

## CRUX MATHEMATICORUM CHALLENGES-(I)

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4073. Solve the following system:

$$\begin{cases} \sin 2x + \cos 3y = -1 \\ \sqrt{\sin^2 x + \sin^2 y} + \sqrt{\cos^2 x + \cos^2 y} = 1 + \sin(x + y) \end{cases}$$

*Daniel Sitaru*

*Solution by Michele Bataille.*

We first show that the second equation is equivalent to  $x + y \equiv \frac{\pi}{2} \pmod{2\pi}$ .

If  $x + y \equiv \frac{\pi}{2} \pmod{2\pi}$ , then  $\sin^2 y = \cos^2 x$  and  $\cos^2 y = \sin^2 x$ . It immediately follows that both sides of the equation equal 2. Conversely, if the equation holds, then squaring gives

$$2\sqrt{(\sin x \cos x - \sin y \cos y)^2 + \sin^2(x + y)} = 2\sin(x + y) - (1 - \sin^2(x + y))$$

and therefore

$$\begin{aligned} 2\sin(x + y) &\leq 2\sqrt{\sin^2(x + y)} \leq 2\sqrt{(\sin x \cos x - \sin y \cos y)^2 + \sin^2(x + y)} \\ &= 2\sin(x + y) - (1 - \sin^2(x + y)) \leq 2\sin(x + y) \end{aligned}$$

Thus, equality must hold throughout and in particular  $\sin(x + y) \geq 0$  and  $\sin^2(x + y) = 1$ . We deduce that  $x + y \equiv \frac{\pi}{2} \pmod{2\pi}$ .

Since  $\cos 3(\frac{\pi}{2} - x) = -\sin 3x$ , we are led to seek the solutions to the equation  $f(x) = 1$  where  $f(x) = \sin 3x - \sin 2x$ . Note that  $f(-\frac{\pi}{2}) = 1$  so that the numbers  $-\frac{\pi}{2} + 2k\pi (k \in \mathbb{Z})$  are solutions. For other solutions note that  $f$  is odd and  $2\pi$ -periodic; consequently, we may restrict the study of  $f$  to the interval  $[0, \pi]$  and look for  $x$  satisfying either  $f(x) = 1$  or  $f(x) = -1$  (the latter since then  $f(-x) = 1$ ). Consider first the interval  $[0, \frac{\pi}{2}]$ . We have  $f(0) = 0$  and if  $x \in (0, \frac{\pi}{2})$ , then  $\sin 2x > 0$  and so  $f(x) < 1$ .

Thus, equality must hold throughout and in particular  $\sin(x + y) \geq 0$  and  $\sin^2(x + y) = 1$ . We deduce that  $x + y \equiv \frac{\pi}{2} \pmod{2\pi}$ .

Since  $\cos 3(\frac{\pi}{2} - x) = -\sin 3x$ , we are led to seek the solutions to the equation  $f(x) = 1$  where  $f(x) = \sin 3x - \sin 2x$ . Note that  $f(-\frac{\pi}{2}) = 1$  so that the numbers  $-\frac{\pi}{2} + 2k\pi (k \in \mathbb{Z})$  are solutions. For other solutions note that  $f$  is odd and  $2\pi$ -periodic; consequently, we may restrict the study of  $f$  to the interval  $[0, \pi]$  and look for  $x$  satisfying either  $f(x) = 1$  or  $f(x) = -1$  (the latter since then  $f(-x) = 1$ ). Consider first the interval  $[0, \frac{\pi}{2}]$ . We have  $f(0) = 0$  and if  $x \in (0, \frac{\pi}{2})$  then  $\sin 2x > 0$  and so  $f(x) < 1$ .

- $x \in (0, \frac{\pi}{3}]$ :  $\sin 3x > 0$  for  $x$  between 0 and  $\frac{\pi}{3}$ , hence  $f(x) > -1$ ; since  $f(\frac{\pi}{3}) > -1$ , there is no  $x \in (0, \frac{\pi}{3}]$  such that  $f(x) = -1$ .
- $x \in (\frac{\pi}{3}, \frac{\pi}{2})$ :  $f''(x) = 4\sin 2x - 9\sin 3x > 0$ , hence  $f'(x) = 3\cos 3x - 2\cos 2x$  is nondecreasing on the interval  $(\frac{\pi}{3}, \frac{\pi}{2})$ . For some  $x_1 \in (\frac{\pi}{3}, \frac{\pi}{2})$ , we have  $f'(x) \leq 0$  for  $x \in (\frac{\pi}{3}, x_1]$  and  $f'(x) > 0$  for  $x \in (x_1, \frac{\pi}{2})$ . Since  $f(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

and  $f\left(\frac{\pi}{2}\right) = -1$ , we have  $f(x_1) < -1$  and  $f(\alpha) = -1$  for a unique  $\alpha$  in  $(\frac{\pi}{3}, \frac{\pi}{2})$ .

In a similar way we treat the interval  $(\frac{\pi}{2}, \pi]$ . We have  $f(\pi) = 0$  and if  $x \in (\frac{\pi}{2}, \pi)$ , then  $\sin 2x < 0$ , hence  $f(x) > -1$ .

