

POWER MEANS INEQUALITY AND APPLICATIONS

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ABSTRACT. In this paper are presented power means concepts, a few connections and applications.

Proposition 1.

If $a, b > 0, a, b$ - fixed, $x \geq y > 0$ then:

$$\left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}} \geq \left(\frac{a^y + b^y}{2} \right)^{\frac{1}{y}}$$

Proof.

Let be $f : \mathbb{R} \rightarrow \mathbb{R}$:

$$\begin{aligned} f(a, b, x) &= \begin{cases} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}}; x \neq 0 \\ \sqrt{ab}; x = 0 \end{cases} \\ \lim_{\substack{x \rightarrow 0 \\ x > 0}} f(a, b, x) &= \lim_{\substack{x \rightarrow 0 \\ x > 0}} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}} = \\ &= e^{\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{1}{x} \log \left(\frac{a^x + b^x}{2} \right)} = e^{\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{a^x \log a + b^x \log b}{2} \cdot \frac{2}{a^x + b^x}} = \\ &= e^{\frac{\log a + \log b}{1+1}} = e^{\log \sqrt{ab}} = \sqrt{ab} = f(a, b, 0) \end{aligned}$$

f continuous

$$\begin{aligned} f'(a, b, x) &= \frac{1}{x} \left(\frac{a^x + b^x}{2} \right)' \cdot \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}-1} - \frac{1}{x^2} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}} \cdot \log \left(\frac{a^x + b^x}{2} \right) \\ f'(a, b, x) &= \frac{1}{x} \cdot \frac{a^x \log a + b^x \log b}{2} \cdot \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}-1} - \frac{1}{x^2} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}} \cdot \log \left(\frac{a^x + b^x}{2} \right) \\ x^2 f'(a, b, x) &= \frac{x(a^x \log a + b^x \log b)}{2} \cdot \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}-1} - \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}} \cdot \log \left(\frac{a^x + b^x}{2} \right) \\ (1) \quad x^2 f'(a, b, x) &= \frac{1}{2} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}-1} \left(a^x \log a^x + b^x \log b^x - (a^x + b^x) \log \left(\frac{a^x + b^x}{2} \right) \right) \end{aligned}$$

Define $g : (0, \infty) \rightarrow \mathbb{R}; g(x) = x \log x$

$$g'(x) = \log x + 1; g''(x) = \frac{1}{x} > 0; g - \text{convexe}$$

By Jensen's inequality:

$$g(u) + g(v) \geq 2g\left(\frac{u+v}{2}\right); u, v > 0$$

For $u = a^x; v = b^x$

$$\begin{aligned} g(a^x) + g(b^x) &\geq 2g\left(\frac{a^x + b^x}{2}\right) \\ a^x \log a^x + b^x \log b^x &\geq 2 \cdot \frac{a^x + b^x}{2} \cdot \log\left(\frac{a^x + b^x}{2}\right) \end{aligned}$$

$$(2) \quad a^x \log a^x + b^x \log b^x - (a^x + b^x) \log \left(\frac{a^x + b^x}{2} \right) \geq 0$$

By (1); (2):

$$\begin{aligned} x^2 f'(a, b, x) &\geq 0 \Rightarrow f \text{ increasing} \\ x \geq y > 0; f \text{ increasing} &\Rightarrow f(a, b, x) \geq f(a, b, y) \\ \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}} &\geq \left(\frac{a^y + b^y}{2} \right)^{\frac{1}{y}} \end{aligned}$$

Corollary 1:

f increasing and $2 > 1 > 0 > -1 \Rightarrow$

$$\begin{aligned} f(a, b, 2) &\geq f(a, b, 1) \geq f(a, b, 0) \geq f(a, b, -1) \\ \left(\frac{a^2 + b^2}{2} \right)^{\frac{1}{2}} &\geq \left(\frac{a^1 + b^1}{2} \right)^{\frac{1}{1}} \geq \sqrt{ab} \geq \left(\frac{a^{-1} + b^{-1}}{2} \right)^{\frac{-1}{-1}} \\ \sqrt{\frac{a^2 + b^2}{2}} &\geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}} \end{aligned}$$

Corollary 2:

If $n \in \mathbb{N}; n \geq 1$;

$$n > n-1 > n-2 > \dots > 3 > 2 > 1 > 0$$

f increasing, then:

$$\begin{aligned} f(a, b, n) &\geq f(a, b, n-1) \geq \dots \geq f(a, b, 1) \geq f(a, b, 0) \\ \left(\frac{a^n + b^n}{2} \right)^{\frac{1}{n}} &\geq \left(\frac{a^{n-1} + b^{n-1}}{2} \right)^{\frac{1}{n-1}} \geq \dots \geq \left(\frac{a^1 + b^1}{2} \right)^{\frac{1}{1}} \geq \sqrt{ab} \\ \sqrt[n]{\frac{a^n + b^n}{2}} &\geq \sqrt[n-1]{\frac{a^{n-1} + b^{n-1}}{2}} \geq \dots \geq \frac{a+b}{2} \geq \sqrt{ab} \end{aligned}$$

