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ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

ABOUT NAGEL'S AND GERGONNE'S CEVIANS-(IX)

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In $\Delta ABC, F$ –area, R –circumradius, a, b, c –lengths sides and $M, P \in Int(\Delta ABC)$.

The following relationship holds:

$$a \cdot AP \cdot AM + b \cdot BP \cdot BM + c \cdot CP \cdot CM \geq abc \ (G.Bennet); (1)$$

Equality holds if and only if P and M are isogonal conjugate.

Let $P = G, G$ –centroid, hence

$$a \cdot AG \cdot AM + b \cdot BG \cdot BM + c \cdot CG \cdot CM \geq abc$$

$$AG = \frac{2}{3}m_a; BG = \frac{2}{3}m_b; CG = \frac{2}{3}m_c$$

$$a \cdot m_a \cdot AM + b \cdot m_b \cdot BM + c \cdot m_c \cdot CM \geq \frac{3}{2}abc = \frac{3}{2} \cdot 4RF = 6RF$$

So, we get:

$$a \cdot m_a \cdot AM + b \cdot m_b \cdot BM + c \cdot m_c \cdot CM \geq 6RF; (2)$$

$$2F = a \cdot h_a = b \cdot h_b = c \cdot h_c$$

$$\frac{a \cdot m_a \cdot AM}{a \cdot h_a} + \frac{b \cdot m_b \cdot BM}{b \cdot h_b} + \frac{c \cdot m_c \cdot CM}{c \cdot h_c} \geq \frac{2RF}{2F}$$

$$\frac{m_a}{h_a} \cdot AM + \frac{m_b}{h_b} \cdot BM + \frac{m_c}{h_c} \cdot CM \geq 3R; (3)$$

$$\frac{R}{2r} \geq \frac{m_a}{h_a} \ (Panaitopol)$$

$$\frac{R}{2r}(AM + BM + CM) \geq 3R \Rightarrow AM + BM + CM \geq 6r; (4)$$

Let K –intersection point of simmedians. If $P = K, AK = \frac{2bc}{a^2+b^2+c^2} \cdot m_a$, then

$$\frac{2abc}{a^2 + b^2 + c^2}(m_a \cdot AM + m_b \cdot BM + m_c \cdot CM) \geq abc$$

$$m_a \cdot AM + m_b \cdot BM + m_c \cdot CM \geq \frac{1}{2}(a^2 + b^2 + c^2); (5)$$

If $P = O \Rightarrow AP = BP = CP = R$,

$$(a\cdot AM+b\cdot BM+c\cdot CM)\cdot R\geq abc$$

$$a\cdot AM+b\cdot BM+c\cdot CM\geq 4F;(abc=4F)\Rightarrow$$

$$\frac{AM}{h_a}+\frac{BM}{h_b}+\frac{CM}{h_c}\geq 2;(6)$$

Let N_a —Nagel's point, $AN_a=\frac{a\cdot n_a}{s}$. If $P=N_a$, then:

$$\frac{a^2\cdot n_a}{s}\cdot AM+\frac{b^2\cdot n_b}{s}\cdot BM+\frac{c^2\cdot n_c}{s}\cdot CM\geq abc$$

$$a^2\cdot n_a\cdot AM+b^2\cdot n_b\cdot BM+c^2\cdot n_c\cdot CM\geq s\cdot abc;(7)$$

$$\frac{a\cdot n_a}{h_a}\cdot AM+\frac{b\cdot n_b}{h_b}\cdot BM+\frac{c\cdot n_c}{h_c}\cdot CM\geq 2Rs;(8)$$

$$\frac{n_a}{h_a}\cdot\frac{AM}{bc}+\frac{n_b}{h_b}\cdot\frac{BM}{ac}+\frac{n_c}{h_c}\cdot\frac{CM}{ab}\geq\frac{2Rs}{abc}=\frac{1}{2r}$$

