R M M

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Let $A(x)$ be the following $n\times n$ real matrix

$$A(x)=\begin{pmatrix} x^2 & 1 & x^2 & 1 & \dots & x^2 & 1\\ 1 & x^{-2} & 1 & x^{-2} & \dots & 1 & x^{-2}\\ x^2 & 1 & x^2 & 1 & \dots & x^2 & 1\\ 1 & x^{-2} & 1 & x^{-2} & \dots & 1 & x^{-2}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ x^2 & 1 & x^2 & 1 & \dots & x^2 & 1\\ 1 & x^{-2} & 1 & x^{-2} & \dots & 1 & x^{-2}\end{pmatrix};n-\text{even and } F(x)=\det(I_n+A(x))$$

Prove that:

$$\int_0^\infty \frac{1}{(1-F(x))\left(1-F\left(e^{\frac{\pi x^2}{8}}\right)\right)}\frac{dx}{x}=\frac{\pi-2\log(1+\sqrt{2})}{n^2\sqrt{2}}$$

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Solution by proposer

Let $X=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ and $||X||^2=\sum_{k=1}^n x_kx_k$. If regard X as an element of $Mat(n\times n,\mathbb{R})$, then $X^T\in Mat(1\times n,\mathbb{R})$ and $XX^T=(x_jx_k)_{1\le j,k\le n}\in Mat(n,\mathbb{R})$. Let

$$J_n=\int_0^\infty \frac{1}{(1-F(x))\left(1-F\left(e^{\frac{\pi x^2}{8}}\right)\right)}\frac{dx}{x}$$

We claim that:

$$\mathbb{A}.\ \det(I_n+XX^T)=1+||X||^2;\forall X\in\mathbb{R}^n$$

$$\mathbb{B}.\ \det(I_n+A(x))=1+\frac{n}{2}\left(x^2+\frac{1}{x^2}\right);\forall x\in\mathbb{R}^*$$

$$\mathbb{C}.\ J_n=\frac{1}{n^2}\int_0^\infty\frac{dx}{(1+x^2)\cosh\left(\frac{\pi x}{4}\right)}=\frac{4}{n^2}\sum_{m=0}^\infty\frac{(-1)^m}{4m+3}$$

Proof of claim $\mathbb{A}$: Let $E=(1,0,\dots,0)\in\mathbb{R}^n$ and $X=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ such that $||X||\neq 0$, if we regard X and E as elements of $Mat(n\times 1,\mathbb{R})$, then there exists an orthogonal matrix $R\in O(n,\mathbb{R}),R^tR=I_n$, such that $RX=||X||E$.

Since $\det(RMR^T)=\det(MR^TR)=\det(M)$, for any $M\in Mat(n,\mathbb{R})$, then

$$\det(I_n + XX^T) = \det[R(I_n + XX^T)R^T] = \det(I_n + (RX)(RX)^T) = \det\left(I_n + ||X||^2 EE^T\right)$$

Since EE^T is the diagonal matrix $diag(1, 0, \dots, 0)$, then

$I_n + ||X||^2 EE^T = diag(1 + ||X||^2, 1, \dots, 1)$ and

$$\det(I_n + XX^T) = \det\left(I_n + ||X||^2 EE^T\right) = 1 + ||X||^2$$

The claim $\mathbb{A}$ is demonstrated.

Proof of claim $\mathbb{B}$: Let $X = \left(x, \frac{1}{x}, x, \frac{1}{x}, \dots, x, \frac{1}{x}\right) \in \mathbb{R}^n, x \in \mathbb{R}$ and n is even, then

$||X||^2 = \frac{n}{2}\left(x^2 + \frac{1}{x^2}\right)$ and $XX^T = A(x)$. It follows from claim $\mathbb{A}$ that

$$\det\big(I_n + A(x)\big) = \det(I_n + XX^T) = 1 + ||X||^2 = 1 + \frac{n}{2}\left(x^2 + \frac{1}{x^2}\right)$$

The claim $\mathbb{B}$ is demonstrated.