Corollary 3:

If $n \in \mathbb{N}; n \geq 1; f$ increasing;

$$\begin{aligned} 0 < \frac{1}{n} < \frac{1}{n-1} < \frac{1}{n-2} < \dots < \frac{1}{3} < \frac{1}{2} < 1 \\ f(a, b, 0) &\leq f\left(a, b, \frac{1}{n}\right) \leq f\left(a, b, \frac{1}{n-1}\right) \leq \dots \leq \\ &\leq f\left(a, b, \frac{1}{3}\right) \leq f\left(a, b, \frac{1}{2}\right) \leq f(a, b, 1) \\ \sqrt{ab} &\leq \left(\frac{a^{\frac{1}{n}} + b^{\frac{1}{n}}}{2} \right)^n \leq \left(\frac{a^{\frac{1}{n-1}} + b^{\frac{1}{n-1}}}{2} \right)^{n-1} \leq \dots \\ \dots &\leq \left(\frac{a^{\frac{1}{3}} + b^{\frac{1}{3}}}{2} \right)^3 \leq \left(\frac{a^{\frac{1}{2}} + b^{\frac{1}{2}}}{2} \right)^2 \leq \frac{a+b}{2} \\ \sqrt{ab} &\leq \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} \right)^n \leq \left(\frac{\sqrt[n-1]{a} + \sqrt[n-1]{b}}{2} \right)^{n-1} \leq \dots \\ \dots &\leq \left(\frac{\sqrt[3]{a} + \sqrt[3]{b}}{2} \right)^3 \leq \left(\frac{\sqrt{a} + \sqrt{b}}{2} \right)^2 \leq \frac{a+b}{2} \end{aligned}$$

Observation:

In corollaries 1,2,3 equality holds for $a = b$

□

Proposition 2.

If $a, b, c > 0$; a, b, c - fixed; $x \geq y > 0$ then:

$$\left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}} \geq \left(\frac{a^y + b^y + c^y}{3} \right)^{\frac{1}{y}}$$

Proof.

Let be $f : \mathbb{R} \rightarrow \mathbb{R}$;

$$\begin{aligned} f(a, b, x) &= \begin{cases} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}}; x \neq 0 \\ \sqrt[3]{abc}; x = 0 \end{cases} \\ \lim_{\substack{x \rightarrow 0 \\ x > 0}} f(a, b, c, x) &= \lim_{\substack{x \rightarrow 0 \\ x > 0}} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}} = \\ &= e^{\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{1}{x} \log \left(\frac{a^x + b^x + c^x}{3} \right)} = \\ &= e^{\lim_{\substack{x \rightarrow 0 \\ x > 0}} \frac{a^x \log a + b^x \log b + c^x \log c}{3} \cdot \frac{3}{a^x + b^x + c^x}} = \\ &= e^{\frac{\log a + \log b + \log c}{1+1+1}} = e^{\log \sqrt[3]{abc}} = f(a, b, c, 0) \end{aligned}$$

f continuous

$$\begin{aligned} f'(a, b, c, x) &= \frac{1}{x} \left(\frac{a^x + b^x + c^x}{2} \right)' \cdot \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}-1} - \\ &\quad - \frac{1}{x^2} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}} \cdot \log \left(\frac{a^x + b^x + c^x}{3} \right) \\ f'(a, b, c, x) &= \frac{1}{x} \cdot \frac{a^x \log a + b^x \log b + c^x \log c}{3} \cdot \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}-1} - \\ &\quad - \frac{1}{x^2} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}} \cdot \log \left(\frac{a^x + b^x + c^x}{3} \right) \\ x^2 f'(a, b, c, x) &= \frac{x(a^x \log a + b^x \log b + c^x \log c)}{3} \cdot \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}-1} - \\ &\quad - \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}} \cdot \log \left(\frac{a^x + b^x + c^x}{3} \right) \\ (3) \quad x^2 f'(a, b, c, x) &= \frac{1}{3} \left(\frac{a^x + b^x + c^x}{3} \right)^{\frac{1}{x}-1} \left(a^x \log a + b^x \log b + c^x \log c - (a^x + b^x + c^x) \log \left(\frac{a^x + b^x + c^x}{3} \right) \right) \end{aligned}$$