But: $bc=2R\cdot h_a, ca=2R\cdot h_b, ab=2R\cdot h_c$, then:

$$\frac{n_a}{h_a^2}\cdot AM+\frac{n_b}{h_b^2}\cdot BM+\frac{n_c}{h_c^2}\cdot CM\geq\frac{R}{r};(9)$$

If $P=I$, I —incenter, we have: $AI=\frac{r}{\sin\frac{A}{2}}$, $a=4R\cdot\sin\frac{A}{2}\cos\frac{A}{2}$.

$$\sum_{cyc}4R\cdot\sin\frac{A}{2}\cos\frac{A}{2}\cdot\frac{r}{\sin\frac{A}{2}}\cdot AM\geq 4RF;$$

$$AM\cdot\cos\frac{A}{2}+BM\cdot\cos\frac{B}{2}+CM\cdot\cos\frac{C}{2}\geq\frac{F}{r}=s;(10)$$

Let Ω —be the first Brocard's point and ω —Brocard's angle.then:

$$A\Omega=2R\cdot\frac{b}{a}\cdot\sin\omega, B\Omega=2R\cdot\frac{c}{b}\cdot\sin\omega, C\Omega=2R\cdot\frac{a}{c}\cdot\sin\omega$$

$$\sin\omega=\frac{2F}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$$

$$a\cdot A\Omega\cdot AM+b\cdot B\Omega\cdot BM+c\cdot C\Omega\cdot CM\geq abc$$

$$2R\cdot\sin\omega\,(b\cdot AM+c\cdot BM+a\cdot CM)\geq abc=4RF$$

$$b\cdot AM+c\cdot CM+a\cdot CM\geq\sqrt{a^2b^2+b^2c^2+c^2a^2};(11)$$

G_a —Gergonne's point, then $AG_e=\frac{g_a(r_b+r_c)}{4R+r}$

From Bennet's inequality, we have that:

$$\sum_{cyc} a\cdot AM\cdot g_a(r_b+r_c)\ge(4R+r)abc;(12)$$

$$bc=2R\cdot h_a;abc=4RF\Rightarrow$$

$$\sum_{cyc}\frac{g_a(r_b+r_c)}{h_a}\cdot AM\ge 2R(4R+r);(13)$$

If $M=\Omega,\Omega$ −the first point of Brocard's, it follows that:

$$\sum_{cyc} bg_a(r_b+r_c)\ge(4R+r)\sqrt{a^2b^2+b^2c^2+c^2a^2};(14)$$

If $M\in Int(\Delta ABC)$ then:

$$AM\cdot\cos\frac{A}{2}+BM\cdot\cos\frac{B}{2}+CM\cdot\cos\frac{C}{2}\ge s$$

Let $M=G\Rightarrow AG=\frac{2}{3}m_a;BG=\frac{2}{3}m_b;CG=\frac{2}{3}m_c$, thus,

$$m_a\cdot\cos\frac{A}{2}+m_b\cdot\cos\frac{B}{2}+m_c\cdot\cos\frac{C}{2}\ge\frac{3}{2}s;(15)$$

If $M=\Omega,\Omega$ −the first point of Brocard's, it follows that:

$$\frac{b}{a}\cdot\cos\frac{A}{2}+\frac{c}{b}\cdot\cos\frac{B}{2}+\frac{a}{c}\cdot\cos\frac{C}{2}\ge\frac{s}{2R}\cdot\frac{1}{\sin\omega}$$

$$\frac{1}{\sin\omega}=\frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{2F};2F=2sr$$

$$\frac{b}{a}\cdot\cos\frac{A}{2}+\frac{c}{b}\cdot\cos\frac{B}{2}+\frac{a}{c}\cdot\cos\frac{C}{2}\ge\frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{4RF};(16)$$

$$\cos\frac{A}{2}=\frac{s}{\sqrt{s^2+r_a^2}};s^2=n_a^2+2r_ah_a\Rightarrow\cos\frac{A}{2}=\frac{s}{\sqrt{n_a^2+r_a^2+2r_ah_a}}$$