- $x \in (\frac{\pi}{2}, \frac{3\pi}{4}) : f'(x) > 0$  and so  $f$  is increasing from  $-1$  to  $1 + \frac{\sqrt{2}}{2}$ . Thus,  $f(\beta) = 1$  for a unique  $\beta \in (\frac{\pi}{2}, \frac{3\pi}{4})$ .
- $x \in (\frac{5\pi}{6}, \pi) : f$  is decreasing from  $1 + \frac{\sqrt{3}}{2}$  to  $0$ , hence  $f(\gamma) = 1$  for a unique  $\gamma$  of  $(\frac{5\pi}{6}, \pi)$ .
- $x \in [\frac{3\pi}{4}, \frac{5\pi}{6}] :$ Resorting to  $f''(x)$ , we see that  $f'(x)$  decreases from positive to negative so that  $f(x) > 1$ .

In conclusion, on the interval  $[-\pi, \pi]$  the solutions  $(x, y)$  of the system are the pairs  $(-\frac{\pi}{2}, \pi), (-\alpha, \frac{\pi}{2} + \alpha), (\beta, \frac{\pi}{2} - \beta)$  and  $(\gamma, \frac{\pi}{2} - \gamma)$ .

All other solutions are obtained by adding multiples of  $2\pi$  to  $x$  or  $y$ .  $\square$

*Solution by Daniel Sitaru.*

From Cauchy-Schwarz inequality:

$$\begin{aligned} \sin x \cos y + \sin^2 y &\leq \sqrt{\sin^2 x + \sin^2 y} \cdot \sqrt{\cos^2 y + \sin^2 y} \\ \sin y \cos x + \sin^2 x &\leq \sqrt{\sin^2 y + \sin^2 x} \cdot \sqrt{\cos^2 x + \sin^2 x} \\ (1) \quad \sin^2 x + \sin^2 y + \sin(x+y) &\leq 2\sqrt{\sin^2 y + \sin^2 x} \end{aligned}$$

Analogous:

$$(2) \quad \cos^2 x + \cos^2 y + \sin(x+y) \leq 2\sqrt{\cos^2 x + \cos^2 y}$$

By adding (1) with (2)

$$\begin{aligned} 2 + 2\sin(x+y) &\leq 2\sqrt{\sin^2 y + \sin^2 x} + 2\sqrt{\cos^2 x + \cos^2 y} \\ 1 + \sin(x+y) &\leq \sqrt{\sin^2 y + \sin^2 x} + \sqrt{\cos^2 x + \cos^2 y} \end{aligned}$$

The inequality holds if:

$$\sin x = \cos y \text{ and } \sin y = \cos x \text{ wherefrom}$$

$$x = m\pi; y = n\pi; m, n \in \mathbb{Z} \text{ or } x + y = \frac{\pi}{2} + 2p\pi; p \in \mathbb{Z}$$

We write the first equation:

$$\sin 2m\pi + \cos 3n\pi = -1 \text{ wherefrom: } m \in \mathbb{Z}$$

$$(-1)^{3n} = -1; n \in \mathbb{Z}; n \text{ odd}$$

solution:  $\begin{cases} x = m\pi; m \in \mathbb{Z} \\ y = (2p+1)\pi; p \in \mathbb{Z} \end{cases}$

$\square$

4081. Determine all  $A, B \in M_2(\mathbb{R})$  such that:

$$\begin{cases} A^2 + B^2 = \begin{pmatrix} 22 & 44 \\ 14 & 28 \end{pmatrix} \\ AB + BA = \begin{pmatrix} 10 & 20 \\ 2 & 4 \end{pmatrix} \end{cases}$$

*Daniel Sitaru*

*Solution by Joseph DiMuro.*

Summing the two equations, we obtain:

$$(A + B)^2 = A^2 + AB + BA + B^2 = \begin{pmatrix} 32 & 64 \\ 16 & 32 \end{pmatrix}$$

We can diagonalize this matrix in order to find its square roots:

$$(A + B)^2 = PDP^{-1} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 64 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

$$A + B = PD^{\frac{1}{2}}P^{-1} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pm 8 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix} = \pm \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}$$

We can also subtract the original two equations to obtain:

$$(A - B)^2 = A^2 - AB - BA + B^2 = \begin{pmatrix} 12 & 24 \\ 12 & 24 \end{pmatrix}$$

As before, we diagonalize this matrix:

$$(A - B)^2 = PDP^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 36 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix},$$

$$A - B = PD^{\frac{1}{2}}P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pm 6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} = \pm \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$$

Now we have the two equations

$$A + B = \pm \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}, A - B = \pm \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix},$$

which can easily be solved to produce four possible pairs of matrices for  $A$  and  $B$ .  
One solution is

$$A = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

The other solutions may be obtained by interchanging  $A$  and  $B$ , and/or replacing  $A$  and  $B$  with their negatives.  $\square$

*Solution by Daniel Sitaru.*

Let be  $X = \frac{1}{2}(A + B)$ ;  $Y = \frac{1}{2}(A - B)$ .