Define $g : (0, \infty) \rightarrow \mathbb{R}$; $g(x) = x \log x$

$$g'(x) = \log x + 1; g''(x) = \frac{1}{x} > 0; g - \text{convexe}$$

By Jensen's inequality:

$$g(u) + g(v) + g(w) \geq 3g\left(\frac{u+v+w}{3}\right); u, v, w > 0$$

For $u = a^x; v = b^x; w = c^x$

$$\begin{aligned} g(a^x) + g(b^x) + g(c^x) &\geq 3g\left(\frac{a^x + b^x + c^x}{3}\right) \\ a^x \log a^x + b^x \log b^x + c^x \log c^x &\geq 3 \cdot \frac{a^x + b^x + c^x}{3} \log \left(\frac{a^x + b^x + c^x}{3} \right) \\ (4) \quad a^x \log a^x + b^x \log b^x + c^x \log c^x - (a^x + b^x + c^x) \log \left(\frac{a^x + b^x + c^x}{3} \right) &\geq 0 \end{aligned}$$

By (3);(4):

$$\begin{aligned} x^2 f'(a, b, c, x) \geq 0 &\Rightarrow f \text{ increasing} \\ x \geq y > 0; f \text{ increasing} &\Rightarrow f(a, b, x) \geq f(a, b, y) \\ \left(\frac{a^x + b^x + c^x}{3}\right)^{\frac{1}{x}} &\geq \left(\frac{a^y + b^y + c^y}{3}\right)^{\frac{1}{y}} \end{aligned}$$

Corollary 4:

If f increasing and $2 > 1 > 0 > -1$

$$\begin{aligned} f(a, b, c, 2) &\geq f(a, b, c, 1) \geq f(a, b, c, 0) \geq f(a, b, c, -1) \\ \left(\frac{a^2 + b^2 + c^2}{3}\right)^{\frac{1}{2}} &\geq \left(\frac{a^1 + b^1 + c^1}{3}\right)^{\frac{1}{1}} \geq \sqrt[3]{abc} \geq \left(\frac{a^{-1} + b^{-1} + c^{-1}}{3}\right)^{\frac{1}{-1}} \\ \sqrt{\frac{a^2 + b^2 + c^2}{3}} &\geq \frac{a + b + c}{3} \geq \sqrt[3]{abc} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \end{aligned}$$

Corollary 5:

If $n \in \mathbb{N}; n \geq 1$; f increasing and:

$$n > n - 1 > n - 2 > \dots > 3 > 2 > 1 > 0$$

$$\begin{aligned} f(a, b, c, n) &\geq f(a, b, c, n - 1) \geq \dots \geq f(a, b, c, 1) \geq f(a, b, c, 0) \\ \left(\frac{a^n + b^n + c^n}{3}\right)^{\frac{1}{n}} &\geq \left(\frac{a^{n-1} + b^{n-1} + c^{n-1}}{3}\right)^{\frac{1}{n-1}} \geq \dots \geq \left(\frac{a^1 + b^1 + c^1}{3}\right)^{\frac{1}{1}} \geq \sqrt[3]{abc} \\ \sqrt[n]{\frac{a^n + b^n + c^n}{3}} &\geq \sqrt[n-1]{\frac{a^{n-1} + b^{n-1} + c^{n-1}}{3}} \geq \dots \geq \sqrt{\frac{a^2 + b^2 + c^2}{3}} \geq \frac{a + b + c}{3} \geq \sqrt[3]{abc} \end{aligned}$$

Corollary 6:

If $n \in \mathbb{N}; n \geq 1$; f increasing;

$$\begin{aligned} 0 < \frac{1}{n} < \frac{1}{n-1} < \frac{1}{n-2} < \dots < \frac{1}{3} < \frac{1}{2} < 1 \\ f(a, b, c, 0) &\leq f\left(a, b, c, \frac{1}{n}\right) \leq f\left(a, b, c, \frac{1}{n-1}\right) \leq \dots \\ &\dots \leq f\left(a, b, c, \frac{1}{3}\right) \leq f\left(a, b, c, \frac{1}{2}\right) \leq f(a, b, c, 1) \\ \sqrt[3]{abc} &\leq \left(\frac{a^{\frac{1}{n}} + b^{\frac{1}{n}} + c^{\frac{1}{n}}}{3}\right)^n \leq \left(\frac{a^{\frac{1}{n-1}} + b^{\frac{1}{n-1}} + c^{\frac{1}{n-1}}}{3}\right)^{n-1} \leq \dots \\ &\dots \leq \left(\frac{a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}}{3}\right)^3 \leq \left(\frac{a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}}}{3}\right)^3 \leq \frac{a + b + c}{3} \\ \sqrt[3]{abc} &\leq \left(\frac{\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c}}{3}\right)^n \leq \left(\frac{\sqrt[n-1]{a} + \sqrt[n-1]{b} + \sqrt[n-1]{c}}{3}\right)^{n-1} \leq \dots \\ &\dots \leq \left(\frac{\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}}{3}\right)^3 \leq \left(\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3}\right)^2 \leq \frac{a + b + c}{3} \end{aligned}$$

Observation:

In corollaries 4,5,6 equality holds for $a = b = c$. □

REFERENCES

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