$$n_a^2+r_a^2\ge 2r_ah_a\Rightarrow\frac{AM}{\sqrt{s^2+r_a^2}}+\frac{BM}{\sqrt{s^2+r_b^2}}+\frac{CM}{\sqrt{s^2+r_c^2}}\ge 1;(17)$$

$$\sum_{cyc}\frac{AM}{\sqrt{r_a(n_a+h_a)}}\ge\sqrt{2};(18)$$

Let $M = G$ and from (18), we get:

$$\sum_{cyc} \frac{m_a}{\sqrt{r_a(n_a + h_a)}} \geq \frac{3\sqrt{2}}{2}; (19)$$

If $M = N_a; AN_a = \frac{a \cdot n_a}{s}$ and from (18), we get:

$$\sum_{cyc} \frac{a \cdot n_a}{\sqrt{r_a(n_a + h_a)}} \geq s\sqrt{2}; (20)$$

If $M = N_a; AN_a = \frac{a \cdot n_a}{s}$ and from (10), we get:

$$\sum_{cyc} an_a \cdot \cos\frac{A}{2} \geq s^2; (21)$$

If $M = G_e, G_e$ −Gergonne's point and from (10), we get:

$$\sum_{cyc} \frac{g_a(r_b + r_c)}{4R + r} \cdot \cos\frac{A}{2} \geq s \Rightarrow \sum_{cyc} g_a(r_b + r_c) \cos\frac{A}{2} \geq (4R + r)s; (22)$$

Using $\cos\frac{A}{2} = \frac{s}{\sqrt{s^2 + r_a^2}}$ and from (21), (22), we get:

$$\sum_{cyc} \frac{an_a}{\sqrt{r_a^2 + s^2}} \geq s; (23) \text{ and } \sum_{cyc} \frac{g_a(r_b + r_c)}{\sqrt{r_a^2 + s^2}} \geq 4R + r; (24)$$

If $M = G_e; AG_e = \frac{g_a(r_b + r_c)}{4R + r}$ and using (18), it follows that:

$$\sum_{cyc} \frac{g_a(r_b + r_c)}{\sqrt{r_a(n_a + h_a)}} \geq (4R + r)\sqrt{2}; (25)$$

Let a, b, c −be lengths sides of a triangle, then the system

$x + y = c, y + z = a, z + x = b$ has unique solution $x = \frac{b+c-a}{2}, y = \frac{a+c-b}{2}, z = \frac{a+b-c}{2}$

$$x = s - a; y = s - b; z = s - c; 2s = a + b + c = 2(x + y + z)$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{x(x + y + z)}{(x + y)(x + z)}}; x, y, z > 0$$

$$AM \cdot \cos\frac{A}{2} + BM \cdot \cos\frac{B}{2} + CM \cdot \cos\frac{C}{2} \geq s \Rightarrow$$

$$\sqrt{x+y+z}\sum_{cyc} AM\sqrt{\frac{x}{(x+y)(x+z)}} \ge x+y+z$$

Finally, for $M \in Int(\Delta ABC); x, y, z > 0$, we have:

$$\sum_{cyc} AM\sqrt{\frac{x}{(x+y)(x+z)}} \ge \sqrt{x+y+z}; (26)$$

$$\cos A\cos B\cos C = \frac{s^2-(2R+r)^2}{4R^2}$$

In acute ΔABC, $\cos A, \cos B, \cos C \ge 0 \Rightarrow s^2-(2R+r)^2 \ge 0 \Rightarrow s \ge 2R+r$. Thus,

$$\sum_{cyc} AM\cdot\cos\frac{A}{2} \ge 2R+r; (27); \{\Delta ABC - acute, M \in Int(\Delta ABC)\}$$