It follows:  $A = X + Y$ ;  $B = X - Y$ .

$$A^2 + B^2 = (X + Y)^2 + (X - Y)^2 = 2(X^2 + Y^2)$$

$$AB + BA = (X + Y)(X - Y) + (X - Y)(X + Y) = 2(X^2 - Y^2)$$

We write the system:

$$\begin{cases} 2(X^2 + Y^2) = \begin{pmatrix} 22 & 44 \\ 14 & 28 \end{pmatrix} \\ 2(X^2 - Y^2) = \begin{pmatrix} 10 & 20 \\ 2 & 4 \end{pmatrix} \end{cases} \Rightarrow \begin{cases} X^2 + Y^2 = \begin{pmatrix} 11 & 22 \\ 7 & 14 \end{pmatrix} \\ X^2 - Y^2 = \begin{pmatrix} 5 & 10 \\ 1 & 2 \end{pmatrix} \end{cases}$$

$$\text{By adding: } 2X^2 = \begin{pmatrix} 16 & 32 \\ 8 & 16 \end{pmatrix} \Rightarrow X^2 = \begin{pmatrix} 8 & 16 \\ 4 & 8 \end{pmatrix}$$

$$\text{If } X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ it follows: } \begin{cases} (a+d)a = 8 \\ (a+d)b = 16 \\ (a+d)c = 4 \\ (a+d)d = 8 \end{cases}$$

$$\text{wherfrom } a = d; b = 2a; c = \frac{a}{2} \Rightarrow X = \begin{pmatrix} a & 2a \\ \frac{a}{2} & a \end{pmatrix} \text{ and then } a^2 = 4 \Rightarrow a = \pm 2$$

$$X_1 = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}; X_2 = \begin{pmatrix} -2 & -4 \\ -1 & -2 \end{pmatrix}$$

Analogous:

$$\begin{cases} X^2 + Y^2 = \begin{pmatrix} 11 & 22 \\ 7 & 14 \end{pmatrix} \\ -X^2 + Y^2 = \begin{pmatrix} -5 & -10 \\ -1 & -2 \end{pmatrix} \end{cases} \Rightarrow 2Y^2 = \begin{pmatrix} 6 & 12 \\ 6 & 12 \end{pmatrix}$$

$$Y^2 = \begin{pmatrix} 3 & 6 \\ 3 & 6 \end{pmatrix}. \text{ If } Y = \begin{pmatrix} u & v \\ p & q \end{pmatrix} \text{ then:}$$

$$\begin{cases} (u+q)u = 3 \\ (u+q)v = 6 \\ (u+q)p = 3 \\ (u+q)q = 6 \end{cases} \Rightarrow \begin{cases} u = p \\ v = q = 2u \end{cases} \Rightarrow Y = \begin{pmatrix} u & 2u \\ u & 2u \end{pmatrix}$$

$$3u^2 = 3 \Rightarrow u^2 = 1 \Rightarrow u = \pm 1$$

$$Y_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}; Y = \begin{pmatrix} -1 & -2 \\ -1 & -2 \end{pmatrix}$$

$$\text{solution: } \begin{cases} A = X_i + Y_j \\ B = X_i - Y_j \end{cases} ; i, j \in \{1, 2\}$$

□

4104. Prove that for  $0 < a \leq b \leq c \leq d < 2$ , we have

$$5(ab^4 + bc^4 + cd^4 + 16d) < 5(b^5 + c^5 + d^5 + 16a) + 128$$

Daniel Sitaru

*Solution 1 by Sefket Arslanagic; and Salem Malikic (independently).*

By the arithmetic-geometric means inequality, we have

$$a^5 + 4b^5 \geq 5ab^4, b^5 + 4c^5 \geq 5bc^4, c^5 + 4d^5 \geq 5cd^4$$

and

$$d^5 + 128 = d^5 + 4 \cdot 2^5 \geq 5 \cdot 2^4 d = 80d$$

Adding these along with the positive  $4a^5$  yields that

$$5(a^5 + b^5 + c^5 + d^5) + 128 > 5(ab^4 + bc^4 + cd^4 + 16d)$$

□

*Solution 2 by the proposer.*

The function  $f(x) = 16 - x^4$  is nonnegative, decreasing and concave on  $[0, 2]$ .

