}\int\frac{x^{-3}}{x^2\sqrt{1-ax^{-2}}}dx=\int\frac{a(1-t^2)\cdot\left(\frac{t}{a}\right)dt}{t}=\int(1-t^2)\,dt=t-\frac{t^3}{3}+C=$$

$$=\frac{1}{a^4}\left(\sqrt{1-\frac{a^2}{x^2}}-\frac{1}{3}\left(\sqrt{1-\frac{a^2}{x^2}}\right)^3\right)+C=$$

$$=\frac{(2x^2+a^2)\sqrt{x^2-a^2}}{3a^4x^3}+C$$

1641. Prove that for $n\geq 1$

$$\int_0^\infty\frac{1}{\sqrt{n+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}=\frac{4}{\pi}\frac{\sinh^{-1}(\sqrt{n})}{\sqrt{n}}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Asmat Qatea-Afghanistan

$$\Omega=\int_0^\infty\frac{1}{\sqrt{n+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}\overset{x\to x^2}{=}\int_0^\infty\frac{2}{\sqrt{n+e^{\frac{\pi}{x}}}}\frac{dx}{x^2}\overset{\frac{\pi}{x}\to x}{=}\frac{2}{\pi}\int_0^\infty\frac{dx}{\sqrt{n+e^x}}=$$

$$(*)\begin{cases}n+e^x=u^2\\ dx=\dfrac{2u}{u^2-n}du\end{cases}$$

$$=\frac{2}{\pi}\int_{\sqrt{n+1}}^\infty\frac{2}{u^2-n}du=\frac{2}{\pi\sqrt{n}}\int_{\sqrt{n+1}}^\infty\left(\frac{1}{u-\sqrt{n}}-\frac{1}{n+\sqrt{n}}\right)du=$$

$$=\frac{2}{\pi\sqrt{u}}\log\left(\frac{u-\sqrt{n}}{u+\sqrt{n}}\right)\Bigg|_{\sqrt{n+1}}^\infty=\frac{2}{\pi\sqrt{n}}\log\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}\right)=\frac{4}{\pi\sqrt{n}}\log(\sqrt{n+1}+\sqrt{n})=$$

$$=\frac{4}{\pi\sqrt{n}}\sinh^{-1}(\sqrt{n})$$

$$\because\ \sinh^{-1}x=\log\left(x+\sqrt{x^2+1}\right)$$

Therefore,

$$\int_0^\infty\frac{1}{\sqrt{n+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}=\frac{4}{\pi}\frac{\sinh^{-1}(\sqrt{n})}{\sqrt{n}}$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega=\int_0^\infty \frac{1}{\sqrt{n+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}\overset{x=k^2}{\Rightarrow}\frac{\Omega}{2}=\int_0^\infty \frac{dk}{k^2\sqrt{n+e^{\frac{\pi}{k}}}}$$

$$\text{Let}\begin{cases}n+e^{\frac{\pi}{k}}=h\\ \frac{dk}{k^2}=-\frac{dh}{\pi(h-n)}\end{cases}, h=\begin{cases}\infty, & if\ k=0\\ n+1, & if\ k=\infty\end{cases}\ \text{then}$$

$$\frac{\pi}{2}\Omega=\int_{n+1}^\infty \frac{dh}{(h-n)\sqrt{h}}$$

$$\text{Let}\begin{cases}h=w^2\\ dh=2wdw\end{cases}, w=\begin{cases}\sqrt{n+1}, & if\ h=n+1\\ \infty, & if\ h=\infty\end{cases}.$$

$$\frac{\pi\sqrt{n}}{4}\Omega=\frac{1}{2}\int_{\sqrt{n+1}}^\infty \frac{dw}{w^2-n}=\log\left(\frac{w-\sqrt{n}}{w+\sqrt{n}}\right)\Bigg|_{\sqrt{n+1}}^\infty=\log\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}\right)=$$

$$=\log\left[\left(\sqrt{n+1}+\sqrt{n}\right)^2\right]$$

$$\because \sinh^{-1}x=\log\left(x+\sqrt{x^2+1}\right)$$

Therefore,

$$\int_0^\infty \frac{1}{\sqrt{n+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}=\frac{4}{\pi}\frac{\sinh^{-1}(\sqrt{n})}{\sqrt{n}}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\int_0^\infty \frac{1}{\sqrt{n+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}\overset{(*)}{=};\quad (*):\begin{cases}\frac{\pi}{\sqrt{x}}=2t\\ -\frac{\pi}{x^{\frac{3}{2}}}dx=4dt\end{cases}$$

$$\Omega=\frac{1}{\pi}\int_\infty^0 \frac{-4dt}{\sqrt{n+e^{2t}}}=\frac{4}{\pi}\int_0^\infty \frac{e^{-t}}{\sqrt{ne^{-2t}+1}}dt=$$

$$=-\frac{4}{\pi\sqrt{n}}\log\left(\sqrt{n}e^{-t}+\sqrt{1+ne^{-2t}}\right)\Bigg|_0^\infty=\frac{4}{\pi\sqrt{n}}\log\left(\sqrt{n}+\sqrt{n+1}\right)$$

$$\because \sinh^{-1}x=\log\left(x+\sqrt{x^2+1}\right)$$

Therefore,

$$\int_0^\infty \frac{1}{\sqrt{n+\exp\left(\frac{\pi}{\sqrt{x}}\right)}}\frac{dx}{\sqrt{x^3}}=\frac{4}{\pi}\frac{\sinh^{-1}(\sqrt{n})}{\sqrt{n}}$$

1642. Prove that:

$$\int_0^\infty \frac{3x-2x^3}{(x^4-7x^2+3)^2+(6x-4x^3)^2}dx=\frac{\pi\sqrt{3}}{24}-\frac{\log 3}{8}$$

Proposed by Angad Singh-India

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^\infty \frac{3x-2x^3}{(x^4-7x^2+3)^2+(6x-4x^3)^2}dx \overset{x^2=t}{=}$$

$$=\frac{1}{2}\int_0^\infty \frac{3-2t}{(t^2-7t+3)^2+4t(3-2t)^2}dt=$$

$$=\frac{1}{2}\int_0^\infty \frac{3-2t}{(t^2-t+1)(t^2+3t+9)}dx=\frac{1}{8}\int_0^\infty \left(\frac{t+3}{t^2+3t+9}-\frac{t-1}{t^2-t+1}\right)dt=$$

$$=\frac{1}{16}\int_0^\infty \left(\frac{2t+3}{t^2+3t+9}-\frac{2t-1}{t^2-t+1}\right)dt+\frac{3}{16}\int_0^\infty \frac{dt}{t^2+3t+9}+\frac{1}{16}\int_0^\infty \frac{dt}{t^2-t+1}=$$

$$=\frac{1}{16}\log\left(\frac{t^2+3t+9}{t^2-t+1}\right)\Bigg|_0^\infty+\frac{3}{16}\int_0^\infty \frac{dt}{\left(t+\frac{3}{2}\right)^2+\frac{27}{4}}+\frac{1}{16}\int_0^\infty \frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}=$$

$$=-\frac{\log 3}{8}+\frac{1}{8\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+3}{3\sqrt{3}}\right)+\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)\right]\Bigg|_0^\infty=-\frac{\log 3}{8}+\frac{\pi}{8\sqrt{3}}$$

Therefore,

$$\int_0^\infty \frac{3x-2x^3}{(x^4-7x^2+3)^2+(6x-4x^3)^2}dx=\frac{\pi\sqrt{3}}{24}-\frac{\log 3}{8}$$

Solution 2 by Amrit Awasthi-India

Consider the closed line integral:

$$I=\oint_C \frac{1}{1+z^2+z^4}dz$$

Where C is a rectangle having vertices $0, 1, 1+iR, 0+iR$ traversed counter clockwise and R approaches infinity. Hence,

$$I = \oint_C \frac{1}{1+z^2+z^4}dz = I_1 + I_2 + I_3 + I_4, \text{where}$$

$$I_1 = \int_0^1 \frac{dx}{1+x^2+x^4}; I_2 = \int_0^R \frac{idy}{1+(1+iy)^2+(1+iy)^4}$$

$$I_3 = \int_1^0 \frac{dx}{1+(x+iR)^2+(x+iR)^4}; I_4 = \int_R^0 \frac{idy}{1+(iy)^2+(iy)^4}$$

Clearly the conditions for applying residuum is satisfied hence using the residue theorem we get:

$$I = 2i\pi \cdot \frac{1}{2\left(e^{-\frac{i\pi}{3}} - 2\right)}$$

Consider only the real part of I, we get:

$$Re(I) = \frac{\pi}{2\sqrt{3}}; (*)$$

Now, we'll evaluate the real parts of our I_1, I_2, I_3, I_4 and then Equate with the real part of I. First note that

$$I_4 = \int_R^0 \frac{idy}{1+(iy)^2+(iy)^4} = -i\int_0^R \frac{dy}{1-y^2+y^4} \Rightarrow Re(I_4) = 0; (i)$$

Now, note that

$$|I_3| = \left|\int_1^0 \frac{dx}{1+(x+iR)^2+(x+iR)^4}\right| \le \int_0^1 \frac{dx}{|1+(x+iR)^2+(x+iR)^4|}$$

Therefore, using properties of modulus and some manipulations it can be shown that I_3 vanishes as R approaches infinity. Hence,

$$Re(I_3) = 0; (ii)$$

Now, using real methods we can easily evaluate I_1,

$$I_1 = \int_0^1 \frac{dx}{1+x^2+x^4} = \left[\frac{1}{4}\log\left|\frac{x^2+x+1}{x^2-x+1}\right| + \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{1-x^2}\right)\right]_0^1$$

$$Re(I_1) = I_1 = \frac{\log 3}{4} + \frac{\pi}{4\sqrt{3}}; (iii)$$

Now, observe that

$$Re(I_2)=\lim_{R\to\infty}\int_0^R \frac{6x-4x^3}{(x^4-7x^2+3)^2+(6x-4x^3)^2}dx=$$

$$=\int_0^\infty \frac{6x-4x^3}{(x^4-7x^2+3)^2+(6x-4x^3)^2}dx\,;(iv)$$

By adding $(i),(ii),(iii),(iv)$ we get:

$$\frac{\log 3}{8}+\frac{\pi}{4\sqrt{3}}=\int_0^\infty \frac{3x-2x^3}{(x^4-7x^2+3)^2+(6x-4x^3)^2}dx=\frac{\pi}{2\sqrt{3}}$$

Therefore,

$$\int_0^\infty \frac{3x-2x^3}{(x^4-7x^2+3)^2+(6x-4x^3)^2}dx=\frac{\pi\sqrt{3}}{24}-\frac{\log 3}{8}$$

1643. Prove that for $n\in\mathbb{N}$

$$\int_0^\infty \frac{(\tan^{-1}x-\cot^{-1}x)^{2n}}{(x^2+1)^2}dx=\frac{2^{-(2n+1)}\pi^{2n+1}}{2n+1}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^\infty \frac{(\tan^{-1}x-\cot^{-1}x)^{2n}}{(x^2+1)^2}dx=\int_0^\infty \frac{\left(2\tan^{-1}x-\frac{\pi}{2}\right)^{2n}}{(1+x^2)^2}dx\overset{x=\tan t}{=}$$

$$=\int_0^{\frac{\pi}{2}}\left(2t-\frac{\pi}{2}\right)^{2n}\cos^2 t\,dt\overset{y=2t-\frac{\pi}{2}}{=}\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y^{2n}\cos^2\left(\frac{y}{2}+\frac{\pi}{4}\right)dy=$$

$$=\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y^{2n}\left(1+\cos\left(y+\frac{\pi}{2}\right)\right)dy=\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y^{2n}(1-\sin y)\,dy=$$

$$=\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y^{2n}\,dy-\frac{1}{4}\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y^{2n}\sin y\,dy}_{=0-odd\ function}=\frac{\pi^{2n+1}}{2^{2n+2}(2n+1)}$$

Therefore,

$$\int_0^\infty \frac{(\tan^{-1}x-\cot^{-1}x)^{2n}}{(x^2+1)^2}dx=\frac{2^{-(2n+1)}\pi^{2n+1}}{2n+1}$$

Solution 2 by Syed Shahabudeen-Kerala-India

$$\Omega=\int_0^\infty \frac{(\tan^{-1}x-\cot^{-1}x)^{2n}}{(x^2+1)^2}dx=\int_0^\infty \frac{\left(2\tan^{-1}x-\frac{\pi}{2}\right)^{2n}}{(x^2+1)^2}dx\overset{t=2\tan^{-1}x-\frac{\pi}{2}}{=}$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{t^{2n}}{\tan^2\left(\frac{\pi}{4}+\frac{t}{2}\right)+1}dt=\frac{1}{2}\int_0^{\frac{\pi}{2}}t^{2n}\left(\cos^2\left(\frac{\pi}{4}+\frac{t}{2}\right)+\cos^2\left(\frac{\pi}{4}-\frac{t}{2}\right)\right)dt=$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}t^{2n}\left(1+\cos^2\left(\frac{\pi}{4}+\frac{t}{2}\right)-\sin^2\left(\frac{\pi}{4}-\frac{t}{2}\right)\right)dt$$

$$\because \cos^2(a+b)-\sin^2(a+b)=\cos 2a\sin 2b$$

$$\Omega=\frac{1}{2}\int_0^{\frac{\pi}{2}}t^{2n}\left(1+\cos\frac{\pi}{2}\cos t\right)dt=\frac{1}{2}\int_0^{\frac{\pi}{2}}t^{2n}\,dt=\frac{2^{-(2n+1)}\pi^{2n+1}}{2n+1}$$

Therefore,

$$\int_0^{\infty}\frac{(\tan^{-1}x-\cot^{-1}x)^{2n}}{(x^2+1)^2}dx=\frac{2^{-(2n+1)}\pi^{2n+1}}{2n+1}$$

Solution 3 by Kartick Chandra Betal-India

$$\Omega=\int_0^{\infty}\frac{(\tan^{-1}x-\cot^{-1}x)^{2n}}{(x^2+1)^2}dx=\int_0^{\infty}\frac{x^2(\cos t^{-1}x-\tan^{-1}x)^{2n}}{(1+x^2)^2}dx$$

$$2\Omega=\int_0^{\infty}\frac{(\tan^{-1}x-\cot^{-1}x)^{2n}}{1+x^2}dx=\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-2x\right)^{2n}dx=$$

$$=\frac{1}{2}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)^{2n}dx=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x^{2n}(-dx)=\int_0^{\frac{\pi}{2}}x^{2n}\,dx=$$

$$=\frac{\left(\frac{\pi}{2}\right)^{2n+1}}{2n+1}=\frac{2^{-(2n+1)}\pi^{2n+1}}{2n+1}$$

Therefore,

$$\int_0^{\infty}\frac{(\tan^{-1}x-\cot^{-1}x)^{2n}}{(x^2+1)^2}dx=\frac{2^{-(2n+1)}\pi^{2n+1}}{2n+1}$$

1644. Find:

$$\Omega=\int_1^{\infty}\frac{\log x}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_1^{\infty}\frac{\log x}{(x+1)(x^2+1)}dx\overset{x=\frac{1}{x}}{=}-\int_0^1\frac{x\log x}{(1+x)(1+x^2)}dx=$$

$$=\frac{1}{2}\left(\int_0^1\frac{\log x}{1+x}dx-\int_0^1\frac{\log x}{1+x^2}dx-\int_0^1\frac{x\log x}{1+x^2}dx\right)=\frac{1}{2}(A-B-C),$$

$$A=\int_0^1\frac{\log x}{1+x}dx=\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{n-1}\log x\,dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\frac{\pi^2}{12}$$

$$B=\int_0^1\frac{\log x}{1+x^2}dx=\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-2}\log x\,dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}=-G$$

$$C=\int_0^1\frac{x\log x}{1+x^2}dx=\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-1}\log x\,dx=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2}=-\frac{\pi^2}{48}$$

Therefore,

$$\Omega=\frac{1}{2}\left(-\frac{\pi^2}{12}+G+\frac{\pi^2}{48}\right)=\frac{G}{2}-\frac{\pi^2}{32}$$

Solution 2 Serlea Kabay-Liberia

$$\Omega=\int_1^{\infty}\frac{\log x}{(x+1)(x^2+1)}dx\overset{x=\frac{1}{x}}{=}-\int_0^1\frac{x\log x}{1+x+x^2+x^3}dx=$$

$$=\int_0^1\frac{(x-x^2)\log x}{1-x^4}dx\overset{u=x^4}{=}-\frac{1}{16}\int_0^1\frac{\left(u^{-\frac{1}{2}}-u^{-\frac{1}{4}}\right)\log u}{1-u}du=$$

$$=\frac{1}{16}\sum_{n=0}^{\infty}\int_0^1 u^{n-\frac{1}{4}}\log u\,du-\frac{1}{16}\sum_{n=0}^{\infty}\int_0^1 u^{n-\frac{1}{2}}\log u\,du$$

$$\because\int_0^1 x^a\log x\,dx\overset{IBP}{=}-\frac{1}{(a+1)^2}$$

$$\Rightarrow\Omega=-\frac{1}{16}\left(\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{3}{4}\right)^2}-\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{2}\right)^2}\right)=$$

$$=-\frac{1}{16}\left(\psi^{(1)}\left(\frac{3}{4}\right)-\psi^{(1)}\left(\frac{1}{2}\right)\right);\left(\because\psi^{(1)}\left(\frac{3}{4}\right)=\pi^2-8G,\psi^{(1)}\left(\frac{1}{2}\right)=\frac{\pi^2}{2}\right)$$

$$\Omega=\frac{1}{16}\left(\frac{\pi^2}{2}-8G\right)=\frac{G}{2}-\frac{\pi^2}{32}$$

Therefore,

$$\Omega = \frac{1}{2}\left(-\frac{\pi^2}{12} + G + \frac{\pi^2}{48}\right) = \frac{G}{2} - \frac{\pi^2}{32}$$

Solution 3 by Kartick Chandra Betal-India

$$\Omega = \int_1^\infty \frac{\log x}{(x+1)(x^2+1)}dx \overset{x=\frac{1}{x}}{=} -\int_0^1 \frac{x\log x}{(1+x)(1+x^2)}dx =$$
$$= \frac{1}{2}\int_0^1 \frac{\log x}{1+x}dx - \frac{1}{2}\int_0^1 \frac{\log x}{1+x^2}dx - \frac{1}{2}\int_0^1 \frac{x\log x}{1+x^2}dx =$$
$$= \frac{1}{2}\int_0^1 \frac{\log x}{1+x}dx - \frac{1}{8}\int_0^1 \frac{\log x}{1+x}dx - \frac{1}{2}\int_0^{\frac{\pi}{4}} \log(\tan x)\,dx =$$
$$= \frac{3}{8}[\log x \log(1+x)]_0^1 - \frac{3}{8}\int_0^1 \frac{\log(1+x)}{x}dx + \frac{G}{2} =$$
$$= -\frac{3}{8}\eta(2) + \frac{G}{2} = \frac{G}{2} - \frac{\pi^2}{32}$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\Omega = \int_1^\infty \frac{\log x}{(x+1)(x^2+1)}dx = \int_0^1 \frac{-u\log u}{(1+u)(1+u^2)}du =$$
$$= -\int_0^1 \frac{u\log u}{(1+u)(1+u^2)}du = -\frac{1}{2}\int_0^1 \left(\frac{1}{u+1} - \frac{u-1}{u^2+1}\right)du =$$
$$= \frac{1}{2}\int_0^1 \frac{\log u}{1+u}du - \frac{1}{2}\int_0^1 \frac{\log u}{1+u^2}du - \frac{1}{2}\int_0^1 \frac{u\log u}{1+u^2}du =$$
$$= \frac{1}{2}\int_0^1 \frac{(1-u)\log u}{1-u^2}du - \frac{1}{2}\int_0^{\frac{\pi}{4}} \log(\tan\alpha)\,d\alpha - \frac{1}{8}\int_0^1 \frac{\log u}{1+u^2}du =$$
$$= \frac{1}{8}\int_0^1 \frac{(v^{1-2}-1)\log v}{1-v}dv + \frac{G}{2} - \frac{1}{32}\int_0^1 \frac{(z^{1-2}-1)\log z}{1-z}dz =$$
$$= \frac{1}{8}[\zeta(2) - 3\zeta(2)] + \frac{G}{2} - \frac{1}{32}[\zeta(2) - 3\zeta(2)] =$$
$$= \frac{G}{2} + \frac{3}{32}\left(-\frac{\pi^2}{3}\right) = \frac{G}{2} - \frac{\pi^2}{32}$$

1645. Let $n \in \mathbb{N}^*$ and $\lambda \geq 0$. Find:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^n x + \lambda \sin^2 x}{\lambda + \sin^n x + \cos^n x}dx$$

Proposed by Marin Chirciu – Romania

Solution 1 by George Florin Serban-Romania

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^n x+\lambda\sin^2 x}{\lambda+\sin^n x+\cos^n x}dx$$

$$\frac{\pi}{2}-x=y\Rightarrow dx=dy$$

$$x=0\Rightarrow y=\frac{\pi}{2},x=\frac{\pi}{2}\Rightarrow y=0$$

$$\Rightarrow I=-\int_{\frac{\pi}{2}}^{0}\frac{\cos^n\left(\frac{\pi}{2}-y\right)+\lambda\sin^2\left(\frac{\pi}{2}-y\right)}{\lambda+\sin^n\left(\frac{\pi}{2}-y\right)+\cos^n\left(\frac{\pi}{2}-y\right)}dy$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin^n y+\lambda\cos^2 y}{\lambda+\cos^n y+\sin^n y}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\frac{\cos^n x+\lambda\sin^2 x+\sin^n x+\lambda\cos^2 x}{\lambda+\sin^n x+\cos^n x}dx$$

$$2I=\int_0^{\frac{\pi}{2}}\frac{-\sin^n x+\cos^n x+\lambda(\sin^2 x+\cos^2 x)}{\lambda+\sin^n x+\cos^n x}dx$$

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^n x+\cos^n x+\lambda}{\sin^n x+\cos^n x+\lambda}dx=\int_0^{\frac{\pi}{2}}1\,dx$$

$$2I=x|_0^{\frac{\pi}{2}}=\frac{\pi}{2}-0=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

$$\Rightarrow\int_0^{\frac{\pi}{2}}\frac{\cos^n x+\lambda\sin^2 x}{\lambda+\sin^n x+\cos^n x}dx=\frac{\pi}{4}$$

Solution 2 by Ankush Kumar Parcha-India

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^n(x)+\lambda\sin^2(x)}{\lambda+\sin^n(x)+\cos^n(x)}dx\quad(1)$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^n(x)+\lambda\sin^2(x)}{\lambda+\sin^n(x)+\cos^n(x)}dx=\int_0^{\frac{\pi}{2}}\frac{\sin^n(x)+\lambda\cos^2(x)}{\lambda+\sin^n(x)+\cos^n(x)}dx$$

$$\left(\because\int_a^b f(x)\,dx=\int_a^b f(a+b-x)dx\right)$$

Adding above equation with equation (1)

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^n(x)+\cos^n(x)+\lambda(\sin^2(x)+\cos^2(x))}{\lambda+\sin^n(x)+\cos^n(x)}dx$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^n(x)+\lambda\sin^2(x)}{\lambda+\sin^n(x)+\cos^n(x)}dx=\frac{\pi}{4}$$

1646. Find:

$$\Omega=\int\frac{1}{1+\sqrt{x}}\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx$$

Proposed by Asliddin Egamberdiyev-Uzbekistan

Solution 1 by Adrian Popa-Romania

$$\Omega=\int\frac{1}{1+\sqrt{x}}\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx\overset{(*)}{=};\quad (*):\begin{cases}\dfrac{\sqrt{x}-1}{\sqrt{x}+1}=t\Rightarrow x=\left(\dfrac{1+t^3}{1-t^3}\right)^2\\ dx=\dfrac{6t^2}{(1-t^3)^2}dt\end{cases}$$

$$\overset{(*)}{=}\int\frac{1-t^3}{2}\cdot t\cdot\frac{6t^2}{(1-t^3)^2}dt=\int\frac{3t^2}{1-t^3}dt=\int\frac{3t^3-3+3}{1-t^2}dt=$$

$$=-t+3\int\frac{dt}{(1-t)(1+t+t^2)}=-t+3\int\left(\frac{A}{1-t}+\frac{Bt+C}{1+t+t^2}\right)dt$$

$$A+At+At^2+Bt+C-Bt^2-Ct=1\Rightarrow A-B=0\Rightarrow A=B$$

$$A+B-C=0\Rightarrow 2A=C$$

$$A+C=1\Rightarrow 3A=1\Rightarrow A=B=\frac{1}{3},C=\frac{2}{3}$$

Hence, we have:

$$\Omega=-t+3\int\left(\frac{1}{3}\cdot\frac{1}{1-t}+\frac{1}{3}\cdot\frac{t+2}{1+t+t^2}\right)dt=-t+\int\frac{dt}{1-t}+\int\frac{t+2}{t^2+t+1}dt=$$

$$=-t+\log|1-t|+\frac{1}{2}\int\frac{2t+1}{t^2+t+1}dt+\frac{3}{2}\int\frac{dt}{t^2+t+1}=$$

$$=-t+\log|1-t|+\frac{1}{2}\log(t^2+t+1)+\frac{3}{2}\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}=$$

$$= -t + \log|1-t| + \frac{3}{2}\cdot\frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2\sqrt{3}}{3}\left(t+\frac{1}{2}\right)\right) + C =$$

$$\overset{(*)}{=} \sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}} + \log\left[\left(1-\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}\right)\left(\left(\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}\right)^2 + \sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}} + 1\right)\right]$$

$$+ 3\tan^{-1}\left[\frac{2\sqrt{3}}{3}\left(\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}} + \frac{1}{2}\right)\right] + C$$

Solution 2 by Ankush Kumar Parcha-India

$$\Omega = \int \frac{1}{1+\sqrt{x}}\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}\,dx \overset{y^2=x}{=} \int \frac{y+1+y-1}{1+y}\sqrt[3]{\frac{y-1}{y+1}}\,dy =$$

$$= \int \sqrt[3]{\frac{y-1}{y+1}}\,dy + \int \sqrt[3]{\left(\frac{y-1}{y+1}\right)^4}\,dy = \Omega_{\frac{1}{3}} + \Omega_{\frac{4}{3}}, \text{ where}$$

$$\Omega_{\frac{1}{3}} = \int \sqrt[3]{\frac{y-1}{y+1}}\,dy \overset{\frac{y-1}{y+1}=t^3}{=} \int 6t\cdot\frac{t^2}{(1-t^3)^2}\,dt = \frac{2t}{1-t^3} - 2\underbrace{\int\frac{dt}{1-t^3}}_{=\omega}$$

Hence, we get:

$$\frac{1}{2}\Omega_{\frac{1}{3}} = \frac{t}{1-t^3} - \omega, \omega = \int\frac{dt}{1-t^3} = \int\frac{dt}{(1-t)(t^2+t+1)}$$

Using partial decomposition, we have:

$$\frac{1}{(1-t)(t^2+t+1)} = \frac{A}{1-t} + \frac{Bx+C}{t^2+t+1}$$

By solving system of equation, we get: $A = B = \frac{1}{3}, C = \frac{2}{3}$

$$3\omega = \int\frac{1}{1-t}\,dt + \int\frac{2t+1}{t^2+t+1}\,dt + \int\frac{3}{t^2+t+1}\,dt =$$

$$= -\log|1-x| + \log|t^2+t+1| + 3\int\frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Omega_{\frac{1}{3}}=\frac{2\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}}{1-\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}}-\log\left(\sqrt[3]{\frac{\left(\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}\right)^2+\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}+1}{\left(1-\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}\right)^2}}\right)$$

$$-\frac{2\sqrt{3}}{3}\left(\frac{2\sqrt{3}}{3}\cdot\left(\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}+\sqrt{3}\right)\right)$$

Consider the integral $\Omega_a=\int\left(\frac{x-1}{x+1}\right)^a dx;\forall a\in\mathbb{R}-\{-1\}$. The recurrence relation of the above integral is:

$$\Omega_a=\frac{x-1}{a+1}\cdot\left(\frac{x-1}{x+1}\right)^a+\frac{a}{a+1}\Omega_{a+1}$$

Put $a=\frac{1}{3}$ in the above recurrence relation, we get

$$\Omega_{\frac{1}{3}}=\frac{3(x-1)}{4}\sqrt[3]{\frac{x-1}{x+1}}+\frac{1}{4}\Omega_{\frac{4}{3}}\Rightarrow\Omega_{\frac{4}{3}}=4\Omega_{\frac{1}{3}}-3(x-1)\left(\frac{x-1}{x+1}\right)^{\frac{4}{3}}$$

Therefore,

$$\Omega=\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}+\log\left[\left(1-\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}\right)\left(\left(\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}\right)^2+\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}+1\right)\right]$$

$$+3\tan^{-1}\left[\frac{2\sqrt{3}}{3}\left(\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}+\frac{1}{2}\right)\right]+C$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\text{Let: }\begin{cases}\dfrac{\sqrt{x}-1}{\sqrt{x}+1}=z^3\Rightarrow\sqrt{x}=\dfrac{z^3+1}{z^3-1}\\ x=\dfrac{(z^3+1)^2}{(z^3-1)^2}\Rightarrow\dfrac{1}{1+\sqrt{x}}=\dfrac{z^3-1}{2z^3}\\ dx=\left[-12z^2\cdot\dfrac{z^3+1}{(z^3-1)^2}\right]dz\end{cases}$$

$$\Omega=\int\frac{1}{1+\sqrt{x}}\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx=\int\frac{z^3-1}{2z^3}\cdot z\cdot\left(-12z^2\cdot\frac{z^3+1}{(z^3-1)^3}\right)dz=$$

$$=-6\int(z^3-1)\cdot\left(\frac{z^3+1}{(z^3-1)^3}\right)dz=-6\int\frac{z^3+1}{z^3-1}dz=-6\int\frac{(z^3-1)+2}{z^3-1}dz=$$

$$=-6z-6\int\frac{2}{(z-1)(z^2+z+1)}dz=-6z-6\left[\frac{2}{3}\int\frac{(z^2+z+1)-(z-1)(z+2)}{(z-1)(z^2+z+1)}dz\right]$$
$$=$$

$$=-6z-6\left[\frac{2}{3}\int\left(\frac{1}{z-1}-\frac{z+2}{z^2+z+1}\right)dz\right]$$

$$=-6z-6[\frac{2}{3}\log|z-1|-\frac{1}{3}\int\frac{\big((2z+1)+3\big)dz}{z^2+z+1}$$

$$=-6z-6\left[\frac{2}{3}\log|z-1|-\frac{1}{3}\log(z^2+z+1)-\frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}(z+1)}{3}+C\right)\right]=$$

$$=-6z-4\log|z-1|+2\log(z^2+z+1)-4\sqrt{3}\tan^{-1}\left(\frac{\sqrt{3}(2z+1)}{3}\right)+C$$

Therefore,

$$\Omega=\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}+\log\left[\left(1-\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}\right)\left(\left(\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}\right)^2+\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}+1\right)\right]$$
$$+3\tan^{-1}\left[\frac{2\sqrt{3}}{3}\left(\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}+\frac{1}{2}\right)\right]+C$$

1647. Find:

$$\Omega=\int_1^4\sqrt{\frac{\sqrt{x}-1}{\sqrt{x}}}dx$$

Proposed by Durmuş Ogmen-Turkiye

Solution 1 by Yen Tung Chung-Taichung-Taiwan

$$\text{Let: }y=\sqrt{1-\frac{1}{\sqrt{x}}}\Rightarrow\frac{1}{\sqrt{x}}=1-y^2\Rightarrow x=\frac{1}{(1-y^2)^2}$$

Hence, we have:

$$\Omega=\int_1^4\sqrt{\frac{\sqrt{x}-1}{\sqrt{x}}}dx=\int_1^4\sqrt{1-\frac{1}{\sqrt{x}}}dx=$$

$$=\int_0^{\frac{1}{\sqrt{2}}}y\,d\left(\frac{1}{(1-y^2)^2}\right)=\frac{y}{(1-y^2)^2}\bigg|_0^{\frac{1}{\sqrt{2}}}-\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{(1-y^2)^2}dy=$$

$$=2\sqrt{2}-\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{(1-y^2)^2}dy\overset{y=\sin\theta}{=}2\sqrt{2}-\int_0^{\frac{\pi}{4}}\frac{1}{(1-\sin^2\theta)^2}\cdot\cos\theta\,d\theta=$$

$$=2\sqrt{2}-\int_0^{\frac{\pi}{4}}\sec^3\theta\,d\theta=2\sqrt{2}-\left(\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\log|\sec\theta+\tan\theta|\right)\bigg|_0^{\frac{\pi}{4}}=$$

$$=2\sqrt{2}-\left(\frac{\sqrt{2}}{2}+\frac{1}{2}\log(1+\sqrt{2})\right)=\frac{3\sqrt{2}}{2}-\frac{1}{2}\log(1+\sqrt{2})$$