In acute $\Delta ABC, M \in Int(\Delta ABC)$, we have:

$$\sum_{cyc}\frac{an_a}{h_a}\cdot AM \ge 2R(2R+r); (28)$$

$$\sum_{cyc} m_a\cdot\cos\frac{A}{2} \ge \frac{3}{2}(2R+r); (29)$$

$$\sum_{cyc}\frac{an_a}{\sqrt{r_a(n_a+h_a)}} \ge (2R+r)\sqrt{2}; (30)$$

$$\sum_{cyc} g_a(r_b+r_c)\cos\frac{A}{2} \ge (4R+r)(2R+r); (31)$$

$$\sum_{cyc}\frac{an_a}{\sqrt{r_a^2+s^2}} \ge 2R+r; (32)$$

$$\sum_{cyc}\frac{an_a}{\sqrt{r_a^2+(2R+r)^2}} \ge s; (33)$$

$$\sum_{cyc}\frac{g_a(r_b+r_c)}{\sqrt{r_a^2+(2R+r)^2}} \ge 4R+r; (34)$$

If in $\Delta ABC, A \ge B \ge \frac{\pi}{3} \ge C$, then $s \ge (R+r)\sqrt{3}$; $\tan\frac{A}{2}, \tan\frac{B}{2} \ge \frac{\sqrt{3}}{3}$ and $\tan C \le \frac{\sqrt{3}}{3}$. So,

$$\prod_{cyc}\left(\tan\frac{A}{2}-\frac{\sqrt{3}}{3}\right)\le 0\Leftrightarrow\prod_{cyc}\tan\frac{A}{2}-\frac{\sqrt{3}}{3}\sum_{cyc}\tan\frac{B}{2}\tan\frac{C}{2}+\frac{1}{3}\sum_{cyc}\tan\frac{A}{2}-\frac{\sqrt{3}}{9}\le 0$$

$$\frac{r}{s}-\frac{\sqrt{3}}{3}+\frac{1}{3}\cdot\frac{4R+r}{s}-\frac{\sqrt{3}}{9}\le 0;\ \sum_{cyc}\tan\frac{B}{2}\tan\frac{C}{2}=1;\ \sum_{cyc}\tan\frac{A}{2}=\frac{4R+r}{s}\Leftrightarrow$$

$$9r+3(4R+r)-4\sqrt{3}s\le 0\Rightarrow s\ge(R+r)\sqrt{3}$$

If in $\Delta ABC, A\ge B\ge\frac{\pi}{3}\ge C$, then

$$\sum_{cyc}AM\cdot\cos\frac{A}{2}\ge(R+r)\sqrt{3};(35)$$

$$\sum_{cyc}\frac{an_a}{h_a}\cdot AM\ge 2R(R+r)\sqrt{3};(36)$$

$$\sum_{cyc}m_a\cdot\cos\frac{A}{2}\ge\frac{3\sqrt{3}}{2}(R+r);(37)$$

$$\sum_{cyc}\frac{an_a}{\sqrt{r_a(n_a+h_a)}}\ge(R+r)\sqrt{6};(38)$$

$$\sum_{cyc}g_a(r_b+r_c)\cos\frac{A}{2}\ge(4R+r)(R+r)\sqrt{3};(39)$$

$$\sum_{cyc}\frac{an_a}{\sqrt{r_a^2+s^2}}\ge(R+r)\sqrt{3};(40)$$

$$\sum_{cyc}\frac{g_a(r_b+r_c)}{\sqrt{r_a^2+3(R+r)^2}}\ge 4R+r;(41)$$

$$\sum_{cyc}\frac{an_a}{\sqrt{r_a^2+3(R+r)^2}}\ge s;(42)$$

$$\sum_{cyc}an_a\cos\frac{A}{2}\ge 3(R+r)^2;(43)$$

In ΔABC the following relationship holds:

$$\frac{(m_a+m_b+m_c)^2}{a^2+b^2+c^2}\le 2+\left(\frac{r}{R}\right)^2;(Sun\ Wen\ Cai)$$

$$(m_a+m_b+m_c)^2\le\frac{2R^2+r^2}{R^2}(a^2+b^2+c^2)$$

Using (26), we can write:

$$\sum_{cyc}AM\sqrt{\frac{a^2}{(a^2+b^2)(a^2+c^2)}}\ge\sqrt{a^2+b^2+c^2}$$

$$\sum_{cyc}\frac{a\cdot AM}{\sqrt{(a^2+b^2)(a^2+c^2)}}\ge\sqrt{\frac{R^2}{2R^2+r^2}(m_a+m_b+m_c)^2}$$