Proof of claim $\mathbb{C}$: Let $F(x) = \det\big(I_n + A(x)\big)$, it follows from claim $\mathbb{B}$ that

$$1 - F(x) = 1 - \det\big(I_n + A(x)\big) = -\frac{n}{2}\left(x^2 + \frac{1}{x^2}\right) \text{ and}$$

$$1 - F\left(e^{\frac{\pi x^2}{8}}\right) = -\frac{n}{2}\left(e^{\frac{\pi x^2}{4}} + e^{-\frac{\pi x^2}{4}}\right) = -n\cosh\left(\frac{\pi x^2}{4}\right)$$

Thus,

$$J_n = \int_0^\infty \frac{1}{\big(1 - F(x)\big)\left(1 - F\left(e^{\frac{\pi x^2}{8}}\right)\right)} \frac{dx}{x} = \frac{2}{n^2}\int_0^\infty \frac{x\,dx}{(1 + x^4)\cosh\left(\frac{\pi x^2}{4}\right)} \overset{x^2 \to x}{=}$$

$$= \frac{1}{n^2}\int_0^\infty \frac{dx}{(1 + x^2)\cosh\left(\frac{\pi x}{4}\right)}$$

Now, use the partial fraction expansion of $\cosh\left(\frac{\pi x}{4}\right)$

$$\frac{1}{\cosh\left(\frac{\pi x}{4}\right)} = \frac{16}{\pi}\sum_{m=0}^\infty (-1)^m \frac{2m + 1}{x^2 + 4(2m + 1)}$$

To rewrite J_n as

$$J_n = \frac{16}{\pi n^2}\sum_{m=0}^\infty (-1)^m (2m + 1)\int_0^\infty \frac{1}{1 + x^2}\frac{dx}{x^2 + 4(2m + 1)^2}dx$$

The integral on the right is equal to

$$\int_0^{\infty} \frac{1}{1+x^2} \frac{dx}{x^2+4(2m+1)^2} dx = \frac{\pi}{4} \frac{1}{(4m+3)(2m+1)}$$

So,

$$J_n = \frac{1}{n^2} \int_0^{\infty} \frac{dx}{(1+x^2)\cosh\left(\frac{\pi x}{n}\right)} = \frac{4}{n^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{4m+3}; (1)$$

The claim $\mathbb{C}$ is then demonstrated.

The summation on the right of Eq. (1) is equal to

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{4m+3} = \sum_{m=0}^{\infty} (-1)^m \int_0^1 x^{4m+2} dx = \int_0^1 \frac{x^2}{1+x^4} dx$$

$$J_n = \frac{4}{n^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{4m+3} = \frac{4}{n^2} \int_0^1 \frac{x^2}{1+x^4} dx$$

Which can be rewritten as

$$J_n = \frac{2}{n^2} \int_0^1 \frac{x^2-1}{1+x^4} dx + \frac{2}{n^2} \int_0^1 \frac{x^2+1}{1+x^4} dx$$

$$J_n = \frac{2}{n^2} \int_0^1 \frac{1+x^{-2}}{(x+x^{-1})^2+2} dx + \frac{2}{n^2} \int_0^1 \frac{1-x^{-2}}{(x+x^{-1})^2-2} dx$$

Making the change of variable $u = -(x - x^{-1})$ in the first integral and $u = x + x^{-1}$ in the second integral, one has

$$J_n = \frac{2}{n^2} \int_0^{\infty} \frac{du}{u^2+2} - \frac{2}{n^2} \int_2^{\infty} \frac{du}{u^2-2} = \frac{\pi}{\sqrt{2}n^2} - \frac{1}{\sqrt{2}n^2} \log\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)$$

Therefore,

$$\int_0^{\infty} \frac{1}{(1-F(x))\left(1-F\left(e^{\frac{\pi x^2}{8}}\right)\right)} \frac{dx}{x} = \frac{\pi - 2\log(1+\sqrt{2})}{n^2\sqrt{2}}$$