Solution 2 by Adrian Popa-Romania

Let: $\sqrt[4]{x}=t\Rightarrow x=t^4,dx=4t^3dt$

$$\Omega=\int_1^4\sqrt{\frac{\sqrt{x}-1}{\sqrt{x}}}dx=\int_1^{\sqrt{2}}\sqrt{\frac{t^2-1}{t^2}}\cdot 4t^3dt=4\int_1^{\sqrt{2}}t^2\sqrt{t^2-1}\,dt=$$

$$=4\int_1^{\sqrt{2}}\frac{t^4dt}{\sqrt{t^2-1}}-4\int_1^{\sqrt{2}}\frac{t^2dt}{\sqrt{t^2-1}}=\Omega_2-\Omega_1,\qquad\text{where}$$

$$\Omega_1=4\int_1^{\sqrt{2}}\frac{t^4dt}{\sqrt{t^2-1}}=4\int_1^{\sqrt{2}}t\left(\sqrt{t^2-1}\right)'dt=4t\sqrt{t^2-1}\Big|_1^{\sqrt{2}}-4\int_1^{\sqrt{2}}\sqrt{t^2-1}\,dt=$$

$$=4\sqrt{2}-4\int_1^{\sqrt{2}}\sqrt{t^2-1}\,dt=4\sqrt{2}-\Omega_{11},\qquad\text{where}$$

$$\Omega_{11}=4\int_1^{\sqrt{2}}\sqrt{t^2-1}\,dt=4\int_1^{\sqrt{2}}\frac{t^2-1}{\sqrt{t^2-1}}dt=4\int_1^{\sqrt{2}}\frac{t^2}{\sqrt{t^2-1}}dt-4\int_1^{\sqrt{2}}\frac{dt}{\sqrt{1-t^2}}=$$

$$=\Omega_1-4\log\left(t+\sqrt{t^2+1}\right)\Big|_1^{\sqrt{2}}=\Omega_1-4\log(\sqrt{2}+1)$$

Hence, we have:

$$\Omega_1=4\sqrt{2}-\Omega_1+4\log(\sqrt{2}+1)\Rightarrow\Omega_1=2\sqrt{2}+2\log(\sqrt{2}+1)$$

$$\Omega_2=4\int_1^{\sqrt{2}}\frac{t^2dt}{\sqrt{t^2-1}}=4\int_1^{\sqrt{2}}t^3\left(\sqrt{t^2-1}\right)'dt=4t^3\sqrt{t^2-1}\Big|_1^{\sqrt{2}}-4\int_1^{\sqrt{2}}3t^2\sqrt{t^2-1}\,dt$$

$$=$$

$$=8\sqrt{2}-12\int_1^{\sqrt{2}}\frac{t^4-t^2}{\sqrt{t^2-1}}dt=8\sqrt{3}-3\Omega$$

Therefore,

$$\Omega=\int_1^4\sqrt{\frac{\sqrt{x}-1}{\sqrt{x}}}dx=\frac{3\sqrt{2}}{2}-\frac{1}{2}\log(1+\sqrt{2})$$

Solution 3 by Ankush Kumar Parcha-India

$$\Omega=\int_1^4\sqrt{\frac{\sqrt{x}-1}{\sqrt{x}}}dx$$

$$\text{Let: } I=\int\sqrt{\frac{\sqrt{x}-1}{\sqrt{x}}}dx\overset{\sqrt{x}=y}{=}\int 2\sqrt{y^2-y}\,dy$$

$$\frac{1}{2}I=\int\sqrt{y^2-y}\,dy=\int\sqrt{\left(y-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dy$$

$$\because\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

$$I=\frac{(2\sqrt{x}-1)\sqrt{\sqrt{x}(\sqrt{x}-1)}}{2}-\frac{1}{4}\log\left|\sqrt{x}-\frac{1}{2}\sqrt{\sqrt{x}(\sqrt{x}-1)}\right|+C$$

Therefore,

$$\Omega=\int_1^4\sqrt{\frac{\sqrt{x}-1}{\sqrt{x}}}dx=\frac{3\sqrt{2}}{2}-\frac{1}{2}\log(1+\sqrt{2})$$

Solution 4 by Ravi Prakash-New Delhi-India

$$\text{Let } \sqrt{x}=t+1\Rightarrow x=(t+1)^2, dx=2(t+1)dt;\ (*)$$

$$\Omega=\int_1^4\sqrt{\frac{\sqrt{x}-1}{\sqrt{x}}}dx\overset{(*)}{=}2\int_0^1\frac{(t+1)\sqrt{t}}{\sqrt{t+1}}dt=2\int_0^1\sqrt{t(t+1)}\,dt=$$

$$=2\int_0^1\sqrt{\left(t+\frac{1}{2}\right)^2-\frac{1}{4}}dt\overset{t+\frac{1}{2}=u}{=}2\int_{\frac{1}{2}}^{\frac{3}{2}}\sqrt{u^2-\frac{1}{4}}du=$$

$$=\left[u\sqrt{u^2-\frac{1}{4}}-\frac{1}{4}\log\left(u+\sqrt{u^2-\frac{1}{4}}\right)\right|_{\frac{1}{2}}^{\frac{3}{2}}=\frac{3\sqrt{2}}{2}-\frac{1}{4}\log\left(\frac{3}{2}+\sqrt{2}\right)+\frac{1}{4}\log\left(\frac{1}{2}\right)=$$

$$=\frac{3\sqrt{2}}{2}-\frac{1}{4}\log(3+2\sqrt{2})$$

1648. Solve the integral for c, if $a+b=c$

$$\int_0^\infty\frac{ax^2+bx+c}{x^4+x^3+x^2+x+1}\frac{dx}{\sqrt{x}}=\pi$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Asmat Qatea-Afghanistan

Let's start with this integral

$$I_n=\int_0^\infty\frac{x^n}{x^4+x^3+x^2+x+1}dx=\int_0^\infty\frac{x^n-x^{n+1}}{1-x^5}=$$

$$=\int_0^1\underbrace{\frac{x^n-x^{n+1}}{1-x^5}}_{x^5\to x}dx+\int_1^\infty\underbrace{\frac{x^n-x^{n+1}}{1-x^5}}_{x\to\frac{1}{x}}dx$$

$$=\frac{1}{5}\left(\int_0^1\frac{x^{\frac{n-4}{5}}-x^{\frac{n-3}{5}}}{1-x}dx\right)+\int_0^1\frac{x^{-n}-x^{-(n+1)}}{1-x^{-5}}\cdot\frac{1}{x^2}dx$$

$$=\frac{1}{5}\left(\psi\left(\frac{n+2}{5}\right)-\psi\left(\frac{n+1}{5}\right)+\int_0^1\frac{x^{\frac{-n-2}{5}}-x^{\frac{-n-1}{5}}}{1-x}dx\right)$$

$$=\frac{1}{5}\left(\psi\left(\frac{n+2}{5}\right)-\psi\left(\frac{n+1}{5}\right)+\psi\left(\frac{-n+4}{5}\right)-\psi\left(\frac{-n+3}{5}\right)\right)$$

$$=\frac{1}{5}\left(\psi\left(\frac{n+2}{5}\right)-\psi\left(\frac{n+1}{5}\right)+\psi\left(1-\frac{n+1}{5}\right)-\psi\left(1-\frac{n+2}{5}\right)\right)$$

$\psi(1-x)-\psi(x)=\pi\cot\pi x$ and $\cot x-\cot y=\frac{1+\cot x\cot y}{\cot(y-x)}$

$$=\frac{\pi}{5}\left(\cot\left(\frac{n+1}{5}\pi\right)-\cot\left(\frac{n+2}{5}\pi\right)\right)=\frac{\pi}{5}\cdot\frac{1+\cot\left(\frac{n+1}{5}\pi\right)\cot\left(\frac{n+2}{5}\pi\right)}{\cot\left(\frac{\pi}{5}\right)}$$

$$\int_0^\infty\frac{x^n}{x^4+x^3+x^2+x+1}dx=\frac{\pi}{5\cot\left(\frac{\pi}{5}\right)}\left(1+\cot\left(\frac{n+1}{5}\pi\right)\cot\left(\frac{n+2}{5}\pi\right)\right)$$

$$\Omega=\int_0^\infty\frac{ax^2+bx+c}{x^4+x^3+x^2+x+1}\frac{dx}{\sqrt{x}}=\pi \text{ and } a+b=c$$

$$a\int_0^\infty\frac{x^{\frac{3}{2}}}{x^4+x^3+x^2+x+1}dx+b\int_0^\infty\frac{x^{\frac{1}{2}}}{x^4+x^3+x^2+x+1}dx+$$

$$+c\int_0^\infty\frac{x^{-\frac{1}{2}}}{x^4+x^3+x^2+x+1}dx=\pi$$

$$a\left(1+\underbrace{\cot\left(\frac{\pi}{2}\right)\cot\left(\frac{7}{5}\pi\right)}_{0}\right)+b\left(1+\underbrace{\cot\left(\frac{3}{10}\pi\right)\cot\left(\frac{\pi}{2}\right)}_{0}\right)+$$

$$+c\left(1+\cot\left(\frac{\pi}{10}\right)\cot\left(\frac{3}{10}\pi\right)\right)=5\cot\left(\frac{\pi}{5}\right)$$

$$c=\frac{5\cot\left(\frac{\pi}{5}\right)}{2+\underbrace{\cot\left(\frac{\pi}{10}\right)\cot\left(\frac{3}{10}\pi\right)}_{\sqrt{5}}}=\frac{\frac{5}{\sqrt{5-2\sqrt{5}}}}{2+\sqrt{5}}\Rightarrow c=\frac{5}{(2+\sqrt{5})\left(\sqrt{5-2\sqrt{5}}\right)}$$

1649. If $0<a\le b$ then:

$$\int_a^b\left(\frac{1}{x}\cdot\tan^{-1}x\right)dx\ge\log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)=\log\left(b+\sqrt{1+b^2}\right)-\log\left(a+\sqrt{1+a^2}\right)=$$

$$=\log\left(x+\sqrt{1+x^2}\right)\Big|_a^b=\int_a^b\frac{dx}{\sqrt{1+x^2}}$$

We must to prove:

$$\frac{\tan^{-1}x}{x} \geq \frac{1}{\sqrt{1+x^2}}, \forall x > 0$$

$$\text{Let: } f(x) = \tan^{-1}x - \frac{x}{\sqrt{1+x^2}}, x > 0 \text{ then } f'(x) = \frac{\sqrt{1+x^2}-1}{1+x^2} > 0, \forall x > 0$$

$$\Rightarrow f \nearrow \text{ and } f(0) = 0 \Rightarrow f(x) > 0, \forall x > 0 \Rightarrow \tan^{-1}x \geq \frac{x}{\sqrt{1+x^2}}, \forall x > 0.$$

Therefore,

$$\int_a^b \left(\frac{1}{x}\cdot\tan^{-1}x\right)dx \geq \log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

Equality holds for $a = b$.

Solution 2 by Amrit Awasthi-India

$$\sin x < x, \forall x > 0 \Rightarrow \sin^{-1}(\sin x) < \sin^{-1}x \Rightarrow x < \sin^{-1}x \Rightarrow$$

$$\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) > \frac{x}{\sqrt{1+x^2}}, \text{but } \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}x > \frac{x}{\sqrt{1+x^2}} \Rightarrow \frac{\tan^{-1}x}{x} \geq \frac{1}{\sqrt{1+x^2}}, \forall x > 0; (1)$$

Integrating both sides w.r.t. x from a to b, it follows

$$\int_a^b \frac{\tan^{-1}x}{x}dx \geq \int_a^b \frac{dx}{\sqrt{1+x^2}} = \log\left(x+\sqrt{1+x^2}\right)\Big|_a^b =$$

$$= \log\left(b+\sqrt{1+b^2}\right) - \log\left(a+\sqrt{1+a^2}\right) = \log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

Therefore,

$$\int_a^b \left(\frac{1}{x}\cdot\tan^{-1}x\right)dx \geq \log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

Equality holds for $a = b$.

Solution 3 by Kamel Gandouli Rezgui-Tunisia

$$\frac{\left(x+\sqrt{1+x^2}\right)'}{x+\sqrt{1+x^2}} = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} = (\sinh^{-1}x)'$$

$$\text{Let: } f(x) = \tan^{-1}x - \frac{x}{\sqrt{1+x^2}}, \text{then } f'(x) = \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}} \geq 0, \forall x > 0$$

$$f\nearrow(0,\infty)\Rightarrow f(x)\geq f(0)=0,\forall x\geq 0$$

$$\frac{\tan^{-1}x}{x}\geq\frac{1}{\sqrt{1+x^2}},\forall x>0$$

$$\int_a^b\frac{\tan^{-1}x}{x}dx\geq\int_a^b\frac{dx}{\sqrt{1+x^2}}=\log\left(x+\sqrt{1+x^2}\right)\Big|_a^b=$$

$$=\log\left(b+\sqrt{1+b^2}\right)-\log\left(a+\sqrt{1+a^2}\right)=\log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

Therefore,

$$\int_a^b\left(\frac{1}{x}\cdot\tan^{-1}x\right)dx\geq\log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

Equality holds for $a=b$.

1650. Prove that:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}=\frac{\pi^{\frac{3}{2}}}{4}$$

where $\Gamma(x)$ –is *Gamma function*.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}B\left(\frac{n}{2},\frac{1}{2}\right)$$

$$\Omega=\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^{\frac{\pi}{2}}\sin^{n-1}x\,dx=\frac{2}{\sqrt{\pi}}\int_0^{\frac{\pi}{2}}\frac{1}{\sin x}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin^n x}{n}\right)dx=$$

$$=\frac{2}{\sqrt{\pi}}\overbrace{\int_0^{\frac{\pi}{2}}\frac{\log(1+\sin x)}{\sin x}dx}^{I}\overset{t=\tan\left(\frac{x}{2}\right)}{=}\int_0^1\frac{\log\left(1+\frac{2t}{1+t^2}\right)}{\frac{2t}{1+t^2}}\frac{2dt}{1+t^2}=$$

$$=\int_0^1\frac{2\log(1+t)-\log(1+t^2)}{t}dt=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

$$I=\frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{8}\Rightarrow\Omega=\frac{2}{\sqrt{\pi}}\cdot\frac{\pi^2}{8}=\frac{\pi^{\frac{3}{2}}}{4}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}=\frac{\pi^{\frac{3}{2}}}{4}$$

1651. Given $u_1=1,u_{n+1}=\frac{n^2-n+1}{n^2}u_n;\forall n\in\mathbb{N}^*$. Find:

$$\Omega=\lim_{n\to\infty}u_n.$$

Proposed by Minh Vu-Vietnam

Solution by Amrit Awasthi-India

$$u_{n+1}=\frac{n^2-n+1}{n^2}u_n\Rightarrow\frac{u_{n+1}}{u_n}=\frac{n^2-n+1}{n^2};u_1=1$$

$$\Omega=\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{u_n}{u_1}=\lim_{n\to\infty}\frac{u_n}{u_2}\cdot\frac{u_2}{u_1}=\lim_{n\to\infty}\frac{u_n}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}=$$

$$=\lim_{n\to\infty}\frac{u_n}{u_{n-1}}\cdot\frac{u_{n-1}}{u_{n-2}}\cdot\ldots\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}=\lim_{n\to\infty}\prod_{k=1}^{n}\frac{k^2-k+1}{k^2}=$$

$$=\lim_{n\to\infty}\prod_{k=1}^{n}\frac{\left(k-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\left(k-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}{k\cdot k}=$$

$$=\lim_{n\to\infty}\frac{\Gamma\left(n+\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\Gamma\left(n+\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}{\Gamma(n+1)\Gamma(n+1)}\cdot\frac{1}{\Gamma\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)}=$$

$$=\frac{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{\pi}\cdot\lim_{n\to\infty}\left(n^{\frac{1}{2}-\frac{\sqrt{3}i}{2}}\cdot n^{-1}\cdot n^{\frac{1}{2}+\frac{\sqrt{3}i}{2}}\cdot n^{-1}\right)=\frac{\cosh\left(\frac{\sqrt{3}\pi}{2}\right)}{\pi}\cdot\lim_{n\to\infty}\frac{1}{n}=0$$

Therefore,

$$\Omega=\lim_{n\to\infty}u_n=0.$$

1652. Given (x_n) such that:

$$x_1 = 1, x_{n+1} = \frac{n^3 - 3n^2 + 4n + 1}{n^3} x_n, \forall n \in \mathbb{N}^*$$

Find:

$$\lim_{n\to\infty} x_n$$

Proposed by Minh Vu-Vietnam

Solution 1 by Amrit Awasthi-India

Note that

$$x_{n+1} = \frac{x_{n+1}}{x_n} \cdot \frac{x_n}{x_{n-1}} \cdot \frac{x_{n-1}}{x_{n-2}} \cdot \ldots \cdot \frac{x_3}{x_2} \cdot \frac{x_2}{x_1}$$

This implies

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{k^3 - 3k^2 + 4k + 1}{k^3}$$

Now follow that for large "n" ($n \geq 2$)

$$4n + 1 \leq 3n^2$$

$$-3n^2 + 4n + 1 \leq 0 \to n^3 - 3n^2 + 4n + 1 \leq n^3 \to \frac{n^3 - 3n^2 + 4n + 1}{n^3} \leq 1$$

This implies that

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{k^3 - 3k^2 + 4k + 1}{k^3} = 0$$

Hence we have

$$\lim_{n\to\infty} x_n = 0$$

Solution 2 by Supriyo Halder-India

We have $x_1 = 1$ and,

$$x_{n+1} = \frac{n^3 - 3n^2 + 4n + 1}{n^3} x_n = \prod_{i=1}^{n} f(i)$$

where $f(i) = \frac{i^3 - 3i^2 + 4i + 1}{i^3}$

Now for sufficiently large $i, 0 < f(i) < 1$. So, (x_n) is ultimately decreasing sequence of positive numbers. So, it must converges. Note that,

$$(n+1)(n^3-3n^2+4n+1) = n^4-2n^3+n^2+5n+1 \le n^4$$

for sufficiently large n

$$\Rightarrow \frac{n^3-3n^2+4n+1}{n^3} \le \frac{n}{n+1}, \text{ for } n \ge k, k \in \mathbb{N}$$

$$\Rightarrow \forall n \ge k, \prod_{i=k}^{n} f(i) \le \frac{k}{n+1} \Rightarrow \prod_{i=1}^{\infty} f(i) = 0$$

Hence,

$$\lim x_n = \lim \prod_{i=1}^{n} f(i) = 0$$

1653. Prove that the following relationship holds:

$$\sum_{n=1}^{\infty} \frac{F_{2n}H_n}{4^n n} = \frac{1}{\sqrt{5}}\left(Li_2\left(\frac{2}{\sqrt{5}}-1\right) - Li_2\left(-1-\frac{2}{\sqrt{5}}\right)\right)$$

where F_n, H_n and $Li_2(x)$ are n^{th} Fibonacci, Harmonic number and dilogarithm function respectively.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Lucas Paes Barreto-Pernambuco-Brazil

$$\sum_{n=1}^{\infty} \frac{F_{2n}H_n}{4^n n} = \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{\varphi^n - (-\varphi)^n}{4^n n} H_n = \frac{1}{\sqrt{5}}\left\{\sum_{n=1}^{\infty} \frac{\varphi^{2n}}{4^n n} H_n - \sum_{n=1}^{\infty} \frac{(-\varphi)^{-2n}}{4^n n} H_n\right\} =$$

$$= \frac{1}{\sqrt{5}}\left\{Li_2\left(\frac{\varphi^2}{4}\right) + \frac{1}{2}\log^2\left(1-\frac{\varphi^2}{4}\right) - Li_2\left(\frac{1}{4\varphi^2}\right) - \frac{1}{2}\log^2\left(1-\frac{1}{4\varphi^2}\right)\right\}; (1)$$

$$= \frac{1}{\sqrt{5}}\left\{Li_2\left(\frac{\varphi^2}{4}\right) + \left[-Li_2\left(\frac{\varphi^2}{4}\right) - Li_2\left(1-\frac{1}{1-\frac{\varphi^2}{4}}\right)\right] - Li_2\left(\frac{1}{4\varphi^2}\right)\right.$$

$$\left. + \left[Li_2\left(\frac{1}{4\varphi^2}\right) + Li_2\left(1-\frac{1}{1-\frac{1}{4\varphi^2}}\right)\right]\right\} =$$

$$=\frac{1}{\sqrt{5}}\left\{-Li_2\left(1-\frac{1}{1-\frac{1}{4\varphi^2}}\right)+Li_2\left(1-\frac{1}{1-\frac{1}{4\varphi^2}}\right)\right\};(2)$$

$$=\frac{1}{\sqrt{5}}\left\{-Li_2\left(1-2-\frac{2}{\sqrt{5}}\right)+Li_2\left(1-2+\frac{2}{\sqrt{5}}\right)\right\}=$$

$$=\frac{1}{\sqrt{5}}\left\{-Li_2\left(-1-\frac{2}{\sqrt{5}}\right)+Li_2\left(-1+\frac{2}{\sqrt{5}}\right)\right\}$$

$$(1):-\frac{1}{2}\log^2 x=Li_2(1-x)+Li_2\left(1-\frac{1}{x}\right)$$

$$(2):\ \varphi=\frac{1+\sqrt{5}}{2}$$

1654. Let f be a class C^3 function defined on $[0,1]$ such that:

$$I_n=\int_0^{\frac{1}{n}} xf'(nx)\,dx+\frac{1}{2n^4}\sum_{n=0}^{n-1}\left(f'\left(\frac{k}{n}\right)+2nf\left(\frac{k}{n}\right)\right)$$

$$I_n=\frac{A}{n^2}+\frac{B}{n^4}+O\left(\frac{1}{n^4}\right)$$

Find A and B.

Proposed by Serlea Kabay-Liberia

Solution by proposer

We have:

$$\int_0^{\frac{1}{n}} xf'(x)\,dx=\frac{1}{n^2}\int_0^1 xf'(x)\,dx\overset{IBP}{=}\frac{1}{n^2}xf(x)|_0^1-\frac{1}{n^2}\int_0^1 f(x)\,dx$$

$$\int_0^{\frac{1}{n}} xf'(x)\,dx=\frac{f(0)}{n^2}-\frac{1}{n^2}\int_0^1 f(x)\,dx$$

$$I_n=\frac{f(0)}{n^2}-\frac{1}{n^2}\int_0^1 f(x)\,dx+\frac{1}{2n^4}\sum_{k=0}^{n-1}\left(f'\left(\frac{k}{n}\right)+2nf\left(\frac{k}{n}\right)\right)=$$

$$=\frac{f(0)}{n^2}-\frac{1}{n^2}\left[\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,dx-\frac{1}{2n^2}\sum_{k=0}^{n-1}\left(f'\left(\frac{k}{n}\right)+2nf\left(\frac{k}{n}\right)\right)\right]$$

Let $f=g'\Rightarrow g=\int f$, then

$$I_n=\frac{f(0)}{n^2}-\frac{1}{n^2}\left[\sum_{k=0}^{n-1}\left(g\left(\frac{k+1}{n}\right)-g\left(\frac{k}{n}\right)-\frac{1}{n}g'\left(\frac{k}{n}\right)-\frac{1}{2n^2}g''\left(\frac{k}{n}\right)\right)\right]$$

Using the Taylor Lagrange Inequality,

$$\left|g\left(\frac{k+1}{n}\right)-g\left(\frac{k}{n}\right)-\frac{1}{n}g'\left(\frac{k}{n}\right)-\frac{1}{2n^2}g''\left(\frac{k}{n}\right)-\frac{1}{6n^3}g^{(3)}\left(\frac{k}{n}\right)\right|=$$

$$=\left|g\left(\frac{k+1}{n}\right)-\sum_{i=1}^{3}\frac{g^{(i)}\left(\frac{k}{n}\right)}{i!}\left(\frac{1}{n}\right)^i\right|\le\frac{M}{24n^4},\text{where } M=\sup|g^{(4)}(x)|,\forall x\in\left[\frac{k+1}{n},\frac{k}{n}\right]$$

Therefore,

$$\left|g\left(\frac{k+1}{n}\right)-\sum_{i=1}^{3}\frac{g^{(i)}\left(\frac{k}{n}\right)}{i!}\left(\frac{1}{n}\right)^i\right|=O\left(\frac{1}{n^4}\right)$$

$$g\left(\frac{k+1}{n}\right)-g\left(\frac{k}{n}\right)-\frac{1}{n}g'\left(\frac{k}{n}\right)-\frac{1}{2n^2}g''\left(\frac{k}{n}\right)-\frac{1}{6n^3}g^{(3)}\left(\frac{k}{n}\right)=O\left(\frac{1}{n^4}\right)$$

$$g\left(\frac{k+1}{n}\right)-g\left(\frac{k}{n}\right)-\frac{1}{n}g'\left(\frac{k}{n}\right)-\frac{1}{2n^2}g''\left(\frac{k}{n}\right)=\frac{1}{6n^3}g^{(3)}\left(\frac{k}{n}\right)+O\left(\frac{1}{n^4}\right)$$

Now,

$$I_n=\frac{f(0)}{n^2}-\frac{1}{n^2}\sum_{k=0}^{n-1}\left(\frac{1}{6n^3}g^{(3)}\left(\frac{k}{n}\right)+O\left(\frac{1}{n^4}\right)\right)=\frac{f(0)}{n^2}-\frac{1}{6n^4}\cdot\frac{1}{n}\sum_{k=0}^{n-1}g^{(3)}\left(\frac{k}{n}\right)$$

$$=\frac{f(0)}{n^2}-\frac{1}{6n^4}g''(x)|_0^1+O\left(\frac{1}{n^4}\right)$$

$$I_n=\frac{f(1)}{n^2}-\frac{f'(1)-f'(0)}{6n^4}+O\left(\frac{1}{n^4}\right)$$

Therefore, $A=f(1),B=\frac{1}{6}[f'(1)-f'(0)]$

1655. Prove that:

$$\int_0^\infty\frac{\cos(bx)\sin(ax)}{x(\cosh(px))}dx=\frac{1}{2}\tan^{-1}\left(\sinh\left(\frac{\pi(a+b)}{2p}\right)\right)+\frac{1}{2}\tan^{-1}\left(\sinh\left(\frac{\pi(a-b)}{2p}\right)\right)$$

Proposed by George Moses-Nigeria

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^\infty\frac{\cos(bx)\sin(ax)}{x(\cosh(px))}dx\overset{x=px}{=}\int_0^\infty\frac{\sin\left(\frac{ax}{p}\right)\cos\left(\frac{bx}{p}\right)}{x\cosh x}dx=$$

$$=\frac{1}{2}\left(\int_0^\infty \frac{\sin\left(\frac{a+b}{p}\right)x}{x\cosh x}dx+\int_0^\infty \frac{\sin\left(\frac{a-b}{p}\right)x}{x\cosh x}dx\right)=\frac{1}{2}(A+B)$$

$$A=\int_0^\infty \frac{\sin\left(\frac{a+b}{p}\right)x}{x\cosh x}dx \overset{k=\frac{a+b}{p}}{=} \int_0^\infty \frac{\sin(kx)}{x\cosh x}dx$$

$$\frac{dA}{dk}=\int_0^\infty \frac{\cos(kx)}{\cosh x}dx=\int_0^\infty \frac{e^{ikx}+e^{-ikx}}{e^x+e^{-x}}dx=\int_0^\infty \frac{e^{-x(1-ik)}+e^{-x(1+ik)}}{1+e^{-2x}}dx=$$

$$=\int_0^\infty \frac{e^{-x(1-ik)}+e^{-x(1+ik)}}{1+e^{-2x}}dx \overset{t=e^{-2x}}{=} \frac{1}{2}\int_0^\infty \frac{t^{\frac{1-ik}{2}-1}+t^{\frac{1+ik}{2}-1}}{1+t}dt$$

$$\because \int_0^\infty \frac{t^{n-1}}{1+t}dt=\frac{1}{2}\left\{\psi\left(\frac{n+1}{2}\right)-\psi\left(\frac{n}{2}\right)\right\}$$

$$\frac{dA}{dk}=\frac{1}{4}\left[\psi\left(\frac{3-ik}{4}\right)-\psi\left(\frac{1+ik}{4}\right)+\psi\left(\frac{3+ik}{4}\right)-\psi\left(\frac{1-ik}{2}\right)\right]=$$

$$=\frac{\pi}{4}\cot\left(\frac{\pi}{4}+\frac{ik\pi}{4}\right)+\frac{\pi}{4}\cot\left(\frac{\pi}{4}-\frac{ik\pi}{4}\right)=\frac{\pi}{2}\sec\left(\frac{ik\pi}{2}\right)=\frac{\pi}{2}\sec\left(\frac{k\pi}{2}\right)$$

$$A=\frac{\pi}{2}\int_0^k \operatorname{sech}\left(\frac{\pi y}{2}\right)dy=\int_0^{\frac{k\pi}{2}}\operatorname{sech} t\,dt=\int_0^{\frac{k\pi}{2}}\frac{\cosh t}{1+\sinh^2 t}dt \overset{u=\sinh t}{=}$$

$$=\int_0^{\sinh\left(\frac{k\pi}{2}\right)}\frac{du}{1+u^2}=\tan^{-1}\left(\sinh\left(\frac{k\pi}{2}\right)\right)=\tan^{-1}\left(\sinh\left(\frac{\pi(a+b)}{2p}\right)\right)$$

Hence,

$$B=\tan^{-1}\left(\sinh\left(\frac{\pi(a+b)}{2p}\right)\right)$$

Therefore,

$$\int_0^\infty \frac{\cos(bx)\sin(ax)}{x(\cosh(px))}dx=\frac{1}{2}\tan^{-1}\left(\sinh\left(\frac{\pi(a+b)}{2p}\right)\right)+\frac{1}{2}\tan^{-1}\left(\sinh\left(\frac{\pi(a-b)}{2p}\right)\right)$$

1656. Prove that for $a>1$, the following relationship holds

$$\int_1^\infty \frac{x^2\tan^{-1}(ax)}{x^4+x^2+1}dx\cong\frac{\pi^2}{8\sqrt{3}}+\frac{\pi}{8}\log 3-\frac{\pi}{6a\sqrt{3}}+\frac{1}{3a^3}\left(\frac{\log 3}{4}-\frac{\pi}{12\sqrt{3}}\right)-$$

$$-\frac{1}{5a^5}\left(\frac{1}{2}-\frac{\pi}{12\sqrt{2}}-\frac{\log 3}{4}\right)+\frac{1}{7a^7}\left(\frac{\pi}{6\sqrt{3}}-\frac{1}{2}\right)-\frac{1}{108a^9}+$$

$$+\frac{1}{9a^9}\left(\frac{\log 3}{9}-\frac{\pi}{12\sqrt{3}}\right)-\frac{1}{11a^{11}}\left(\frac{11}{24}-\frac{\pi}{12\sqrt{3}}-\frac{\log 3}{4}\right)+\cdots$$

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Surjeet Singhania-India

For every $a>1$ and $x\in(1,\infty)$ it is known that $\tan^{-1}(ax)=\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{ax}\right)$

$$0<\frac{1}{ax}<1\Rightarrow\tan^{-1}(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^{2n-1}}{2n-1},\text{for }|z|<1$$

$$\int_1^\infty\frac{x^2\tan^{-1}(ax)}{x^4+x^2+1}dx=\frac{\pi}{2}\int_1^\infty\frac{x^2}{x^4+x^2+1}dx-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{a^{2n-1}(2n-1)}\underbrace{\int_0^\infty\frac{x^{3-2n}}{x^4+x^2+1}dx}_{I_n}$$

$$I_n=\int_0^\infty\frac{x^{3-2n}}{x^4+x^2+1}dx=\frac{1}{2}\int_0^\infty\frac{x^{-n+1}}{x^2+x+1}dx=\frac{1}{2}\int_0^1\frac{(1-x)x^{n-1}}{1-x^3}dx$$

$$I_n=\frac{1}{6}\psi^{(0)}\left(\frac{n+1}{3}\right)-\frac{1}{6}\psi^{(0)}\left(\frac{n}{3}\right)$$

Also,

$$\int_0^\infty\frac{x^2}{x^4+x^2+1}dx=\frac{\log 3}{4}+\frac{\pi}{4\sqrt{3}}$$

$$\int_1^\infty\frac{x^2\tan^{-1}(ax)}{x^4+x^2+1}dx=$$

$$=\frac{\pi}{8}\log 3+\frac{\pi^2}{8\sqrt{3}}-\frac{1}{6}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{a^{2n-1}(2n-1)}\left(\frac{1}{6}\psi^{(0)}\left(\frac{n+1}{3}\right)-\frac{1}{6}\psi^{(0)}\left(\frac{n}{3}\right)\right)$$