$$\sum_{cyc}\frac{a\cdot AM}{\sqrt{(a^2+b^2)(a^2+c^2)}}\ge\frac{R(m_a+m_b+m_c)}{\sqrt{2R^2+r^2}}$$

$$\because a=2R\cdot\sin A$$

$$\sum_{cyc}\frac{\sin A}{\sqrt{(a^2+b^2)(a^2+c^2)}}\cdot AM\ge\frac{1}{2}\cdot\frac{m_a+m_b+m_c}{\sqrt{2R^2+r^2}};(44)$$

$$s^2=n_a^2+2r_ah_a\Rightarrow s^2-n_a^2=2r_ah_a\Rightarrow(s+n_a)(s+n_a)+\frac{2r_ah_a}{n_a+s}$$

$$\Rightarrow 3s=n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s}\overset{(10)}{\Longrightarrow}$$

$$3\sum_{cyc}AM\cdot\cos\frac{A}{2}\ge n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s};(45)$$

$$2\sum_{cyc}m_a\cdot\cos\frac{A}{2}\ge n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s};(46)$$

$$3\sum_{cyc}\frac{an_a}{\sqrt{r_a(n_a+h_a)}}\ge\sqrt{2}\left(n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s}\right);(47)$$

$$3\sum_{cyc}g_a(r_b+r_c)\cos\frac{A}{2}\ge(4R+r)\left(n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s}\right);(48)$$

$$3\sum_{cyc}\frac{an_a}{\sqrt{r_a^2+s^2}}\ge n_a+n_b+n_c+2\sum_{cyc}\frac{r_ah_a}{n_a+s};(49)$$

$$9\sum_{cyc}an_a\cos\frac{A}{2}\ge\left(n_a+n_b+n_c+2\sum_{cyc}\frac{n_ah_a}{n_a+s}\right)^2;(50)$$

From (5) and Sun Wen Cai's inequality, we get:

$$m_aAM+m_bBM+m_cCM\ge\frac{R^2}{2(2R^2+r^2)}(m_a+m_b+m_c);(51)$$

In $\Delta ABC,\Delta A_1B_1C_1,F-$ area of $\Delta ABC,F_1-$ area of $\Delta A_1B_1C_1,M\in Int(\Delta ABC)$, holds:

$$a_1\cdot AM+b_1\cdot BM+c_1\cdot CM\ge\sqrt{\frac{1}{2}\sum_{cyc}a^2(b_1^2+c_1^2-a_1^2)+8FF_1};(Bottema)$$

$$a_1\cdot AM+b_1\cdot BM+c_1\cdot CM\ge\sqrt{\frac{1}{2}\sum_{cyc}a_1^2(b^2+c^2-a^2)+8FF_1};(Bottema)$$

$$b^2+c^2=n_a^2+g_a^2+2rr_a$$

$$2rr_a=h_a(r-a-r)$$

$$2F=ah_a=bh_b=ch_c=2sr$$

$$r_a+r_b+r_c=4R+r$$

$$a^2=2R\cdot\frac{h_bh_c}{h_a}$$

$$ah_a=(a+b+c)r\Rightarrow\frac{h_a}{r}=1+\frac{b+c}{a}$$

$$b^2+c^2=n_a^2+g_a^2+2rr_a\ge 2n_ag_a+2rr_a$$

$$b^2+c^2=2Rh_a\left(\frac{h_b}{h_c}+\frac{h_c}{h_b}\right)\ge 2n_ag_a+h_a(r_a-r)$$

$$\frac{b}{c}+\frac{c}{b}\ge\frac{2n_ag_a+h_a(r_a-r)}{2Rh_a}\Rightarrow\frac{b}{c}+\frac{c}{b}\ge\frac{1}{R}\left(\frac{n_ag_a}{h_a}+\frac{r_a-r}{2}\right)$$