Therefore

$$(b-a)f(b) + (c-b)f(b) + (d-c)f(d) < \int_0^2 f(x)dx$$

Hence

$$16(d-a) - (b^5 + c^5 + d^5) + (ab^4 + bc^4 + cd^4) < \frac{128}{5}$$

Multiplying by 5 and rearranging the terms gives the result. □

4122. Prove that for  $n \in \mathbb{N}$ , the following holds:

$$\left(\frac{e^n - 1}{n}\right)^{2n+1} \leq \frac{(e-1)(e^2-1)(e^3-1) \cdots (e^{2n}-1)}{(2n)!}$$

*Daniel Sitaru*

*Solution by Angel Plaza.*

Note that the inequality in the statement can be rewritten as

$$(1) \quad \left(\frac{e^n - 1}{n}\right)^{2n+1} \leq \left(\frac{e-1}{1}\right)\left(\frac{e^2-1}{2}\right) \cdots \left(\frac{e^{2n}-1}{2n}\right)$$

Consider the function

$$f(x) = \ln\left(\frac{e^x - 1}{x}\right)$$

defined for  $x > 0$  and set  $f(0) = 0$ . Then  $f$  is continuous for  $x \geq 0$  and has second derivative

$$f''(x) = \frac{(e^x - 1)^2 - x^2 e^x}{x^2 (e^x - 1)^2}$$

To show that  $f(x)$  is convex it suffices to prove that  $(e^x - 1)^2 - x^2 e^x > 0$ . This can be reformulated to  $e^x(e^x + e^{-x} - (2 + x^2)) > 0$ . But we have

$$e^x + e^{-x} = \sum_{k=0}^{\infty} \frac{2x^{2k}}{(2k)!} > 2 + x^2$$

Therefore, the second derivative of  $f$  is positive and  $f(x)$  is convex for  $x > 0$ .

Rephrasing inequality (0.1) by taking logarithms we obtain

$$(2n+1)f(n) \leq \sum_{k=0}^{2n} f(k)$$

which follows from Jensen's inequality. □

*Solution by Daniel Sitaru.*

Let be  $f : [0, 1] \rightarrow \mathbb{R}$ ,  $f(x) = e^x$  and  $I_n = \int_0^1 e^{nx} dx$ .

$$\begin{aligned} I_n^2 &= \left( \int_0^1 e^{nx} dx \right)^2 = \left( \int_0^1 \sqrt{e^{(n-k)x}} \cdot \sqrt{e^{(n+k)x}} dx \right)^2 \leq \\ &\leq \left( \int_0^1 e^{(n-k)x} dx \right) \left( \int_0^1 e^{(n+k)x} dx \right); 0 \leq k \leq n; k \in \mathbb{N} \end{aligned}$$

(Cauchy - Schwarz - integral form)

$$\begin{aligned} I_n^2 &\leq I_{n-k} \cdot I_{n+k}; \quad 0 \leq k \leq n \\ I_n^2 &\leq I_{n-1} \cdot I_{n+1} \\ I_n^2 &\leq I_{n-2} \cdot I_{n+2} \\ &\cdots \\ I_n^2 &\leq I_0 \cdot I_{2n} \end{aligned}$$

By multiplying:

$$\begin{aligned} I_n^{2n} &\leq I_0 I_1 \cdots I_{n-1} I_{n+1} \cdots I_{2n} \\ I_n^{2n+1} &\leq I_0 I_1 \cdots I_{2n} \\ \left( \frac{e^{nx}}{n} \Big|_0^1 \right)^{2n+1} &\leq \left( e^x \Big|_0^1 \right) \cdot \left( \frac{e^{2x}}{2} \Big|_0^1 \right) \cdots \left( \frac{e^{2nx}}{2n} \Big|_0^1 \right) \\ \left( \frac{e^n - 1}{n} \right)^{2n+1} &\leq \frac{(e-1)(e^2-1)(e^3-1) \cdots (e^{2n}-1)}{(2n)!} \end{aligned}$$

□

4135. Let  $ABC$  be a triangle with  $BC = a, AC = b, AB = c$ . Prove that the following relationship holds:

$$\sqrt{ab} + \sqrt{b} + \sqrt{c} \leq \sqrt{3 \left( \frac{a^2}{b+c-a} + \frac{b^2}{a+c-b} + \frac{c^2}{a+b-c} \right)}$$

Daniel Sitaru

*Solution by Dionne Bailey, Elsie Campbell and Charles R. Diminnie.*

Since  $f(x) = \sqrt{x}$  is concave on  $(0, \infty)$ , Jensen's theorem implies that

$$(2) \quad \sqrt{a} + \sqrt{b} + \sqrt{c} = f(a) + f(b) + f(c) \leq 3f\left(\frac{a+b+c}{3}\right) = \sqrt{3(a+b+c)}$$

By the Cauchy-Schwarz inequality, writing  $a = \frac{a}{\sqrt{b+c-a}}\sqrt{b+c-a}$  and similarly for  $b$  and  $c$ , we get

$$a+b+c \leq \left( \frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} \right)^{\frac{1}{2}} (a+b+c)^{\frac{1}{2}}$$

which (dividing both sides by  $(a+b+c)^{\frac{1}{2}}$  and multiplying by  $\sqrt{3}$ ) yields

$$(3) \quad \sqrt{3(a+b+c)} \leq \sqrt{3 \left( \frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} \right)}$$

Combining (0.2) and (3), we get the desired inequality; note that equality holds if and only if  $a = b = c$ , in other words if and only if  $\Delta ABC$  is equilateral. □

*Solution by Daniel Sitaru.*

The following relationships are considered to be known:

$$\sum \sin A \leq \frac{3\sqrt{3}}{2}; \sum \cos A \leq \frac{3}{2}$$

We replace  $A, B, C$  with  $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}, \frac{\pi}{2} - \frac{C}{2}$  and we rewrite the relationships:

$$\sum \cos \frac{A}{2} \leq \frac{3\sqrt{3}}{2} \text{ and } \sum \sin \frac{A}{2} \leq \frac{3}{2}$$

From Cauchy - Schwarz inequality:

$$\left( \sum \sqrt{\sin \frac{A}{2} \cos \frac{A}{2}} \right)^2 \leq \left( \sum \sin \frac{A}{2} \right) \left( \sum \cos \frac{A}{2} \right) \leq \frac{3\sqrt{3}}{2} \cdot \frac{3}{2} = \frac{9\sqrt{3}}{4}$$

$$\sum \sqrt{\sin \frac{A}{2} \cos \frac{A}{2}} \leq \frac{3\sqrt[4]{3}}{2} \text{ which can be written successively:}$$

$$\sum \sqrt{\sqrt{\frac{(p-b)(p-c)}{bc}} \sqrt{\frac{p(p-a)}{bc}}} \leq \frac{3\sqrt[4]{3}}{2}$$

$$\sum \sqrt{\frac{S}{bc}} \leq \frac{3\sqrt[4]{3}}{2}; \sum \sqrt{\frac{abc}{4Rbc}} \leq \frac{3\sqrt[4]{3}}{2};$$

$$\sum \sqrt{\frac{a}{R}} \leq \frac{3\sqrt[4]{3}}{2}; \sum \sqrt{\frac{a}{R}} \leq 3\sqrt[4]{3};$$

$$(1) \quad \sum \sqrt{a} \leq 3\sqrt[4]{3}\sqrt{R}$$

On the other hand:

$$\begin{aligned} \frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} &= \frac{1}{2} \left( \frac{a^2}{p-a} + \frac{b^2}{p-b} + \frac{c^2}{p-c} \right) = \\ &= \frac{1}{2} \left[ \left( \frac{pa}{p-a} - a \right) + \left( \frac{pb}{p-b} - b \right) + \left( \frac{pc}{p-c} - c \right) \right] = \\ &= \frac{p}{2} \left( \frac{p}{p-a} + \frac{b}{p-b} + \frac{c}{p-c} \right) - p = \\ &= p \left( \frac{2R}{r} - 1 \right) - p = 2p \left( \frac{R}{r} - 1 \right) = \frac{2p(R-r)}{r} \geq \\ &\geq \frac{2 \cdot 3\sqrt{3}r \cdot (R-r)}{r} = 3\sqrt{3}(2R-2r) \geq 3\sqrt{3}R \end{aligned}$$

because  $R - 2r \geq 0$  (Euler)

We proved that:

$$\begin{aligned} 3\sqrt{3}R &\leq \frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} \\ \sqrt{3} \cdot \sqrt[4]{3}\sqrt{R} &\leq \sqrt{\sum \left( \frac{a^2}{b+c-a} \right)} \\ (2) \quad 3\sqrt[4]{3} \cdot \sqrt{R} &\leq \sqrt{3 \sum \left( \frac{a^2}{b+c-a} \right)} \end{aligned}$$

From (1);(2) it follows:

$$\begin{aligned}\sum \sqrt{a} &\leq \sqrt{3 \sum \left( \frac{a^2}{b+c-a} \right)} \\ \sqrt{a} + \sqrt{b} + \sqrt{c} &\leq \sqrt{3 \left( \frac{a^2}{b+c-a} + \frac{b^2}{a+c-b} + \frac{c^2}{a+b-c} \right)}\end{aligned}$$

□

4136. Prove that if  $a, b, c \in (0, \infty)$  then:

$$b \int_0^a e^{-t^2} dt + c \int_0^b e^{-t^2} dt + a \int_0^c e^{-t^2} dt < \frac{\pi}{2} \sqrt{3(a^2 + b^2 + c^2)}$$

*Daniel Sitaru, Mihály Bencze*

*Solution by Arkady Alt, Sefket Arslanagic, Paul Bracken and Digby Smith (independently).*

We use  $S$  to denote the left side of the given inequality. Since

$$\int_0^x e^{-t^2} dt \leq \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

for  $x = a, b$  and  $c$ , we have by the Cauchy-Schwarz Inequality that

$$\begin{aligned}S &\leq \sqrt{b^2 + c^2 + a^2} \cdot \sqrt{\left( \int_0^a e^{-t^2} dt \right)^2 + \left( \int_0^b e^{-t^2} dt \right)^2 + \left( \int_0^c e^{-t^2} dt \right)^2} \\ &\leq \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{3 \left( \frac{\sqrt{\pi}}{2} \right)^2} = \frac{\sqrt{\pi}}{2} \cdot \sqrt{3(a^2 + b^2 + c^2)} \\ &< \frac{\pi}{2} \cdot \sqrt{3(a^2 + b^2 + c^2)}\end{aligned}$$

□

*Solution by Daniel Sitaru.*

Let be  $f : [0, \infty) \rightarrow \mathbb{R}; f(x) = \arctan x - \int_0^x e^{-t^2} dt$

$f'(x) = \frac{e^{x^2} - (1+x^2)}{(1+x^2)e^{x^2}} \geq 0; (\forall)x \geq 0$  because  $e^x \geq 1+x^2$ . It follows  $f(x) \geq f(0) = 0 \Rightarrow$

$$\arctan x - \int_0^x e^{-t^2} dt \geq 0 \Rightarrow \int_0^x e^{-t^2} dt \leq \arctan x$$

Let be  $x = a \Rightarrow \int_0^a e^{-t^2} dt \leq \arctan a$

By multiplying with  $b$  it follows:

$$b \int_0^a e^{-t^2} dt \leq b \arctan a$$

and analogous:

$$c \int_0^b e^{-t^2} dt \leq c \arctan b; a \int_0^c e^{-t^2} dt \leq a \arctan c$$

By adding:

$$(0.1) \quad \sum b \int_0^a e^{-t^2} dt \leq \sum b \arctan a$$

From Cauchy - Schwarz inequality:

$$\left( \sum b \arctan a \right)^2 \leq \left( \sum a^2 \right) \left( \sum \arctan^2 a \right) < \frac{3\pi^2}{4} \sum a^2$$

because  $\arctan^2 a < \frac{\pi^2}{4}$ . It follows:

$$(0.2) \quad \sum b \arctan a < \frac{\pi}{2} \sqrt{3 \sum a^2}$$

From 0.1 and 0.2 it follows:

$$\sum b \int_0^a e^{-t^2} dt < \frac{\pi}{2} \sqrt{3 \sum a^2}$$

□

4142. Prove that  $a, b, c \in (0, \infty)$  then:

$$\left( 1 + \frac{a^2 + b^2 + c^2}{ab + bc + ca} \right)^{\frac{(a+b+c)^2}{a^2+b^2+c^2}} \leq \left( 1 + \frac{a}{b} \right) \left( 1 + \frac{b}{c} \right) \left( 1 + \frac{c}{a} \right)$$

*Daniel Sitaru*

*Solution by Arkady Alt.*

Assuming due to the homogeneity of the original inequality, that  $a + b + c = 1$  and denoting

$$p = ab + bc + ca, q = abc,$$

we obtain

$$\begin{aligned} a^2 + b^2 + c^2 &= 1 - 2p, \\ \left( 1 + \frac{a}{b} \right) \left( 1 + \frac{b}{c} \right) \left( 1 + \frac{c}{a} \right) &= \frac{(a+b)(b+c)(c+a)}{abc} = \frac{p-q}{q}, \end{aligned}$$

and

$$\begin{aligned} 1 + \frac{a^2 + b^2 + c^2}{ab + bc + ca} &= 1 + \frac{1-2p}{p} = \frac{1-p}{p} \\ \left( 1 + \frac{a}{b} \right) \left( 1 + \frac{b}{c} \right) \left( 1 + \frac{c}{a} \right) &= \frac{(a+b)(b+c)(c+a)}{abc} = \frac{p-q}{q} \end{aligned}$$

and

$$1 + \frac{a^2 + b^2 + c^2}{ab + bc + ca} = 1 + \frac{1-2p}{p} = \frac{1-p}{p}$$

The original inequality thus becomes

$$\left( \frac{1-p}{p} \right)^{\frac{1}{1-2p}} \leq \frac{p}{q} - 1$$

Since  $0 < q \leq \frac{p^2}{3}$ , we have  $\frac{p}{q} \geq \frac{3}{p}$ , and it suffices to prove the inequality

$$\left( \frac{1-p}{p} \right)^{\frac{1}{1-2p}} \leq \frac{3}{p} - 1$$

For  $0 < p \leq \frac{1}{3}$ , this is successively equivalent to

$$\begin{aligned} \frac{1-p}{p} &\leq \left( \frac{3-p}{p} \right)^{1-2p} \\ \left( \frac{3-p}{p} \right)^{2p} &\leq \frac{3-p}{1-p}, \end{aligned}$$

$$\left(\frac{3}{p} - 1\right)^2 \leq \left(\frac{\frac{3}{p} - 1}{\frac{1}{p} - 1}\right)^{\frac{1}{p}}.$$

Denoting  $t = \frac{1}{p} \in [3, \infty)$ , we obtain the following more convenient equivalent form of the latter inequality.

$$(3t - 1)^2 \leq \left(\frac{3t - 1}{t - 1}\right)^t \Leftrightarrow t \ln\left(\frac{3t - 1}{t - 1}\right) \geq 2 \ln(3t - 1)$$

Let

$$h(t) = t[\ln(3t - 1) - \ln(t - 1)] - 2 \ln(3t - 1)$$

Then

$$\begin{aligned} h'(t) &= \ln(3t - 1) - \ln(t - 1) + t\left(\frac{3}{3t - 1} - \frac{1}{t - 1}\right) - \frac{6}{3t - 1} \\ &= \ln(3t - 1) - \ln(t - 1) - \frac{1}{t - 1} - \frac{5}{3t - 1} \end{aligned}$$

and

$$h''(t) = \frac{3}{3t - 1} - \frac{1}{t - 1} + \frac{1}{(t - 1)^2} + \frac{15}{(3t - 1)^2} = \frac{2(9t^2 - 14t + 7)}{(3t - 1)^2(t - 1)^2}$$

Since  $h''(t) > 0$  for  $t \geq 3$ ,  $h'(t)$  increases on  $[3, \infty)$  and, therefore,

$$h'(t) \geq h'(3) = \ln 8 - \ln 2 - \frac{1}{2} - \frac{5}{8} = 2 \ln 2 - \frac{9}{8} > 0.$$

Hence,  $h(t)$  increases on  $[3, \infty)$  and, therefore,

$$h(t) \geq h(3) = 3(\ln 8 - \ln 2) - 2 \ln 8 = 0.$$

Thus,  $t \ln\left(\frac{3t - 1}{t - 1}\right) \geq 2 \ln(3t - 1)$ , as desired.  $\square$

*Solution by Daniel Sitaru.*

From Hölder's inequality:

$$\begin{aligned} (a+b+c)\left(\frac{a}{ax+b} + \frac{b}{bx+c} + \frac{c}{cx+a}\right)(a(ax+b)+b(bx+c)+c(cx+a)) &\geq (a+b+c)^3 \\ \sum \frac{a}{ax+b} &\geq \frac{(a+b+c)^2}{a(ax+b)+b(bx+c)+c(cx+a)} \\ \sum \int_0^1 \frac{a}{ax+b} dx &\geq (a+b+c)^2 \int_0^1 \frac{dx}{x(a^2+b^2+c^2)+ab+bc+ca} \\ \sum \ln(ax+b) \Big|_0^1 &\geq \frac{(a+b+c)^2}{a^2+b^2+c^2} \ln\left(x(a^2+b^2+c^2)+ab+bc+ca\right) \Big|_0^1 \\ \sum \ln\left(\frac{a+b}{b}\right) &\geq \frac{(a+b+c)^2}{a^2+b^2+c^2} \ln \frac{a^2+b^2+c^2+ab+bc+ca}{ab+bc+ca} \\ \ln \prod \left(1 + \frac{a}{b}\right) &\geq \frac{(a+b+c)^2}{a^2+b^2+c^2} \ln\left(1 + \frac{a^2+b^2+c^2}{ab+bc+ca}\right) \\ \left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) &\geq \left(1 + \frac{a^2+b^2+c^2}{ab+bc+ca}\right)^{\frac{(a+b+c)^2}{a^2+b^2+c^2}} \end{aligned}$$

$\square$

4149. Prove that if  $[a, b] \subset [0, \frac{\pi}{4}]$  then:

$$3(a \tan b + b \tan a) \geq ab(6 + a \tan a + b \tan b)$$

*Daniel Sitaru*

*Solution by Digby Smith.*

We first prove that if  $x$  is a real number such that  $0 \leq x \leq 1$ , then

$$(4) \quad (3 - x^2) \tan x \geq 3x$$

From the Maclaurin series expansion for  $\tan x$ , we have that

$$\tan x \geq x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7$$

Hence,

$$\begin{aligned} & (3 - x^2) \tan x - 3x \\ & \geq (3 - x^2) \left( x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 \right) - 3x \\ & = \left( 3x + x^3 + \frac{6}{15}x^5 + \frac{51}{315}x^7 \right) - \left( x^3 + \frac{1}{3}x^5 + \frac{2}{15}x^7 + \frac{17}{315}x^9 \right) - 3x \\ & = \frac{1}{15}x^5 + \frac{9}{315}x^7 - \frac{17}{315}x^9 \\ & = \frac{1}{315}x^5(21 + 9x^2 - 17x^4) \geq 0, \end{aligned}$$

which establishes (4).

Applying (4) with  $x = a$  and  $b$ , respectively, we then have

$$(3 - a^2) \tan a \geq 3a \text{ and } (3 - b^2) \tan b \geq 3b.$$

Therefore,

$$a(3 - b^2) \tan b + b(3 - a^2) \tan a \geq 6ab,$$

from which the given inequality follows immediately.  $\square$

*Solution by Daniel Sitaru.*

We prove that if  $x \in [0, \frac{\pi}{4}]$  then:  $3 \tan x - 3x - x^2 \tan x \geq 0$

Let be  $f : [0, \frac{\pi}{4}] \rightarrow \mathbb{R}; f(x) = 3 \tan x - 3x - x^2 \tan x$

$$\begin{aligned} f'(x) &= \frac{3}{\cos^2 x} - 3 - 2x \tan x - \frac{x^2}{\cos^2 x} \\ f'(x) &= \frac{3(1 - \cos^2 x)}{\cos^2 x} - 2x \tan x - \frac{x^2}{\cos^2 x} = \frac{3 \sin^2 x}{\cos^2 x} - 2x \tan x - \frac{x^2}{\cos^2 x} \end{aligned}$$

Let be  $g : [0, \frac{\pi}{4}] \rightarrow \mathbb{R}; g(x) = \cos^2 x f'(x)$

$$g(x) = 3 \sin^2 x - 2x \sin x \cos x - x^2 = 3 \sin^2 x - x \sin^2 x - x^2$$

$$g'(x) = 3 \sin 2x - \sin^2 x - 2x \cos 2x - 2x$$

$$g'(x) = 2 \sin 2x - 2x \cos 2x - 2x$$

$$g''(x) = 4 \cos 2x - 2 \cos 2x + 4x \sin 2x - 2 = 2 \cos 2x + 4x \sin 2x - 2$$

$$g'''(x) = -4 \sin 2x + 4 \sin 2x + 8x \cos 2x$$

$$g'''(x) = 8x \cos 2x \geq 0; (\forall)x \in [0, \frac{\pi}{4}]$$

$$g(0) = g'(0) = g''(0) = g'''(0) = 0 \text{ and } g'''(x) \geq 0 \Rightarrow$$

$$\Rightarrow g''(x) \geq 0; g'(x) \geq 0; g(x) \geq 0. \text{ It follows:}$$

$$g(x) = x f'(x) \geq 0 \Rightarrow f'(x) \geq 0, (\forall)x \in [0, \frac{\pi}{4}]$$

From  $f(0) = 0$  it follows  $f(x) \geq 0 \Rightarrow x^2 \tan x \leq 3 \tan x - 3x$   
 It follows:  $a^2 \tan a \leq 3 \tan a - 3a$ . By multiplying with  $b$ :

$$a^2 b \tan a \leq 3b \tan a - 3ab$$

Analogous:

$$b^2 a \tan b \leq 3a \tan b - 3ba$$

By adding:

$$\begin{aligned} ab(a \tan a + b \tan b) &\leq 3(b \tan a + a \tan b - ab - ba) \\ ab(a \tan a + b \tan b) &\leq 3(b \tan a + a \tan b) - 6ab \\ ab(a \tan a + b \tan b + 6) &\leq 3(b \tan a + a \tan b) \\ 3(a \tan b + b \tan a) &\geq ab(6 + a \tan a + b \tan b) \end{aligned}$$

□

4152. Prove that if  $a, b, c \in (0, \infty)$  then:

$$\ln(1+a)^{\ln(1+b)^{\ln(1+c)}} \leq \ln^3(1 + \sqrt[3]{abc})$$

Daniel Sitaru

*Solution by Ali Adnan.*

The left side of the inequality should be bracketed

$$\ln \left[ (1+a)^{\ln[(1+b)^{\ln(1+c)}]} \right]$$

so that it is equal to  $\ln(1+a) \ln(1+b) \ln(1+c)$ .

Let  $f(x) = \ln(\ln(1+e^x))$ . Then

$$f''(x) = e^x [(1+e^x) \ln(1+e^x)]^{-2} [\ln(1+e^x) - e^x] < 0,$$

so that  $f(x)$  is concave. By Jensen's theorem, for each  $x, y, z$ ,

$$f(x) + f(y) + f(z) \leq 3f\left(\frac{x+y+z}{3}\right)$$

Setting  $(x, y, z) = (\ln a, \ln b, \ln c)$  yields that

$$\ln[\ln(1+a) \ln(1+b) \ln(1+c)] \leq 3 \ln \left[ \ln \left( 1 + e^{\frac{\ln abc}{3}} \right) \right]$$

Exponentiating yields

$$\ln(1+a) \ln(1+b) \ln(1+c) \leq \ln^3(1 + \sqrt[3]{abc})$$

as desired. □

*Solution by Daniel Sitaru.*

Let be  $f : \mathbb{R} \rightarrow \mathbb{R}; f(x) = \ln(\ln(1+e^x))$

$$f'(x) = \frac{e^x}{(1+e^x) \ln(1+e^x)}; f''(x) = \frac{e^x ((\ln(1+e^x) - e^x))}{(1+e^x)^2 \cdot \ln^2(1+e^x)} < 0$$

$\Rightarrow f$  concave.

From Jensen inequality:

$$\frac{1}{3}(\ln(\ln(1+e^x)) + \ln(\ln(1+e^y)) + \ln(\ln(1+e^z))) \leq \ln \left( \ln(1 + e^{\frac{x+y+z}{3}}) \right)$$

We choose:  $x = \ln a; y = \ln b; z = \ln c$

$$\begin{aligned} \frac{1}{3}(\ln(\ln(1+a)) + \ln(\ln(1+b)) + \ln(\ln(1+c))) &\leq \ln(\ln(1 + \sqrt[3]{abc})) \\ \ln(\ln(1+a) \ln(1+b) \ln(1+c)) &\leq \ln(\ln^3(1 + \sqrt[3]{abc})) \\ \ln(1+a) \ln(1+b) \ln(1+c) &\leq \ln^3(1 + \sqrt[3]{abc}) \\ \ln(1+a)^{\ln(1+b)^{\ln(1+c)}} &\leq \ln^3(1 + \sqrt[3]{abc}) \end{aligned}$$

□

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