Therefore,

$$\int_1^\infty\frac{x^2\tan^{-1}(ax)}{x^4+x^2+1}dx\cong\frac{\pi^2}{8\sqrt{3}}+\frac{\pi}{8}\log 3-\frac{\pi}{6a\sqrt{3}}+\frac{1}{3a^3}\left(\frac{\log 3}{4}-\frac{\pi}{12\sqrt{3}}\right)-$$

$$-\frac{1}{5a^5}\left(\frac{1}{2}-\frac{\pi}{12\sqrt{2}}-\frac{\log 3}{4}\right)+\frac{1}{7a^7}\left(\frac{\pi}{6\sqrt{3}}-\frac{1}{2}\right)-\frac{1}{108a^9}+$$

$$+\frac{1}{9a^9}\left(\frac{\log 3}{9}-\frac{\pi}{12\sqrt{3}}\right)-\frac{1}{11a^{11}}\left(\frac{11}{24}-\frac{\pi}{12\sqrt{3}}-\frac{\log 3}{4}\right)+\cdots$$

1657. Find a closed form:

$$\Omega(a)=\int_0^\infty \frac{x\sqrt{x}}{(x^2+1)(1+ax)}dx, a>0$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Kartick Chandra Betal-India

$$\Omega(a)=\int_0^\infty \frac{x\sqrt{x}}{(x^2+1)(1+ax)}dx=\int_0^\infty \frac{dx}{\sqrt{x}(1+x^2)(x+a)}=$$

$$=\frac{1}{1+a^2}\int_0^\infty \frac{1}{\sqrt{x}}\left(\frac{a-x}{1+x^2}+\frac{1}{a+x}\right)dx=$$

$$=\frac{1}{1+a^2}\int_0^\infty \left(\frac{a}{\sqrt{x}(1+x^2)}-\frac{\sqrt{x}}{1+x^2}+\frac{1}{\sqrt{x}(a+x)}\right)dx=$$

$$=\frac{2a}{1+a^2}\int_0^\infty \frac{dx}{1+x^4}-\frac{1}{1+a^2}\int_0^\infty \frac{x\cdot 2x}{1+x^4}dx+\frac{2}{1+a^2}\int_0^\infty \frac{dx}{a+x^2}=$$

$$=\frac{2a}{1+a^2}\cdot\frac{1}{4}\int_0^\infty \frac{x^{\frac{1}{4}-1}dx}{1+x}-\frac{2}{1+a^2}\cdot\frac{1}{4}\int_0^\infty \frac{x^{\frac{3}{4}-1}}{1+x}dx+\frac{2}{(1+a^2)\sqrt{a}}\int_0^\infty \frac{dx}{x^2+1}=$$

$$=\frac{a}{2(1+a^2)}\cdot\frac{\pi}{\sin\frac{\pi}{4}}-\frac{1}{2(1+a^2)}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{1}+\frac{\pi}{\sqrt{a}(1+a^2)}=$$

$$=\frac{\pi a}{\sqrt{2}(1+a^2)}-\frac{\pi}{\sqrt{2}(1+a^2)}+\frac{\pi}{\sqrt{a}(1+a^2)}=\frac{\pi(a-1)}{\sqrt{2}(1+a^2)}+\frac{\pi}{\sqrt{a}(1+a^2)}=$$

$$=\frac{\pi\sqrt{a}(a-1)+\sqrt{2}\pi}{\sqrt{2a}(1+a^2)}=\frac{\pi}{a\sqrt{2}(1+a^2)}\left(a(a-1)+\sqrt{2a}\right)$$

Solution 2 by Ose Favour-Nigeria

$$\Omega(a)=\int_0^\infty \frac{x\sqrt{x}}{(x^2+1)(1+ax)}dx \overset{x=\frac{1}{u}}{=} \int_0^\infty \frac{1}{x^3\sqrt{x}\left(1+\frac{1}{x^2}\right)\left(1+\frac{a}{x}\right)}dx=$$

$$=\int_0^\infty \frac{dx}{\sqrt{x}(1+x^2)(a+x)}$$

$$\frac{1}{(1+x^2)(1+ax)}=\frac{a}{(a^2+1)(1+x^2)}-\frac{x}{(a^2+1)(1+x^2)}+\frac{1}{(a^2+1)(x+a)}$$

$$\Omega=\frac{1}{a^2+1}\left(a\int_0^\infty\frac{dx}{\sqrt{x}(1+x^2)}-\int_0^\infty\frac{x}{\sqrt{x}(1+x^2)}dx+\int_0^\infty\frac{dx}{\sqrt{x}(x+a)}\right)=$$

$$=\frac{1}{a^2+1}(a\Omega_1-\Omega_2+\Omega_3),\text{ where}$$

$$\Omega_1=\int_0^\infty\frac{dx}{\sqrt{x}(1+x^2)}\overset{x=u^2}{=}2\int_0^\infty\frac{du}{1+u^4}\overset{u=y^{\frac{1}{4}}}{=}\frac{1}{2}\int_0^\infty\frac{y^{\frac{1}{4}-1}}{1+y}dy=\frac{\pi\sqrt{2}}{2}$$

$$\Omega_2=\int_0^\infty\frac{x}{\sqrt{x}(1+x^2)}dx\overset{x=u^2}{=}\frac{1}{2}\int_0^\infty\frac{y^{\frac{1}{2}+\frac{1}{4}-1}}{1+y}dy=\frac{\pi\sqrt{2}}{2}$$

$$\Omega_3=\int_0^\infty\frac{dx}{\sqrt{x}(x+a)}\overset{x=u^2}{=}2\int_0^\infty\frac{du}{a+u^2}\overset{u=\sqrt{a}\tan^{-1}\theta}{=}\frac{2\sqrt{a}}{a}\int_0^{\frac{\pi}{2}}1\,d\theta=\frac{\pi\sqrt{a}}{a}$$

Therefore,

$$\Omega=\frac{\pi}{2(a^2+1)}\left(\sqrt{2}(a-1)+\frac{\sqrt{a}}{a}\right)$$

Solution 3 by Hikmat Mammadov-Azerbaijan

$$\Omega(a)=\int_0^\infty\frac{x\sqrt{x}}{(x^2+1)(1+ax)}dx\overset{x=u^2}{=}\int_0^\infty\frac{2u^4}{(u^4+1)(1+au^2)}du$$

$$\frac{u^4}{(u^4+1)(1+au^2)}=\frac{Au^2+B}{u^4+1}+\frac{C}{1+au^2}\Rightarrow u^4=u^4(aA+C)+u^2(A+aB)+B+C$$

$$\Rightarrow A=\frac{a}{1+a^2};B=-\frac{1}{1+a^2};C=\frac{1}{1+a^2}$$

$$\Rightarrow\Omega=2\int_0^\infty\left(\frac{Au^2+B}{u^4+1}+\frac{C}{1+au^2}\right)du=$$

$$=2A\int_0^\infty\frac{u^2}{u^4+1}du+2B\int_0^\infty\frac{du}{u^4+1}+\frac{2C}{\sqrt{a}}\int_0^\infty\frac{d(\sqrt{a}u)}{1+(\sqrt{a}u)^2}=$$

$$=A\Omega_1+B\Omega_2+2C\cdot\tan^{-1}(\sqrt{a}u)\Big|_0^\infty=$$

$$=A\Omega_1+B\Omega_2+2C\left[\lim_{u\to\infty}\tan^{-1}(\sqrt{a}u)-\tan^{-1}0\right]=A\Omega_1+B\Omega_2+\pi C$$

$$\Omega_2=\int_0^\infty\frac{2du}{u^4+1}=\int_0^\infty\frac{(u^2+1)-(u^2-1)}{u^4+1}du=\int_0^\infty\left(\frac{u^2+1}{u^4+1}-\frac{u^2-1}{u^4+1}\right)du=$$

$$=\int_0^{\infty}\left(\frac{1+u^{-2}}{u^2+u^{-2}}-\frac{1-u^{-2}}{u^2+u^{-2}}\right)du$$

$$=\int_0^{\infty}\frac{1+u^{-2}}{u^2-2+u^{-2}+2}du-\int_0^{\infty}\frac{1-u^{-2}}{u^2+2+u^{-2}-2}du$$

$$=\int_0^{\infty}\frac{d(u-u^{-1})}{(u-u^{-1})^2+\left(\sqrt{2}\right)^2}-\int_0^{\infty}\frac{d(u+u^{-1})}{(u+u^{-1})^2-\left(\sqrt{2}\right)^2}=$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u-u^{-1}}{\sqrt{2}}\right)\Big|_0^{\infty}-\frac{1}{2\sqrt{2}}\log\left|\frac{u+u^{-1}-\sqrt{2}}{u+u^{-1}+\sqrt{2}}\right|\Big|_0^{\infty}=$$

$$=\frac{1}{\sqrt{2}}\left[\lim_{n\to\infty}\tan^{-1}\left(\frac{u-u^{-1}}{\sqrt{2}}\right)-\lim_{n\to 0_+}\tan^{-1}\left(\frac{u-u^{-1}}{\sqrt{2}}\right)\right]-$$

$$-\frac{1}{2\sqrt{2}}\left[\lim_{n\to\infty}\log\left|\frac{u+u^{-1}-\sqrt{2}}{u+u^{-1}+\sqrt{2}}\right|-\lim_{n\to 0_+}\log\left|\frac{u+u^{-1}-\sqrt{2}}{u+u^{-1}+\sqrt{2}}\right|\right]\Rightarrow\Omega_2=\frac{\pi}{\sqrt{2}}$$

$$\Omega_1=\int_0^{\infty}\frac{2u^2}{u^4+1}du=\int_0^{\infty}\frac{(u^2+1)+(u^2-1)}{u^4+1}du=\int_0^{\infty}\frac{u^2+1}{u^4+1}du+\int_0^{\infty}\frac{u^2-1}{u^4+1}du$$

$$\Rightarrow\Omega_1=\frac{\pi}{\sqrt{2}}\Rightarrow\Omega=A\Omega_1+B\Omega_2+\pi C=\frac{\pi}{\sqrt{2}}\cdot\frac{a-1+\sqrt{2}}{1+a^2}$$

Therefore,

$$\Omega=\frac{\pi}{2(a^2+1)}\left(\sqrt{2}(a-1)+\frac{\sqrt{a}}{a}\right)$$

1658. Prove that:

$$\int_0^{2\pi}\log(10+6\cos\theta)\,d\theta=4\pi\log 3$$

Proposed by Surjeet Singhania-India

Solution 1 by Kartick Chandra Betal-India

$$\Omega=\int_0^{2\pi}\log(10+6\cos\theta)\,d\theta=2\int_0^{\frac{\pi}{2}}\log(10+6\cos\theta)\,d\theta=$$

$$=2\int_0^{\pi}\log(10-6\cos\theta)\,d\theta$$

$$\Rightarrow 2\Omega = 2\int_0^{\frac{\pi}{2}} \log(10^2 - 6^2\cos^2\theta)\,d\theta = 2\int_0^{\frac{\pi}{2}} \log(10^2\sin^2\theta + 8^2\cos^2\theta)\,d\theta =$$

$$= 2\pi\log\left(\frac{10+8}{2}\right) = 2\pi\log 9 = 4\pi\log 3$$

Solution 2 by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^{2\pi} \log(10 + 6\cos\theta)\,d\theta = 2\int_0^{\pi} \log(10 + 6\cos\theta)\,d\theta \overset{\theta=2x}{=}$$

$$= 4\int_0^{\frac{\pi}{2}} \log(10 + 6\cos 2x)\,dx = 4\int_0^{\frac{\pi}{2}} \log(4 + 12\cos^2 x)\,dx =$$

$$= 4\int_0^{\frac{\pi}{2}} \log(4\sin^2 x + 16\cos^2 x)\,dx$$

$$\because \int_0^{\frac{\pi}{2}} \log(a^2\sin^2 x + b^2\cos^2 x)\,dx = \pi\log\left(\frac{a+b}{2}\right)$$

$$\Omega = 4\pi\log\left(\frac{2+4}{2}\right) = 4\pi\log 3$$

Therefore,

$$\int_0^{2\pi} \log(10 + 6\cos\theta)\,d\theta = 4\pi\log 3$$

1659. $f(\cot x) = \sin 2x + \cos 2x\,, 0 < x < \pi$. **Find:**

$$\Omega = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x \cdot f(\sin x)\,dx$$

Proposed by Neculai Stanciu-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$f(\cot x) = \frac{2\tan x}{1+\tan^2 x} + \frac{1-\tan^2 x}{1+\tan^2 x} = \frac{2\cot x}{1+\cot^2 x} + \frac{\cot^2 x - 1}{\cot^2 x + 1} =$$

$$= \frac{2\cot x + \cot^2 x - 1}{\cot^2 x + 1} \Rightarrow f(\sin x) = \frac{2\sin x + \sin^2 x - 1}{\sin^2 x} + 1$$

$$\Omega = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x \cdot f(\sin x)\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x\,(2\sin x + \sin^2 x - 1)}{1+\sin^2 x}\,dx =$$

$$=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{t^2+2t-1}{t^2+1}dt=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\left(1+\frac{2t}{t^2+1}-\frac{2}{t^2+1}\right)dt=$$

$$=\left(t+\log(t^2+1)-2\tan^{-1}(t)\right)\Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}=$$

$$=\frac{\sqrt{3}-1}{2}+\log\left(\frac{7}{5}\right)-2\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)+2\tan^{-1}\left(\frac{1}{2}\right)$$

Solution 2 by Kartick Chandra Betal-India

$$f(\cot x)=\sin 2x+\cos 2x=\frac{2\tan x}{1+\tan^2 x}+\frac{1-\tan^2 x}{1+\tan^2 x}$$

$$f\left(\frac{1}{x}\right)=\frac{2x}{1+x^2}+\frac{1-x^2}{1+x^2}\Rightarrow f(x)=\frac{2x}{1+x^2}+\frac{x^2-1}{x^2+1}=\frac{x^2+2x-1}{1+x^2}$$

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\cos x\cdot f(\sin x)\,dx=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}f(x)\,dx=$$

$$=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{x^2+2x-1}{x^2+1}dx=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\left(1+\frac{2x}{1+x^2}-\frac{2}{1+x^2}\right)dx=$$

$$=\frac{\sqrt{3}-1}{2}+\log\left(\frac{7}{5}\right)-2\tan^{-1}\left(\frac{2\sqrt{3}-2}{4+\sqrt{3}}\right)$$

Solution 3 by Myagmasuren Yadamsuren-Darkhan-Mongolia

$$\sin 2x=\frac{2\cot x}{1+\cot^2 x},\cos 2x=\frac{\cot^2 x-1}{1+\cot^2 x}$$

$$f(\cot x)=\frac{\cot^2 x+2\cot x-1}{1+\cot^2 x}\Rightarrow f(t)=\frac{t^2+2t-1}{1+t^2};(*)$$

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\cos x\cdot f(\sin x)\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}f(\sin x)d(\sin x)=$$

$$=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}f(t)\,dt=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{t^2+2t-1}{1+t^2}dt=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\left(1+\frac{2t-2}{1+t^2}\right)dt=$$

$$=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\left(1+\frac{2t}{1+t^2}-\frac{2}{1+t^2}\right)dt=\left(t+\log(1+t^2)-2\tan^{-1}t\right)\Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}=$$

$$=\frac{\sqrt{3}-1}{2}+\log\left(\frac{7}{5}\right)-2\tan^{-1}\frac{2}{13}(5\sqrt{3}-7)$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$f(\cot x)=\sin 2x+\cos 2x=\frac{2\tan x}{1+\tan^2 x}+\frac{1-\tan^2 x}{1+\tan^2 x}=$$

$$=\frac{\cot^2 x+2\cot x-1}{\cot^2 x+1}\Rightarrow f(x)=\frac{x^2+2x-1}{x^2+1}=1+\frac{2x-2}{x^2+1}$$

$$\Omega=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\cos x\cdot f(\sin x)\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\cos x\cdot\left(1+\frac{2\sin x-2}{\sin^2 x+1}\right)dx=$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\cos x\,dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos x\cdot\sin x-2\cos x}{2-\cos^2 x}dx=$$

$$=\sin x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sin 2x}{2-\frac{1+\cos 2x}{2}}dx-2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos x}{2-\cos^2 x}dx=$$

$$=\frac{\sqrt{3}-1}{2}+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{d(3-\cos 2x)}{3-\cos 2x}-2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{d(\sin x)}{\sin^2 x+1}=$$

$$=\frac{\sqrt{3}-1}{2}+\left(\log(3-\cos 2x)-2\tan^{-1}(\sin x)\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=$$

$$=\frac{\sqrt{3}-1}{2}+\log\left(\frac{7}{5}\right)-2\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)-2\tan^{-1}\left(\frac{1}{2}\right)$$

1660. If $0<a\le b\le 1$ then:

$$\int_a^b\int_a^b\int_a^b\frac{dx\,dy\,dz}{x^2+y^2+z^2+3}\le\frac{(b-a)^2}{3}\int_a^b\frac{dx}{x+x^x}$$

Proposed by Daniel Sitaru-Romania

Solution by Kamel Gandouli Rezgui-Tunisia

$x,y,z\in[a,b]^3, 0<a\le b\le 1$ and $f(t)=t^2+1-t-t^t=$

$$=t^2-t+1-t^t\ge 0 \text{ in } [0,1] \text{ and } t^t\le 1$$

$$x^2+y^2+z^2+3=x^2+1+y^2+1+z^2+1\ge x^x+x+y^y+y+z^z+z\Rightarrow$$

$$\frac{1}{x^2+y^2+z^2+3} \le \frac{1}{x^x+x+y^y+y+z^z+z}$$

$$\because (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \overset{CBS}{\ge} 9 \Rightarrow \frac{1}{a+b+c} \le \frac{1}{9a}+\frac{1}{9b}+\frac{1}{9c}, \forall a,b,c>0$$

Hence, we have:

$$\frac{1}{x^2+y^2+z^2+3} \le \frac{1}{9(x^x+x)}+\frac{1}{9(y^y+y)}+\frac{1}{9(z^z+z)}$$

and then

$$\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{x^2+y^2+z^2+3} \le \frac{1}{9}\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{x^x+x}+\frac{1}{9}\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{y^y+y}+$$

$$+\frac{1}{9}\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{z^z+z} = \frac{(b-a)^2}{9}\int_a^b \frac{dx}{x^x+x}+\frac{(b-a)^2}{9}\int_a^b \frac{dy}{y^y+y}+\frac{(b-a)^2}{9}\int_a^b \frac{dz}{z^z+z}$$

$$= 3\cdot\frac{(b-a)^2}{9}\int_a^b \frac{dx}{x^x+x} = \frac{(b-a)^2}{3}\int_a^b \frac{dx}{x^x+x}$$

Therefore,

$$\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{x^2+y^2+z^2+3} \le \frac{(b-a)^2}{3}\int_a^b \frac{dx}{x+x^x}$$

1661. Find:

$$\Omega = \lim_{x\to 0}\frac{\left(e^{3x\log(1+x)\cdot\cot x}-e^{5x}\right)\cdot\log(1+\sin x)}{x^2}$$

Proposed by Dang Le Gia Khanh-Vietnam

Solution 1 by Kamel Gandouli Rezgui-Tunisia

$$\Omega = \lim_{x\to 0}\frac{\left(e^{3x\cdot\log(1+x)\cdot\cot x}-e^{5x}\right)\cdot\log(1+\sin x)}{x^2} =$$

$$= \lim_{x\to 0}\frac{\sin x\left(e^{3x\cdot\log(1+x)\cdot\cot x}-e^{5x}\right)\cdot\log(1+\sin x)}{x^2\cdot\sin x}$$

$$\lim_{x\to 0}\frac{\log(1+\sin x)}{\sin x} = 1, \lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\lim_{x\to 0}\frac{3x}{\sin x}\cdot\log(1+x)\cdot\cot x = 0$$

$$e^{3x\cdot\log(1+x)\cdot\cot x}-e^{5x} \cong 1+3x\cdot\log(1+x)\cdot\cot x-1-5x$$

$$\frac{e^{3x\cdot\log(1+x)\cdot\cot x}-e^{5x}}{x}\cong 3\log(1+x)\cdot\cot x-5$$

$$3\log(1+x)\cdot\cot x=3\log(1+x)\cdot\frac{\cos x}{\sin x}=3\frac{\log(1+x)}{x}\cdot\frac{x}{\sin x}\cdot\cos x\to 3$$

$$\lim_{x\to0}\frac{e^{3x\cdot\log(1+x)\cdot\cot x}-e^{5x}}{x}=-2$$

Therefore,

$$\Omega=\lim_{x\to0}\frac{\left(e^{3x\cdot\log(1+x)\cdot\cot x}-e^{5x}\right)\cdot\log(1+\sin x)}{x^2}=-2$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega=\lim_{x\to0}\frac{\left(e^{3x\cdot\log(1+x)\cdot\cot x}-e^{5x}\right)\cdot\log(1+\sin x)}{x^2}$$

$$\lim_{x\to0}\frac{\log(1+\sin x)}{x}=\lim_{x\to0}\frac{\log(1+\sin x)}{\sin x}\cdot\frac{\sin x}{x}=1$$

Hence,

$$\Omega=\lim_{x\to0}\frac{\left(e^{3x\cdot\log(1+x)\cdot\cot x}-e^{5x}\right)\cdot\log(1+\sin x)}{x^2}=$$

$$=\lim_{x\to0}\frac{e^{3x\cdot\log(1+x)\cdot\cot x}-e^{5x}}{x}\cdot\lim_{x\to0}\frac{\log(1+\sin x)}{\sin x}\cdot\frac{\sin x}{x}=$$

$$=\lim_{x\to0}\frac{e^{3x\cdot\log(1+x)\cdot\cot x}-e^{5x}}{x}=\lim_{x\to0}\left(\frac{e^{3\log(1+x)}-1}{3\log(1+x)}\cdot\frac{3\log(1+x)}{x}-\frac{e^{5x}-1}{5x}\cdot5\right)=$$

$$=1\cdot3-5=-2$$

Solution 3 by Yen Tung Chung-Taichung-Taiwan

$$3x\cdot\cot x\cdot\log(1+x)=\frac{3x\cdot\cos x\cdot\log(1+x)}{\sin x}=\frac{3x\left(1-\frac{1}{2}x^2+\cdots\right)\left(x-\frac{1}{2}x^2+\cdots\right)}{x-\frac{1}{6}x^2}=$$

$$=\frac{3x^2-\frac{3}{2}x^3+\cdots}{x-\frac{1}{6}x^2+\cdots}=3x-\frac{3}{2}x^2+o(x^4)$$

Thus,

$$\Omega=\lim_{x\to0}\frac{\left(e^{3x\log(1+x)\cdot\cot x}-e^{5x}\right)\cdot\log(1+\sin x)}{x^2}=$$

$$=\lim_{x\to 0}\frac{\left(e^{3x-\frac{3}{2}x^2+o(x^4)}-e^{5x}\right)\log\left(1+x+o(x^3)\right)}{x^2}=$$

$$=\lim_{x\to 0}\frac{\left(1+\left(3x+o(x^2)\right)-\left(1+5x+o(x^2)\right)\right)\log\left(1+x+o(x^3)\right)}{x^2}=$$

$$=\lim_{x\to 0}\frac{\left(-2x+o(x^2)\right)\left(x+o(x^3)\right)}{x^2}=\lim_{x\to 0}\frac{-2x^2+o(x^3)}{x^2}=-2$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{x\to 0}3x\cdot\log(1+x)\cdot\cot x=\lim_{x\to 0}\frac{3\log(1+x)\cdot\cos x}{\frac{\sin x}{x}}=\frac{0}{1}=0$$

$$\Omega=\lim_{x\to 0}\frac{\left(e^{3x\log(1+x)\cdot\cot x}-e^{5x}\right)\cdot\log(1+\sin x)}{x^2}=$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\log\frac{1+\sin x}{\sin x}\cdot\left(\lim_{x\to 0}\frac{3^{3x\cdot\log(1+x)\cdot\cot x}-1}{x}-\lim_{x\to 0}\frac{e^{5x}-1}{x}\right)=$$

$$=1\cdot 1\cdot\left(3\lim_{x\to 0}\log(1+x)\cdot\cot x\cdot\lim_{x\to 0}\frac{e^{3x\cdot\log(1+x)\cdot\cot x-1}}{3x\cdot\log(1+x)\cdot\cot x}-5\cdot\lim_{x\to 0}\frac{e^{5x}-1}{5x}\right)=$$

$$=3\lim_{x\to 0}\frac{x}{\tan x}\cdot\lim_{x\to 0}\frac{\log(1+x)}{x}-5=3-5=-2.$$

1662. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(n-k+1)H_k}{k(n-k+1)^2+k}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Kamel Gandouli Rezgui-Tunisia

$$\frac{(n-k+1)H_k}{k(n-k+1)^2+k}=\frac{n-k+1}{k[(n-k+1)^2+1]}\le\frac{1}{k},\forall k\le n$$

Hence,

$$0\le\frac{(n-k+1)H_k}{k(n-k+1)^2+k}\le\frac{H_k}{k}$$

and then

$$0\le\frac{1}{n}\sum_{k=1}^{n}\frac{(n-k+1)H_k}{k(n-k+1)^2+k}\le\frac{1}{n}\sum_{k=1}^{n}\frac{H_k}{k}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{H_k}{k}\cong\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\log k}{k}=0;\ (\because H_n\cong\log n)$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(n-k+1)H_k}{k(n-k+1)^2+k}=0$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{For } 1\le k\le n,\text{ let } a_k=\frac{(n-k+1)H_k}{k(n-k+1)^2+k}=\frac{(n-k+1)H_k}{((n-k+1)^2+1)k}=$$

$$=\frac{n-k+1}{(n-k+1)^2+1}\cdot\frac{H_k}{k}<\frac{1}{n-k+1}\cdot 1=\frac{1}{n-k+1}$$

Hence,

$$0<\frac{1}{n}\sum_{k=1}^{n}a_k<\frac{1}{n}\sum_{k=1}^{n}\frac{1}{n-k+1}=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}$$

$$\text{As }\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k}\overset{L.C-S}{=}0$$

By the sandwich theorem, we get

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{(n-k+1)H_k}{k(n-k+1)^2+k}=0$$

1663. Find:

$$\Omega(k)=\lim_{n\to\infty}n^{2k}\left(\frac{\tan^{-1}(n^k)}{n^k}-\frac{\tan^{-1}(1+n^k)}{1+n^k}\right),k\in\mathbb{N}^*$$

Proposed by Ovidiu Gabriel Dinu-Romania

Solution by Ravi Prakash-New Delhi-India

$$\text{Let } a=n^k \text{ and } f(x)=\frac{\tan^{-1}x}{x},x\in[a,a+1],\text{then } f'(x)=\frac{1}{x(1+x^2)}-\frac{\tan^{-1}x}{x^2}$$

By the Lagrange's mean value theorem, we have:

$$f(a+1)-f(a)=f'(a+\theta)\text{ for some }\theta\in(0,1).$$

Hence,

$$n^{2k}\left(\frac{\tan^{-1}(n^k)}{n^k}-\frac{\tan^{-1}(1+n^k)}{1+n^k}\right)=n^{2k}\left(\frac{1}{(n^k+\theta)[1+(n^k+\theta)^2]}-\frac{\tan^{-1}(n^k+\theta)}{(n^k+\theta)^2}\right)$$

Taking limit as $n\to\infty$, we get:

$$\Omega(k)=\lim_{n\to\infty}n^{2k}\left(\frac{\tan^{-1}(n^k)}{n^k}-\frac{\tan^{-1}(1+n^k)}{1+n^k}\right)=$$

$$=\lim_{n\to\infty}n^{2k}\left(\frac{1}{(n^k+\theta)[1+(n^k+\theta)^2]}-\frac{\tan^{-1}(n^k+\theta)}{(n^k+\theta)^2}\right)=$$

$$=\lim_{n\to\infty}\left(-\frac{1}{n^k+\theta}\cdot\frac{1}{\frac{1}{n^{2k}}+\left(1+\frac{\theta}{n^k}\right)^2}+\frac{\tan^{-1}(n^k+\theta)}{\left(1+\frac{\theta}{n^k}\right)^2}\right)=\frac{\pi}{2}$$

1664. For all $k\in\mathbb{N}^*$ let $S(k)$ be the k^{th} term in the sequence

$$1,2,5,12,29,70,169,408,985,2378,\ldots$$

Then prove the following summations

$$\sum_{n=1}^{\infty}\frac{S(n)}{n!}=\frac{e\sinh(\sqrt{2})}{\sqrt{2}}$$

$$\sum_{n=1}^{\infty}\frac{S(3n)}{(3n)!}=\frac{1}{3\sqrt{2}}\left(\frac{1}{\sqrt{e^{\sqrt{2}+1}}}\cos\left(\frac{\sqrt{6}+\sqrt{3}}{2}\right)-\sqrt{e^{\sqrt{2}-1}}\cos\left(\frac{\sqrt{6}-\sqrt{3}}{2}\right)+e\sinh(\sqrt{2})\right)$$

Proposed by Amrit Awasthi-India

Solution 1 by Ahmed Yacoube Chach-Mauritania

$$\begin{cases}S(1)=1;S(2)=2\\S(n+2)=2S(n+1)+S(n)\end{cases}$$

$x^2-2x-1=0$ equation associate of $S(n)=\alpha(1+\sqrt{2})^n+\beta(1-\sqrt{2})^n$ and from initial conditions, we get: $S(n)=\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$

$$\sum_{n=1}^{\infty}\frac{S(n)}{n!}=\frac{1}{2\sqrt{2}}\left(\sum_{n=1}^{\infty}\frac{(1+\sqrt{2})^n}{2}-\sum_{n=1}^{\infty}\frac{(1-\sqrt{2})^n}{2}\right)=$$

$$=\frac{1}{2\sqrt{2}}\left(e^{1+\sqrt{2}}-1-(e^{1-\sqrt{2}}-1)\right)=\frac{e}{\sqrt{2}}\cdot\frac{e^{\sqrt{2}}-e^{-\sqrt{2}}}{2}=\frac{e\sinh(\sqrt{2})}{\sqrt{2}}$$

$$\text{Let } f(x)=\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n)!}\Rightarrow f^{(3)}(x)=f(x) \text{ and } f(0)=1, f'(0)=f''(0)=0$$

$$x^3-1=0$$

$$f(x)=Ae^x+Be^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)+Ce^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)\Rightarrow$$

$$f(x)=\frac{1}{3}e^x+\frac{2}{3}e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$\sum_{n=1}^{\infty}\frac{S(3n)}{(3n)!}=\frac{1}{2\sqrt{2}}\left(\sum_{n=1}^{\infty}\frac{\left(1+\sqrt{3}\right)^{3n}}{(3n)!}-\sum_{n=1}^{\infty}\frac{\left(1-\sqrt{2}\right)^{3n}}{(3n)!}\right)=$$

$$=\frac{1}{2\sqrt{2}}\left(\frac{1}{3}e^{1+\sqrt{2}}+\frac{2}{3}e^{-\frac{1+\sqrt{2}}{2}}\cos\left(\frac{\sqrt{3}}{2}\left(1+\sqrt{2}\right)\right)-1\right)-$$

$$-\frac{1}{2\sqrt{2}}\left(\frac{1}{3}e^{1-\sqrt{2}}+\frac{2}{3}e^{-\frac{1-\sqrt{2}}{2}}\cos\left(\frac{\sqrt{3}}{2}\left(1-\sqrt{2}\right)\right)-1\right)=$$

$$=\frac{1}{3\sqrt{2}}\left(\frac{1}{2}e^{1+\sqrt{2}}+e^{-\frac{1+\sqrt{2}}{2}}\cos\left(\frac{\sqrt{6}+\sqrt{3}}{2}\right)\right)-$$

$$-\frac{1}{3\sqrt{2}}\left(\frac{1}{2}e^{1-\sqrt{2}}+e^{-\frac{1-\sqrt{2}}{2}}\cos\left(\frac{\sqrt{6}-\sqrt{3}}{2}\right)\right)=$$

$$\frac{1}{3\sqrt{2}}\left(\frac{1}{\sqrt{e^{\sqrt{2}+1}}}\cos\left(\frac{\sqrt{6}+\sqrt{3}}{2}\right)-\sqrt{e^{\sqrt{2}-1}}\cos\left(\frac{\sqrt{6}-\sqrt{3}}{2}\right)+e\sinh\left(\sqrt{2}\right)\right)$$