$$\sum_{cyc}\frac{b+c}{a}\ge\frac{1}{R}\left(\sum_{cyc}\frac{n_ag_a}{h_a}+2R-r\right)$$

$$\sum_{cyc}\frac{b+c}{a}\geq 2+\frac{1}{R}\sum_{cyc}\frac{n_a g_a}{h_a}-\frac{r}{R}$$

$$\frac{h_a+h_b+h_c-3r}{r}\geq 2+\frac{1}{R}\sum_{cyc}\frac{n_a g_a}{h_a}-\frac{r}{R}$$

$$\frac{h_a+h_b+h_c}{r}\geq\frac{5R-r}{R}+\frac{1}{R}\sum_{cyc}\frac{n_a g_a}{h_a}$$

$$\frac{R}{r}\geq\frac{5R-r+\sum\frac{n_a g_a}{h_a}}{h_a+h_b+h_c};(51)$$

$$5R-r\geq 4R+r\Rightarrow 5R-4R\geq r+r\Rightarrow R\geq 2r\ (Euler)$$

$$\frac{R}{r}\geq\frac{r_a+r_b+r_c+\sum\frac{n_a g_a}{h_a}}{h_a+h_b+h_c};(52)$$

$$g_a\geq h_a\Rightarrow\frac{R}{r}\geq\frac{r_a+r_b+r_c+n_a+n_b+n_c}{h_a+h_b+h_c};(53)$$

$$\frac{R}{r}\geq\frac{5R-r+n_a+n_b+n_c}{h_a+h_b+h_c};(54)$$

$$n_a g_a\geq m_a w_a\Rightarrow\frac{R}{r}\geq\frac{5R-r+\sum\frac{m_a w_a}{h_a}}{h_a+h_b+h_c};(55)$$

$$\frac{R}{r}\geq\frac{r_a+r_b+r_c+\sum\frac{m_a w_a}{h_a}}{h_a+h_b+h_c};(56)$$

$$m_a w_a\geq s(s-a)=r_b r_c=\frac{h_a}{2}(r_b+r_c);(Panaitopol)$$

$$\frac{R}{r}\geq\frac{5R-r+r_a+r_b+r_c}{h_a+h_b+h_c}=\frac{9R}{h_a+h_b+h_c};(57)$$

$$\sum_{cyc}\frac{n_a}{h_a^2}AM\geq\frac{5R-r+\sum\frac{n_a g_a}{h_a}}{h_a+h_b+h_c};(58)$$

$$\sum_{cyc}\frac{n_a}{h_a^2}AM \geq \frac{r_a+r_b+r_c+\sum\frac{n_ag_a}{h_a}}{h_a+h_b+h_c};(59)$$

$$\sum_{cyc}\frac{n_a}{h_a^2}AM \geq \frac{r_a+r_b+r_c+n_a+n_b+n_c}{h_a+h_b+h_c};(60)$$

$$\sum_{cyc}\frac{n_a}{h_a^2}AM \geq \frac{5R-r+n_a+n_b+n_c}{h_a+h_b+h_c};(61)$$

$$\sum_{cyc}\frac{n_a}{h_a^2}AM \geq \frac{5R-r+\sum\frac{m_aw_a}{h_a}}{h_a+h_b+h_c};(63)$$

$$\sum_{cyc}\frac{n_a}{h_a^2}AM \geq \frac{9R}{h_a+h_b+h_c};(64)$$

REFERENCES:

[1]. DANIEL SITARU-*A type of useful substitutions in triangle geometry-www.ssmrmh.ro*

[2]. 584 G. BENNETT-*Multiple triangle inequalities.*

[3]. BOGDAN FUŞTEI-*About Nagel's and Gergonne's cevian (I-VIII)*

[4]. BOGDAN FUŞTEI-*Oppenheim's inequality revisited.*

[5]. JIAN LIU-*On a sharp inequality for the medians of a triangle.*

[6]. G.BOTTEMA-*Geometric Inequalities.*

[7]. ROMANIAN MATHEMATICAL MAGAZINE- *www.ssmrmh.ro*