Solution 2 by Kamel Rezgui Gandouli-Tunisia

$$s_k\in\{0,1,2,5,12,29,\dots\}\Rightarrow\begin{cases}s_0=0,s_1=1\\ s_{n+2}=2s_{n+1}+s_n\end{cases}$$

$$x^2=2x+1\Rightarrow s_n=\alpha\left(1+\sqrt{2}\right)^n+\beta\left(1-\sqrt{2}\right)^n$$

$$s_0=0\Rightarrow\alpha=-\beta\Rightarrow\alpha=\frac{1}{2\sqrt{2}}\Rightarrow s_n=\frac{1}{2\sqrt{2}}\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}\Rightarrow$$

$$\sum_{n=1}^{\infty}\frac{S(n)}{n!}=\frac{1}{2\sqrt{2}}\sum_{n=1}^{\infty}\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}=\frac{1}{2\sqrt{2}}e^{1+\sqrt{2}}-\frac{1}{2\sqrt{2}}e^{1-\sqrt{2}}=$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{e^{\sqrt{2}}-e^{-\sqrt{2}}}{2}=\frac{e\sinh\left(\sqrt{2}\right)}{\sqrt{2}}$$

$$\text{Let } s(x)=\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n)!}\Rightarrow s^{(3)}(x)=\sum_{n=1}^{\infty}\frac{x^{3n-3}}{(3n-3)!}=s(x)\Rightarrow s(x)=ae^{x}+be^{jx}+ce^{j^2x}$$

$$s(0)=1, s'(0)=s''(0)=0$$

$$\begin{cases} a+b+c=1\\ a+bj+cj^2=0\\ a+bj^2+cj=0\end{cases}\Rightarrow a+b+c=\frac{1}{3}\Rightarrow$$

$$s(x)=\frac{1}{3}\left(e^{x}+e^{jx}+e^{j^2x}\right)=\frac{e^x}{3}+\frac{2}{3}e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$\sum_{n=0}^{\infty}\frac{s(3n)}{(3n)!}=\frac{1}{2\sqrt{2}}s(1+\sqrt{2})-\frac{1}{2\sqrt{2}}s(1-\sqrt{2})=$$

$$=\frac{1}{6\sqrt{2}}e^{1+\sqrt{2}}+\frac{1}{3\sqrt{2}}\frac{1}{\sqrt{e^{1+\sqrt{2}}}}\cos\left(\frac{\sqrt{3}}{2}(1+\sqrt{2})\right)-$$

$$-\frac{1}{6\sqrt{2}}e^{1-\sqrt{2}}+\frac{1}{3\sqrt{2}}\frac{1}{\sqrt{e^{1-\sqrt{2}}}}\cos\left(\frac{\sqrt{3}}{2}(1-\sqrt{2})\right)=$$

$$=\frac{1}{3\sqrt{2}}\left(\frac{1}{\sqrt{e^{\sqrt{2}+1}}}\cos\left(\frac{\sqrt{6}+\sqrt{3}}{2}\right)-\sqrt{e^{\sqrt{2}-1}}\cos\left(\frac{\sqrt{6}-\sqrt{3}}{2}\right)+e\sinh(\sqrt{2})\right)$$

1665. For all $k\in\mathbb{N}^*$ let $S(k)$ be the k^{th} term in the sequence

$1,2,5,12,29,70,169,408,985,2378,\ldots$Then prove the following summations

$$\sum_{n=1}^{\infty}\frac{S(n)}{n!}=\frac{e\sinh(\sqrt{2})}{\sqrt{2}}$$

$$\sum_{n=1}^{\infty}\frac{S(3n)}{(3n)!}=\frac{1}{3\sqrt{2}}\left(\frac{1}{\sqrt{e^{\sqrt{2}+1}}}\cos\left(\frac{\sqrt{6}+\sqrt{3}}{2}\right)-\sqrt{e^{\sqrt{2}-1}}\cos\left(\frac{\sqrt{6}-\sqrt{3}}{2}\right)+e\sinh(\sqrt{2})\right)$$

Proposed by Amrit Awasthi-India

Solution 1 by Ahmed Yacoube Chach-Mauritania

$$\begin{cases} S(1)=1; S(2)=2\\ S(n+2)=2S(n+1)+S(n)\end{cases}$$

$x^2-2x-1=0$ equation associate of $S(n)=\alpha(1+\sqrt{2})^n+\beta(1-\sqrt{2})^n$ and from

initial conditions, we get: $S(n)=\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$

$$\sum_{n=1}^{\infty}\frac{S(n)}{n!}=\frac{1}{2\sqrt{2}}\left(\sum_{n=1}^{\infty}\frac{\left(1+\sqrt{2}\right)^n}{2}-\sum_{n=1}^{\infty}\frac{\left(1-\sqrt{2}\right)^n}{2}\right)=$$

$$=\frac{1}{2\sqrt{2}}\left(e^{1+\sqrt{2}}-1-\left(e^{1-\sqrt{2}}-1\right)\right)=\frac{e}{\sqrt{2}}\cdot\frac{e^{\sqrt{2}}-e^{-\sqrt{2}}}{2}=\frac{e\sinh(\sqrt{2})}{\sqrt{2}}$$

$$\text{Let } f(x)=\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n)!}\Rightarrow f^{(3)}(x)=f(x)\text{ and } f(0)=1, f'(0)=f''(0)=0$$

$$x^3-1=0$$

$$f(x)=Ae^x+Be^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)+Ce^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)\Rightarrow$$

$$f(x)=\frac{1}{3}e^x+\frac{2}{3}e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$\sum_{n=1}^{\infty}\frac{S(3n)}{(3n)!}=\frac{1}{2\sqrt{2}}\left(\sum_{n=1}^{\infty}\frac{\left(1+\sqrt{3}\right)^{3n}}{(3n)!}-\sum_{n=1}^{\infty}\frac{\left(1-\sqrt{2}\right)^{3n}}{(3n)!}\right)=$$

$$=\frac{1}{2\sqrt{2}}\left(\frac{1}{3}e^{1+\sqrt{2}}+\frac{2}{3}e^{-\frac{1+\sqrt{2}}{2}}\cos\left(\frac{\sqrt{3}}{2}\left(1+\sqrt{2}\right)\right)-1\right)-$$

$$-\frac{1}{2\sqrt{2}}\left(\frac{1}{3}e^{1-\sqrt{2}}+\frac{2}{3}e^{-\frac{1-\sqrt{2}}{2}}\cos\left(\frac{\sqrt{3}}{2}\left(1-\sqrt{2}\right)\right)-1\right)=$$

$$=\frac{1}{3\sqrt{2}}\left(\frac{1}{2}e^{1+\sqrt{2}}+e^{-\frac{1+\sqrt{2}}{2}}\cos\left(\frac{\sqrt{6}+\sqrt{3}}{2}\right)\right)-$$

$$-\frac{1}{3\sqrt{2}}\left(\frac{1}{2}e^{1-\sqrt{2}}+e^{-\frac{1-\sqrt{2}}{2}}\cos\left(\frac{\sqrt{6}-\sqrt{3}}{2}\right)\right)=$$

$$\frac{1}{3\sqrt{2}}\left(\frac{1}{\sqrt{e^{\sqrt{2}+1}}}\cos\left(\frac{\sqrt{6}+\sqrt{3}}{2}\right)-\sqrt{e^{\sqrt{2}-1}}\cos\left(\frac{\sqrt{6}-\sqrt{3}}{2}\right)+e\sinh(\sqrt{2})\right)$$

Solution 2 by Kamel Rezgui Gandouli-Tunisia

$$s_k\in\{0,1,2,5,12,29,\dots\}\Rightarrow\begin{cases}s_0=0, s_1=1\\ s_{n+2}=2s_{n+1}+s_n\end{cases}$$

$$x^2=2x+1\Rightarrow s_n=\alpha\left(1+\sqrt{2}\right)^n+\beta\left(1-\sqrt{2}\right)^n$$

$$s_0=0\Rightarrow\alpha=-\beta\Rightarrow\alpha=\frac{1}{2\sqrt{2}}\Rightarrow s_n=\frac{1}{2\sqrt{2}}\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}\Rightarrow$$

$$\sum_{n=1}^{\infty}\frac{S(n)}{n!}=\frac{1}{2\sqrt{2}}\sum_{n=1}^{\infty}\left\{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right\}=\frac{1}{2\sqrt{2}}e^{1+\sqrt{2}}-\frac{1}{2\sqrt{2}}e^{1-\sqrt{2}}=$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{e^{\sqrt{2}}-e^{-\sqrt{2}}}{2}=\frac{e\sinh\left(\sqrt{2}\right)}{\sqrt{2}}$$

$$\text{Let } s(x)=\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n)!}\Rightarrow s^{(3)}(x)=\sum_{n=1}^{\infty}\frac{x^{3n-3}}{(3n-3)!}=s(x)\Rightarrow s(x)=ae^x+be^{ix}+ce^{j^2x}$$

$$s(0)=1,s'(0)=s''(0)=0$$

$$\begin{cases}a+b+c=1\\a+bj+cj^2=0\\a+bj^2+cj=0\end{cases}\Rightarrow a+b+c=\frac{1}{3}\Rightarrow$$

$$s(x)=\frac{1}{3}\left(e^x+e^{jx}+e^{j^2x}\right)=\frac{e^x}{3}+\frac{2}{3}e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$\sum_{n=0}^{\infty}\frac{s(3n)}{(3n)!}=\frac{1}{2\sqrt{2}}s\left(1+\sqrt{2}\right)-\frac{1}{2\sqrt{2}}s\left(1-\sqrt{2}\right)=$$

$$=\frac{1}{6\sqrt{2}}e^{1+\sqrt{2}}+\frac{1}{3\sqrt{2}}\frac{1}{\sqrt{e^{1+\sqrt{2}}}}\cos\left(\frac{\sqrt{3}}{2}\left(1+\sqrt{2}\right)\right)-$$

$$-\frac{1}{6\sqrt{2}}e^{1-\sqrt{2}}+\frac{1}{3\sqrt{2}}\frac{1}{\sqrt{e^{1-\sqrt{2}}}}\cos\left(\frac{\sqrt{3}}{2}\left(1-\sqrt{2}\right)\right)=$$

$$=\frac{1}{3\sqrt{2}}\left(\frac{1}{\sqrt{e^{\sqrt{2}+1}}}\cos\left(\frac{\sqrt{6}+\sqrt{3}}{2}\right)-\sqrt{e^{\sqrt{2}-1}}\cos\left(\frac{\sqrt{6}-\sqrt{3}}{2}\right)+e\sinh\left(\sqrt{2}\right)\right)$$

1666. If we defined the sequence $\forall k,n\in\mathbb{N}$ with recursive relation $M_k(n+1)=k\cdot M_k(n)+M_k(n-1)$ and $M_k(0)=0,M_k(1)=1.$ Prove the following summations:

$$\sum_{n=0}^{\infty}\frac{M_k(n)}{n!}=\frac{2\sqrt{e^k}}{\sqrt{k^2+4}}\sinh\left(\frac{\sqrt{k^2+4}}{2}\right)$$

$$\sum_{n=0}^{\infty}\frac{M_k(3n)}{(3n)!}=\frac{2}{3\sqrt{k^2+4}}\left(\frac{1}{\sqrt[4]{e^{\sqrt{k^2+4}+k}}}\cos\left(\frac{\sqrt{3k^2+12}+\sqrt{3}k}{4}\right)\right.$$
$$\left.-\sqrt[4]{e^{\sqrt{k^2+4}-k}}\cos\left(\frac{\sqrt{3k^2+12}-\sqrt{3}k}{4}\right)+\frac{2\sqrt{e^k}}{3\sqrt{k^2+4}}\sinh\left(\frac{\sqrt{k^2+4}}{2}\right)\right)$$

Proposed by Amrit Awasthi-India

Solution by Kamel Gandouli Rezgui-Tunisia

$M_k(n+1)=k\cdot M_k(n)+M_k(n-1)$ and

$M_k(0)=0, M_k(1)=1\Rightarrow M_k(n)=\alpha x_1^n+\beta x_2^n,$

x_1, x_2 are solutions of the equation $x^2=kx+1$

$$x=\frac{k\pm\sqrt{k^2+4}}{2}, M_k(0)=0\Rightarrow\alpha=-\beta$$

$$M_k(1)=1\Rightarrow\alpha=-\beta=-\frac{1}{\sqrt{k^2+4}}\Rightarrow M_n(k)=\frac{1}{\sqrt{k^2+4}}x_1^n-\frac{1}{\sqrt{k^2+4}}x_2^n$$

$$\sum_{n=0}^{\infty}\frac{M_k(n)}{n!}=\frac{1}{\sqrt{k^2+4}}e^{x_1}-\frac{1}{\sqrt{k^2+4}}e^{x_2}=\frac{1}{\sqrt{k^2+4}}\left(e^{\frac{k+\sqrt{k^2+4}}{2}}-e^{\frac{k-\sqrt{k^2+4}}{2}}\right)=$$

$$=\frac{2e^{\frac{k}{2}}}{\sqrt{k^2+4}}\frac{1}{2}\left(e^{\frac{\sqrt{k^2+4}}{2}}-e^{-\frac{\sqrt{k^2+4}}{2}}\right)=\frac{2\sqrt{e^k}}{\sqrt{k^2+4}}\sinh\left(\frac{\sqrt{k^2+4}}{2}\right)$$

$$f(x)=\left(\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n)!}\right)^{(3)}=\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n)!}$$

$$t^3-1=0\Rightarrow t\in\{1,j,j^2\}, f'(0)=1, f''(0)=f^{(3)}(0)=0$$

$$\Rightarrow\begin{cases}a+b+c=1\\a+bj+cj^2=0\\a+bj^2+cj=0\end{cases}\Rightarrow a=b=\frac{1}{3}\Rightarrow f(x)=\frac{1}{3}\left(e^x+e^{ix}+e^{j^2x}\right)=$$

$$=\frac{e^x}{3}+\frac{2}{3}e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}x\right)=\frac{e^x}{3}+\frac{2}{3}\sqrt{e^{-x}}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$\sum_{n=0}^{\infty}\frac{M_k(3n)}{(3n)!}=\frac{1}{\sqrt{k^2+4}}\left(f\left(\frac{k+\sqrt{k^2+4}}{2}\right)-f\left(\frac{k-\sqrt{k^2+4}}{2}\right)\right)=$$

$$=\frac{1}{3\sqrt{k^2+4}}\left(e^{\frac{k+\sqrt{k^2+4}}{2}}+2\sqrt{e^{-\frac{k+\sqrt{k^2+4}}{2}}}\cos\frac{\sqrt{3}\left(k+\sqrt{k^2+4}\right)}{4}\right)-$$

$$-\frac{1}{3\sqrt{k^2+4}}\left(e^{\frac{k-\sqrt{k^2+4}}{2}}+2\sqrt{e^{-\frac{\mp\sqrt{k^2+4}}{2}}}\cos\frac{\sqrt{3}\left(k-\sqrt{k^2+4}\right)}{4}\right)=$$

$$=\frac{1}{3\sqrt{k^2+4}}\left(e^{\frac{k+\sqrt{k^2+4}}{2}}+\frac{2}{\sqrt[4]{e^{\sqrt{k^2+4}+k}}}\cos\frac{\sqrt{3}k+\sqrt{3k^2+12}}{4}\right)-$$

$$-\frac{1}{3\sqrt{k^2+4}}\left(e^{\frac{k-\sqrt{k^2+4}}{2}}+\frac{2}{\sqrt[4]{e^{\sqrt{k^2+4}-k}}}\cos\frac{\sqrt{3}k-\sqrt{3k^2+12}}{4}\right)=$$

$$=\frac{2}{3\sqrt{k^2+4}}\left(\frac{1}{\sqrt[4]{e^{\sqrt{k^2+4}+k}}}\cos\left(\frac{\sqrt{3k^2+12}+\sqrt{3}k}{4}\right)\right.$$

$$\left.-\sqrt[4]{e^{\sqrt{k^2+4}-k}}\cos\left(\frac{\sqrt{3k^2+12}-\sqrt{3}k}{4}\right)+\frac{2\sqrt{e^k}}{3\sqrt{k^2+4}}\sinh\left(\frac{\sqrt{k^2+4}}{2}\right)\right)$$

1667. Prove that:

$$\gamma=\lim_{n\to\infty}\frac{(-1)^{n+1}n^{n+1}}{n!}\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)\Big|_{x=n}$$

where γ is Euler-Mascheroni constant.

Proposed by Amrit Awasthi-India

Solution 1 by Serlea Kabay-Liberia

$$y=\frac{1}{x}\log x,u(x)=\frac{1}{x}\text{ and }v(x)=\log x$$

$$u^{(n)}(x)=\frac{(-1)^n n!}{x^{n+1}}\text{ and }v^{(n)}(x)=\frac{(-1)^n n!}{nx^n}$$

$$y^{(n)}=\frac{(-1)^n n!}{x^{n+1}}\log x-\frac{(-1)^n n!}{x^{n+1}}-\frac{(-1)^n n!}{2x^{n+1}}-\frac{(-1)^n n!}{3x^{n+1}}+\cdots+\frac{(-1)^n n!}{nx^{n+1}}=$$

$$=\frac{(-1)^n n!}{x^{n+1}}\left[\log x-1-\frac{1}{2}-\frac{1}{3}-\cdots-\frac{1}{n}\right]=\frac{(-1)^n n!}{x^{n+1}}(\log x-H_n)$$

Therefore,

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)\Big|_{x=n} = \frac{(-1)^n n!}{n^{n+1}}(\log n - H_n)$$

$$\lim_{n\to\infty}\frac{(-1)^{n+1}n^{n+1}}{n!}\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)\Big|_{x=n} = -\lim_{n\to\infty}\frac{(-1)^{n+1}n^{n+1}}{n!}\cdot\frac{(-1)^n n!}{n^{n+1}}(\log n - H_n) =$$

$$= -\lim_{n\to\infty}(\log n - H_n) = \gamma$$

Solution 2 by proposer

$$y = \frac{1}{x}\log x, u(x) = \frac{1}{x} \text{ and } v(x) = \log x$$

$$u^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \text{ and } v^{(n)}(x) = \frac{(-1)^n n!}{n x^n}$$

$$y = u(x)\cdot v(x) = \frac{\log x}{x}$$

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)\Big|_{x=n} = \frac{(-1)^n n!}{x^{n+1}}\log x - \frac{(-1)^n n!}{x^{n+1}} - \frac{(-1)^n n!}{2x^{n+1}} - \frac{(-1)^n n!}{3x^{n+1}} + \cdots + \frac{(-1)^n n!}{n x^{n+1}}$$

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)\Big|_{x=n} = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right] = \frac{(-1)^n n!}{x^{n+1}}(\log x - H_n)$$

Therefore,

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)\Big|_{x=n} = \frac{(-1)^n n!}{n^{n+1}}(\log n - H_n)$$

$$\lim_{n\to\infty}\frac{(-1)^{n+1}n^{n+1}}{n!}\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)\Big|_{x=n} = -\lim_{n\to\infty}\frac{(-1)^{n+1}n^{n+1}}{n!}\cdot\frac{(-1)^n n!}{n^{n+1}}(\log n - H_n) =$$

$$= -\lim_{n\to\infty}(\log n - H_n) = \gamma$$

1668. **Let the sequence $a(n-2)+a(n-1)+a(n+1)+a(n) = \varphi^{-n}$**

$$a(0) = \varphi^{-1}, a(1) = \varphi, a(2) = \varphi^2$$

Evaluate the sum:

$$\Omega = \sum_{m=0}^{\infty}\frac{a(m)}{\varphi^m}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Kamel Gandouli Rezgui-Tunisia

$$a(n-2)+a(n-1)+a(n+1)+a(n)=\varphi^{-n}$$

$$a(0)=\frac{1}{\varphi},a(1)=\varphi,a(2)=\varphi^2\Rightarrow a(m)+a(m+1)+a(m+2)+a(m+3)=\frac{1}{\varphi^{m+2}}$$

$$\Omega=\sum_{m=0}^{\infty}\frac{a(m)}{\varphi^m}+\sum_{m=0}^{\infty}\frac{a(m+1)}{\varphi^m}+\sum_{m=0}^{\infty}\frac{a(m+2)}{\varphi^m}+\sum_{m=0}^{\infty}\frac{a(m+3)}{\varphi^m}=\sum_{m=0}^{\infty}\frac{1}{(\varphi^2)^{m+1}}$$

$$\sum_{m=0}^{\infty}\frac{a(m+1)}{\varphi^m}=\varphi\sum_{m=0}^{\infty}\frac{a(m+1)}{\varphi^{m+1}}=\varphi\sum_{m=1}^{\infty}\frac{a(m)}{\varphi^m}=\varphi\sum_{m=0}^{\infty}\frac{a(m)}{\varphi^m}-1$$

$$\sum_{m=0}^{\infty}\frac{a(m+2)}{\varphi^m}=\varphi^2\sum_{m=0}^{\infty}\frac{a(m+2)}{\varphi^{m+2}}=\varphi^2\sum_{m=2}^{\infty}\frac{a(m)}{\varphi^m}=\varphi^2\sum_{m=0}^{\infty}\frac{a(m)}{\varphi^m}-\varphi-\varphi^2$$

$$\sum_{m=0}^{\infty}\frac{a(m+3)}{\varphi^m}=\varphi^3\sum_{m=3}^{\infty}\frac{a(m)}{\varphi^m}=\varphi^3\sum_{m=0}^{\infty}\frac{a(m)}{\varphi^m}-\varphi^2-\varphi^3-\varphi^3$$

$$\text{Let } x=\sum_{m=0}^{\infty}\frac{a(m)}{\varphi^m},\varphi^2=1+\varphi,\varphi^3=2\varphi+1,\frac{1}{\varphi}=\varphi-1$$

$$x+\varphi x+\varphi^2x+\varphi^3x-1-\varphi-2\varphi^2-2\varphi^3=\sum_{m=0}^{\infty}\frac{1}{(\varphi^2)^{m+1}}$$

$$(3+4\varphi)x-5-7\varphi=\frac{1}{\varphi^2}\sum_{m=0}^{\infty}\frac{1}{\varphi^{2m}}=\frac{1}{\varphi^2}\frac{\varphi^2}{\varphi^2-1}=\frac{1}{\varphi}=\varphi-1$$

$$\Rightarrow(3+4\varphi)x=8\varphi+4\Rightarrow x=\frac{8+4\sqrt{5}}{5+2\sqrt{5}}=\frac{4\sqrt{5}}{5}$$

Therefore,

$$\Omega=\sum_{m=0}^{\infty}\frac{a(m)}{\varphi^m}=\frac{4\sqrt{5}}{5}$$

1669. If we define

$$S(x,n)=\frac{(-1)^n\cdot n!}{x^{n+1}}\cdot\sum_{k=0}^{n}\frac{(-1)^k}{k!}\cdot\sin\left(\frac{k\pi}{2}+x\right)\cdot x^k$$

Then prove:

$$\lim_{n\to\infty}\sum_{n=0}^{2m}\lim_{x\to 0}S(x,n)=\frac{\pi}{4}$$

Proposed by Amrit Awasthi-India

Solution by Ravi Prakash-New Delhi-India

$$\text{For } x\neq 0, S(x,n)=\frac{(-1)^n\cdot n!}{x^{n+1}}\cdot\sum_{k=0}^{n}\frac{(-1)^k}{k!}\cdot\sin\left(\frac{k\pi}{2}+x\right)\cdot x^k=$$

$$=\sum_{k=0}^{n}\binom{n}{k}\frac{d^k(\sin x)}{dx^k}\frac{d^{n-k}}{dx^{n-k}}\left(\frac{1}{x}\right)=\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right);(\text{by Leibniz's theorem})$$

$$=\frac{d^n}{dx^n}\left[\frac{1}{x}\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7\dots\dots\right)\right]=\frac{d^n}{dx^n}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots\right)$$

$$\lim_{x\to 0}S(x,n)=\begin{cases}1,\text{ if } n=0\\ 0,\text{ if } n=1\\ -\frac{1}{3},\text{ if } n=2\\ 0,\text{ if } n=3\\ \frac{1}{5},\text{ if } n=4\\ 0,\text{ if } n=5\\ -\frac{1}{7},\text{ if } n=6\end{cases}$$

Thus,

$$\sum_{n=0}^{2m}\lim_{x\to 0}S(x,n)=\sum_{n=0}^{m}\lim_{x\to 0}S(x,2n)=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+\frac{(-1)^m}{2m+1}$$

$$\lim_{n\to\infty}\sum_{n=0}^{2m}\lim_{x\to 0}S(x,n)=\lim_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+\frac{(-1)^m}{2m+1}\right)=\frac{\pi}{4}$$

1670. Find:

$$\Omega=\int\cosh(3x)\cdot\cosh\left(\frac{3x}{2}\right)\cdot\cosh\left(\frac{13x}{2}\right)dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ruxandra Daniela Tonilă-Romania

$$\Omega=\int\cosh(3x)\cdot\cosh\left(\frac{3x}{2}\right)\cdot\cosh\left(\frac{13x}{2}\right)dx=$$

$$=\int\frac{e^{3x}+e^{-3x}}{2}\cdot\frac{e^{\frac{3x}{2}}+e^{-\frac{3x}{2}}}{2}\cdot\frac{e^{\frac{13x}{2}}+e^{-\frac{13x}{2}}}{2}dx=$$

$$=\frac{1}{8}\int\left(e^{3x}+e^{-3x}\right)\left(e^{\frac{3x}{2}}+e^{-\frac{3x}{2}}\right)\left(e^{\frac{13x}{2}}+e^{-\frac{13x}{2}}\right)dx=$$

$$=\frac{1}{8}\int\left(e^{\frac{9x}{2}}+e^{\frac{3x}{2}}+e^{-\frac{3x}{2}}+e^{-\frac{9x}{2}}\right)\left(e^{\frac{13x}{2}}+e^{-\frac{13x}{2}}\right)dx=$$

$$=\frac{1}{8}\int\left(e^{11x}+e^{-2x}+e^{8x}+e^{-5x}+e^{5x}+e^{-8x}+e^{2x}+e^{-11x}\right)dx=$$

$$=\frac{1}{8}\left(\frac{e^{11x}-e^{-11x}}{2}+\frac{e^{2x}-e^{-2x}}{2}+\frac{e^{5x}-e^{-5x}}{5}+\frac{e^{8x}-e^{-8x}}{8}\right)+C=$$

$$=\frac{\sinh(11x)}{44}+\frac{\sinh(2x)}{8}+\frac{\sinh(5x)}{20}+\frac{\sinh(8x)}{32}+C$$

Solution 2 by Rana Ranino-Setif-Algerie

$$(\because)\ \cosh(a)+\cosh(b)=2\cosh\left(\frac{a-b}{2}\right)\cosh\left(\frac{a+b}{2}\right)$$

$$\cosh\left(\frac{3x}{2}\right)\cdot\cosh\left(\frac{13x}{2}\right)=\frac{1}{2}\left(\cosh(8x)+\cosh(5x)\right)$$

$$\cosh(3x)\cosh(8x)=\frac{1}{2}\left(\cosh(11x)+\cosh(5x)\right)$$

$$\cosh(3x)\cosh(5x)=\frac{1}{2}\left(\cosh(8x)+\cosh(2x)\right)$$

Hence,

$$\cosh(3x)\cdot\cosh\left(\frac{3x}{2}\right)\cdot\cosh\left(\frac{13x}{2}\right)$$

$$=\frac{1}{4}\left(\cosh(2x)+\cosh(5x)+\cosh(8x)+\cosh(11x)\right)$$

Therefore,

$$\int\cosh(3x)\cdot\cosh\left(\frac{3x}{2}\right)\cdot\cosh\left(\frac{13x}{2}\right)dx=$$

$$=\frac{\sinh(11x)}{44}+\frac{\sinh(2x)}{8}+\frac{\sinh(5x)}{20}+\frac{\sinh(8x)}{32}+C$$

Solution 3 by Ankush Kumar Parcha-India

$$\Omega=\int\cosh(3x)\cdot\cosh\left(\frac{3x}{2}\right)\cdot\cosh\left(\frac{13x}{2}\right)dx=$$

$$=\int \frac{e^{3x}+e^{-3x}}{2}\cdot\frac{e^{\frac{3x}{2}}+e^{-\frac{3x}{2}}}{2}\cdot\frac{e^{\frac{13x}{2}}+e^{-\frac{13x}{2}}}{2}dx=$$

$$=\frac{1}{8}\int\left(e^{11x}+e^{-2x}+e^{8x}+e^{-5x}+e^{5x}+e^{-8x}+e^{2x}+e^{-11x}\right)dx=$$

$$=\frac{1}{4}\int(\cosh(11x)+\cosh(2x)+\cosh(11x)+\cosh(5x))dx=$$

$$=\frac{\sinh(11x)}{44}+\frac{\sinh(2x)}{8}+\frac{\sinh(5x)}{20}+\frac{\sinh(8x)}{32}+C$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\Omega=\int\cosh(3x)\cdot\cosh\left(\frac{3x}{2}\right)\cdot\cosh\left(\frac{13x}{2}\right)dx\overset{x=2t}{=}$$

$$=2\int\cosh(6t)\cdot\cosh(3t)\cdot\cosh(13t)\,dt=$$

$$=\int\frac{e^{6t}+e^{-6t}}{2}\cdot\frac{e^{3t}+e^{-3t}}{2}\cdot\frac{e^{13t}+e^{-13t}}{2}dt=$$

$$=\frac{1}{2}\int\frac{e^{22t}+e^{-22t}+e^{16t}+e^{-16t}+e^{10t}+e^{-10t}+e^{4t}+e^{-4t}}{2}dt=$$

$$=\frac{1}{2}\int\cosh(22t)\,dt+\frac{1}{2}\int\cosh(16t)\,dt+\frac{1}{2}\int\cosh(10t)\,dt+\frac{1}{2}\int\cosh(4t)\,dt=$$

$$=\frac{1}{44}\int\cosh(11x)\,d(11x)+\frac{1}{32}\int\cosh(8x)\,d(8x)+\frac{1}{20}\int\cosh(5x)\,d(5x)$$

$$+\frac{1}{8}\int\cosh(4t)\,d(4t)=$$

$$=\frac{\sinh(11x)}{44}+\frac{\sinh(2x)}{8}+\frac{\sinh(5x)}{20}+\frac{\sinh(8x)}{32}+C$$

Solution 5 by Nelson Javier Villaherrera Lopez-El Salvador

$$\Omega=\int\cosh(3x)\cdot\cosh\left(\frac{3x}{2}\right)\cdot\cosh\left(\frac{13x}{2}\right)dx=$$

$$=\int \frac{e^{3x}+e^{-3x}}{2}\cdot\frac{e^{\frac{3x}{2}}+e^{-\frac{3x}{2}}}{2}\cdot\frac{e^{\frac{13x}{2}}+e^{-\frac{13x}{2}}}{2}dx=$$

$$=\frac{1}{8}\int\left(e^{11x}+e^{-2x}+e^{8x}+e^{-5x}+e^{5x}+e^{-8x}+e^{2x}+e^{-11x}\right)dx=$$

$$=\frac{1}{4}\int(\cosh(11x)+\cosh(2x)+\cosh(11x)+\cosh(5x))dx=$$

$$=\frac{\sinh(11x)}{44}+\frac{\sinh(2x)}{8}+\frac{\sinh(5x)}{20}+\frac{\sinh(8x)}{32}+C$$

1671. Find a closed form:

$$\Omega=\int_0^{\frac{\pi}{4}}\sqrt{\tan x\,(1-\tan x)}\,dx$$

Proposed by Abdul Mukhtar-Nigeria

Solution 1 by Ose Favour-Nigeria

$$\Omega=\int_0^{\frac{\pi}{4}}\sqrt{\tan x\,(1-\tan x)}\,dx\overset{u^2=\tan x}{=}2\int_0^1\frac{u^2\sqrt{1-u^2}}{1+u^4}du\overset{u=\sin y}{=}$$

$$=2\int_0^{\frac{\pi}{2}}\frac{\sin^2 y\cos^2 y}{1+\sin^4 y}dy=2\int_0^{\frac{\pi}{2}}\frac{\csc^2 y\cot^2 y}{\csc^6 y+\csc^2 y}dy=$$

$$=2\int_0^{\frac{\pi}{2}}\frac{\csc^2 y\cot^2 y}{(1+\cot^2 y)((1+\cot^2 y)^2+1)}dy\overset{u=\cot y}{=}2\int_0^{\infty}\frac{u^2du}{((1+u^2)^2+1)(1+u^2)}=$$

$$\overset{P.F.}{=}2\left(\int_0^{\infty}\frac{u^2+2}{u^4+2u^2+2}du-\int_0^{\infty}\frac{du}{1+u^2}\right)=2\left(\Phi-\frac{\pi}{2}\right)$$

$$\Phi=\int_0^{\infty}\frac{u^2+2}{u^4+2u^2+2}du=\frac{1}{2}\int_{-\infty}^{\infty}\frac{u^2+2}{u^4+2u^2+2}du$$

$$\varphi(z)=\frac{z^2+2}{z^4+2z^2+2}\ \text{poles of }\varphi$$

$$z^4+2z^2+2=0\Rightarrow t^2+2t+2=0;(t=x^2)$$

$$t_1=-1+i=\sqrt{2}\left(-\tfrac{1}{\sqrt{2}}+\tfrac{i}{\sqrt{2}}\right)\text{ and }t_2=-1-i=\sqrt{2}e^{-\frac{3i\pi}{4}}$$

$$\varphi(z)=\frac{z^2+2}{\left(z^2-\sqrt{2}e^{\frac{3i\pi}{4}}\right)\left(z^2-\sqrt{2}e^{-\frac{3i\pi}{4}}\right)}\overset{\alpha=\sqrt[4]{2}}{=}$$

$$=\frac{z^2+2}{\left(z-\alpha e^{\frac{3i\pi}{8}}\right)\left(z+\alpha e^{\frac{3i\pi}{8}}\right)\left(z-\alpha e^{-\frac{3i\pi}{8}}\right)\left(z-\alpha e^{-\frac{3i\pi}{8}}\right)}$$

$$\Rightarrow \int_R \varphi(z)\,dz = 2i\pi\left\{Res\left(\varphi, \alpha e^{\frac{3i\pi}{8}}\right) + Res\left(\varphi, -\alpha e^{-\frac{3i\pi}{8}}\right)\right\}$$

$$Res\left(\varphi, \alpha e^{\frac{3i\pi}{8}}\right) = \frac{\alpha^2 e^{\frac{3i\pi}{4}}}{2\alpha e^{\frac{3i\pi}{8}}\sqrt{2}\left(2i\sin\frac{3\pi}{4}\right)} = \frac{\alpha^2 e^{\frac{3i\pi}{4}} + 2}{4i\alpha} e^{-\frac{3i\pi}{8}}$$

$$Res\left(\varphi, -\alpha e^{-\frac{3i\pi}{8}}\right) = \frac{\alpha^2 e^{-\frac{3i\pi}{4}}}{-2\alpha e^{\frac{3i\pi}{8}}(-\sqrt{2})\left(2i\sin\frac{3\pi}{4}\right)} = \frac{\alpha^2 e^{-\frac{3i\pi}{4}} + 2}{4i\alpha} e^{\frac{3i\pi}{8}}$$

$$\Rightarrow \int_R \varphi(z)\,dz = 2i\pi\left\{\frac{\alpha^2 e^{\frac{3i\pi}{4}} + 2}{4i\alpha} e^{-\frac{3i\pi}{8}} + \frac{\alpha^2 e^{-\frac{3i\pi}{4}} + 2}{4i\alpha} e^{\frac{3i\pi}{8}}\right\} =$$

$$= \frac{\pi}{2\alpha}\left\{\alpha^2 e^{\frac{3i\pi}{8}} + 2e^{-\frac{3i\pi}{8}} + \alpha^2 e^{-\frac{3i\pi}{8}} + 2e^{\frac{3i\pi}{8}}\right\} = \frac{\pi}{\sqrt[4]{2}}(2+\sqrt{2})\cos\frac{3\pi}{8}$$

$$\Phi = \frac{1}{2}\int_R \varphi(z)\,dz = \frac{\pi}{2\sqrt[4]{2}}(2+\sqrt{2})\cos\frac{3\pi}{8}$$

$$\Omega = 2\left(\frac{\pi}{2\sqrt[4]{2}}(2+\sqrt{2})\cos\frac{3\pi}{8} - \frac{\pi}{2}\right) = \pi\left(\frac{2+\sqrt{2}}{\sqrt[4]{2}}\cos\frac{3\pi}{8} - 1\right)$$

Solution 2 by Kartick Chandra Betal-India

$$\Omega = \int_0^{\frac{\pi}{4}} \sqrt{\tan x\,(1-\tan x)}\,dx = \int_0^1 \frac{\sqrt{x(1-x)}}{1+x^2}dx = \int_0^\infty \frac{\sqrt{x-1}}{x(1+x^2)}dx =$$

$$= \int_0^\infty \frac{\sqrt{x}dx}{(1+x)(x^2+2x+2)} = 2\int_0^\infty \frac{x^2dx}{(1+x^2)(x^4+2x^2+2)} =$$

$$= 2\int_0^\infty \left\{\frac{1}{1+x^2} - \frac{1+x^2}{x^4+2x^2+2}\right\}dx = 2\cdot\frac{\pi}{2} - \int_0^\infty \frac{A\left(1+\frac{\sqrt{2}}{x^2}\right) + B\left(1-\frac{\sqrt{2}}{x^2}\right)}{x^2+\frac{2}{x^2}+2}dx =$$

$$= \pi - A\int_0^\infty \frac{d\left(x-\frac{\sqrt{2}}{x}\right)}{\left(x-\frac{\sqrt{2}}{x}\right)^2 + \left(\sqrt{2+2\sqrt{2}}\right)^2} - B\int_0^\infty \frac{d\left(x+\frac{\sqrt{2}}{x}\right)}{\left(x+\frac{\sqrt{2}}{x}\right)^2 - \left(\sqrt{2-2\sqrt{2}}\right)^2} =$$

$$= \pi - \frac{A}{\sqrt{2+2\sqrt{2}}}\cdot\pi - \frac{B}{2\sqrt{2\sqrt{2}-2}}\left[\log\left(\frac{x^2 - x\sqrt{2\sqrt{2}-2}+\sqrt{2}}{x^2 - x\sqrt{2\sqrt{2}-2}+\sqrt{2}}\right)\right]_0^\infty =$$

$$=\pi\left(1-\frac{A}{\sqrt{2+2\sqrt{2}}}\right)=\pi\left(1-\frac{2+\sqrt{2}}{2\sqrt{2+2\sqrt{2}}}\right)=\pi\left(1-\frac{\sqrt{1+\sqrt{2}}}{2}\right)$$

$$A+B=2, A-B=\sqrt{2}; A=\frac{2+\sqrt{2}}{2}$$

1672. Find:

$$\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx$$

Proposed by Hussain Reza Zadah-Afghanistan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx=\frac{1}{8}\int_0^1\frac{\log^2 x}{x^4+x^2+1}dx=\frac{1}{8}\int_0^1\frac{(1-x^2)\log^2 x}{1-x^6}dx\overset{t=x^6}{=}$$

$$=\frac{1}{1728}\int_0^1\frac{\left(x^{\frac{1}{6}-1}-x^{\frac{1}{2}-1}\right)\log^2 t}{1-t}dt=\frac{1}{1728}\sum_{n=0}^{\infty}\left\{\psi^{(2)}\left(\frac{1}{2}\right)-\psi^{(2)}\left(\frac{1}{6}\right)\right\}=$$

$$=\frac{1}{1728}\left\{168\zeta(3)+4\sqrt{3}\pi^3\right\}$$

Therefore,

$$\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx=\frac{7\zeta(3)}{72}+\frac{\pi^3}{144\sqrt{3}}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx=\int_0^1\frac{(x-x^5)\log^2 x}{1-x^{12}}dx=\Omega_1-\Omega_2$$

$$\Omega_1=\int_0^1\frac{x\log^2 x}{1-x^{12}}dx\overset{x=\sqrt[12]{y}}{=}\int_0^1\frac{\sqrt[12]{y}}{1-y}\log^2\sqrt[12]{y}\,\frac{1}{12\sqrt[12]{y^{11}}}dy=$$

$$=\frac{1}{1728}\int_0^1\frac{y^{-\frac{5}{6}}}{1-y}\log^2 y\,dy=\frac{1}{1728}\sum_{k=0}^{\infty}\int_0^1 y^{k-\frac{5}{6}}\log^2 y\,dy$$

$$\because\int_0^1 y^{k-\frac{5}{6}}\log^2 y\,dy=\frac{2}{\left(k+\frac{1}{6}\right)^3}$$

$$\Omega_1=\frac{1}{864}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{6}\right)^3};\left(\because \psi_n(z)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}}\right.$$

$$\psi_2\left(\frac{1}{6}\right)=(-1)^{2+1}2!\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{6}\right)^3}\Rightarrow\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{6}\right)^3}=-\frac{\psi_2\left(\frac{1}{6}\right)}{2}$$

$$\Omega_1=-\frac{\psi_2\left(\frac{1}{6}\right)}{1728}$$

$$\Omega_2=\int_0^1\frac{x^5\log^2 x}{1-x^{12}}dx=\int_0^1\frac{\sqrt[12]{y^5}\log^2\sqrt[12]{y}}{1-y}\cdot\frac{1}{12\sqrt[12]{y^{11}}}dy=\frac{1}{1728}\int_0^1\frac{\log^2 y}{(1-y)\sqrt{y}}dy=$$

$$=\frac{1}{1728}\sum_{k=0}^{\infty}\int_0^1 y^{k-\frac{1}{2}}\log^2 y\,dy$$

$$\because\int_0^1 y^{k-\frac{1}{2}}\log^2 y\,dy=\frac{2}{\left(k+\frac{1}{2}\right)^3}$$

$$\Omega_2=\frac{1}{864}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{2}\right)^3}=-\frac{\psi_2\left(\frac{1}{2}\right)}{1728};\left(\because\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{2}\right)^3}=-\frac{\psi_2\left(\frac{1}{2}\right)}{2}\right.$$

$$\Omega_2=-\frac{\psi_2\left(\frac{1}{2}\right)}{1728}$$

$$\Omega=\frac{\psi_2\left(\frac{1}{2}\right)-\psi_2\left(\frac{1}{6}\right)}{1728}=\frac{168\zeta(3)+4\sqrt{3}\pi^2}{1728}$$

Solution 3 by Ose Favour-Nigeria

$$\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx\overset{u=x^2}{=}\frac{1}{8}\int_0^1\frac{\log^2 u}{u^4+u^2+1}du$$

$$\Omega(a)=\int_0^1\frac{u^a}{u^4+u^2+1}du=\int_0^1\frac{u^a(1-u^2)}{1-u^6}du$$

$$\Omega(a)=\Phi(a)-\Psi(a)$$

$$\Phi(a)=\int_0^1\frac{u^a}{1-u^6}du=\int_0^1\frac{u^a}{1-u^6}du=\sum_{n=0}^{\infty}\int_0^1 u^{a+6n}\,du=\sum_{n=0}^{\infty}\frac{1}{a+6n+1}$$

$$\Psi(a)=\int_0^1\frac{u^{a+2}}{1-u^6}du=\sum_{n=0}^{\infty}\int_0^1 u^{a+2+6n}du=\sum_{n=0}^{\infty}\frac{1}{a+6n+3}$$

$$\Omega(a)=-\sum_{n=0}^{\infty}\frac{1}{a+6n+3}+\sum_{n=0}^{\infty}\frac{1}{a+6n+1}$$

$$\because\sum_{n=0}^{\infty}\frac{1}{n+a}=-\psi^{(0)}(a)$$

$$\Omega(a)=-\frac{1}{6}\sum_{n=0}^{\infty}\frac{1}{n+\frac{a+3}{6}}+\frac{1}{6}\sum_{n=0}^{\infty}\frac{1}{n+\frac{a+1}{6}}=\frac{1}{6}\psi^{(0)}\left(\frac{a+3}{6}\right)-\frac{1}{6}\psi^{(0)}\left(\frac{a+1}{6}\right)$$

$$\Omega'(a)=\frac{1}{36}\left(\psi^{(1)}\left(\frac{a+3}{6}\right)-\psi^{(1)}\left(\frac{a+1}{6}\right)\right)$$

$$\Omega''(a)=\frac{1}{216}\left(\psi^{(2)}\left(\frac{a+3}{6}\right)-\psi^{(2)}\left(\frac{a+1}{6}\right)\right)$$

$$\Omega=\frac{1}{8}\Omega''(0)=\frac{\psi_2\left(\frac{1}{2}\right)-\psi_2\left(\frac{1}{6}\right)}{1728}=\frac{168\zeta(3)+4\sqrt{3}\pi^2}{1728}$$

Solution 4 by Yen Tung Chung-Taichung-Taiwan

$$\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx\overset{y=x^2}{=}\int_0^1\frac{\log^2(\sqrt{y})}{y^4+y^2+1}\cdot\frac{1}{2}dy=\frac{1}{8}\int_0^1\frac{\log^2 y}{y^4+y^2+1}dy\overset{y=e^{-z}}{=}$$

$$=\frac{1}{8}\int_0^{\infty}\frac{z^2e^{-z}}{1+e^{-2z}+e^{-4z}}dz=\frac{1}{8}\int_0^{\infty}\frac{z^2(e^{-z}-e^{-3z})}{1-e^{-6z}}dz=$$

$$=\frac{1}{8}\int_0^{\infty}z^2(e^{-z}+e^{-7z}+e^{-13z}+\cdots)\,dz-\frac{1}{8}\int_0^{\infty}z^2\left(e^{-3z}+e^{-9z}+e^{-15z}+\cdots\right)dz=$$

$$=\frac{1}{8}\left(\frac{\Gamma(3)}{1^3}+\frac{\Gamma(3)}{7^3}+\frac{\Gamma(3)}{13^3}+\cdots\right)-\frac{1}{8}\left(\frac{\Gamma(3)}{3^3}+\frac{\Gamma(3)}{9^3}+\frac{\Gamma(3)}{15^3}+\cdots\right)=$$

$$=\frac{1}{4}\left(\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{7^3}-\frac{1}{9^3}+\frac{1}{13^3}-\frac{1}{15^3}+\cdots\right)\cong 0.24117$$

Solution 5 by Abdul Mukhtar-Nigeria

$$x^8+x^4+1=\frac{1-x^{12}}{1-x^4}\Rightarrow\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx=\int_0^1\frac{x\log^2 x}{1-x^{12}}dx-\int_0^1\frac{x^5\log^2 x}{1-x^{12}}dx=$$

$$=H-J$$

$$\int_0^1\frac{x^{p-1}\log^2 x}{1-x^{12}}dx\overset{x=e^{-t}}{=}\int_0^{\infty}\frac{e^{-tp}t^2}{1-e^{12t}}dt=\int_0^{\infty}t^2e^{-tp}\sum_{k\geq 0}e^{-12kt}\,dt=$$

$$=\sum_{k\geq 0}\left[\int_0^\infty t^2e^{-t(p+12k)}\,dt\right]=\sum_{k\geq 0}\left[\frac{1}{(p+12k)^3}\int_0^\infty t^2e^{-t}\,dt\right]=\Gamma(3)=2$$

$$\Rightarrow 2\sum_{k\geq 0}\frac{1}{(p+12k)^3}=\frac{1}{864}\sum_{k\geq 0}\frac{1}{\left(k+\frac{p}{12}\right)^3}$$

$$\zeta(s,p)=\sum_{n>0}\frac{1}{(n+p)^s}$$

$$\Omega=\frac{1}{864}\left[\zeta\left(3,\frac{1}{6}\right)-\zeta\left(3,\frac{1}{2}\right)\right]=\frac{1}{432}(42\zeta(3)-53\pi^3)$$

1673. If we have $\int_{-\pi}^{\pi}\frac{\cos^2 2x+1}{a\cos^2 2x+1}dx=\pi a$, then find the value of

$$\Phi=a^5+a^4+8a^3+8a^2-32a.$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_{-\pi}^{\pi}\frac{\cos^2 2x+1}{a\cos^2 2x+1}dx=4\int_0^{\frac{\pi}{2}}\frac{\cos^2 2x+1}{a\cos^2 2x+1}dx\overset{t=\tan x;a+1=b^2}{=}$$

$$=8\int_0^\infty\frac{1+t^4}{(b^2+t^2)(1+t^2)^2}dt=$$

$$=\frac{8(b^4+1)}{(b^2-1)^2}\int_0^\infty\frac{dt}{b^2+t^2}+\frac{16}{b^2-1}\int_0^\infty\frac{dt}{(1+t^2)^2}-\frac{16b^2}{(b^2-1)^2}\int_0^\infty\frac{dt}{1+t^2}=$$

$$=\frac{4\pi(b^4+1)}{b(b^2-1)^2}+\frac{4\pi}{b^2-1}-\frac{8\pi^2}{(b^2-1)^2}=\frac{4\pi(b^2+b+1)}{b(b+1)^2}$$

$$\frac{4\pi(b^2+b+1)}{b(b+1)^2}=\pi(b^2-1)\Rightarrow 4b^2+4b+4=b^5+2b^4-6b^2-5b-4=0$$

$$b=\frac{2b^4-6b^2-4}{5-b^4};\sqrt{a+1}=\frac{2a^2-2a-8}{4-a^2-2a}$$

$$a+1=\frac{4a^4-8a^3-28a^2+32a+64}{a^4+4a^3-4a^2-16a+16}$$

$$\Phi=a^5+a^4+8a^3+8a^2-32a=48$$

1674. If $0<a\leq b<\frac{\pi}{2}$ then:

$$3\int_a^b\sin x\cdot\sinh x\,dx\leq b^3-a^3$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$b^3 - a^3 = x^3\Big|_a^b = \int_a^b 3x^2\, dx$$

We must to prove:

$$3\int_a^b \sin x \cdot \sinh x\, dx \le 3\int_a^b x^2\, dx$$

$$\sin x \cdot \sinh x \le x^2, x \in [a, b]$$

Let $f(x) = \sin x \cdot \sinh x - x^2; f(0) = 0$**, then**

$$f'(x) = \cos x \cdot \sinh x + \sin x \cdot \cosh x - 2x; f'(0) = 0$$

$$f''(x) = 2\cos x \cdot \cosh x - 2; f''(0) = 0$$

$$f'''(x) = 2(-\sin x \cdot \cosh x + \cos x \cdot \sinh x); f'''(x) = 0$$

$$f^{(iv)}(x) = -2\sin x \cdot \sinh x \le 0; f^{(iv)}(0) = 0$$

x	0		$\frac{\pi}{2}$
$f^{(iv)}(x)$	0	$-\ -\ -\ -\ -$	
$f'''(x)$	0	$-\ -\ -\ -\ -$ $\searrow$	
$f''(x)$	0	$-\ -\ -\ -\ -$ $\searrow$	
$f'(x)$	0	$-\ -\ -\ -\ -$ $\searrow$	
$f(x)$	0	$-\ -\ -\ -\ -$ $\searrow$	

Thus, $f(x) \le 0, \forall x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin x \cdot \sinh x \le x^2, x \in [a, b]$

$$3\int_a^b \sin x \cdot \sinh x\, dx \le 3\int_a^b x^2\, dx$$

1675. $f \in C^2([a, b]), 0 < a \le b, \int_a^b f(x)\, dx = 0$

Prove that:

$$\left(f(b)+f(a)\right)^2 \le \frac{b-a}{3}\int_a^b \left(f'(x)\right)^2 dx$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Chris Kyriazis-Greece

$$\int_a^b \left(x-\frac{a+b}{2}\right)^2 dx = \left[\frac{1}{3}\left(x-\frac{a+b}{2}\right)^3\right]_a^b = \frac{(b-a)^3}{24}-\frac{(a-b)^3}{24}=\frac{(b-a)^3}{12};(1)$$

$$\int_a^b \left(f'(x)\right)^2 dx \cdot \int_a^b \left(x-\frac{a+b}{2}\right)^2 dx \overset{CBS}{\ge} \left[\int_a^b f'(x)\left(x-\frac{a+b}{2}\right)dx\right]^2 \overset{(1)}{\Rightarrow}$$

$$\frac{(b-a)^3}{12}\int_a^b \left(f'(x)\right)^2 dx \ge \left[\left(x-\frac{a+b}{2}\right)f(x)\Big|_a^b - \underbrace{\int_a^b f(x)\,dx}_{0}\right]^2$$

$$\frac{(b-a)^3}{12}\int_a^b \left(f'(x)\right)^2 dx \ge \left[\frac{b-a}{2}f(b)+\frac{b-a}{2}f(a)\right]^2$$

$$\frac{(b-a)^3}{12}\int_a^b \left(f'(x)\right)^2 dx \ge \frac{(b-a)^2}{4}\left(f(a)+f(b)\right)^2$$

$$\left(f(b)+f(a)\right)^2 \le \frac{b-a}{3}\int_a^b \left(f'(x)\right)^2 dx$$

1676. If $f\in C^2([a,b]), 0<a<b, f(a)=f(b)$ then:

$$\left(f'(b)\right)^2 \le \frac{b-a}{3}\int_a^b \left(f''(x)\right)^2 dx$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by proposer

$$\int_a^b (x-a)f''(x)\,dx = \int_a^b (x-a)d\left(f'(x)\right) =$$

$$= (x-a)f'(x)|_a^b - \int_a^b f'(x)d(x-a) =$$

$$= (b-a)f'(b) - \int_a^b f'(x)\,dx = (b-a)f'(b) - f(x)|_a^b =$$

$$= (b-a)f'(b) - f(b) + f(a) = (b-a)f'(b); (1)$$

$$\left(\int_a^b (x-a)f''(x)\,dx\right)^2 \overset{CBS}{\leq} \int_a^b (x-a)^2\,dx \cdot \int_a^b \left(f''(x)\right)^2 dx =$$

$$= \frac{(x-a)^3}{3}\Bigg|_a^b \cdot \int_a^b \left(f''(x)\right)^2 dx = \frac{(b-a)^3}{3} \cdot \int_a^b \left(f''(x)\right)^2 dx; (2)$$

From (1) and (2), we get:

$$(b-a)^2\left(f'(b)\right)^2 \leq \frac{(b-a)^3}{3} \cdot \int_a^b \left(f''(x)\right)^2 dx$$

Therefore,

$$\left(f'(b)\right)^2 \leq \frac{b-a}{3}\int_a^b \left(f''(x)\right)^2 dx$$

1677. $f \in C^2([a,b]), a < b, f(a) = f(b)$. **Prove that:**

$$3 \cdot \max\left(\left(f'(a)\right)^2, \left(f'(b)\right)^2\right) \leq (b-a)\int_a^b \left(f''(x)\right)^2 dx$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Kamel Gandouli Rezgui-Tunisia

$$\int_a^b (x-a)f''(x)\,dx = [(x-a)f'(x)]_a^b - \int_a^b f'(x)\,dx =$$

$$= (b-a)f'(b) - f(b) + f(a) = (b-a)f'(b)$$

$$\left(\int_a^b (x-a)f''(x)dx\right)^2 \overset{CBS}{\leq} \int_a^b (x-a)^2\,dx \cdot \int_a^b \left(f''(x)\right)^2 dx$$

$$\left(\int_a^b (x-a)f''(x)dx\right)^2 \leq \frac{(b-a)^3}{3}\int_a^b \left(f''(x)\right)^2 dx$$

$$(b-a)^2\left(f'(b)\right)^2 \leq \frac{(b-a)^3}{3}\int_a^b \left(f''(x)\right)^2 dx$$

$$\left(f'(b)\right)^2 \leq \frac{b-a}{3}\int_a^b \left(f''(x)\right)^2 dx; (1)$$

$$\int_a^b (b-x)f''(x)\,dx = [(b-x)f'(x)]_a^b + \int_a^b f'(x)\,dx =$$

$$= -(b-a)f'(x) + f(b) - f(a) = -(b-a)f'(a)$$

$$\left(\int_a^b (b-x)f''(x)dx\right)^2 \overset{CBS}{\leq} \int_a^b (b-x)^2\,dx \cdot \int_a^b \left(f''(x)\right)^2 dx$$

$$(b-a)^2\left(f'(a)\right)^2 \leq \int_a^b (b-x)^2 dx \cdot \int_a^b \left(f''(x)\right)^2 dx$$

$$(b-a)\left(f'(a)\right)^2 \leq \frac{(b-a)^3}{3}\int_a^b \left(f''(x)\right)^2 dx$$

$$\left(f'(a)\right)^2 \leq \frac{b-a}{3}\int_a^b \left(f''(x)\right)^2 dx; (2)$$

From (1) and (2), we get:

$$3 \cdot \max\left(\left(f'(a)\right)^2, \left(f'(b)\right)^2\right) \leq (b-a)\int_a^b \left(f''(x)\right)^2 dx$$

1678. Prove that the following equality holds:

$$\sum_{n=1}^{\infty} \frac{H_{2n}}{n2^{4n}}\binom{4n}{2n} = \frac{5\pi^2}{12} - \frac{\log^2 2}{2} - 4Li_2\left(\frac{1}{\sqrt{2}}\right) + 2\log^2(\delta_S) + \frac{1}{2}\log\left(\frac{1}{2}\right)\log\left(17+12\sqrt{2}\right)$$

where H_n is Skew Harmonic sum, $Li_2(x)$ is dilogarithm function and δ_S is Silver ratio.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Rana Ranino-Setif-Algerie

$$\Omega = \sum_{n=1}^{\infty} \frac{H_{2n}}{n2^{4n}}\binom{4n}{2n}$$

$$(\because)\ H_{2n} = \log 2 - \int_0^1 \frac{x^{2n}}{1+x}dx\,;\ \frac{1}{2^{4n}}\binom{4n}{2n} = \frac{B\left(2n+\frac{1}{2},\frac{1}{2}\right)}{\pi} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^{4n} x\,dx$$

$$\Omega = \overbrace{\log 2\sum_{n=1}^{\infty}\frac{\binom{4n}{2n}}{n2^{4n}}}^{A} - \overbrace{\sum_{n=1}^{\infty}\frac{\binom{4n}{2n}}{n2^{4n}}\int_0^1 \frac{x^{2n}}{1+x}dx}^{B}$$

$$A = \frac{2\log 2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} \sin^{4n} x\,dx = -\frac{2\log 2}{\pi}\int_0^{\frac{\pi}{2}} \log(1-\sin^4 x)\,dx =$$

$$= -\frac{2\log 2}{\pi}\int_0^{\frac{\pi}{2}}\log(\cos x)\,dx - \frac{2\log 2}{\pi}\int_0^{\frac{\pi}{2}}\log(1+\sin^2 x)\,dx =$$

$$= 2\log^2 2 - \frac{2\log 2}{\pi}\int_0^{\frac{\pi}{2}}\log(\cos^2 x + 2\sin^2 x)\,dx = 2\log^2 2 - 2\log 2\log\left(\frac{1+\sqrt{2}}{2}\right) =$$

$$= 4\log^2 2 + \frac{1}{2}\log\left(\frac{1}{2}\right)\log\left[\left(1+\sqrt{2}\right)\right]^4 = 4\log^2 2 + \frac{1}{2}\log\left(\frac{1}{2}\right)\log\left(17+12\sqrt{2}\right)$$

$$\sum_{n=1}^{\infty}\binom{4n}{2n}x^{2n} = \frac{1}{2}\left(\frac{1}{\sqrt{1+4x}}+\frac{1}{\sqrt{1-4x}}\right)-1;\ \left(|x|<\frac{1}{4}\right).\ \text{Take } 4x=t,$$

$$\sum_{n=1}^{\infty}\binom{4n}{2n}\frac{t^{2n-1}}{2^{4n}} = \frac{1}{2t}\left(\frac{1}{\sqrt{1+t}}+\frac{1}{\sqrt{1-t}}\right)-\frac{1}{t}$$

Integrating both sides, we have:

$$\sum_{n=1}^{\infty}\binom{4n}{2n}\frac{x^{2n}}{n2^{4n}} = \int_0^x \frac{1}{t}\left(\frac{1}{\sqrt{1+t}}+\frac{1}{\sqrt{1-t}}-2\right)dt =$$

$$= \left[\log\left|\frac{\sqrt{t+1}-1}{\sqrt{t+1}+1}\right| + \log\left|\frac{\sqrt{1-t}-1}{\sqrt{1-t}+1}\right| - 2\log t\right]_0^x$$

$$\sum_{n=1}^{\infty}\binom{4n}{2n}\frac{x^{2n}}{n2^{4n}} = \log\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + \log\left|\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}\right| - 2\log x + 4\log 2$$

$$B = \int_0^1 \frac{1}{x+1}\cdot\log\left(\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right)dx + \int_0^1 \frac{1}{x+1}\cdot\log\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)dx -$$

$$-2\int_0^1 \frac{\log x - 2\log 2}{x+1}\,dx = B_1 + B_2 + B_3$$

$$B_1 = \int_0^1 \frac{1}{x+1}\cdot\log\left(\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right)dx \overset{y^2=\frac{1}{x+1}}{=} 2\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{y}\log\left(\frac{1-y}{1+y}\right)dy =$$

$$= 2\left[-2Li_2(y) + \frac{1}{2}Li_2(y^2)\right]_{\frac{1}{\sqrt{2}}}^{1} = -3Li_2(1) + 4Li_2\left(\frac{1}{\sqrt{2}}\right) - Li_2\left(\frac{1}{2}\right)$$

$$B_2 = \int_0^1 \frac{1}{x+1}\cdot\log\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)dx \overset{IBP}{=}$$

$$=\left[\log\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)\log(1+x)\right]_0^1-\int_0^1\frac{\log(1+x)}{x\sqrt{1-x}}dx=-\int_0^1\frac{\log(1+x)}{x\sqrt{1-x}}dx\overset{x=\sin^2 x}{=}$$

$$=-2\int_0^1\frac{\log(1+\sin^2 x)}{\sin x}dx=2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\frac{\pi}{2}}\sin^{2k-1}x\,dx=$$

$$=\sqrt{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k\Gamma(k)}{k\Gamma\left(k+\frac{1}{2}\right)}=\sum_{k=1}^{\infty}\frac{(-4)^k\Gamma^2(k)}{2k\Gamma(2k)}=-2\sum_{k=0}^{\infty}\frac{(-4)^k(k!)^2}{(k+1)(2k+1)!}=-2(\sinh^{-1}(1))^2$$

$$B_2=-2\log^2(\delta_S)$$

$$B_3=-2\int_0^1\frac{\log x-2\log 2}{x+1}dx=\frac{\pi^2}{6}+4\log^2 2$$

$$B=4Li_2\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\log^2 2-\frac{7\pi^2}{12}-2\log^2(\delta_S)+\frac{\pi^2}{6}+4\log^2 2$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{H_{2n}}{n2^{4n}}\binom{4n}{2n}=\frac{5\pi^2}{12}-\frac{\log^2 2}{2}-4Li_2\left(\frac{1}{\sqrt{2}}\right)+2\log^2(\delta_S)+\frac{1}{2}\log\left(\frac{1}{2}\right)\log(17+12\sqrt{2})$$

1679. If $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is continuous function, $(x_n)_{n\ge1}, x_n=\sum_{k=1}^n\frac{1}{k}$, then prove:

$$\lim_{n\to\infty}\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx=\lim_{n\to\infty}(e^{x_n}-ne^{\gamma})$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Kamel Gandouli Rezgui-Tunisia

$$I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx\overset{x=e^{x_n}+e^{x_{n+1}}-y}{=}$$

$$=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f\left((e^{x_n}+e^{x_{n+1}}-y)-e^{x_n}\right)}{f\left(e^{x_{n+1}}-(e^{x_n}+e^{x_{n+1}}-y)\right)+f\left((e^{x_n}+e^{x_{n+1}}-y)-e^{x_n}\right)}dy=$$

$$=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(y-e^{x_n})}{f(e^{x_{n+1}}-y)+f(y-e^{x_n})}dy$$

$$2I_n=\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(y-e^{x_n})+f(e^{x_{n+1}}-y)}{f(e^{x_{n+1}}-y)+f(y-e^{x_n})}dy=\int_{e^{x_n}}^{e^{x_{n+1}}}dy$$

$$\Rightarrow I_n=\frac{e^{x_{n+1}}-e^{x_n}}{2}=e^{x_n}-\frac{3e^{x_n}-e^{x_{n+1}}}{2}$$

$$x_n\cong\log n+\gamma$$

$$\lim_{n\to\infty}I_n=\lim_{n\to\infty}\left(e^{x_n}-\frac{3e^{x_n}-e^{x_{n+1}}}{2}\right)=\lim_{n\to\infty}\frac{3e^{x_n}-e^{x_{n+1}}}{2}=$$

$$=\lim_{n\to\infty}ne^{\gamma}=\lim_{n\to\infty}(e^{x_n}-ne^{\gamma})$$

Solution 2 by Marian Ursărescu-Romania

$$\text{Let } I=\int_a^b\frac{f(x-a)}{f(b-x)+f(x-a)}dx\overset{x=a+b-t}{=}\int_a^b\frac{f(b-t)}{f(b-t)+f(t-a)}dt;(1)$$

$$\text{Let } J=\int_a^b\frac{f(b-x)}{f(b-x)+f(x-a)}dx\overset{(1)}{=}I$$

$$I+J=\int_a^b dx=b-a\Rightarrow I=J=\frac{b-a}{2}$$

$$\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx=\frac{e^{x_{n+1}}-e^{x_n}}{2}$$

$$\lim_{n\to\infty}\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx=\lim_{n\to\infty}\frac{e^{x_{n+1}}-e^{x_n}}{2}=$$

$$=\frac{1}{2}\lim_{n\to\infty}e^{x_n}\left(e^{x_{n+1}-x_n}-1\right)=\frac{1}{2}\lim_{n\to\infty}\frac{e^{x_n}\left(e^{\frac{1}{n+1}}-1\right)}{\frac{1}{n+1}\cdot(n+1)}=\frac{1}{2}\lim_{n\to\infty}\frac{e^{x_n}}{n+1}=$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{e^{x_n}}{n}=\frac{1}{2}\lim_{n\to\infty}\frac{e^{x_n}}{e^{\log n}}=\frac{1}{2}\lim_{n\to\infty}e^{x_n-\log n}=\frac{1}{2}e^{\gamma};(3)$$

Now, we have:

$$\lim_{n\to\infty}(e^{x_n}-ne^{\gamma})=\lim_{n\to\infty}\left(e^{x_n}-e^{\log n}e^{\gamma}\right)=\lim_{n\to\infty}e^{\log n}\left(e^{x_n-\log n}-e^{\gamma}\right)=$$

$$=\lim_{n\to\infty}n\left(e^{x_n-\log n}-e^{\gamma}\right)=\lim_{n\to\infty}\frac{e^{x_n-\log n}-e^{\gamma}}{\frac{1}{n}}\overset{C-S}{=}$$

$$=\lim_{n\to\infty}\frac{e^{x_{n+1}-\log(n+1)}-e^{x_n-\log n}}{\frac{1}{n+1}-\frac{1}{n}}=\lim_{n\to\infty}\frac{e^{x_n-\log n}\left(e^{x_{n+1}-\log(n+1)-x_n+\log n}-1\right)}{-\frac{1}{n(n+1)}}=$$

$$= \lim_{n\to\infty} \frac{e^{x_n}}{n} \cdot \frac{e^{\frac{1}{n+1} - \log(n+1) + \log n} - 1}{-\frac{1}{n(n+1)}} =$$

$$= -\lim_{n\to\infty} e^{x_n} \cdot \frac{e^{\frac{1}{n+1} - \log(n+1) + \log n} - 1}{\frac{1}{n+1} \cdot \left(\frac{1}{n+1} - \log(n+1) + \log n\right)} \cdot \left(\frac{1}{n+1} - \log(n+1) + \log n\right) =$$

$$= -\lim_{n\to\infty} (n+1)e^{x_n} \cdot \frac{1 - (n+1)\log\left(\frac{n+1}{n}\right)}{n+1} = -\lim_{n\to\infty} \frac{e^{x_n}}{e^{\log n}} \cdot \frac{1 - (n+1)\log\left(1 + \frac{1}{n}\right)}{\frac{1}{n}}$$

$$= -\lim_{n\to\infty} e^{x_n - \log n} \cdot \frac{1 - (n+1)\log\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = -e^{\gamma} \cdot \lim_{x\to 0} \frac{x + (x+1)\log(1+x)}{x^2} \overset{L'H}{=}$$

$$= -e^{\gamma} \lim_{x\to 0} \frac{1 - \log(1+x) - 1}{2x} = -e^{\gamma} \lim_{x\to 0} \frac{-\log(1+x)}{2x} = \frac{e^{\gamma}}{2}; (4)$$

From (3) and (4), we get:

$$\lim_{n\to\infty} \int_{e^{x_n}}^{e^{x_{n+1}}} \frac{f(x - e^{x_n})}{f(e^{x_{n+1}} - x) + f(x - e^{x_n})} dx = \lim_{n\to\infty} (e^{x_n} - ne^{\gamma})$$

1680. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{(2n-1)!!} \left(\sin\left(\frac{\pi \sqrt[n+1]{(n+1)!}}{2\sqrt[n]{n!}} \right) - 1 \right)$$

Proposed by Shivam Sharma, Anisha Garg-India

Solution 1 by Asmat Qatea-Afghanistan

$$(2n-1)!! = \frac{(2n)!}{2^n n!}, \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e} \text{ and } \lim_{n\to\infty} n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\Omega = \frac{1}{2} \lim_{n\to\infty} \frac{\sqrt[n]{(2n)!}}{\sqrt[n]{n!}} \left(\sin\left(\frac{\pi \frac{\sqrt[n+1]{n+1}}{n+1} \cdot (n+1)}{2 \cdot \frac{\sqrt[n]{n}}{n} \cdot n} \right) - 1 \right) =$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{\left(\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}\right)^{\frac{1}{n}}}{\left(\sqrt{2n\pi}\left(\frac{n}{e}\right)^{n}\right)^{\frac{1}{n}}}\left(\sin\left(\frac{\pi(n+1)}{2n}\right)-1\right)=$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{\frac{4n^2}{e^2}}{\frac{n}{e}}\left(\sin\left(\frac{\pi(n+1)}{2n}\right)-1\right)=\frac{2}{e}\lim_{n\to\infty}n\left(\sin\left(\frac{\pi}{2}+\frac{\pi}{2n}\right)-1\right)=$$

$$=\frac{2}{e}\lim_{n\to\infty}n\left(\cos\left(\frac{\pi}{2n}\right)-1\right)=\frac{2}{e}\lim_{n\to\infty}n\left(1-\frac{\left(\frac{\pi}{2n}\right)^2}{2!}+\frac{\left(\frac{\pi}{2n}\right)^4}{4!}-\cdots-1\right)=$$

$$=\frac{2}{e}\lim_{n\to\infty}\left(-\frac{\pi^2}{2!\cdot 4n}+\frac{\pi^4}{4!\cdot 2^4n^3}-\cdots\right)=0$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\left(\sin\left(\frac{\pi\sqrt[n+1]{(n+1)!}}{2\sqrt[n]{n!}}\right)-1\right)=0$$

Solution 2 by Kamel Gandouli Rezgui-Tunisia

$$\Omega=\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\left(\sin\left(\frac{\pi\sqrt[n+1]{(n+1)!}}{2\sqrt[n]{n!}}\right)-1\right)=$$

$$=\lim_{n\to\infty}\sqrt[n]{\frac{(2n)!}{2^n n!}}\left(\sin\left(\frac{\pi}{2}\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)-1\right)=$$

$$=\lim_{n\to\infty}\sqrt[n]{\frac{(2n)!}{2^n n!}}\frac{\left(\sin\left(\frac{\pi}{2}\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)-1\right)}{\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-1}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-1\right)$$

$$n!\cong\sqrt{2n\pi}\left(\frac{n}{e}\right)^n;\ \lim_{x\to 1}\frac{\sin\frac{\pi}{2}x-1}{(x-1)^2}=-1\Rightarrow\lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{2}\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)-1}{\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-1}=-1$$

$$\sqrt[n]{\frac{(2n)!}{2^n n!}}\cong\frac{\sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)}{\sqrt{2n\pi}\left(\frac{n}{e}\right)}=\left(\frac{n+1}{n}\right)^{\frac{3}{2}}=\left(1+\frac{1}{n}\right)^{\frac{3}{2}}$$

$$\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-1\right)^2 \cong \frac{\left(\left(1+\frac{1}{n}\right)^{\frac{3}{2}}-1\right)}{\frac{1}{n}}\cdot\frac{\left(\left(1+\frac{1}{n}\right)^{\frac{3}{2}}-1\right)}{\frac{1}{n}}\cdot\frac{1}{n^2}$$

$$\sqrt[n]{\frac{(2n)!}{2^n n!}}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-1\right)^2 \cong \frac{2n}{e}\frac{\left(\left(1+\frac{1}{n}\right)^{\frac{3}{2}}-1\right)}{\frac{1}{n}}\cdot\frac{\left(\left(1+\frac{1}{n}\right)^{\frac{3}{2}}-1\right)}{\frac{1}{n}}\cdot\frac{1}{n^2}=$$

$$=\frac{2n}{e}\frac{\left(\left(1+\frac{1}{n}\right)^{\frac{3}{2}}-1\right)}{\frac{1}{n}}\cdot\frac{\left(\left(1+\frac{1}{n}\right)^{\frac{3}{2}}-1\right)}{\frac{1}{n}}\cdot\frac{2}{ne}\to 0$$

Because:

$$\lim_{n\to\infty}\frac{\left(\left(1+\frac{1}{n}\right)^{\frac{3}{2}}-1\right)}{\frac{1}{n}}=\lim_{x\to 0}\frac{(1+x)^{\frac{3}{2}}-1}{x}\overset{L'H}{=}1$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\left(\sin\left(\frac{\pi\sqrt[n+1]{(n+1)!}}{2\sqrt[n]{n!}}\right)-1\right)=0$$

1681. Find:

$$\Omega=\lim_{x\to 0_+}\frac{\log(\sin x)+\log(\cos x)+\log(\tan x)+\log(\cot x)}{\log(\sin 2x)+\log(\cos 3x)+\log(\tan 4x)+\log(\cot 5x)}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Kamel Gandouli Rezgui-Tunisia

$$\Omega=\lim_{x\to 0_+}\frac{\log(\sin x)+\log(\cos x)+\log(\tan x)+\log(\cot x)}{\log(\sin 2x)+\log(\cos 3x)+\log(\tan 4x)+\log(\cot 5x)}=$$

$$=\lim_{x\to 0_+}\frac{\log(\cos x\cdot\sin x)}{\log(\cos 3x\cdot\sin 2x\cdot\tan 4x\cdot\cot 5x)}=$$

$$=\lim_{x\to 0_+}\frac{\log\left(\frac{\sin x\cdot\cos x}{x}\right)+\log x}{\log\left(\frac{\sin 2x\cdot\cos 3x\cdot\tan 4x\cdot\cot 5x}{2x\cdot 4x}\right)+\log 8+2\log x}=$$

$$= \lim_{x\to 0_+} \frac{\log x}{\log(\cot 5x) + \log 8 + 2\log x} = \lim_{x\to 0_+} \frac{1}{\frac{\log(\cot 5x)}{\log x} + 2} =$$

$$= \lim_{x\to 0_+} \frac{1}{\frac{\log(\cot 5x) - \log(\sin 5x)}{\log x} + 2} = \lim_{x\to 0_+} \frac{1}{\frac{-\log\left(\frac{\sin 5x}{5x}\right) - \log 5x}{\log x} + 2} =$$

$$= \lim_{x\to 0_+} \frac{1}{-\frac{\log 5x}{\log x} + 2} = \lim_{x\to 0_+} \frac{1}{\frac{-\log 5 - \log x}{\log x} + 2} = 1$$

1682. If $x, y, z \in \mathbb{R}; m, n \in \mathbb{N} \cup \{0\}$ and $\phi(x, y, z; m, n) =$

$\frac{1}{y}\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial x^n}\left(\frac{\Gamma\left(\frac{x}{y}\right)\Gamma\left(z-\frac{x}{y}\right)}{\Gamma(z)}\right)$ then find the conditions imposed on x, y and z such that

the following relationship holds:

$$\sum_{k=0}^{n} \binom{n}{k} (-y)^k \phi(yz - x, y, z; m + k, n - k) = (-1)^m \phi(x, y, z; m, n)$$

Proposed by Angad Singh-India

Solution by proposer

It is known from the properties of Beta and Gamma function that,

$$I = \int_0^\infty \frac{t^{x-1}}{(1+t^y)^z} dt = \frac{\Gamma\left(\frac{x}{y}\right)\Gamma\left(z-\frac{x}{y}\right)}{y\Gamma(z)} \text{ thus,}$$

$$\phi(x, y, z; m, n) = (-1)^n \int_0^\infty \frac{t^{x-1}\log^m(t)\log^n(1+t^y)}{(1+t^y)^z} dt$$

Substituting $t = \frac{1}{x}$ in the above integral and then expanding $(\log(1+t^y) - \log(t^y))^n$,

using binomial theorem and simplifying in further, we obtain the desired series in L.H.S..

Notice that ϕ will exists iff the above integral converges. Observe that, converges of

$$\int_0^\infty \frac{t^{x-1}}{(1+t^y)^z} dt \Rightarrow \int_0^\infty (-1)^n \frac{t^{x-1}\log^m(t)\log^n(1+t^y)}{(1+t^y)^z} dt \text{, converges. Now,}$$

$$\frac{t^{x-1}}{(1+t^y)^z} = O\left(\frac{1}{t^{yz-x+1}}\right) \Rightarrow \int_0^\infty \frac{t^{x-1}}{(1+t^y)^z} dt = O\left(\int_0^\infty \frac{1}{t^{yz-x+1}} dt\right)$$

Thus, integral I will converge if $yz - x + 1 > 1 \Rightarrow x < yz$.

Observe again that,

$$I = \int_0^\infty \frac{t^{x-1}}{(1+t^y)^z}dt = \int_0^\infty \frac{t^{yz-x-1}}{(1+t^y)^z}dt = O\left(\int_0^\infty \frac{1}{t^{x+1}}dt\right)$$

Thus, integral I will converge if $x + 1 > 1 \Rightarrow x > 0$.

Now, y and z cannot be less than zero since that will lead to a contradiction.

Let $y = -u$ and $z = -v$, where u and v are positive real numbers, following the same line of proof we obtain, $x < 0$ and $x > uv$, which is not possible. Therefore, the integral I converges and the series exists if $0 < x < yz; y > 0, z > 0$.

1683. Prove that :

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(2\sin x + 2\cos x + \sqrt{2}\right)^5 dx > 243\sqrt{2}$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Lemma: \ \forall a,b,c > 0, \qquad (a+b+c)^5 \geq 81abc(a^2+b^2+c^2).$$

$$Proof: \ (a+b+c)^5 = \frac{(a+b+c)^6}{a+b+c}$$

$$= \frac{[(a^2+b^2+c^2)+(ab+bc+ca)+(ab+bc+ca)]^3}{a+b+c} \geq$$

$$\overset{AM-GM}{\geq} \frac{3^3.(a^2+b^2+c^2).(ab+bc+ca).(ab+bc+ca)}{a+b+c}$$

$$= \frac{3^3.(a^2+b^2+c^2).(ab+bc+ca)^2}{a+b+c} \geq$$

$$\geq \frac{3^3.(a^2+b^2+c^2).3(ab.bc+bc.ca+ca.ab)}{a+b+c} = 81abc(a^2+b^2+c^2)$$

$$\rightarrow \ (a+b+c)^5 \geq 81abc(a^2+b^2+c^2), \forall a,b,c > 0 \ with\ equality\ iff\ a=b=c.$$

$$For\ a = 2\sin x, b = 2\cos x, c = \sqrt{2}, where\ x \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right], we\ have:$$

$$\left(2\sin x + 2\cos x + \sqrt{2}\right)^5 \geq 81.2\sin x.2\cos x.\sqrt{2}.(4\sin^2 x + 4\cos^2 x + 2)$$

$$= 972\sqrt{2}.\sin 2x$$

$$With\ equality\ iff\ 2\sin x = 2\cos x = \sqrt{2}\ or\ x = \frac{\pi}{4}.$$

$$Now, \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(2\sin x + 2\cos x + \sqrt{2}\right)^5 dx > 972\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin 2x\, dx =$$

$$= 972\sqrt{2}\left(\frac{-\cos\left(2.\frac{\pi}{3}\right)}{2} - \frac{-\cos\left(2.\frac{\pi}{4}\right)}{2}\right) = 972\sqrt{2}.\frac{1}{4} = 243\sqrt{2}.$$

$$Therefore, \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(2\sin x + 2\cos x + \sqrt{2}\right)^5 dx > 243\sqrt{2}.$$

Solution 2 by proposer

Lemma. If $x, y, z > 0$, then $\sqrt[5]{81xyz(x^2+y^2+z^2)} \le x+y+z$

$$\text{Proof. } (x+y+z)^5 = \frac{(x+y+z)^6}{x+y+z} = \frac{((x+y+z)^2)^3}{x+y+z} =$$

$$= \frac{\left(x^2+y^2+z^2+2(xy+yz+zx)\right)^3}{x+y+z} \overset{AGM}{\geq} \frac{\left(3\cdot\sqrt[3]{(x^2+y^2+z^2)(xy+yz+zx)^2}\right)^3}{x+y+z} =$$

$$= \frac{27(x^2+y^2+z^2)(xy+yz+zx)^2}{x+y+z} = \frac{27(x^2+y^2+z^2)\cdot 3(xy^2z+yz^2x+x^2yz)}{x+y+z} =$$

$$= \frac{81xyz(x^2+y^2+z^2)(x+y+z)}{x+y+z} = 81xyz(x^2+y^2+z^2)$$

$$\sqrt[5]{81xyz(x^2+y^2+z^2)} \le x+y+z$$

Equality holds for $x = y = z$.

$$x \to \sin x\,;\, y \to \cos x\,;\, z \to \frac{\sqrt{2}}{2};\, x \in \left[0, \frac{\pi}{2}\right] \Rightarrow$$

$$81\sin x\cos x - \frac{\sqrt{2}}{2}\left(\sin^2 x + \cos^2 x + \left(\frac{\sqrt{2}}{2}\right)^2\right) \le \left(\sin x + \cos x + \frac{\sqrt{2}}{2}\right)^5$$

$$\left(\sin x + \cos x + \frac{\sqrt{2}}{2}\right)^5 \ge 60,75\cdot\sqrt{2}\cdot\sin x\cdot\cos x \Rightarrow$$

$$\left(2\sin x + 2\cos x + \sqrt{2}\right)^5 \ge 1944\cdot\sqrt{2}\cdot\sin x\cdot\cos x$$

Equality holds for $x = \frac{\pi}{4}$. Now,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(2\sin x+2\cos x+\sqrt{2}\right)^5dx>1944\sqrt{2}\cdot\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(\sin x\cdot\cos x)dx$$

$$972\cdot\sqrt{2}\cdot\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sin 2x\,dx=-486\cdot\sqrt{2}\cos 2x|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=243\sqrt{2}$$

Therefore,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(2\sin x+2\cos x+\sqrt{2}\right)^5dx>243\sqrt{2}$$

1684. Let $(y_n)_{n\geq1}$ be the sequence such that $y_n=\sum_{k=1}^{n^2+n}\left[\sqrt{k}+\frac{1}{2}\right]$, then prove the following holds

$$\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=m}^{\infty}\frac{1}{3y_n}=4\zeta(3)-\frac{2\pi^2}{3}\log 2$$
$$\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m^2}\sum_{n=m}^{\infty}\frac{1}{3y_n}=4G\pi-\frac{15}{2}\zeta(3)-\frac{\pi^2}{3}\log 2$$

where $\zeta(z)$ is Riemann zeta function, G is Catalan's Constant and $[\cdot]$ is greatest integer function.

Proposed by Naren Bhandari-Bajura-Nepal
Inspired by Florică Anastase's limit

Solution by Rana Ranino-Setif-Algerie

$$y_n=\sum_{k=1}^{n^2+n}\left[\sqrt{k}+\frac{1}{2}\right]=\frac{1}{3n(n+1)(2n+1)}$$

$$\Omega=\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=m}^{\infty}\frac{1}{3y_n}=\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=m}^{\infty}\frac{1}{n(n+1)(2n+1)}=$$

$$=\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=m}^{\infty}\left(\frac{1}{n}+\frac{1}{n+1}-\frac{4}{2n+1}\right)=\sum_{m=1}^{\infty}\frac{2\psi\left(m+\frac{1}{2}\right)-\psi(m)-\psi(m+1)}{m^2}=$$

$$=\sum_{m=1}^{\infty}\frac{2\psi\left(m+\frac{1}{2}\right)-2\psi(m)-\frac{1}{m}}{m^2}=-\zeta(3)+2\sum_{m=1}^{\infty}\frac{1}{m^2}\int_0^1\frac{t^{m-1}-t^{m-\frac{1}{2}}}{1-t}dt=$$

$$= -\zeta(3) + 2\int_0^1 \frac{\left(t^{-1} - t^{-\frac{3}{2}}\right) Li_2(t)}{1-t} dt \overset{t=x^2}{=} -\zeta(3) + 4\int_0^1 \frac{Li_2(x^2)}{x(1+x)} dx - 4\int_0^1 \frac{Li_2(x^2)}{1+x} dx$$

$$= -\zeta(3) + 2Li_3(1) - 4[\log(1+x)Li(x^2)]_0^1 - 8\int_0^1 \frac{\log(1-x^2)\log(1+x)}{x} dx =$$

$$= \zeta(3) - \frac{2\pi^2}{3}\log 2 - 4\underbrace{\int_0^1 \frac{2\log(1-x^2)\log(1+x)}{x} dx}_{I}$$

$$I = \int_0^1 \frac{\log^2(1+x)}{x} dx + \int_0^1 \frac{\log^2(1-x^2)}{x} dx - \int_0^1 \frac{\log^2(1-x)}{x} dx =$$

$$= \int_0^1 \frac{\log^2(1+x)}{x} dx - \frac{1}{2}\int_0^1 \frac{\log^2(1-x)}{x} dx$$

$$\int_0^1 \frac{\log^2(1+x)}{x} dx = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} H_n}{n+1} \int_0^1 x^n \, dx = 2\sum_{n=1}^{\infty} \frac{(-1)^n H_{n-1}}{n^2} =$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^2} - \frac{(-1)^n}{n^3} = \frac{\zeta(3)}{4}$$

$$\int_0^1 \frac{\log^2(1-x)}{x} dx = \int_0^1 \frac{\log^2 x}{1-x} dx = \sum_{n=1}^{\infty} \int_0^1 x^{n-1} \log^2 x \, dx = 2\sum_{n=1}^{\infty} \frac{1}{n^3} = 2\zeta(3)$$

$$I = \frac{\zeta(3)}{4} - \zeta(3) = -\frac{3\zeta(3)}{4}$$

Hence,

$$\Omega = \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=m}^{\infty} \frac{1}{3y_n} = 4\zeta(3) - \frac{2\pi^2}{3}\log 2$$

Now,

$$\Lambda = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2} \sum_{n=m}^{\infty} \frac{1}{3y_n} = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} \sum_{n=m}^{\infty} \frac{1}{n(n+1)(2n+1)} =$$

$$= -\sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} \sum_{n=m}^{\infty} \left(\frac{1}{n} + \frac{1}{n+1} - \frac{4}{2n+1}\right) =$$

$$= -\sum_{m=1}^{\infty} \frac{(-1)^m \left[2\psi\left(m+\frac{1}{2}\right) - \psi(m) - \psi(m+1)\right]}{m^2} =$$

$$= -\sum_{m=1}^{\infty} \frac{(-1)^m \left[2\psi\left(m+\frac{1}{2}\right) - 2\psi(m) - \frac{1}{m}\right]}{m^2} =$$

$$= -\frac{3}{4}\zeta(3) - 2\sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} \int_0^1 \frac{t^{m-1} - t^{m-\frac{1}{2}}}{1-t} dt =$$

$$= -\frac{3}{4}\zeta(3) - 2\int_0^1 \frac{\left(\frac{1}{t} - \frac{1}{\sqrt{t}}\right) Li_2(-t)}{1-t} dt \overset{t=x^2}{=} -\frac{3}{4}\zeta(3) - 4\int_0^1 \frac{Li_2(-x^2)}{x(1+x)} dx =$$

$$= -\frac{3}{4}\zeta(3) - 4\int_0^1 \frac{Li_2(-x^2)}{x} dx + 4\int_0^1 \frac{Li_2(-x^2)}{1+x} dx$$

$$\Lambda = -\frac{3}{4}\zeta(3) - 2Li_3(-1) + 4\int_0^1 \frac{Li_2(-x^2)}{1+x} dx = \frac{3}{4}\zeta(3) + 4\overbrace{\int_0^1 \frac{Li_2(-x^2)}{1+x} dx}^{A}$$

$$A = \int_0^1 \int_0^1 \frac{x^2 \log y}{(1+x^2 y)(1+x)} dy\, dx = \int_0^1 \frac{\log y}{1+y} \int_0^1 \left(\frac{1}{1+x} - \frac{1-x}{1+x^2 y}\right) dxdy =$$

$$= \int_0^1 \frac{\log y}{1+y} \left[\log(1+x) + \frac{1}{2y}\log(1+x^2 y) - \frac{\tan^{-1}(x\sqrt{y})}{\sqrt{y}}\right]_0^1 dy =$$

$$= \log 2 \int_0^1 \frac{\log y}{y+1} dy + \frac{1}{2}\int_0^1 \frac{\log y \log(1+y)}{y(y+1)} dy - \overbrace{\int_0^1 \frac{\log y \tan^{-1}(\sqrt{y})}{(y+1)\sqrt{y}} dy}^{B}$$

$$\int_0^1 \frac{\log y}{y+1} dy = -\sum_{n=1}^{\infty} (-1)^n \int_0^1 y^{n-1} \log y\, dy = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$\int_0^1 \frac{\log y \log(1+y)}{y(y+1)} dy = \int_0^1 \frac{\log y \log(y+1)}{y} dy - \int_0^1 \frac{\log y \log(1+y)}{y+1} dy$$

$$\int_0^1 \frac{\log y \log(1+y)}{y} dy = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^1 y^{n-1} \log y\, dy = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = -\frac{3\zeta(3)}{4}$$

$$\int_0^1 \frac{\log y \log(1+y)}{y+1} dy = \left[\frac{1}{2}\log y \log^2(1+y)\right]_0^1 - \frac{1}{2}\int_0^1 \frac{\log^2(1+y)}{y} dy = -\frac{\zeta(3)}{8}$$

Hence, we get:

$$\int_0^1 \frac{\log y \log(1+y)}{y(y+1)} dy = -\frac{3\zeta(3)}{4} + \frac{\zeta(3)}{8} = -\frac{5\zeta(3)}{8}$$

$$B = \int_0^1 \frac{\log y \tan^{-1}(\sqrt{y})}{(y+1)\sqrt{y}} dy \overset{y=\tan^2\theta}{=} 4\int_0^{\frac{\pi}{4}} \theta \log(\tan\theta)\, d\theta$$

$$B = -8\sum_{n=1}^{\infty} \frac{1}{2n-1} \int_0^{\frac{\pi}{4}} \theta \cos((4n-2)\theta)\, d\theta =$$

$$= -8\sum_{n=1}^{\infty} \frac{1}{2n-1} \left[\frac{\theta \sin((4n-2)\theta)}{2(2n-1)} + \frac{\cos((4n-2)\theta)}{4(2n-1)^2}\right]_0^{\frac{\pi}{4}}$$

$$B = -\pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} + 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} = -\pi G + \frac{7\zeta(3)}{4}$$

$$A = -\frac{\pi^2}{12}\log 2 - \frac{5\zeta(3)}{16} + \pi G - \frac{7\zeta(3)}{4} = \pi G - \frac{\pi^2}{12}\log 2 - \frac{33\zeta(3)}{16}$$

$$\Lambda = \frac{3}{4}\zeta(3) + 4\pi G - \frac{\pi^2}{3}\log 2 - \frac{33\zeta(3)}{16} = 4\pi G - \frac{\pi^2}{3}\log 2 - \frac{15\zeta(3)}{2}$$

Therefore,

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2} \sum_{n=m}^{\infty} \frac{1}{3y_n} = 4G\pi - \frac{15}{2}\zeta(3) - \frac{\pi^2}{3}\log 2$$

$$y_n = \sum_{k=1}^{n^2+n} \left[\sqrt{k} + \frac{1}{2}\right] = \sum_{k=1}^{2} \left[\sqrt{k} + \frac{1}{2}\right] + \sum_{k=3}^{6} \left[\sqrt{k} + \frac{1}{2}\right] + \sum_{k=7}^{12} \left[\sqrt{k} + \frac{1}{2}\right] + \cdots =$$

$$= \sum_{i=1}^{n} \sum_{k=i^2-i+1}^{i^2+i} \left[\sqrt{k} + \frac{1}{2}\right] = \sum_{i=1}^{n} \sum_{k=i^2-i+1}^{i^2+i} i = \sum_{i=1}^{n} 2i^2 = \frac{1}{3n(n+1)(2n+1)}$$

1685. Find:

$$\Omega = \int_0^1 \frac{x \cdot \log^2 x}{x^8 + x^4 + 1} dx$$

Proposed by Hussain Reza Zadah-Afghanistan

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx=\frac{1}{8}\int_0^1\frac{\log^2 x}{x^4+x^2+1}dx=\frac{1}{8}\int_0^1\frac{(1-x^2)\log^2 x}{1-x^6}dx\overset{t=x^6}{=}$$

$$=\frac{1}{1728}\int_0^1\frac{\left(x^{\frac{1}{6}-1}-x^{\frac{1}{2}-1}\right)\log^2 t}{1-t}dt=\frac{1}{1728}\sum_{n=0}^{\infty}\left\{\psi^{(2)}\left(\frac{1}{2}\right)-\psi^{(2)}\left(\frac{1}{6}\right)\right\}=$$

$$=\frac{1}{1728}\left\{168\zeta(3)+4\sqrt{3}\pi^3\right\}$$

Therefore,

$$\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx=\frac{7\zeta(3)}{72}+\frac{\pi^3}{144\sqrt{3}}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx=\int_0^1\frac{(x-x^5)\log^2 x}{1-x^{12}}dx=\Omega_1-\Omega_2$$

$$\Omega_1=\int_0^1\frac{x\log^2 x}{1-x^{12}}dx\overset{x=\sqrt[12]{y}}{=}\int_0^1\frac{\sqrt[12]{y}}{1-y}\log^2\sqrt[12]{y}\,\frac{1}{12\sqrt[12]{y^{11}}}dy=$$

$$=\frac{1}{1728}\int_0^1\frac{y^{-\frac{5}{6}}}{1-y}\log^2 y\,dy=\frac{1}{1728}\sum_{k=0}^{\infty}\int_0^1 y^{k-\frac{5}{6}}\log^2 y\,dy$$

$$\because\int_0^1 y^{k-\frac{5}{6}}\log^2 y\,dy=\frac{2}{\left(k+\frac{1}{6}\right)^3}$$

$$\Omega_1=\frac{1}{864}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{6}\right)^3};\ \left(\because\psi_n(z)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}}\right.$$

$$\psi_2\left(\frac{1}{6}\right)=(-1)^{2+1}2!\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{6}\right)^3}\Rightarrow\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{6}\right)^3}=-\frac{\psi_2\left(\frac{1}{6}\right)}{2}$$

$$\Omega_1=-\frac{\psi_2\left(\frac{1}{6}\right)}{1728}$$

$$\Omega_2=\int_0^1\frac{x^5\log^2 x}{1-x^{12}}dx=\int_0^1\frac{\sqrt[12]{y^5}\log^2\sqrt[12]{y}}{1-y}\cdot\frac{1}{12\sqrt[12]{y^{11}}}dy=\frac{1}{1728}\int_0^1\frac{\log^2 y}{(1-y)\sqrt{y}}dy=$$

$$=\frac{1}{1728}\sum_{k=0}^{\infty}\int_0^1 y^{k-\frac{1}{2}}\log^2 y\,dy$$

$$\because\int_0^1 y^{k-\frac{1}{2}}\log^2 y\,dy=\frac{2}{\left(k+\frac{1}{2}\right)^3}$$

$$\Omega_2=\frac{1}{864}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{2}\right)^3}=-\frac{\psi_2\left(\frac{1}{2}\right)}{1728};(\because\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{2}\right)^3}=-\frac{\psi_2\left(\frac{1}{2}\right)}{2}$$

$$\Omega_2=-\frac{\psi_2\left(\frac{1}{2}\right)}{1728}$$

$$\Omega=\frac{\psi_2\left(\frac{1}{2}\right)-\psi_2\left(\frac{1}{6}\right)}{1728}=\frac{168\zeta(3)+4\sqrt{3}\pi^2}{1728}$$

Solution 3 by Ose Favour-Nigeria

$$\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx\overset{u=x^2}{=}\frac{1}{8}\int_0^1\frac{\log^2 u}{u^4+u^2+1}du$$

$$\Omega(a)=\int_0^1\frac{u^a}{u^4+u^2+1}du=\int_0^1\frac{u^a(1-u^2)}{1-u^6}du$$

$$\Omega(a)=\Phi(a)-\Psi(a)$$

$$\Phi(a)=\int_0^1\frac{u^a}{1-u^6}du=\int_0^1\frac{u^a}{1-u^6}du=\sum_{n=0}^{\infty}\int_0^1 u^{a+6n}\,du=\sum_{n=0}^{\infty}\frac{1}{a+6n+1}$$

$$\Psi(a)=\int_0^1\frac{u^{a+2}}{1-u^6}du=\sum_{n=0}^{\infty}\int_0^1 u^{a+2+6n}\,du=\sum_{n=0}^{\infty}\frac{1}{a+6n+3}$$

$$\Omega(a)=-\sum_{n=0}^{\infty}\frac{1}{a+6n+3}+\sum_{n=0}^{\infty}\frac{1}{a+6n+1}$$

$$\because\sum_{n=0}^{\infty}\frac{1}{n+a}=-\psi^{(0)}(a)$$

$$\Omega(a)=-\frac{1}{6}\sum_{n=0}^{\infty}\frac{1}{n+\frac{a+3}{6}}+\frac{1}{6}\sum_{n=0}^{\infty}\frac{1}{n+\frac{a+1}{6}}=\frac{1}{6}\psi^{(0)}\left(\frac{a+3}{6}\right)-\frac{1}{6}\psi^{(0)}\left(\frac{a+1}{6}\right)$$

$$\Omega'(a)=\frac{1}{36}\left(\psi^{(1)}\left(\frac{a+3}{6}\right)-\psi^{(1)}\left(\frac{a+1}{6}\right)\right)$$

$$\Omega''(a)=\frac{1}{216}\left(\psi^{(2)}\left(\frac{a+3}{6}\right)-\psi^{(2)}\left(\frac{a+1}{6}\right)\right)$$

$$\Omega=\frac{1}{8}\Omega''(0)=\frac{\psi_2\left(\frac{1}{2}\right)-\psi_2\left(\frac{1}{6}\right)}{1728}=\frac{168\zeta(3)+4\sqrt{3}\pi^2}{1728}$$

Solution 4 by Yen Tung Chung-Taichung-Taiwan

$$\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx\overset{y=x^2}{=}\int_0^1\frac{\log^2(\sqrt{y})}{y^4+y^2+1}\cdot\frac{1}{2}dy=\frac{1}{8}\int_0^1\frac{\log^2 y}{y^4+y^2+1}dy\overset{y=e^{-z}}{=}$$

$$=\frac{1}{8}\int_0^{\infty}\frac{z^2e^{-z}}{1+e^{-2z}+e^{-4z}}dz=\frac{1}{8}\int_0^{\infty}\frac{z^2(e^{-z}-e^{-3z})}{1-e^{-6z}}dz=$$

$$=\frac{1}{8}\int_0^{\infty}z^2(e^{-z}+e^{-7z}+e^{-13z}+\cdots)\,dz-\frac{1}{8}\int_0^{\infty}z^2\left(e^{-3z}+e^{-9z}+e^{-15z}+\cdots\right)dz=$$

$$=\frac{1}{8}\left(\frac{\Gamma(3)}{1^3}+\frac{\Gamma(3)}{7^3}+\frac{\Gamma(3)}{13^3}+\cdots\right)-\frac{1}{8}\left(\frac{\Gamma(3)}{3^3}+\frac{\Gamma(3)}{9^3}+\frac{\Gamma(3)}{15^3}+\cdots\right)=$$

$$=\frac{1}{4}\left(\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{7^3}-\frac{1}{9^3}+\frac{1}{13^3}-\frac{1}{15^3}+\cdots\right)\cong 0.24117$$

Solution 5 by Abdul Mukhtar-Nigeria

$$x^8+x^4+1=\frac{1-x^{12}}{1-x^4}\Rightarrow\Omega=\int_0^1\frac{x\cdot\log^2 x}{x^8+x^4+1}dx=\int_0^1\frac{x\log^2 x}{1-x^{12}}dx-\int_0^1\frac{x^5\log^2 x}{1-x^{12}}dx=$$

$$=H-J$$

$$\int_0^1\frac{x^{p-1}\log^2 x}{1-x^{12}}dx\overset{x=e^{-t}}{=}\int_0^{\infty}\frac{e^{-tp}t^2}{1-e^{12t}}dt=\int_0^{\infty}t^2e^{-tp}\sum_{k\ge0}e^{-12kt}\,dt=$$

$$=\sum_{k\ge0}\left[\int_0^{\infty}t^2e^{-t(p+12k)}\,dt\right]=\sum_{k\ge0}\left[\frac{1}{(p+12k)^3}\int_0^{\infty}t^2e^{-t}\,dt\right]=\Gamma(3)=2$$

$$\Rightarrow 2\sum_{k\ge0}\frac{1}{(p+12k)^3}=\frac{1}{864}\sum_{k\ge0}\frac{1}{\left(k+\frac{p}{12}\right)^3}$$

$$\zeta(s,p)=\sum_{n>0}\frac{1}{(n+p)^s}$$

$$\Omega=\frac{1}{864}\left[\zeta\left(3,\frac{1}{6}\right)-\zeta\left(3,\frac{1}{2}\right)\right]=\frac{1}{432}(42\zeta(3)-53\pi^3)$$

1686.

If we have $\int_{-\pi}^{\pi}\frac{\cos^2 2x+1}{a\cos^2 2x+1}dx=\pi a,$ **then find the value of**

$$\Phi=a^5+a^4+8a^3+8a^2-32a.$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_{-\pi}^{\pi}\frac{\cos^2 2x+1}{a\cos^2 2x+1}dx=4\int_0^{\frac{\pi}{2}}\frac{\cos^2 2x+1}{a\cos^2 2x+1}dx\overset{t=\tan x;a+1=b^2}{=}$$

$$=8\int_0^{\infty}\frac{1+t^4}{(b^2+t^2)(1+t^2)^2}dt=$$

$$=\frac{8(b^4+1)}{(b^2-1)^2}\int_0^{\infty}\frac{dt}{b^2+t^2}+\frac{16}{b^2-1}\int_0^{\infty}\frac{dt}{(1+t^2)^2}-\frac{16b^2}{(b^2-1)^2}\int_0^{\infty}\frac{dt}{1+t^2}=$$

$$=\frac{4\pi(b^4+1)}{b(b^2-1)^2}+\frac{4\pi}{b^2-1}-\frac{8\pi^2}{(b^2-1)^2}=\frac{4\pi(b^2+b+1)}{b(b+1)^2}$$

$$\frac{4\pi(b^2+b+1)}{b(b+1)^2}=\pi(b^2-1)\Rightarrow 4b^2+4b+4=b^5+2b^4-6b^2-5b-4=0$$

$$b=\frac{2b^4-6b^2-4}{5-b^4};\sqrt{a+1}=\frac{2a^2-2a-8}{4-a^2-2a}$$

$$a+1=\frac{4a^4-8a^3-28a^2+32a+64}{a^4+4a^3-4a^2-16a+16}$$

$$\Phi=a^5+a^4+8a^3+8a^2-32a=48$$

1687.

$$R(t)=\begin{pmatrix}0&t&0&0&0&0&0&0\\t&0&t&0&0&0&0&0\\0&t&0&t&0&0&0&0\\0&0&t&0&t&0&0&0\\0&0&0&t&0&t&0&0\\0&0&0&0&t&0&t&0\\0&0&0&0&0&t&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}$$

Prove the relation for $n > 1$

$$\int_0^\infty Tr\left[e^{R(t)}\right]e^{-nt}\,dt = \frac{8n^7 - 42n^5 + 56n^3 - 16n}{n^8 - 7n^6 + 14n^4 - 8n^2 + 1}$$

$Tr(*)$ – is the Trace of matrix and $e^{R(t)}$ –is matrix exponential.

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Farid Khelili-Algerie

Let $P(z) = z^4 - 7z^3 + 14z^2 - 8z + 1$ and $\{\lambda_k^{\pm}(t), k = 1, 2, \dots, 4\}$ be the eigenvalues of the matrix $R(t)$, we claim that

$$A.\ \det(R(t) - \lambda I_8) = [(y-3)(y-1)y + 1]t^8; \lambda^2 = (y+1)t^2$$

$$B.\ \lambda_k^{\pm}(t) = \pm\alpha_k t; 0 < \alpha_k < 2; P\left(\alpha_k^2\right) = 0; k = 1, 2, \dots, 4$$

$$C.\ \int_0^\infty Tr\left(e^{R(t)}\right)e^{-nt}dt = \sum_{k=1}^{4}\frac{2n}{n^2 - \alpha_k^2} = \frac{8n^7 - 42n^5 + 56n^3 - 16n}{n^8 - 7n^6 + 14n^4 - 8n^2 + 1}$$

We first prove claim C then we prove claim B and at the end we prove claim A.

Proof of claim C: It follows from claim (B) thet

$$J_n = \int_0^\infty Tr\left(e^{R(t)}\right)e^{-nt}dt = \sum_{k=1}^{4}\int_0^\infty e^{-\lambda_k^{+}(t)}e^{-nt}dt + \sum_{k=1}^{4}\int_0^\infty e^{-\lambda_k^{-}(t)}e^{-nt}dt$$

$$J_n = \sum_{k=1}^{4}\int_0^\infty e^{-\alpha_k t}e^{-nt}\,dt + \sum_{k=1}^{4}\int_0^\infty e^{\alpha_k t}e^{-nt}\,dt =$$

$$= \sum_{k=1}^{4}\int_0^\infty e^{-(\alpha_k+n)t}\,dt + \sum_{k=1}^{4}\int_0^\infty e^{(\alpha_k+n)t}\,dt$$

$$J_n = \sum_{k=1}^{4}\left(\frac{1}{n+\alpha_k} + \frac{1}{n-\alpha_k}\right) = \sum_{k=1}^{4}\frac{2n}{n^2 - \alpha_k^2}$$

Since, by claim B, $P\left(\alpha_k^2\right) = 0, k = 1, 2, \dots, 4$, then

$$P(z) = z^4 - 7z^3 + 14z^2 - 8z + 1 = \left(z - \alpha_1^2\right)\left(z - \alpha_2^2\right)\left(z - \alpha_3^2\right)\left(z - \alpha_4^2\right)$$

and

$$\sum_{k=1}^{4}\frac{1}{z-\alpha_k^2}=\frac{P'(z)}{P(z)}=\frac{4z^3-21z^2+28z-8}{z^4-7z^3+14z^2-8z+1}$$

It follows that

$$J_n=\sum_{k=1}^{4}\frac{2n}{n^2-\alpha_k^2}=2n\frac{P'(n^2)}{P(n^2)}=\frac{8n^7-42n^5+56n^3-16n}{n^8-7n^6+14n^4-8n^2+1}$$

The claim C is then demonstrated.

$$\int_0^{\infty}Tr\left[e^{R(t)}\right]e^{-nt}\,dt=\frac{8n^7-42n^5+56n^3-16n}{n^8-7n^6+14n^4-8n^2+1}$$

Proof of claim B: If follows from claim (A) that the eigenvalues $\{\lambda_k^{\pm}(t), k=1,2,\dots,4\}$ of the matrix $R(t)$ are given by $\lambda_k^{\pm}(t)=\pm\sqrt{y_k+1}t$, where the real numbers $\{y_k, k=1,2,\dots,4\}$ are the roots of the equation

$$(y-3)(y-1)(y+1)+1=0.$$

Let $\alpha_k=\sqrt{y_k+1}$, then the real numbers $\{\alpha_k^2, k=1,2,\dots,4\}$ are the roots of the equation

$$(z-4)(z-2)(z-1)z+1=0.$$

Let $P(z)=(z-4)(z-2)(z-1)z+1$, then

$$P(z)=z^4-7z^3+14z^2-8z+1;\ P(\alpha_k^2)=0, k=1,2,\dots,4$$

Let γ be a root of $P(z)$, it is easy to see that $P(\gamma)\ge 1$ if $\gamma\ge 4$, then $0<\gamma<4$, and $0<\alpha_k<2$. The claim of B is then demonstrated.

$$B.\ \alpha_k^{\pm}(t)=\pm\alpha_k t;\ 0<\alpha_k<2;\ P(\alpha_k^2)=0;\ k=1,2,\dots,4$$

Proof of claim A: Let $A(\mu)$ and $B_n(\mu)$ the 8×8 real matrix and $n\times n$ real matrix, respectively.

$$A(\mu)=\begin{pmatrix}\mu&1&0&0&0&0&0&0\\1&\mu&1&0&0&0&0&0\\0&1&\mu&1&0&0&0&0\\0&0&1&\mu&1&0&0&0\\0&0&0&1&\mu&1&0&0\\0&0&0&0&1&\mu&1&0\\0&0&0&0&0&1&\mu&0\\0&0&0&0&1&0&0&\mu\end{pmatrix};B_n(\mu)=\begin{pmatrix}\mu&1&0&\dots&0&0\\1&\ddots&\ddots&\ddots&\ddots&0\\0&\ddots&\ddots&\ddots&\ddots&\vdots\\\vdots&\ddots&\ddots&\ddots&\ddots&0\\0&\ddots&\ddots&\ddots&\ddots&1\\0&0&\dots&0&1&\mu\end{pmatrix}$$

Let $\mu = -\frac{\lambda}{t}$, then $\det(R(t) - \lambda I_8) = t^8 \det A(\mu)$ and

$\det A(\mu) = \mu D_7 - (\mu^2 - 1)D_4$, where D_n –is the determinant of the $n \times n$ matrix $B_n(\mu)$.

It is easy to show that D_n satisfies the recurrence relations.

$$D_n = \mu D_{n-1} - D_{n-2} = (\mu^2 - 1)D_{n-2} - \mu D_{n-3}$$

$$D_n = [(\mu^2 - 1)^2 - \mu^2)D_{n-4} - \mu(\mu^2 - 2)D_{n-5}$$

Since, $D_1 = \mu; D_2 = \mu^2 - 1; D_3 = \mu(\mu^2 - 2)$, then $D_4 = (\mu^2 - 1)^2 - \mu^2$;

$D_7 = \mu(\mu^2 - 2)((\mu^2 - 1)^2 - 2\mu^2 + 1)$ and

$\det A(\mu) = \mu D_7 - (\mu^2 - 1)D_4 = (y - 3)(y - 1)(y + 1)y + 1$, where $y = \mu^2 - 1$.

It follows that:

$\det(R(t) - \lambda I_8) = t^8 \det A(\mu) = [(y - 3)(y - 1)(y + 1)y + 1]t^8$, where

$\lambda^2 = \mu^2 t^2 (y + 1)t^2$. The claim A is then demonstrated.

$A. \det(R(t) - \lambda I_8) = [(y - 3)(y - 1)y + 1]t^8; \lambda^2 = (y + 1)t^2$

1688. Find a closed form:

$$\Omega = \int_0^\infty \frac{\log(1 + x)}{x^2 - x + 1} dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Rana Ranino-Setif-Algerie

$$\Omega = \int_0^\infty \frac{\log(1 + x)}{x^2 - x + 1} dx = \int_0^1 \int_0^\infty \frac{x}{(1 + xy)(1 - x + x^2)} dx\, dy =$$

$$\int_0^1 \frac{1}{1 + y + y^2} \int_0^\infty \left(\frac{x + y}{1 - x + x^2} - \frac{y}{1 + xy} \right) dx\, dy =$$

$$= \int_0^1 \frac{1}{1 + y + y^2} \left[\frac{1}{2} \log(1 - x + x^2) - \log(1 + xy) + \frac{2y + 1}{\sqrt{3}} \tan^{-1}\left(\frac{2x - 1}{\sqrt{3}} \right) \right]_0^\infty dy =$$

$$= \frac{2\pi}{3\sqrt{3}} \int_0^1 \frac{2y + 1}{1 + y + y^2} dy - \int_0^1 \frac{\log y}{1 + y + y^2} dy =$$

$$= \frac{2\pi \log 3}{3\sqrt{3}} - \int_0^1 \frac{(1 - y)\log y}{1 - y^3} dy = \frac{2\pi \log 3}{3\sqrt{3}} - \frac{1}{9} \int_0^1 \frac{\left(t^{\frac{1}{3}-1} - t^{\frac{2}{3}-1} \right) \log t}{1 - t} dy =$$

$$=\frac{2\pi\log 3}{3\sqrt{3}}-\frac{1}{9}\left\{\psi^{(1)}\left(\frac{2}{3}\right)-\psi^{(1)}\left(\frac{1}{3}\right)\right\}=$$

$$=\frac{2\pi\log 3}{3\sqrt{3}}-\frac{1}{9}\left\{\psi^{(1)}\left(\frac{2}{3}\right)+\psi^{(1)}\left(\frac{1}{3}\right)\right\}+\frac{2}{9}\psi^{(1)}\left(\frac{1}{3}\right)$$

Therefore,

$$\Omega=\int_0^\infty\frac{\log(1+x)}{x^2-x+1}dx=\frac{2\pi\log 3}{3\sqrt{3}}-\frac{4\pi^2}{27}+\frac{2}{9}\psi^{(1)}\left(\frac{1}{3}\right)$$

1689. Show that:

$$\int_0^1\log\left(\frac{\left(\sqrt{1-x}-1\right)\left(\sqrt{1-x^2}+1\right)}{\left(\sqrt{1-x}+1\right)\left(\sqrt{1-x^2}-1\right)}\right)\sin^{-1}(x)^2\,dx=$$

$$=\frac{\pi^3}{12}-\pi\left(6\sqrt{2}+2+\log\left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right)\right)-4(2C-7)$$

C –Catalan's constant.

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\log\left(\frac{\left(\sqrt{1-x}-1\right)\left(\sqrt{1-x^2}+1\right)}{\left(\sqrt{1-x}+1\right)\left(\sqrt{1-x^2}-1\right)}\right)\sin^{-1}(x)^2\,dx=$$

$$=\int_0^1\log\left(\frac{\left(1-\sqrt{1-x}\right)\left(1+\sqrt{1-x^2}\right)}{\left(1+\sqrt{1-x}\right)\left(1-\sqrt{1-x^2}\right)}\right)\sin^{-1}(x)^2\,dx$$

$$=\overbrace{\int_0^1\log\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)\sin^{-1}(x)^2\,dx}^{A}+\overbrace{\int_0^1\log\left(\frac{1+\sqrt{1-x\text{^}2}}{1+\sqrt{1-x^2}}\right)\sin^{-1}(x)^2\,dx}^{B}$$

$$B\overset{x=\sin t}{=}-2\int_0^{\frac{\pi}{2}}t^2\log\left(\tan\frac{t}{2}\right)\cos t\,dt=-16\int_0^{\frac{\pi}{4}}t^2\log(\tan t)\cos 2t\,dt=$$

$$=-16\left[\left(\frac{1}{2}t^2\sin 2t+\frac{1}{2}t\sin 2t-\frac{1}{4}\sin 2t\right)\log(\tan t)\right]_0^{\frac{\pi}{2}}+8\int_0^{\frac{\pi}{4}}(2t^2+2t\cos 2t-1)dt$$

$$=\frac{\pi^3}{12}+2\pi\log 2-2\pi$$

$$A=\left[\left(2\sqrt{1-x^2}\sin^{-1}x-2x+x\sin^{-1}(x)^2\right)\log\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)\right]_0^1-$$

$$-\int_0^1\frac{2\sqrt{1-x^2}\sin^{-1}x-2x+x\sin^{-1}(x)^2}{x\sqrt{1-x}}dx=$$

$$=2\int_0^1\frac{dx}{\sqrt{1-x}}-\int_0^1\frac{\sin^{-1}(x)^2}{\sqrt{1-x}}dx-2\int_0^1\frac{\sqrt{1+x}\sin^{-1}x}{x}dx-\int_0^1\frac{\sin^{-1}(x)^2}{\sqrt{1-x}}dx=$$

$$=\left[\sin^{-1}(x)^2\sqrt{1-x}\right]_0^1-4\int_0^1\frac{\sin^{-1}x}{\sqrt{1+x}}dx=-8\left[\sqrt{1+x}\sin^{-1}x\right]_0^1+8\int_0^1\frac{dx}{\sqrt{1-x}}dx=$$

$$=16-4\pi\sqrt{2}-2\int_0^1\frac{\sqrt{1+x}\sin^{-1}x}{x}dx=$$

$$=-2\left[\sqrt{1+x}\sin^{-1}x\log x\right]_0^1+2\int_0^1\frac{\log x}{\sqrt{1-x}}dx+\int_0^1\frac{\log x\sin^{-1}x}{\sqrt{1+x}}dx$$

Now, we have:

$$2\int_0^1\frac{\log x}{\sqrt{1-x}}dx=2\int_0^1 x^{\frac{1}{2}-1}\log(1-x)\,dx=-4H_{\frac{1}{2}}=8\log 2-8$$

$$\int_0^1\frac{\log x\sin^{-1}x}{\sqrt{1+x}}dx\overset{x=\cos t}{=}\sqrt{2}\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-t\right)\log(\cos t)\sin\left(\frac{t}{2}\right)dt=$$

$$=\overbrace{\frac{\pi}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\log(\cos t)\sin\left(\frac{t}{2}\right)dt}^{I}-\overbrace{\sqrt{2}\int_0^{\frac{\pi}{2}}t\log(\cos t)\sin\left(\frac{t}{2}\right)dt}^{J}$$

Where,

$$I=\frac{\pi}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\log\left(2\cos^2\left(\frac{t}{2}\right)-1\right)\sin\left(\frac{t}{2}\right)dt\overset{y=\sqrt{2}\cos\left(\frac{t}{2}\right)}{=}\pi\int_1^{\sqrt{2}}\log(y^2-1)\,dy=$$

$$=\pi\left[(y+1)\log(y+1)+(y-1)\log(y-1)-2y\right]_1^{\sqrt{2}}=2\pi-2\pi\sqrt{2}+2\pi\log\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)$$

and

$$J=4\log 2\int_0^{\frac{\pi}{4}}t\left(\log 2+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(4nt)\right)\sin t\,dt=$$

$$= 4\sqrt{2}\log 2\int_0^{\frac{\pi}{4}} t\sin t\,dt + 4\sqrt{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} t\sin t\cos(4nt)\,dt =$$

$$= 4\log 2 - \pi\log 2 +$$

$$+2\sqrt{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\sin(4nt+t)}{(4n+1)^2} - \frac{\sin(4nt-t)}{(4n-1)^2} + \frac{t\cos(4nt-t)}{4n-1} - \frac{t\cos(4nt+t)}{4n+1}\right]_0^{\frac{\pi}{4}} =$$

$$= 4\log 2 - \pi\log 2 +$$

$$+2\sqrt{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{(-1)^n}{\sqrt{2}(4n+1)^2} + \frac{(-1)^n}{\sqrt{2}(4n-1)^2} + \frac{(-1)^n\pi}{4\sqrt{2}(4n-1)} - \frac{(-1)^n\pi}{4\sqrt{2}(4n+1)}\right) =$$

$$= 4\log 2 - \pi\log 2 + 2\overbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n(4n+1)^2} + \frac{1}{n(4n-1)^2}\right)}^{S_1}$$

$$+\frac{\pi}{2}\overbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n(4n-1)} - \frac{1}{n(4n+1)}\right)}^{S_2}$$

Where,

$$S_1 = \sum_{n=1}^{\infty}\left(\frac{2}{n} + \frac{4}{(4n-1)^2} - \frac{4}{(4n+1)^2} - \frac{4}{4n-1} - \frac{4}{4n+1}\right) =$$

$$= \sum_{n=1}^{\infty}\left(\frac{2}{n+1} + \frac{1}{4\left(n+\frac{3}{2}\right)^2} - \frac{1}{4\left(n+\frac{5}{4}\right)^2} - \frac{1}{n+\frac{3}{4}} - \frac{1}{n+\frac{5}{4}}\right) =$$

$$= -2\psi(1) + \psi\left(\frac{5}{4}\right) + \psi\left(\frac{3}{4}\right) + \frac{1}{4}\psi^{(1)}\left(\frac{3}{4}\right) - \frac{1}{4}\psi^{(1)}\left(\frac{5}{4}\right) =$$

$$= 2\gamma - \gamma + \frac{\pi}{2} - 3\log 2 + 4 - \gamma - \frac{\pi}{2} - 3\log 2 + \frac{\pi^2}{4} - 2C - 2C + 4 - \frac{\pi^2}{4} =$$

$$= 8 - 6\log 2 - 4C$$

and

$$S_2 = \sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{3}{4}} + \frac{1}{n+\frac{5}{4}} - \frac{2}{n}\right) = 2\psi(1) - \psi\left(\frac{3}{4}\right) - \psi\left(\frac{5}{4}\right) =$$

$$= -2\gamma - 4 + \gamma + \frac{\pi}{2} + 3\log 2 + \gamma - \frac{\pi}{2} + 3\log 2 = 6\log 2 - 4$$

$$J = 4\log 2 - \pi\log 2 + 16 - 12\log 2 - 8C + 3\pi\log 2 - 2\pi =$$

$$= 16 + 2\pi\log 2 - 8\log 2 - 2\pi - 8C$$

$$\int_0^1 \frac{\log x \sin^{-1} x}{\sqrt{1+x}}dx = 16 - 2\pi\sqrt{2} + 2\pi\log(1+\sqrt{2}) - 8\log 2 - 8C$$

$$A = 4 + 16 - 4\pi\sqrt{2} + 8\log 2 - 8 + 16 - 2\pi\sqrt{2} + 2\pi\log(1+\sqrt{2}) - 8\log 2 - 8C$$

$$= 28 - 6\pi\sqrt{2} + 2\pi\log(1+\sqrt{2}) - 8C$$

$$\Omega = 28 - 6\pi\sqrt{2} + 2\pi\log(1+\sqrt{2}) - 8C + \frac{\pi^3}{12} + 2\pi\log 2 - 2\pi$$

$$= \frac{\pi^3}{12} - \pi\left(6\sqrt{2} + 2 + \log\left(\frac{3}{4} - \frac{1}{\sqrt{2}}\right)\right) - 4(2C - 7)$$

Therefore,

$$\int_0^1 \log\left(\frac{(\sqrt{1-x}-1)(\sqrt{1-x^2}+1)}{(\sqrt{1-x}+1)(\sqrt{1-x^2}-1)}\right)\sin^{-1}(x)^2\,dx =$$

$$= \frac{\pi^3}{12} - \pi\left(6\sqrt{2} + 2 + \log\left(\frac{3}{4} - \frac{1}{\sqrt{2}}\right)\right) - 4(2C - 7)$$

1690. Prove that:

$$\Omega = \int_0^{2\pi} \frac{(\sin(2\pi - z)^{2l} \cdot (2\pi - z)}{(\sin(2\pi - z))^{2l} + (\cos(2\pi - z))^{2l}}dz = \pi^2$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution 1 by Togrul Ehmedov-Azerbaijan

$$\Omega = \int_0^{2\pi} \frac{(\sin(2\pi - z)^{2l} \cdot (2\pi - z)}{(\sin(2\pi - z))^{2l} + (\cos(2\pi - z))^{2l}}dz =$$

$$= \int_0^{2\pi} \frac{\sin^{2l} z\,(2\pi - z)}{\sin^{2l} z + \cos^{2l} z}dz = \int_0^{2\pi} \frac{z \cdot \sin^{2l} z}{\sin^{2l} z + \cos^{2l} z}dz$$

$$2\Omega = 2\pi \int_0^{2\pi} \frac{\sin^{2l} z}{\sin^{2l} z + \cos^{2l} z}dz$$

$$\Omega=\pi\int_0^{2\pi}\frac{\sin^{2l}z}{\sin^{2l}z+\cos^{2l}z}dz=4\pi\int_0^{\frac{\pi}{2}}\frac{\sin^{2l}z}{\sin^{2l}z+\cos^{2l}z}dz=$$

$$=4\pi\int_0^{\frac{\pi}{2}}\frac{\cos^{2l}z}{\sin^{2l}z+\cos^{2l}z}dz$$

Therefore,

$$2\Omega=4\pi\int_0^{\frac{\pi}{2}}dz\Rightarrow\Omega=\pi^2$$

Solution 2 by Kamel Gandouli Rezgui-Tunisia

$$\Omega=\int_0^{2\pi}\frac{(\sin(2\pi-z)^{2l}\cdot(2\pi-z)}{(\sin(2\pi-z))^{2l}+(\cos(2\pi-z))^{2l}}dz=\int_0^{2\pi}\frac{(\sin z)^{2l}\cdot(2\pi-z)}{(\sin z)^{2l}+(\cos z)^{2l}}dz\Rightarrow$$

$$2\Omega=2\pi\int_0^{2\pi}\frac{\sin^{2l}t}{\sin^{2l}t+\cos^{2l}t}dt=\int_0^{\frac{\pi}{2}}\frac{\sin^{2l}t}{\sin^{2l}t+\cos^{2l}t}dt+\int_{\frac{\pi}{2}}^{\pi}\frac{\sin^{2l}t}{\sin^{2l}t+\cos^{2l}t}dt+$$

$$+\int_{\pi}^{\frac{3\pi}{2}}\frac{\sin^{2l}t}{\sin^{2l}t+\cos^{2l}t}dt+\int_{\frac{3\pi}{2}}^{2\pi}\frac{\sin^{2l}t}{\sin^{2l}t+\cos^{2l}t}dt=\Omega_1+\Omega_2+\Omega_3+\Omega_4$$

$$\Omega_2\overset{t\to t-\frac{\pi}{2}}{=}\int_0^{\frac{\pi}{2}}\frac{\sin^{2l}t}{\sin^{2l}t+\cos^{2l}t}dt$$

$$\Omega_3\overset{t\to t-\pi}{=}\int_0^{\frac{\pi}{2}}\frac{\sin^{2l}t}{\sin^{2l}t+\cos^{2l}t}dt$$

$$\Omega_4\overset{t=t-\frac{3\pi}{2}}{=}\int_0^{\frac{\pi}{2}}\frac{\sin^{2l}t}{\sin^{2l}t+\cos^{2l}t}dt$$

Therefore,

$$2\Omega=2\pi(\Omega_1+\Omega_2+\Omega_3+\Omega_4)=2\pi\cdot\left(2\int_0^{\frac{\pi}{2}}dt\right)=2\pi^2\Rightarrow\Omega=\pi^2$$

1691. Find:

$$\Omega=\lim_{n\to\infty}\frac{n(1\cdot n+2\cdot(n-1)+3\cdot(n-2)+\cdots+n\cdot 1)}{1\cdot n^2+2\cdot(n-1)^2+3\cdot(n-2)^2+\cdots+n\cdot 1^2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Sergio Esteban-Buenos Aires-Argentina

$$S_1=\sum_{k=1}^{n}k(n-k+1)=\sum_{k=1}^{n}(nk-k^2+k)=n\sum_{k=1}^{n}k-\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k=$$

$$=n\cdot\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{6}$$

$$S_2=\sum_{k=1}^{n}k(n+1)^2=\sum_{k=1}^{n}k[(n+1)^2+k^2-2(n+1)k]=$$

$$=\sum_{k=1}^{n}k(n+1)^2+\sum_{k=1}^{n}k^3-\sum_{k=1}^{n}2(n+1)k^2=$$

$$=\frac{n(n+1)}{2}\cdot(n+1)^2+\left(\frac{n(n+1)}{2}\right)^2-2(n+1)\cdot\frac{n(n+1)(2n+1)}{6}$$

$$\Omega=\lim_{n\to\infty}\frac{nS_1}{S_2}=2$$

Solution 2 by Adrian Popa-Romania

$$S_1=\sum_{k=1}^{n}k(n-k+1)=\sum_{k=1}^{n}(nk-k^2+k)=n\sum_{k=1}^{n}k-\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k=$$

$$=n\cdot\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}+n$$

$$S_2=\sum_{k=1}^{n}k(n+1)^2=\sum_{k=1}^{n}k[(n+1)^2+k^2-2(n+1)k]=$$

$$=n^2\sum_{k=1}^{n}k+\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k-2n\sum_{k=1}^{n}k^2+2n\sum_{k=1}^{n}k-2\sum_{k=1}^{n}k^2$$

$$=n^2\cdot\frac{n(n+1)}{2}+\left(\frac{n(n+1)}{2}\right)^2+\frac{n(n+1)}{2}-2n\cdot\frac{n(n+1)(2n+1)}{6}+2n\cdot\frac{n(n+1)}{2}$$

$$-\frac{2n(n+1)(2n+1)}{6},\qquad \Omega=\frac{\frac{1}{2}-\frac{2}{6}}{\frac{1}{2}+\frac{1}{4}-\frac{4}{6}}=2$$

Solution 3 by Ravi Prakash-New Delhi-India

$$S_1=n\sum_{k=1}^{n}k(n-k+1)=n\sum_{k=1}^{n}k(n+1-k)=n(n+1)\sum_{k=1}^{n}k-\sum_{k=1}^{n}k^2=$$

$$=n(n+1)\cdot\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}=\frac{n^2(n+1)(n+2)}{6}$$

$$S_2=\sum_{k=1}^{n}k^2(n+1-k)=(n+1)\frac{1}{6}n(n+1)(2n+1)-\frac{1}{4}n^2(n+1)^2=$$

$$=\frac{1}{12}n(n+1)^2(4n+2-3n)=\frac{1}{12}n(n+1)^2(n+2)$$

$$\Omega=\lim_{n\to\infty}\frac{\frac{n^2(n+1)(n+2)}{6}}{\frac{n(n+1)^2(n+2)}{12}}=2\lim_{n\to\infty}\frac{n}{n+1}=2$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$S_1=\sum_{k=1}^{n}k(n-k+1)=\sum_{k=1}^{n}(nk-k^2+k)=n\sum_{k=1}^{n}k-\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k=$$

$$=n\cdot\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}+n=\frac{n(n+1)}{6}(3n+3-2n-1)=$$

$$=\frac{n(n+1)(n+2)}{6}$$

$$S_2=\sum_{k=1}^{n}k(n+1)^2=\sum_{k=1}^{n}k[(n+1)^2+k^2-2(n+1)k]=$$

$$=n^2\sum_{k=1}^{n}k+\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k-2n\sum_{k=1}^{n}k^2+2n\sum_{k=1}^{n}k-2\sum_{k=1}^{n}k^2$$

$$=n^2\cdot\frac{n(n+1)}{2}+\left(\frac{n(n+1)}{2}\right)^2+\frac{n(n+1)}{2}-2n\cdot\frac{n(n+1)(2n+1)}{6}+2n\cdot\frac{n(n+1)}{2}-$$

$$-\frac{2n(n+1)(2n+1)}{6}=\frac{n(n+1)}{12}[6n+6-4(2n+1)+3n]=\frac{n(n+1)^2(n+2)}{12}$$

$$\Omega=\lim_{n\to\infty}\frac{n^2(n+1)(n+2)}{6}\cdot\frac{12}{n(n+1)^2(n+2)}=2$$

1692. If $A(n)$ denotes the n^{th} term of the sequence $A344317$ in OEIS and defined as $\forall n\in\mathbb{N},\ A(n)=n\cdot A(n-1)+n^{(1+n)\bmod 2}$; $A(0)=1$. Prove that:

$$\lim_{n\to\infty}\frac{A(n)}{n!}=1+2\sinh(1)=1-\frac{1}{e}+e$$

Proposed by Amrit Awasthi-India

Solution by Kamel Gandouli Rezgui-Tunisia

Let $v_n=\frac{A(n)}{n!}\Rightarrow v_k=v_{k-1}+\frac{k^{(k+1)\bmod 2}}{k!}\Rightarrow v_k-v_{k-1}=\frac{k^{(k+1)\bmod 2}}{k!}$

$$\sum_{k=1}^{n}(v_k-v_{k-1})=\sum_{k=1}^{n}\frac{k^{(k+1)\bmod 2}}{k!}\Rightarrow v_n-v_0=\sum_{k=1}^{n}\frac{k^{(k+1)\bmod 2}}{k!}\Rightarrow$$

$$v_n=\sum_{k=1}^{n}\frac{k^{(k+1)\bmod 2}}{k!}+1$$

$$\lim_{n\to\infty}\frac{A(n)}{n!}=\lim_{n\to\infty}v_n=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k^{(k+1)\bmod 2}}{k!}+1\right)=$$

$$=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}+\sum_{k=1}^{\infty}\frac{2k}{(2k)!}+1=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}+\sum_{k=1}^{\infty}\frac{1}{(2k-1)!}+1=$$

$$=\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}=\sinh 1$$

Therefore,

$$\lim_{n\to\infty}\frac{A(n)}{n!}=1+2\sinh(1)=1-\frac{1}{e}+e$$

1693.

$$\Omega_1(n)=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|(i-j)\left(\frac{1}{2n-i+1}-\frac{1}{2n-j+1}\right)\right|$$

$$\Omega_2=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{1}{2n-i+1}-\frac{1}{2n-j+1}\right|$$

Find:

$$\Omega=\lim_{n\to\infty}\frac{\Omega_1(n)}{\Omega_2(n)}$$

Proposed by Daniel Sitaru-Romania

Solution by Kamel Gandouli Rezgui-Tunisia

Let $k = n - i + 1$ and $p = n - j + 1$, then

$$\Omega_1(n) = \sum_{k=1}^{n}\sum_{p=1}^{n}\left|(p-k)\left(\frac{1}{n+k} - \frac{1}{n+p}\right)\right| = \sum_{k=1}^{n}\sum_{p=1}^{n}\frac{(p-k)^2}{(n+p)(n+k)} =$$

$$= \sum_{k=1}^{n}\frac{1}{n+k}\sum_{p=1}^{n}\frac{(p-k)^2}{n+p} \Rightarrow$$

$$\Omega_1(n) \geq \sum_{k=1}^{n}\frac{1}{n+k}\sum_{p=1}^{n}\frac{(k-1)^2}{2n} \geq \sum_{k=1}^{n}\frac{(k-1)^2}{2(n+k)}$$

Because:

$$\sum_{p=1}^{n}\frac{(p-k)^2}{n+p}(p-k)^2 \geq (k-1)^2; p \geq 1 \text{ and } \frac{1}{n+p} \geq \frac{1}{2n}; \forall p \leq n$$

$$\sum_{p=1}^{n}\frac{(k-1)^2}{2n} = n\frac{(k-1)^2}{2n} = \frac{(k-1)^2}{2}$$

$$\frac{1}{n+k} \geq \frac{1}{2n} \Rightarrow \sum_{k=1}^{n}\frac{(k-1)^2}{2(n+k)} \geq \sum_{k=1}^{n}\frac{(k-1)^2}{4n}$$

$$\Omega_1(n) \geq \sum_{k=1}^{n}\frac{(k-1)^2}{4n} = \sum_{k=0}^{n-1}\frac{k^2}{4n} = \frac{1}{12}n^2 - \frac{1}{8}n + \frac{1}{24}$$

$$\Omega_2(n) = \sum_{k=1}^{n}\sum_{p=1}^{n}\left|\left(\frac{1}{n+k} - \frac{1}{n+p}\right)\right| = \sum_{k=1}^{n}\sum_{p=1}^{n}\frac{|p-k|}{(n+p)(n+k)}$$

$$\Rightarrow \Omega_2(n) = \sum_{k=1}^{n}\frac{1}{n+k}\sum_{p=1}^{n}\frac{|p-k|}{n+p} \leq \sum_{k=1}^{n}\frac{1}{n+k}\sum_{p=1}^{n}\frac{|n-k|}{n+1}$$

$$|p-k| \leq |n-k| \leq n-1 \text{ and } \frac{1}{n+p} \leq \frac{1}{n+1}$$

$$\Omega_2(n) \leq \sum_{k=1}^{n}\frac{1}{n+k}\sum_{k=1}^{n}\frac{n-k}{n+1} \leq \sum_{k=1}^{n}\frac{1}{n+k}\sum_{k=1}^{n}\frac{n-1}{n+1} \leq \frac{n(n-1)}{n+1}\sum_{k=1}^{n}\frac{1}{n+k} \leq$$

$$\leq \frac{n(n-1)}{n+1}\cdot\frac{n}{n+1} \leq \frac{n^3}{(n+1)^2} \leq n$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{\Omega_1(n)}{\Omega_2(n)}\geq\lim_{n\to\infty}\frac{\frac{n^2}{12}-\frac{n}{8}+\frac{1}{24}}{n}=+\infty$$

1694. Let $A(x)$ be the following $n\times n$ real matrix

$$A(x)=\begin{pmatrix} x^2 & 1 & x^2 & 1 & \dots & x^2 & 1\\ 1 & x^{-2} & 1 & x^{-2} & \dots & 1 & x^{-2}\\ x^2 & 1 & x^2 & 1 & \dots & x^2 & 1\\ 1 & x^{-2} & 1 & x^{-2} & \dots & 1 & x^{-2}\\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\ x^2 & 1 & x^2 & 1 & \dots & x^2 & 1\\ 1 & x^{-2} & 1 & x^{-2} & \dots & 1 & x^{-2}\end{pmatrix};n-\text{even and }F(x)=\det\big(I_n+A(x)\big)$$

Prove that:

$$\int_0^\infty\frac{1}{\big(1-F(x)\big)\left(1-F\left(e^{\frac{\pi x^2}{8}}\right)\right)}\frac{dx}{x}=\frac{\pi-2\log(1+\sqrt{2})}{n^2\sqrt{2}}$$

Proposed by Farid Khelili-Algerie

Solution by proposer

Let $X=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ and $\left||X|\right|^2=\sum_{k=1}^n x_kx_k$. If regard X as an element of $Mat(n\times n,\mathbb{R})$, then $X^T\in Mat(1\times n,\mathbb{R})$ and $XX^T=(x_jx_k)_{1\leq j,k\leq n}\in Mat(n,\mathbb{R})$. Let

$$J_n=\int_0^\infty\frac{1}{\big(1-F(x)\big)\left(1-F\left(e^{\frac{\pi x^2}{8}}\right)\right)}\frac{dx}{x}$$

We claim that:

$$\mathbb{A}.\det(I_n+XX^T)=1+\left||X|\right|^2;\forall X\in\mathbb{R}^n$$

$$\mathbb{B}.\det\big(I_n+A(x)\big)=1+\frac{n}{2}\left(x^2+\frac{1}{x^2}\right);\forall x\in\mathbb{R}^*$$

$$\mathbb{C}.J_n=\frac{1}{n^2}\int_0^\infty\frac{dx}{(1+x^2)\cosh\left(\frac{\pi x}{4}\right)}=\frac{4}{n^2}\sum_{m=0}^\infty\frac{(-1)^m}{4m+3}$$

Proof of claim $\mathbb{A}$: Let $E=(1,0,\dots,0)\in\mathbb{R}^n$and $X=(x_1,x_2,\dots.,x_n)\in\mathbb{R}^n$ such that $\left||X|\right|\neq 0$, if we regard X and E as elements of $Mat(n\times 1,\mathbb{R})$, then there exists an orthogonal matrix $R\in O(n,\mathbb{R}),R^tR=I_n$, such that $RX=\left||X|\right|E$.

Since $\det(RMR^T) = \det(MR^TR) = \det(M)$, for any $M \in Mat(n, \mathbb{R})$, then

$$\det(I_n + XX^T) = \det[R(I_n + XX^T)R^T] = \det(I_n + (RX)(RX)^T) = \det\left(I_n + ||X||^2 EE^T\right)$$

Since EE^T is the diagonal matrix $diag(1, 0, \dots, 0)$, then

$$I_n + ||X||^2 EE^T = diag(1 + ||X||^2, 1, \dots, 1) \text{ and}$$

$$\det(I_n + XX^T) = \det\left(I_n + ||X||^2 EE^T\right) = 1 + ||X||^2$$

The claim $\mathbb{A}$ is demonstrated.

Proof of claim $\mathbb{B}$: Let $X = \left(x, \frac{1}{x}, x, \frac{1}{x}, \dots, x, \frac{1}{x}\right) \in \mathbb{R}^n$, $x \in \mathbb{R}$ and n is even, then

$||X||^2 = \frac{n}{2}\left(x^2 + \frac{1}{x^2}\right)$ and $XX^T = A(x)$. It follows from claim $\mathbb{A}$ that

$$\det\left(I_n + A(x)\right) = \det(I_n + XX^T) = 1 + ||X||^2 = 1 + \frac{n}{2}\left(x^2 + \frac{1}{x^2}\right)$$

The claim $\mathbb{B}$ is demonstrated.

Proof of claim $\mathbb{C}$: Let $F(x) = \det\left(I_n + A(x)\right)$, it follows from claim $\mathbb{B}$ that

$$1 - F(x) = 1 - \det\left(I_n + A(x)\right) = -\frac{n}{2}\left(x^2 + \frac{1}{x^2}\right) \text{ and}$$

$$1 - F\left(e^{\frac{\pi x^2}{8}}\right) = -\frac{n}{2}\left(e^{\frac{\pi x^2}{4}} + e^{-\frac{\pi x^2}{4}}\right) = -n\cosh\left(\frac{\pi x^2}{4}\right)$$

Thus,

$$J_n = \int_0^\infty \frac{1}{\left(1 - F(x)\right)\left(1 - F\left(e^{\frac{\pi x^2}{8}}\right)\right)} \frac{dx}{x} = \frac{2}{n^2}\int_0^\infty \frac{x\,dx}{(1 + x^4)\cosh\left(\frac{\pi x^2}{4}\right)} \overset{x^2 \to x}{=}$$

$$= \frac{1}{n^2}\int_0^\infty \frac{dx}{(1 + x^2)\cosh\left(\frac{\pi x}{4}\right)}$$

Now, use the partial fraction expansion of $\cosh\left(\frac{\pi x}{4}\right)$

$$\frac{1}{\cosh\left(\frac{\pi x}{4}\right)} = \frac{16}{\pi}\sum_{m=0}^\infty (-1)^m \frac{2m + 1}{x^2 + 4(2m + 1)}$$

To rewrite J_n as

$$J_n = \frac{16}{\pi n^2}\sum_{m=0}^\infty (-1)^m (2m + 1)\int_0^\infty \frac{1}{1 + x^2}\frac{dx}{x^2 + 4(2m + 1)^2} dx$$

The integral on the right is equal to

$$\int_0^\infty \frac{1}{1+x^2}\frac{dx}{x^2+4(2m+1)^2}dx=\frac{\pi}{4}\frac{1}{(4m+3)(2m+1)}$$

So,

$$J_n=\frac{1}{n^2}\int_0^\infty \frac{dx}{(1+x^2)\cosh\left(\frac{\pi x}{n}\right)}=\frac{4}{n^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{4m+3};(1)$$

The claim $\mathbb{C}$ is then demonstrated.The summation on the right of Eq. (1) is equal to

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{4m+3}=\sum_{m=0}^{\infty}(-1)^m\int_0^1 x^{4m+2}\,dx=\int_0^1\frac{x^2}{1+x^4}dx$$

$$J_n=\frac{4}{n^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{4m+3}=\frac{4}{n^2}\int_0^1\frac{x^2}{1+x^4}dx$$

Which can be rewritten as

$$J_n=\frac{2}{n^2}\int_0^1\frac{x^2-1}{1+x^4}dx+\frac{2}{n^2}\int_0^1\frac{x^2+1}{1+x^4}dx$$

$$J_n=\frac{2}{n^2}\int_0^1\frac{1+x^{-2}}{(x+x^{-1})^2+2}dx+\frac{2}{n^2}\int_0^1\frac{1-x^{-2}}{(x+x^{-1})^2-2}dx$$

Making the change of variable $u=-(x-x^{-1})$ in the first integral and $u=x+x^{-1}$ in the second integral, one has

$$J_n=\frac{2}{n^2}\int_0^\infty\frac{du}{u^2+2}-\frac{2}{n^2}\int_2^\infty\frac{du}{u^2-2}=\frac{\pi}{\sqrt{2}n^2}-\frac{1}{\sqrt{2}n^2}\log\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)$$

Therefore,

$$\int_0^\infty \frac{1}{(1-F(x))\left(1-F\left(e^{\frac{\pi x^2}{8}}\right)\right)}\frac{dx}{x}=\frac{\pi-2\log(1+\sqrt{2})}{n^2\sqrt{2}}$$

1695. Find:

$$\Omega=\int_0^\infty\frac{\tan^{-1}x}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Rana Ranino-Setif-Algerie

$$\int \frac{dx}{(1+x)(1+x^2)} = \frac{1}{2}\int \left(\frac{1}{1+x} + \frac{1-x}{1+x^2}\right) dx = \frac{1}{4}\log\left(\frac{x^2+2x+1}{x^2+1}\right) + \frac{1}{2}\tan^{-1} x$$

$$\Omega \overset{IBP}{=} \frac{1}{4}\left[\log\left(\frac{x^2+2x+1}{x^2+1}\right)\tan^{-1} x + 2(\tan^{-1} x)^2\right]_0^{\infty} -$$

$$-\frac{1}{2}\int_0^{\infty} \frac{\log(1+x)}{1+x^2} dx + \frac{1}{4}\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx - \frac{1}{2}\int_0^{\infty} \frac{\tan^{-1} x}{1+x^2} dx =$$

$$= \frac{\pi^2}{8} - \frac{A}{2} + \frac{B}{4} - \frac{C}{2}$$

$$A = \int_0^{\infty} \frac{\log(1+x)}{1+x^2} dx = \int_0^1 \int_0^{\infty} \frac{x}{(1+xy)(1+x^2)} dxdy =$$

$$= \int_0^1 \frac{1}{1+y^2} \int_0^{\infty} \left(\frac{x+y}{1+x^2} - \frac{y}{1+xy}\right) dxdy =$$

$$= \int_0^1 \frac{dy}{1+y^2}\left[y \cdot \tan^{-1} x + \frac{1}{2}\log\left(\frac{1+x^2}{1+2xy+x^2y^2}\right)\right]_0^{\infty} =$$

$$= \frac{\pi}{2}\int_0^1 \frac{y}{1+y^2} dy - \frac{1}{2}\int_0^1 \frac{\log y}{1+y^2} dy = \frac{\pi}{4}\log 2 + \frac{G}{2}$$

$$B = \int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx \overset{x=\tan\theta}{=} -2\int_0^{\frac{\pi}{2}} \log(\cos\theta)\, d\theta = \pi \log 2$$

$$C = \int_0^{\infty} \frac{\tan^{-1} x}{1+x^2} dx = \frac{1}{2}(\tan^{-1}\infty)^2 = \frac{\pi^2}{8}$$

$$\Omega = \int_0^{\infty} \frac{\tan^{-1} x}{(x+1)(x^2+1)} dx = \frac{\pi^2}{16} + \frac{\pi}{8}\log 2 - \frac{G}{2}$$

1696. In $\Delta ABC, \omega = tan^{-1}\left(\sqrt{\frac{rr_a}{r_br_c}}\right) + tan^{-1}\left(\sqrt{\frac{rr_b}{r_cr_a}}\right) + tan^{-1}\left(\sqrt{\frac{rr_c}{r_ar_b}}\right)$. Find:

$$\Omega = \int_0^{\omega} \frac{3sin^2x + cosx + 2}{sinx + cosx + 7} dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\sqrt{\frac{rr_a}{r_br_c}}=\sqrt{\frac{\frac{F^2}{s(s-a)}}{\frac{F^2}{(s-b)(s-c)}}}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\tan\frac{A}{2}$$

$$\tan^{-1}\sqrt{\frac{rr_a}{r_br_c}}+\tan^{-1}\sqrt{\frac{rr_b}{r_cr_a}}+\tan^{-1}\sqrt{\frac{rr_c}{r_ar_b}}=\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$$

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{3\sin^2x+\cos x+2}{\sin x+\cos x+7}dx;(1)$$

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{3\cos^2x+\sin x+2}{\cos x+\sin x+7}dx;(2)$$

By adding (1) and (2), we get:

$$2\Omega=\int_0^{\frac{\pi}{2}}\frac{3+\cos x+\sin x+4}{\cos x+\sin x+7}dx=\frac{\pi}{2}$$

Therefore,

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{3\sin^2x+\cos x+2}{\sin x+\cos x+7}dx=\frac{\pi}{4}$$

Solution 2 by Syed Shahabudeen-Kerala-India

$$\omega=\sum_{cyc}\tan^{-1}\sqrt{\frac{rr_a}{r_br_c}}=\sum_{cyc}\tan^{-1}\left(\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\right)=$$

$$=\sum_{cyc}\tan^{-1}\left(\tan\left(\frac{A}{2}\right)\right)=\frac{1}{2}\sum_{cyc}A=\frac{\pi}{2}\Rightarrow\omega=\frac{\pi}{2}$$

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{3\sin^2x+\cos x+2}{\sin x+\cos x+7}dx$$

$$2\Omega=\int_0^{\frac{\pi}{2}}dx$$

Therefore,

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{3\sin^2 x+\cos x+2}{\sin x+\cos x+7}dx=\frac{\pi}{4}$$

1697. Prove that:

$$\int_0^1\frac{(\tan^{-1}x)^5}{(1+x)(1+x^2)}dx=\frac{225\pi}{256}\zeta(5)+\frac{\pi^6}{49152}-\frac{15\pi^3}{512}\zeta(3)+\frac{\pi^5}{2048}\log 2+$$
$$+\frac{1}{262144}\left(\psi_5\left(\frac{3}{4}\right)-\psi_5\left(\frac{1}{4}\right)\right)$$

where ζ is Zeta function and $\psi_n(z)$ is n^{th} polygamma function.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^1\frac{(\tan^{-1}x)^5}{(1+x)(1+x^2)}dx\overset{x=\tan x}{=}\int_0^{\frac{\pi}{4}}\frac{x^5}{1+\tan x}dx\overset{IBP}{=}$$

$$=\frac{1}{2}\left[x^6+x^5\log(\sin x+\cos x)\right]_0^{\frac{\pi}{4}}-\frac{5}{2}\int_0^{\frac{\pi}{4}}x^4\log(\sin x+\cos x)\,dx$$

$$\Omega=\frac{\pi^6}{49152}-\frac{5}{2}\int_0^{\frac{\pi}{4}}x^4\log\left(\cos\left(\frac{\pi}{4}-x\right)\right)dx=\frac{\pi^6}{49152}-\frac{5}{2}\int_0^{\frac{\pi}{4}}\left(\frac{\pi}{4}-x\right)^4\log(\cos x)\,dx$$

$$\Omega=\frac{\pi^6}{49152}+\frac{5}{2}\log 2\int_0^{\frac{\pi}{4}}\left(\frac{\pi}{4}-x\right)^4dx+\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}\left(\frac{\pi}{4}-x\right)^4\cos(2nx)\,dx$$

$$\int_0^{\frac{\pi}{4}}\left(\frac{\pi}{4}-x\right)^4dx=\int_0^{\frac{\pi}{4}}x^4\,dx=\frac{\pi^5}{5120}$$

$$\int_0^{\frac{\pi}{4}}\left(\frac{\pi}{4}-x\right)^4\cos(2nx)\,dx=$$

$$=\left[\left(\frac{3}{4n^5}-\frac{3\left(\frac{\pi}{4}-x\right)^2}{2n^3}+\frac{3\left(\frac{\pi}{4}-x\right)^4}{2n}\right)\sin(2nx)-\left(\frac{\left(\frac{\pi}{4}-x\right)^3}{n^2}-\frac{3\left(\frac{\pi}{4}-x\right)}{2n^4}\right)\cos(2nx)\right]_0^{\frac{\pi}{4}}$$

$$=\frac{3}{4n^5}\sin\left(\frac{n\pi}{2}\right)+\frac{\pi^3}{64n^2}-\frac{3\pi}{n^4}$$

$$\Omega=\frac{\pi^6}{49152}+\frac{\pi^5}{2048}\log 2+\frac{5\pi^3}{128}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}+\frac{15}{8}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{n\pi}{2}\right)}{n^6}$$

$$\Omega=\frac{\pi^6}{49152}+\frac{\pi^5}{2048}\log 2-\frac{5\pi^3}{128}\zeta(3)+\frac{225\pi}{256}\zeta(5)-\frac{15}{8}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^6}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^6}=\frac{1}{4096}\sum_{n=0}^{\infty}\left(\frac{1}{\left(n+\frac{1}{4}\right)^6}-\frac{1}{\left(n+\frac{3}{4}\right)^6}\right)=-\frac{1}{491520}\left(\psi_5\left(\frac{3}{4}\right)-\psi_5\left(\frac{1}{4}\right)\right)$$

Therefore,

$$\int_0^1\frac{(\tan^{-1}x)^5}{(1+x)(1+x^2)}dx=\frac{225\pi}{256}\zeta(5)+\frac{\pi^6}{49152}-\frac{15\pi^3}{512}\zeta(3)+\frac{\pi^5}{2048}\log 2+$$

$$+\frac{1}{262144}\left(\psi_5\left(\frac{3}{4}\right)-\psi_5\left(\frac{1}{4}\right)\right)$$

1698. Let the matrix: $R(x)=\begin{pmatrix} e^{-\pi x} & e^{-\frac{\pi}{x}} & e^{-\pi x} & e^{-\frac{i\pi}{x}} \\ e^{-\frac{\pi}{x}} & e^{-i\pi x} & e^{-\frac{i\pi}{x}} & e^{-\pi x} \\ e^{-i\pi x} & e^{-\frac{i\pi}{x}} & e^{-\pi x} & e^{-\frac{\pi}{x}} \\ e^{-\frac{i\pi}{x}} & e^{-\pi x} & e^{-\frac{\pi}{x}} & e^{-i\pi x} \end{pmatrix}$ then prove that

$$\int_{-\infty}^{\infty}e^{-\pi x^2}Tr[R(x)\otimes R(ix)]\,dx=4(1+e^{-\pi})$$

where Tr is trace and $\otimes$ is the Kronecker Product

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Mohammad Rostami-Afghanistan

$$\Omega=Tr\begin{bmatrix} e^{-\pi x}R(ix) & e^{-\frac{\pi}{x}}R(ix) & e^{-\pi x}R(ix) & e^{-\frac{i\pi}{x}}R(ix) \\ e^{-\frac{\pi}{x}}R(ix) & e^{-i\pi x}R(ix) & e^{-\frac{i\pi}{x}}R(ix) & e^{-\pi x}R(ix) \\ e^{-i\pi x}R(ix) & e^{-\frac{i\pi}{x}}R(ix) & e^{-\pi x}R(ix) & e^{-\frac{\pi}{x}}R(ix) \\ e^{-\frac{i\pi}{x}}R(ix) & e^{-\pi x}R(ix) & e^{-\frac{\pi}{x}}R(ix) & e^{-i\pi x}R(ix) \end{bmatrix}=$$

$$=\sum_{i=1}^{16}a_{ii}=a_{11}+a_{22}+\cdots+a_{1616}=$$

$$=2\left[2e^{-(1+i)\pi x}+2\right]+2\left[2e^{-2i\pi x}+2e^{(1-i)\pi x}\right]$$

$$I=\int_{-\infty}^{\infty}e^{-\pi x^2}\cdot\Omega\,dx=4\int_{-\infty}^{\infty}e^{-\pi[x^2+(1+i)x]}\,dx+8\int_0^{\infty}e^{-\pi x^2}\,dx+$$

$$+4\int_{-\infty}^{\infty}e^{-\pi(x^2+2ix)}\,dx+4\int_{-\infty}^{\infty}e^{-\pi[x^2+(i-1)x]}\,dx=$$

$$=4\int_{-\infty}^{\infty}e^{-\pi\left[\left(x+\frac{1+i}{2}\right)^2-\frac{i}{2}\right]}dx+8\left[\frac{1}{2\pi}\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)\right]+4\int_{-\infty}^{\infty}e^{-\pi[(x+i)^2+1]}\,dx$$

$$+4\int_{-\infty}^{\infty}e^{-\pi\left[\left(x+\frac{i-1}{2}\right)^2+\frac{i}{2}\right]}dx=$$

$$=4e^{i\frac{\pi}{2}}\int_{\infty}^{\infty}e^{-\pi u^2}\,du+8\cdot\frac{1}{2\pi}\sqrt{\pi}\sqrt{\pi}+4e^{-\pi}\int_{-\infty}^{\infty}e^{-\pi u^2}\,du+4e^{-i\frac{\pi}{2}}\int_{-\infty}^{\infty}e^{-\pi u^2}\,du=$$

$$=4i\left(2\cdot\frac{1}{2\pi}\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)\right)+4+4e^{-\pi}\left(2\cdot\frac{1}{2\pi}\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)\right)-4i\left(2\cdot\frac{1}{2\pi}\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)\right)=$$

$$=4i+4+4e^{-\pi}-4i=4(1+e^{-\pi})$$

$$I=4(1+e^{-\pi})$$

1699. Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences of real numbers defined as

$$x_n=\sum_{k=1}^{n}\sin\left(\frac{1}{k}\right)+\log\left(\sin\frac{1}{n}\right),y_n=\sum_{k=1}^{n^2+n}\left[\sqrt{k}+\frac{1}{2}\right]$$

then prove that $\lim_{n\to\infty}\frac{n^3x_n}{y_n}=\gamma$, where γ is Euler Mascheroni constant and $[\cdot]$ is greatest integer function.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Kamel Gandouli Rezgui-Tunisia

$$y_n=\sum_{k=1}^{n^2+n}\left[\sqrt{k}+\frac{1}{2}\right]=\sum_{k=1}^{2}\left[\sqrt{k}+\frac{1}{2}\right]+\sum_{k=3}^{6}\left[\sqrt{k}+\frac{1}{2}\right]+\sum_{k=7}^{12}\left[\sqrt{k}+\frac{1}{2}\right]+$$

$$+\sum_{k=13}^{20}\left[\sqrt{k}+\frac{1}{2}\right]+\sum_{k=21}^{30}\left[\sqrt{k}+\frac{1}{2}\right]+\cdots=$$

$$=\sum_{i=1}^{n}\sum_{k=i^2-i+1}^{i^2+i}\left[\sqrt{k}+\frac{1}{2}\right]=\sum_{k=1}^{n}\sum_{k=i^2-i+1}^{i^2+i}i=\sum_{i=1}^{n}2i^2=\frac{n(n+1)(2n+1)}{3}$$

Let $f(x)=\sin\frac{1}{x}-\frac{1}{x}+\frac{1}{3x}-\frac{1}{3}\log\left(\frac{x+2}{x+1}\right)\geq 0$ and $g(x)=\sin\frac{1}{x}-\frac{1}{x}+\frac{1}{3x}-\frac{1}{3}\log\frac{x}{x-1}\leq 0$,

then

$$\sum_{k=1}^{n}\left(-\frac{1}{3k}-\frac{1}{3}\log\left(\frac{2+k}{k+1}\right)\right)\leq\sum_{k=1}^{n}\left(\sin\frac{1}{k}-\frac{1}{k}\right)\leq\sum_{k=1}^{n}\left(-\frac{1}{3k}-\frac{1}{3}\log\left(\frac{k}{k-1}\right)\right)$$

$$-\frac{1}{3}(H_n(\log n+2))\leq\sum_{k=1}^{n}\left(\sin\frac{1}{k}-\frac{1}{k}\right)\leq-\frac{1}{3}(H_n-\log n)$$

$$\Rightarrow\sum_{k=1}^{\infty}\left(\sin\frac{1}{k}-\frac{1}{k}\right)=-\frac{\gamma}{3}$$

$$x_n=\sum_{k=1}^{n}\sin\left(\frac{1}{k}\right)+\log\left(\sin\frac{1}{n}\right)=\sum_{k=1}^{n}\sin\left(\frac{1}{k}\right)+\log n+\log\left(\sin\frac{1}{n}\right)-\log n=$$

$$=\sum_{k=1}^{n}\left(\sin\frac{1}{k}-\frac{1}{k}\right)+\log n+\log\left(\sin\frac{1}{n}\right)-\log n+H_n$$

$$\Rightarrow\lim_{n\to\infty}x_n=-\frac{\gamma}{3}+\lim_{n\to\infty}\left(\log n+\log\left(\sin\frac{1}{n}\right)\right)+\lim_{n\to\infty}(-\log n+H_n)=$$

$$=-\frac{\gamma}{3}+\gamma=\frac{2}{3}\gamma$$

$$\because\lim_{n\to\infty}\left(\log n+\log\left(\sin\frac{1}{n}\right)\right)=\lim_{n\to\infty}\log\frac{\sin\frac{1}{n}}{\frac{1}{n}}=\log 1=0$$

Therefore,

$$\lim_{n\to\infty}\frac{n^3x_n}{y_n}=\gamma$$

1700. Prove that:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{2n}{n}\binom{2m}{m}\frac{m+n+3}{8^{m+n}(m+n+2)^3}=$$

$$=\frac{7}{2}\zeta(3)+\frac{\pi^2}{3}+\frac{2}{3}\log^3(2)-2\log^2(2)-\frac{\pi^2}{3}\log(2)-4$$

Proposed by Syed Shahabudeen-India

Solution by Naren Bhandari-Bajura-Nepal

Let S be the sum and since

$$\frac{m+n+3}{(m+n+2)^3}=\frac{1}{(m+n+2)^2}+\frac{1}{(m+n+2)^3}=$$

$$=\int_0^1 x^{m+n+1}\left(-\log x+\frac{\log^2 x}{2}\right)dx$$

$$S=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{2n}{n}\binom{2m}{m}\frac{1}{8^{m+n}}\int_0^1 x^{m+n+1}\left(-\log x+\frac{\log^2 x}{2}\right)dx=S_1+S_2$$

It is easy to see:

$$\sum_{m,n\geq 0}\binom{2m}{m}\binom{2n}{n}x^{n+m}=\left(\sum_{j\geq 0}\binom{2j}{j}y^j\right)^2=\frac{1}{1-4y}$$

for $|x|\leq\frac{1}{4}$ we used then generating function of central binomial coefficients,

namely

$$\sum_{n\geq 0}\binom{2n}{n}y^n=\frac{1}{\sqrt{1-4y}}$$

Putting $y=\frac{1}{8}$, we see that

$$S_1+S_2=-\int_0^1\frac{x\log x}{1-\frac{x}{2}}dx+\frac{1}{2}\int_0^1\frac{x\log^2 x}{1-\frac{x}{2}}dx=$$

$$=\sum_{n\geq 1}\frac{1}{2^{n-1}}\left(-\frac{1}{(n+1)^2}+\frac{1}{2(n+1)^3}\right)=-2+4Li_2\left(\frac{1}{2}\right)-2+4Li_3\left(\frac{1}{2}\right)$$

$$\because Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{1}{2}\log^2(2)\ \text{ and } Li_3\left(\frac{1}{2}\right)=\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}\log(2)+\frac{\log^3(3)}{6}$$

Plugging these values in the latter expression and simplifying gives the announced result.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru