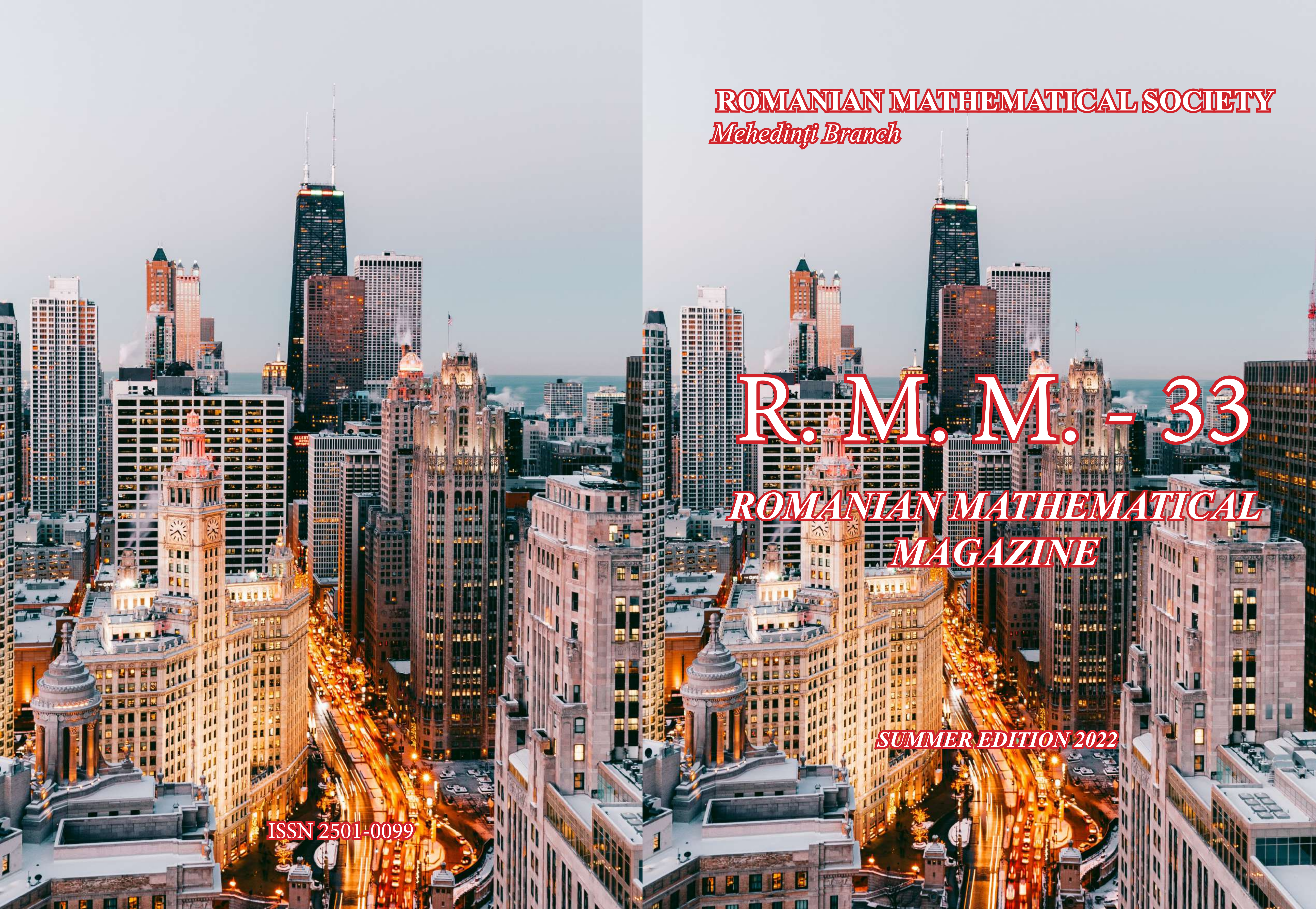

ISSN 2501-0099

ROMANIAN MATHEMATICAL SOCIETY

Mehedinţi Branch

R. M. M. - 33

ROMANIAN MATHEMATICAL MAGAZINE

SUMMER EDITION 2022

ROMANIAN MATHEMATICAL SOCIETY

Mehedinți Branch

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R.M.M.

Nr.33-SUMMER EDITION 2022

ROMANIAN MATHEMATICAL SOCIETY

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CONTENT

A NEW PROOF FOR MITRINOVIC'S INEQUALITY USING TOSCANO'S IDENTITY.

Daniel Sitaru, Claudia Nănuți – Romania

Abstract: In this paper it's proved Mitrinovic's inequality using a famous identity.

Notations: r – inradii; R – circumradii; r_a, r_b, r_c – exradii; s – semiperimeter; F – area

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{s-a}{F}+\frac{s-b}{F}+\frac{s-c}{F}=\frac{3s-(a+b+c)}{F}=$$

$$=\frac{3s-2s}{F}=\frac{s}{F}=\frac{1}{F}=\frac{1}{r}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\quad (1)$$

In the well known identity:

$$(x+y+z)^3=x^3+y^3+z^3+3(x+y)(y+z)(z+x); x,y,z\in\mathbb{R}$$

We take:

$$x=\frac{1}{r_a}; y=\frac{1}{r_b}; z=\frac{1}{r_c}$$

$$\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)^3=\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}+3\prod_{cyc}\left(\frac{1}{r_a}+\frac{1}{r_b}\right)$$

By (1):

$$\left(\frac{1}{r}\right)^3=\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}+3\prod_{cyc}\left(\frac{s-a}{F}+\frac{s-b}{F}\right)$$

$$\frac{1}{r^3}-\frac{1}{r_a^3}-\frac{1}{r_b^3}-\frac{1}{r_c^3}=\frac{3}{F^3}\prod_{cyc}(s-a+s-b)$$

$$\frac{1}{r^3}-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)=\frac{3}{F^3}\prod_{cyc}(a+b+c-a-b)$$

$$\frac{1}{r^3}-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)=\frac{3}{F^3}\cdot abc,\qquad \frac{1}{r^3}-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)=\frac{3\cdot 4RF}{F^3}$$

$$\frac{1}{r^3}-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)=\frac{12R}{F^2}\quad (2)$$

Relation (2) it's called Toscano's identity.

Let be $f:(0,\infty)\to\mathbb{R}; f(x)=x^3; f'(x)=3x^2; f''(x)=6x>0; f$ convexe.

By Jensen's inequality:

$$f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right)$$

Again: $x=\frac{1}{r_a}; y=\frac{1}{r_b}; z=\frac{1}{r_c}$

$$f\left(\frac{1}{r_a}\right)+f\left(\frac{1}{r_b}\right)+f\left(\frac{1}{r_c}\right)\geq 3f\left(\frac{1}{3}\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\right)$$

By (1):

$$\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\geq 3f\left(\frac{1}{3r}\right)=3\cdot\frac{1}{27r^3}=\frac{1}{9r^3}$$

$$-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)\leq-\frac{1}{9r^3},\qquad \frac{1}{r^3}-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)\leq\frac{1}{r^3}-\frac{1}{9r^3}$$

By (2):

$$\frac{12R}{F^2}\le\frac{1}{r^3}-\frac{1}{9r^3},\qquad\frac{12R}{r^2s^2}\le\frac{8}{9r^3},\qquad\frac{3R}{s^2}\le\frac{2}{9r}$$

$$2s^2\ge 27Rr\overset{EULER}{\ge}27\cdot 2r\cdot r=54r^2, 2s^2\ge 54r^2\Rightarrow s^2\ge 27r^2,\qquad s\ge 3\sqrt{3}r$$

which it's Mitrinovic's inequality.

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[1] ROMANIAN MATHEMATICAL MAGAZINE – www.ssmrmh.ro

THE HEPTAGONAL TRIANGLE REVISITED

By Daniel Sitaru, Claudia Nănuți – Romania

Abstract: In this paper are proved the characteristic metric relationships in the heptagonal triangle.

We call heptagonal triangle the obtuse scalene triangle whose vertices coincide with the first, second and fourth vertices of a regular heptagon (from an arbitrary starting vertex). Thus its sides coincide with one side and the adjacent shorter and longer diagonals of the regular heptagon.

Lemma 1

If in $\Delta ABC, a<b<c; \mu(B)=2\mu(A); \mu(C)=2\mu(B)$ then: $b^2=a^2+ac; c^2=b^2+a$.

Proof.

$$\mu(B)=2\mu(A)\Rightarrow \sin B=\sin 2A$$

$$\sin B=2\sin A\cos A\Rightarrow\frac{b}{2R}=2\cdot\frac{a}{2R}\cdot\frac{b^2+c^2-a^2}{2bc}$$

$$b^2c=ab^2+ac^2-a^3\Rightarrow b^2(c-a)-a(c^2-a^2)=0$$

$$b^2(c-a)-a(c-a)(c+a)=0$$

$$(c-a)\big(b^2-a(c+a)\big)=0; c-a\ne 0,\qquad b^2=a^2+ac$$

Analogous: $\mu(C)=2\mu(B)\Rightarrow c^2=b^2+ba$

Lemma 2

If in $\Delta ABC; a<b<c; \mu(C)=2\mu(A)+\mu(B)$ then: $a^2=c^2-bc$.

Proof.

$$\mu(C)=2\mu(A)+\mu(B)\Rightarrow 2\mu(C)=2\mu(A)+\mu(B)+\mu(C)$$

$$2\mu(C)=\pi+\mu(A),\qquad \sin 2C=\sin(\pi+A)$$

$$2\sin C\cos C=-\sin A,\qquad 2\cdot\frac{c}{2R}\cdot\frac{a^2+b^2-c^2}{2ab}=-\frac{a}{2R}$$

$$ca^2+cb^2-c^3=-a^2b,\qquad a^2(b+c)-c(c^2-b^2)=0$$

$$a^2(b+c)-c(b+c)(c-b)=0,\qquad a^2-c^2+bc=0,\qquad a^2=c^2-bc$$

If $\mu(A)=\frac{\pi}{7}; \mu(B)=\frac{2\pi}{7}; \mu(C)=\frac{4\pi}{7}$ by lemma 1 and lemma 2 we obtain that for the triangle with sides $a,b,c; a<b<c$ the relationships:

$$b^2=a^2+ac; c^2=b^2+ba; a^2=c^2-bc\qquad (1)$$

The heptagonal triangle with sides $a,b,c,a<b<c$ and angles A,B,C verify (1).

By adding relationships (1) we obtain:

$$b^2+c^2+a^2=a^2+ac+b^2+ba+c^2-bc$$

$$bc=ba+ac,\qquad\frac{1}{a}=\frac{b+c}{bc}$$

$$\frac{1}{a}=\frac{1}{b}+\frac{1}{c} \quad (2)$$

By multiplying (2) with $2F$:

$$\frac{2F}{a}=\frac{2F}{b}+\frac{2F}{c} \Rightarrow h_a=h_b+h_c$$

Lemma 3:

$$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}=\frac{\sqrt{7}}{8} \quad \textbf{(3)}$$

Proof.

We will use the well known identity:

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}=\frac{n}{2^{n-1}}; n\in\mathbb{N}; n\geq 2$$

For $n=7$:

$$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\sin\frac{4\pi}{7}\sin\frac{5\pi}{7}\sin\frac{6\pi}{7}=\frac{7}{2^6}$$

$$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\sin\left(\pi-\frac{4\pi}{7}\right)\sin\left(\pi-\frac{5\pi}{7}\right)\sin\left(\pi-\frac{6\pi}{7}\right)=\frac{7}{64}$$

$$\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)^2=\frac{7}{64}\Rightarrow \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}=\frac{\sqrt{7}}{8}$$

Lemma 4:

The equation:

$$\boldsymbol{64y^3-112y^2+56y-7=0} \quad \textbf{(4)}$$

has the roots:

$$\boldsymbol{x_1=\sin^2\frac{\pi}{7}; x_2=\sin^2\frac{2\pi}{7}; x_3=\sin^2\frac{4\pi}{7}}$$

Proof.

We will prove that $x_1=\sin^2\frac{\pi}{7}$ is a solution for the equation (4):

$$\sin\frac{4\pi}{7}=\sin\left(\pi-\frac{4\pi}{7}\right)=\sin\frac{3\pi}{7}, \qquad \sin^2\frac{4\pi}{7}=\sin^2\frac{3\pi}{7}$$

$$\left(2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\right)^2=\left(\sin\frac{\pi}{7}\left(3-4\sin^2\frac{\pi}{7}\right)\right)^2$$

$$4\sin^2\frac{2\pi}{7}\cos^2\frac{2\pi}{7}=\sin^2\frac{\pi}{7}\left(3-4\sin^2\frac{\pi}{7}\right)^2$$

$$4\cdot 4\sin^2\frac{\pi}{7}\cos^2\frac{\pi}{7}\left(1-2\sin^2\frac{\pi}{7}\right)^2=\sin^2\frac{\pi}{7}\left(3-4\sin^2\frac{\pi}{7}\right)^2$$

Denote: $\sin^2\frac{\pi}{7}=y$

$$16\left(1-\sin^2\frac{\pi}{7}\right)\left(1-2\sin^2\frac{\pi}{7}\right)^2=\left(3-4\sin^2\frac{\pi}{7}\right)^2$$

$$16(1-y)(1-2y)^2=(3-4y)^2, \qquad 16(1-y)(1-4y+4y^2)=(3-4y)^2$$

$$16(1-4y+4y^2-y+4y^2-4y^3)=(3-4y)^2$$

$$16-80y+128y^2-64y^3=9+16y^2-24y$$

$$7-56y+112y^2-64y^3=0, \qquad 64y^3-112y^2+56y-7=0$$

$$S_1=x_1+x_2+x_3=-\frac{-112}{64}=\frac{14}{8}=\frac{7}{4}$$

$$\sin^2\frac{\pi}{7}+\sin^2\frac{2\pi}{7}+\sin^2\frac{4\pi}{7}=\frac{7}{4} \quad (5)$$

$$S_2 = x_1x_2 + x_2x_3 + x_3x_1 = \frac{56}{64} = \frac{14}{16} = \frac{7}{8}$$

$$\sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7} + \sin^2\frac{2\pi}{7}\sin^2\frac{4\pi}{7} + \sin^2\frac{4\pi}{7}\sin^2\frac{2\pi}{7} = \frac{7}{8} \quad (6)$$

$$S_3 = x_1x_2x_3 = \left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)^2 = \frac{7}{64} \quad (7)$$

R – circumradii; F – area;

$$a = 2R\sin A = 2R\sin\frac{\pi}{7}, b = 2R\sin B = 2R\sin\frac{2\pi}{7}, c = 2R\sin C = 2R\sin\frac{4\pi}{7}$$

$$F = \frac{abc}{4R} = \frac{8R^3\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}}{4R} \overset{(4)}{=} 2R^2\cdot\frac{\sqrt{7}}{8} = \frac{\sqrt{7}}{4}R^2$$

$$a^2 + b^2 + c^2 = 4R^2\left(\sin^2\frac{\pi}{7} + \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7}\right) \overset{(5)}{=} 4R^2\cdot\frac{7}{4} = 7R^2$$

$$a^2 + b^2 + c^2 = 7R^2 \quad (8)$$

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{4R^2}\left(\frac{1}{\sin^2\frac{\pi}{7}} + \frac{1}{\sin^2\frac{2\pi}{7}} + \frac{1}{\sin^2\frac{4\pi}{7}}\right) =$$

$$= \frac{1}{4R^2}\cdot\frac{\sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7} + \sin^2\frac{2\pi}{7}\sin^2\frac{4\pi}{7} + \sin^2\frac{4\pi}{7}\sin^2\frac{\pi}{7}}{\sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7}\sin^2\frac{4\pi}{7}} =$$

$$\overset{(6);(7)}{=} \frac{1}{4R^2}\cdot\frac{\frac{7}{8}}{\frac{7}{64}} = \frac{1}{4R^2}\cdot 8 = \frac{2}{R^2}$$

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{2}{R^2} \quad (9)$$

$$h_a^2 + h_b^2 + h_c^2 = 4F^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = 4\cdot\frac{7}{16}R^4\cdot\frac{2}{R^2} = \frac{7}{2}R^2 = \frac{7R^2}{2} \overset{(8)}{=} \frac{a^2 + b^2 + c^2}{2}$$

$$h_a^2 + h_b^2 + h_c^2 = \frac{a^2 + b^2 + c^2}{2}$$

In conclusion, in a heptagonal triangle with sides $a < b < c; \mu(A) = \frac{\pi}{7}; \mu(B) = \frac{2\pi}{7}$; $\mu(C) = \frac{4\pi}{7}$ the following relationship holds:

$$a^2 = c^2 - bc; b^2 = a^2 + ac; c^2 = b^2 + ba, \qquad \frac{1}{a} = \frac{1}{b} + \frac{1}{c}$$

$$h_a = h_b + h_c; h_a^2 + h_b^2 + h_c^2 = \frac{a^2 + b^2 + c^2}{2}, \qquad F = \frac{\sqrt{7}}{4}R^2$$

$$a^2 + b^2 + c^2 = 7R^2; \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{2}{R^2}$$

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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-II

By Marin Chirciu – Romania

1) Prove that in any acute-angled triangle the following inequality holds:

$$\frac{a^6}{w_bw_c} + \frac{b^6}{w_cw_a} + \frac{c^6}{w_aw_b} \ge [4R(R + r)]^2$$

Proposed by Marian Ursărescu – Romania

Solution.*Using Bergström's inequality we obtain:*

$\sum\frac{a^6}{w_bw_c}=\sum\frac{(a^3)^2}{w_bw_c}\ge\frac{(\sum a^3)^2}{\sum w_bw_c}\ge\frac{(\sum a^3)^2}{s^2}$, *which follows from* $\sum w_bw_c\le\sum w_a^2\le\sum s(s-a)=s^2$

It suffices to prove that: $\frac{(\sum a^3)^2}{s^2}\ge[4R(R+r)]^2\Leftrightarrow\frac{\sum a^3}{s}\ge 4R(4R+r)\Leftrightarrow$

$\Leftrightarrow\frac{2s(s^2+r^2+4Rr)}{s}\ge 4R(4R+r)\Leftrightarrow s^2\ge 2R^2+8Rr+3r^2$ *(Walker's inequality, acute)*

Equality holds if and only if the triangle is equilateral.

Remark: *In the same way we propose:*

2) Prove that in any acute-angled triangle the following inequality holds:

$$\frac{a^6}{r_br_c}+\frac{b^6}{r_cr_a}+\frac{c^6}{r_ar_b}\ge[4R(R+r)]^2$$

Marin Chirciu

Solution:*Using Bergström's inequality we obtain:*

$\sum\frac{a^6}{r_br_c}=\sum\frac{(a^3)^2}{r_br_c}\ge\frac{(\sum a^3)^2}{\sum r_br_c}=\frac{(\sum a^3)^2}{s^2}$, *which follows from* $\sum r_br_c=s^2$.

It suffices to prove that: $\frac{(\sum a^3)^2}{s^2}\ge[4R(R+r)]^2\Leftrightarrow\frac{\sum a^3}{s}\ge 4R(4R+r)\Leftrightarrow$

$\Leftrightarrow\frac{2s(s^2+r^2+4Rr)}{s}\ge 4R(4R+r)\Leftrightarrow s^2\ge 2R^2+8Rr+3r^2$ *(Walker's inequality, acute)*

Equality holds if and only if the triangle is equilateral.

3) Prove that in any acute-angled triangle the following inequality holds:

$$\frac{a^6}{h_bh_c}+\frac{b^6}{h_ch_a}+\frac{c^6}{h_ah_b}\ge[4R(R+r)]^2$$

Solution: *Using Bergström's inequality we obtain:*

$\sum\frac{a^6}{h_bh_c}=\sum\frac{(a^3)^2}{h_bh_c}\ge\frac{(\sum a^3)^2}{\sum h_bh_c}\ge\frac{(\sum a^3)^2}{s^2}$, *which follows from*

$$\sum h_bh_c\le\sum w_bw_c\le\sum w_a^2\le\sum s(s-a)=s^2$$

It suffices to prove that: $\frac{(\sum a^3)^2}{s^2}\ge[4R(R+r)]^2\Leftrightarrow\frac{\sum a^3}{s}\ge 4R(4R+r)\Leftrightarrow$

$\Leftrightarrow\frac{2s(s^2+r^2+4Rr)}{s}\ge 4R(4R+r)\Leftrightarrow s^2\ge 2R^2+8Rr+3r^2$ *(Walker's inequality, acute)*

Equality holds if and only if the triangle is equilateral.

4) In ΔABC the following inequality holds:

$$\frac{a^6}{m_bm_c}+\frac{b^6}{m_cm_a}+\frac{c^6}{m_am_b}\ge\frac{1}{3}(8S)^2$$

Marin Chirciu

Solution: *Using Bergström's inequality we obtain:*

$\sum\frac{a^6}{m_bm_c}=\sum\frac{(a^3)^2}{m_bm_c}\ge\frac{(\sum a^3)^2}{\sum m_bm_c}\ge\frac{(\sum a^3)^2}{27R^2}$, *which follows from*

$$\sum m_bm_c\le\sum m_a^2=\frac{3}{4}\sum a^2\le\frac{3}{4}\cdot 9R^2=\frac{27R^2}{4}$$

It suffices to prove that: $\frac{(\sum a^3)^2}{\frac{27R^2}{4}}\ge\frac{1}{3}(8S)^2\Leftrightarrow\frac{\sum a^3}{\frac{3R}{2}}\ge 8S\Leftrightarrow\sum a^3\ge 12SR\Leftrightarrow$

$\Leftrightarrow 2s(s^2-3r^2-6Rr)\ge 12Rrs\Leftrightarrow s^2\ge 12Rr+3r^2$, *which follows from Gerretsen's inequality* $s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$16Rr-5r^2\ge 12Rr+3r^2\Leftrightarrow R\ge 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-II

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\mathbf{2 \le \sum \frac{1}{\sin^2 A + \sin B \sin C} \le 2\left(\frac{R}{2r}\right)^2}$$

Proposed by George Apostolopoulos – Greece

Solution: *LHS inequality. Using Bergström's inequality, we obtain:*

$$\sum \frac{1}{\sin^2 A + \sin B + \sin C} \ge \frac{9}{\sum(\sin^2 A + \sin B \sin C)} = \frac{9}{\sum \sin^2 A + \sum \sin B \sin C} =$$

$$= \frac{9}{\frac{s^2-r^2-4Rr}{2R^2}+\frac{s^2+r^2+4Rr}{4R^2}} = \frac{9}{\frac{3s^2-r^2-4Rr}{4R^2}} = \frac{36R^2}{3s^2-r^2-4Rr} \overset{(1)}{\ge} 2, \text{ where (1)} \Leftrightarrow$$

$\Leftrightarrow 18R^2 \ge 3s^2 - r^2 - 4Rr$, *it follows from Gerretsen's inequality* $s^2 \le 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$\Leftrightarrow 18R^2 \ge 3(4R^2 + 4Rr + 3r^2) - r^2 - 4Rr \Leftrightarrow 3R^2 - 4Rr - 4r^2 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(3R+2r) \ge 0$, *obviously from Euler's inequality* $R \ge 2r$.

Above we've used the known inequalities in triangle:

$\sum \sin^2 A = \frac{s^2-r^2-4Rr}{2R^2}$ *and* $\sum \sin B \sin C = \frac{s^2+r^2+4Rr}{4R^2}$

Equality holds if and only if the triangle is equileral.RHS inequality.

Using means inequalities: $\sin^2 A + \sin B \sin C \ge 2\sqrt{\sin^2 A \sin B \sin C}$, *we obtain:*

$$\sum \frac{1}{\sin^2 A + \sin B \sin C} \le \sum \frac{1}{2\sqrt{\sin^2 A \sin B \sin C}} = \frac{1}{2\sqrt{\prod \sin A}} \sum \frac{1}{\sqrt{\sin A}} \overset{(1)}{\le} 2\left(\frac{R}{2r}\right)^2$$

where (1) $\Leftrightarrow \frac{1}{\sqrt{\prod \sin A}} \sum \frac{1}{\sqrt{\sin A}} \le \left(\frac{R}{r}\right)^2$, *which follows from:*

CBS inequality: $\sum \frac{1}{\sqrt{\sin A}} \le \sqrt{3 \sum \frac{1}{\sin A}}$ *and the identites* $\sum \frac{1}{\sin A} = \frac{s^2+r^2+4Rr}{2rs}$, $\prod \sin A = \frac{rs}{2R^2}$

We obtain $\frac{1}{\sqrt{\prod \sin A}} \sum \frac{1}{\sqrt{\sin A}} \le \frac{1}{\sqrt{\frac{rs}{2R^2}}} \sqrt{3 \cdot \frac{s^2+r^2+4Rr}{2rs}} = \frac{R}{rs}\sqrt{3(s^2+r^2+4Rr)} \overset{(2)}{\le} \left(\frac{R}{r}\right)^2$, *where (2)*

$$\Leftrightarrow r\sqrt{3(s^2+r^2+4Rr)} \le sR \Leftrightarrow 3r^2(s^2+r^2+4Rr) \le s^2R^2 \Leftrightarrow$$

$\Leftrightarrow s^2(R^2 - 3r^2) \ge 3r^3(4R + r)$, *which follows from Gerretsen's inequality*

$s^2 \ge 16Rr - 5r^2$. *It remains to prove:*

$$(16Rr - 5r^2)(R^2 - 3r^2) \ge 3r^3(4R + r) \Leftrightarrow 16R^3 - 5R^2r - 60Rr^2 + 12r^3 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(16R^2 + 27Rr - 6r^2) \ge 0$, *obviously from Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark: *We propose in the same way:*

2) In ΔABC the following relationship holds:

$$\mathbf{\sum \frac{1}{\cos^2 A + \cos B \cos C} \ge \frac{12r}{R}}$$

Marin Chirciu

Solution: *Using Bergström's inequality we obtain:*

$$\sum \frac{1}{\cos^2 A + \cos B \cos C} \ge \frac{9}{\sum(\cos^2 A + \cos B \cos C)} = \frac{9}{\sum \cos^2 A + \sum \cos B \cos C} =$$

$$= \frac{9}{\frac{6R^2+4Rr+r^2-s^2}{2R^2}+\frac{s^2+r^2-4R^2}{4R^2}} = \frac{9}{\frac{8R^2+8Rr+3r^2-s^2}{4R^2}} = \frac{36R^2}{8R^2+8Rr+3r^2-s^2} \overset{(1)}{\ge} \frac{12r}{R} \text{ where (1)} \Leftrightarrow$$

$\Leftrightarrow 36R^3 \geq 96R^2r + 96Rr^2 + 36r^3 - 12s^2r$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$$36R^3 \geq 96R^2r + 96Rr^2 + 36r^3 - 12r(16Rr - 5r^2) \Leftrightarrow$$

$\Leftrightarrow 3R^3 - 8R^2r + 8Rr^2 - 8r^3 \geq 0 \Leftrightarrow (R - 2r)(3R^2 - 2Rr + 4r^2) \geq 0$, *obviously Euler's inequality. Above we have used the known inequalities in triangle:*

$\sum \cos^2 A = \frac{6R^2+4Rr+r^2-s^2}{2R^2}$ *and* $\sum \cos B \cos C = \frac{s^2+r^2-4R^2}{4R^2}$.

Equality holds if and only if the triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\frac{1}{2R^2} \leq \sum \frac{1}{a^2 + bc} \leq \frac{1}{8r^2}$$

LHS inequality. Using Bergström's inequality, we obtain:

$$\sum \frac{1}{a^2 + bc} \geq \frac{9}{\sum(a^2 + bc)} = \frac{9}{\sum a^2 + \sum bc} = \frac{9}{2(s^2 - r^2 - 4Rr) + s^2 + r^2 + 4Rr} =$$

$= \frac{9}{3s^2-r^2-4Rr} \overset{(1)}{\geq} \frac{1}{2R^2}$, *where (1)* $\Leftrightarrow 18R^2 \geq 3s^2 - r^2 - 4Rr$, *which follows from Gerrtsen's inequality* $s^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$18R^2 \geq 3(4R^2 + 4Rr + 3r^2) - r^2 - 4Rr \Leftrightarrow 3R^2 - 4Rr - 4r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R - 2r)(3R + 2r) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Above we have used the known inequalities in triangle:

$\sum a^2 = 2(s^2 - r^2 - 4Rr)$ *and* $\sum bc = s^2 + r^2 + 4Rr$.

The equality holds if and only if the triangle is equilateral.

RHS inequality. Using the means inequality: $a^2 + bc \geq 2\sqrt{a^2bc}$, *we obtain:*

$$\sum \frac{1}{a^2 + bc} \leq \sum \frac{1}{2\sqrt{a^2bc}} = \frac{1}{2\sqrt{\prod a}} \sum \frac{1}{\sqrt{a}} \overset{(1)}{\leq} \frac{1}{8r^2}$$

where (1) $\Leftrightarrow \frac{1}{\sqrt{\prod a}} \sum \frac{1}{\sqrt{a}} \leq \left(\frac{1}{2r}\right)^2$, *which follows from CBS inequality* $\sum \frac{1}{\sqrt{a}} \leq \sqrt{3 \sum \frac{1}{a}}$ *and the identities* $\sum \frac{1}{a} = \frac{s^2+r^2+4Rr}{4Rrs}$, $\prod a = 4Rrs$. *We obtain*

$\frac{1}{\sqrt{\prod a}} \sum \frac{1}{\sqrt{a}} \leq \frac{1}{\sqrt{4Rrs}} \sqrt{3 \cdot \frac{s^2+r^2+4Rr}{4Rrs}} = \frac{1}{4Rrs} \sqrt{3(s^2 + r^2 + 4Rr)} \overset{(2)}{\leq} \left(\frac{1}{2r}\right)^2$, *where (2)*

$$\Leftrightarrow r\sqrt{3(s^2 + r^2 + 4Rr)} \leq sR \Leftrightarrow 3r^2(s^2 + r^2 + 4Rr) \leq s^2R^2 \Leftrightarrow$$

$\Leftrightarrow s^2(R^2 - 3r^2) \geq 3r^3(4R + r)$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$$(16Rr - 5r^2)(R^2 - 3r^2) \geq 3r^3(4R + r) \Leftrightarrow 16R^3 - 5R^2r - 60Rr^2 + 12r^3 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R - 2r)(16R^2 + 27Rr - 6r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

4) In ΔABC the following inequality holds:

$$\frac{9r}{R} \leq \sum \frac{1}{\tan^2 \frac{A}{2} + \tan \frac{B}{2} \tan \frac{C}{2}} \leq \frac{9R}{4r}$$

Marin Chirciu

Solution: *LHS inequality. Using Bergström's inequality we obtain:*

$$\sum \frac{1}{\tan^2 \frac{A}{2} + \tan \frac{B}{2} \tan \frac{C}{2}} \geq \frac{9}{\sum\left(\tan^2 \frac{A}{2} + \tan \frac{B}{2} \tan \frac{C}{2}\right)} = \frac{9}{\sum \tan^2 \frac{A}{2} + \sum \tan \frac{B}{2} \tan \frac{C}{2}} =$$

$= \frac{9}{\frac{(4R+r)^2-2s^2}{s^2}+1} = \frac{9}{(4R+r)^2-s^2} \overset{(1)}{\geq} \frac{9r}{R}$, *where (1)* $\Leftrightarrow s^2(R+r) \geq r(4R+r)^2$, *which follows from Gerretsen's inequality* $s^2 \leq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$.

Above we've used the known inequaity in triangle:

$\sum \tan^2\frac{A}{2} = \frac{(4R+r)^2-2s^2}{s^2}$ *and* $\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1$.

Equality holds if and only if the triangle is equilateral.

RHS inequality. Using the means inequality: $\tan^2\frac{A}{2} + \tan\frac{B}{2}\tan\frac{C}{2} \geq 2\sqrt{\tan^2\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}$, *we obtain:* $\sum \frac{1}{\tan^2\frac{A}{2}+\tan\frac{B}{2}\tan\frac{C}{2}} \leq \sum \frac{1}{2\sqrt{\tan^2\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}} = \frac{1}{2\sqrt{\prod \tan\frac{A}{2}}}\sum \frac{1}{\sqrt{\tan\frac{A}{2}}} \overset{(1)}{\leq} \frac{9R}{4r}$, *where (1)* $\Leftrightarrow \frac{1}{\sqrt{\prod \tan\frac{A}{2}}}\sum \frac{1}{\sqrt{\tan\frac{A}{2}}} \leq \frac{9R}{2r}$, *which follows from CBS inequalit:* $\sum \frac{1}{\sqrt{\tan\frac{A}{2}}} \leq \sqrt{3\sum \frac{1}{\tan\frac{A}{2}}}$ *and the identites* $\sum \frac{1}{\tan\frac{A}{2}} = \frac{s}{r}, \prod \tan\frac{A}{2} = \frac{r}{s}$. *We obtain* $\frac{1}{\sqrt{\prod \tan\frac{A}{2}}}\sum \frac{1}{\sqrt{\tan\frac{A}{2}}} \leq \frac{1}{\sqrt{\frac{r}{s}}}\sqrt{3\cdot\frac{s}{r}} = \frac{s}{r}\sqrt{3} \overset{(2)}{\leq} \frac{9R}{2r}$, *where*

(2) $\Leftrightarrow s \leq \frac{R\sqrt{3}}{2}$ *(Mitrinovic's inequality)*

Equality holds if and only if the triangle is equilateral.

5) In ΔABC ***the following inequality holds:***

$$2\left(\frac{r}{R}\right)^2 \leq \sum \frac{1}{\cot^2\frac{A}{2} + \cot\frac{B}{2}\cot\frac{C}{2}} \leq \frac{R}{4r}$$

Marin Chirciu

Solution: *LHS inequality. Using Bergström's inequality obtain:*

$$\sum \frac{1}{\cot^2\frac{A}{2} + \cot\frac{B}{2}\cot\frac{C}{2}} \geq \frac{9}{\sum\left(\cot^2\frac{A}{2} + \cot\frac{B}{2}\cot\frac{C}{2}\right)} = \frac{9}{\sum \cot^2\frac{A}{2} + \sum \cot\frac{B}{2}\cot\frac{C}{2}} =$$

$= \frac{9}{\frac{s^2-2r^2-8Rr}{r^2}+\frac{4R+r}{r}} = \frac{9r^2}{s^2-r^2-4Rr} \overset{(1)}{\geq} \frac{2r^2}{R^2}$, *where (1)* $\Leftrightarrow 2s^2 \leq 9R^2 + 8Rr + 2r^2$, *which follows from Gerretsen's inequality* $s^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove:* $2(4R^2 + 4Rr + 3r^2) \leq 9R^2 + 8Rr + 2r^2 \Leftrightarrow R^2 \geq 4r^2$, *obviously from Euler's inequality* $R \geq 2r$. *Above we've used the known inequalities in triangle:*

$\sum \cot^2\frac{A}{2} = \frac{s^2-2r^2-8Rr}{r^2}$ *and* $\sum \cot\frac{B}{2}\cot\frac{C}{2} = \frac{4R+r}{r}$. *Equality holds if and only if the triangle is equilateral.*

RHS inequality. Using the means inequality $\cot^2\frac{A}{2} + \cot\frac{B}{2}\cot\frac{C}{2} \geq 2\sqrt{\cot^2\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}$ *we obtain that:* $\sum \frac{1}{\cot^2\frac{A}{2}+\cot\frac{B}{2}\cot\frac{C}{2}} \leq \sum \frac{1}{2\sqrt{\cot^2\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}} = \frac{1}{2\sqrt{\prod \cot\frac{A}{2}}}\sum \frac{1}{\sqrt{\cot\frac{A}{2}}} \overset{(1)}{\leq} \frac{R}{4r}$ *where (1)* $\Leftrightarrow \frac{1}{\sqrt{\prod \tan\frac{A}{2}}}\sum \frac{1}{\sqrt{\tan\frac{A}{2}}} \leq \frac{R}{2r}$, *which follows from CBS inequality* $\sum \frac{1}{\sqrt{\cot\frac{A}{2}}} \leq \sqrt{3\sum \frac{1}{\cot\frac{A}{2}}}$ *and the identites* $\sum \frac{1}{\cot\frac{A}{2}} = \frac{4R+r}{s} \prod \cot\frac{A}{2} = \frac{s}{r}$. *We obtain*

$$\frac{1}{\sqrt{\prod\cot\frac{A}{2}}}\sum\frac{1}{\sqrt{\cot\frac{A}{2}}}\le\frac{1}{\sqrt{\frac{s}{r}}}\sqrt{3\cdot\frac{4R+r}{s}}=\sqrt{3\cdot\frac{r(4R+r)}{s^2}}\overset{(2)}{\le}\frac{R}{2r},$$ *where (2)*

$\Leftrightarrow s^2R^2\ge 12r^3(4R+r)$, *which follows from Gerretsen's inequality* $s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$$(16Rr-5r^2)R^2\ge 12r^3(4R+r)\Leftrightarrow 16R^3-5R^2r-48Rr^2-12r^3\Leftrightarrow$$

$\Leftrightarrow (R-2r)(16R^2+27Rr+6r^2)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$. *Equality holds if and only if the triangle is equilateral.*

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ABOUT AN INEQUALITY BY HOANG LE NHAT TUNG-I

By Marin Chirciu-Romania

1.Let $a,b,c>0$ such that $(a+b)(b+c)(c+a)=1$. Find the minimum value of expression

$$P=\frac{a}{b(b+2c)(a+3c)^2}+\frac{b}{c(c+2a)(b+3a)^2}+\frac{c}{a(a+2b)(c+3b)^2}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by Marin Chirciu-Romania

$$P=\sum_{cyc}\frac{a}{b(b+2c)(a+3c)^2}=\sum_{cyc}\frac{\left(\frac{a}{a+3c}\right)^2}{ab(b+2c)}\overset{BCS}{\ge}\frac{\left(\sum_{cyc}\frac{a}{a+3c}\right)^2}{\sum_{cyc}ab(b+2c)}\overset{(1)}{\ge}$$

$\ge\frac{\left(\frac{3}{4}\right)^2}{\sum_{cyc}a^2b+6abc}\overset{(2)}{\ge}\frac{\frac{9}{16}}{\frac{9}{8}}=\frac{1}{2}$. Let's proof inequality (1): $\sum_{cyc}\frac{a}{a+3c}\ge\frac{3}{4}$

$$\sum_{cyc}\frac{a}{a+3c}=\sum_{cyc}\frac{a^2}{a^2+3ac}\overset{BCS}{\ge}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}(a^2+3ac)}=\frac{\sum_{cyc}a^2+2\sum_{cyc}bc}{\sum_{cyc}a^2+3\sum_{cyc}bc}\overset{(3)}{\ge}\frac{3}{4}$$

Where (3) $\Leftrightarrow\sum_{cyc}a^2\ge\sum_{cyc}bc\Leftrightarrow\sum_{cyc}(b-c)^2\ge 0$ true, with equality when $a=b=c$.

Let's proof inequality (2): $\sum_{cyc}a^2b+6abc\le\frac{9}{8}$

From AM-GM we have: $a+b\ge 2\sqrt{ab}$ and analogously, then

$(a+b)(b+c)(c+a)\ge 8abc$ and from $(a+b)(b+c)(c+a)=1$ we get

$abc\le\frac{1}{8}$; (4) with equality for $a=b=c=\frac{1}{2}$.Now,

$$(a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc=1+abc\overset{(4)}{\le}1+\frac{1}{8}=\frac{9}{8}$$

$$(a+b+c)(ab+bc+ca)\le\frac{9}{8}\Leftrightarrow\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\le\frac{9}{8};\quad(5)$$

Applying again AM-GM, we have $\sum_{cyc}a^2b\ge 3abc$, then from (5) it follows that

$$\frac{9}{8}\ge\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\ge\sum_{cyc}ab^2+3abc+3abc=\sum_{cyc}ab^2+6abc$$

Hence $\sum_{cyc}ab^2+6abc\le\frac{9}{8}$. From (1),(2) it follows that $P\ge\frac{1}{2}$.

So, minimum of expression $P=\sum_{cyc}\frac{a}{b(b+2c)(a+3c)^2}$ is $\frac{1}{2}$ attained for $a=b=c=\frac{1}{2}$.

2. Let $a, b, c > 0$ such that $(a+b)(b+c)(c+a) = 1$ and $\lambda \geq 2$. Find the minimum value of expression

$$P = \frac{a}{b(b+2c)(a+\lambda c)^2} + \frac{b}{c(c+2a)(b+\lambda a)^2} + \frac{c}{a(a+2b)(c+\lambda b)^2}$$

Marin Chirciu

Solution by proposer

$$P = \sum_{cyc} \frac{a}{b(b+2c)(a+\lambda c)^2} = \sum_{cyc} \frac{\left(\frac{a}{a+\lambda c}\right)^2}{ab(b+2c)} \overset{BCS}{\geq} \frac{\left(\sum_{cyc} \frac{a}{a+\lambda c}\right)^2}{\sum_{cyc} ab(b+2c)} \overset{(1)}{\geq}$$

$$\geq \frac{\left(\frac{3}{\lambda+1}\right)^2}{\sum_{cyc} a^2b + 6abc} \overset{(2)}{\geq} \frac{\frac{9}{(\lambda+1)^2}}{\frac{9}{8}} = \frac{8}{(\lambda+1)^2}.$$

Let's proof inequality (1): $\sum_{cyc} \frac{a}{a+\lambda c} \geq \frac{3}{\lambda+1}$

$$\sum_{cyc} \frac{a}{a+\lambda c} = \sum_{cyc} \frac{a^2}{a^2+\lambda ac} \overset{BCS}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc}(a^2+\lambda ac)} = \frac{\sum_{cyc} a^2 + 2\sum_{cyc} bc}{\sum_{cyc} a^2 + \lambda \sum_{cyc} bc} \overset{(3)}{\geq} \frac{3}{4}$$

Where (3) $\Leftrightarrow (\lambda-2)\sum_{cyc} a^2 \geq (\lambda-2)\sum_{cyc} bc \Leftrightarrow (\lambda-2)\sum_{cyc}(b-c)^2 \geq 0$ true for $\lambda \geq 2$ and $\sum_{cyc}(b-c)^2 \geq 0$, with equality when $a = b = c$.

Let's proof inequality (2): $\sum_{cyc} a^2b + 6abc \leq \frac{9}{8}$

From AM-GM we have: $a + b \geq 2\sqrt{ab}$ and analogously, then $(a+b)(b+c)(c+a) \geq 8abc$ and from $(a+b)(b+c)(c+a) = 1$ we get $abc \leq \frac{1}{8}$; (4) with equality for $a = b = c = \frac{1}{2}$. Now,

$$(a+b+c)(ab+bc+ca) = (a+b)(b+c)(c+a) + abc = 1 + abc \overset{(4)}{\leq} 1 + \frac{1}{8} = \frac{9}{8}$$

$$(a+b+c)(ab+bc+ca) \leq \frac{9}{8} \Leftrightarrow \sum_{cyc} ab^2 + \sum_{cyc} a^2b + 3abc \leq \frac{9}{8}; \quad (5)$$

Applying again AM-GM, we have $\sum_{cyc} a^2b \geq 3abc$, then from (5) it follows that

$$\frac{9}{8} \geq \sum_{cyc} ab^2 + \sum_{cyc} a^2b + 3abc \geq \sum_{cyc} ab^2 + 3abc + 3abc = \sum_{cyc} ab^2 + 6abc$$

Hence $\sum_{cyc} ab^2 + 6abc \leq \frac{9}{8}$. From (1),(2) it follows that $P \geq \frac{8}{(\lambda+1)^2}$.

So, minimum of expression $P = \sum_{cyc} \frac{a}{b(b+2c)(a+\lambda c)^2}$ is $\frac{8}{(\lambda+1)^2}$ attained for $a = b = c = \frac{1}{2}$.

Note: For $\lambda = 2$ we get problem SP.304 from Number 22-RMM Autumn Edition 2021, proposed by Hoang Le Nhat Tung, Hanoi, Vietnam.

3. Let $a, b, c > 0$ such that $(a+b)(b+c)(c+a) = 1$ and $\lambda \geq 2, \mu \geq 2$. Find the minimum value of expression

$$P = \frac{a}{b(b+\mu c)(a+\lambda c)^2} + \frac{b}{c(c+\mu a)(b+\lambda a)^2} + \frac{c}{a(a+\mu b)(c+\lambda b)^2}$$

Marin Chirciu

Solution by proposer

$$P=\sum_{cyc}\frac{a}{b(b+\mu c)(a+\lambda c)^2}=\sum_{cyc}\frac{\left(\frac{a}{a+\lambda c}\right)^2}{ab(b+\mu c)}\overset{BCS}{\geq}\frac{\left(\sum_{cyc}\frac{a}{a+\lambda c}\right)^2}{\sum_{cyc}ab(b+\mu c)}\overset{(1)}{\geq}$$

$$\geq\frac{\left(\frac{3}{\lambda+1}\right)^2}{\sum_{cyc}a^2b+3\mu abc}\overset{(2)}{\geq}\frac{\frac{9}{(\lambda+1)^2}}{\frac{3(\mu+1)}{8}}=\frac{24}{(\mu+1)(\lambda+1)^2}.$$

Let's proof inequality (1): $\sum_{cyc}\frac{a}{a+\lambda c}\geq\frac{3}{\lambda+1}$

$$\sum_{cyc}\frac{a}{a+\lambda c}=\sum_{cyc}\frac{a^2}{a^2+\lambda ac}\overset{BCS}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}(a^2+\lambda ac)}=\frac{\sum_{cyc}a^2+2\sum_{cyc}bc}{\sum_{cyc}a^2+\lambda\sum_{cyc}bc}\overset{(3)}{\geq}\frac{3}{4}$$

Where (3) $\Leftrightarrow(\lambda-2)\sum_{cyc}a^2\geq(\lambda-2)\sum_{cyc}bc\Leftrightarrow(\lambda-2)\sum_{cyc}(b-c)^2\geq 0$ true for $\lambda\geq 2$ and $\sum_{cyc}(b-c)^2\geq 0$, with equality when $a=b=c$.

Let's proof inequality (2): $\sum_{cyc}a^2b+3\mu abc\leq\frac{3(\mu+1)}{8}$

From AM-GM we have: $a+b\geq 2\sqrt{ab}$ and analogously, then $(a+b)(b+c)(c+a)\geq 8abc$ and from $(a+b)(b+c)(c+a)=1$ we get $abc\leq\frac{1}{8}$; (4) with equality for $a=b=c=\frac{1}{2}$. Now,

$$(a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc=1+abc\overset{(4)}{\leq}1+\frac{1}{8}=\frac{9}{8}$$

$$(a+b+c)(ab+bc+ca)\leq\frac{9}{8}\Leftrightarrow\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\leq\frac{9}{8};\ \ (5)$$

Applying again AM-GM, we have $\sum_{cyc}a^2b\geq 3abc$, then from (5) it follows that

$$\frac{9}{8}\geq\sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc\geq\sum_{cyc}ab^2+3abc+3abc=\sum_{cyc}ab^2+6abc$$

Hence $\sum_{cyc}ab^2+6abc\leq\frac{9}{8}$; (6)

$\sum_{cyc}a^2b+3\mu abc\leq\frac{3(\mu+1)}{8}$ true from (6) and (4), for $\mu\geq 2$ we get (2).

From (1),(2) it follows that $P\geq\frac{24}{(\mu+1)(\lambda+1)^2}$.

So, minimum of expression $P=\sum_{cyc}\frac{a}{b(b+\mu c)(a+\lambda c)^2}$ is $\frac{24}{(\mu+1)(\lambda+1)^2}$ attained for $a=b=c=\frac{1}{2}$.

4. Let $a,b,c>0$ such that $(a+b)(b+c)(c+a)=1$ and $\lambda\geq 2, n\in\mathbb{N}^*$. Find the minimum value of expression

$$\boldsymbol{P=\frac{a^n}{b(b+2c)(a+\lambda c)^{n+1}}+\frac{b^n}{c(c+2a)(b+\lambda a)^{n+1}}+\frac{c^n}{a(a+2b)(c+\lambda b)^{n+1}}}$$

Marin Chirciu

Solution by proposer

$$P=\sum_{cyc}\frac{a^n}{b(b+2c)(a+\lambda c)^{n+1}}=\sum_{cyc}\frac{\left(\frac{a}{a+\lambda c}\right)^{n+1}}{ab(b+2c)}\overset{Holder}{\geq}\frac{\left(\sum_{cyc}\frac{a}{a+\lambda c}\right)^{n+1}}{3^{n-1}\sum_{cyc}ab(b+2c)}\overset{(1)}{\geq}$$

$$\geq \frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{3^{n-1}\sum_{cyc}a^2b+6abc} \overset{(2)}{\geq} \frac{\frac{9}{(\lambda+1)^{n+1}}}{\frac{9}{8}} = \frac{8}{(\lambda+1)^{n+1}}.$$

Let's proof inequality (1): $\sum_{cyc}\frac{a}{a+\lambda c} \geq \frac{3}{\lambda+1}$

$$\sum_{cyc}\frac{a}{a+\lambda c} = \sum_{cyc}\frac{a^2}{a^2+\lambda ac} \overset{BCS}{\geq} \frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}(a^2+\lambda ac)} = \frac{\sum_{cyc}a^2+2\sum_{cyc}bc}{\sum_{cyc}a^2+\lambda\sum_{cyc}bc} \overset{(3)}{\geq} \frac{3}{4}$$

Where (3) $\Leftrightarrow (\lambda-2)\sum_{cyc}a^2 \geq (\lambda-2)\sum_{cyc}bc \Leftrightarrow (\lambda-2)\sum_{cyc}(b-c)^2 \geq 0$ true for $\lambda \geq 2$ and $\sum_{cyc}(b-c)^2 \geq 0$, with equality when $a=b=c$.

Let's proof inequality (2): $\sum_{cyc}a^2b+6abc \leq \frac{9}{8}$

From AM-GM we have: $a+b \geq 2\sqrt{ab}$ and analogously, then $(a+b)(b+c)(c+a) \geq 8abc$ and from $(a+b)(b+c)(c+a)=1$ we get $abc \leq \frac{1}{8}$; (4) with equality for $a=b=c=\frac{1}{2}$. Now,

$$(a+b+c)(ab+bc+ca) = (a+b)(b+c)(c+a)+abc = 1+abc \overset{(4)}{\leq} 1+\frac{1}{8} = \frac{9}{8}$$

$$(a+b+c)(ab+bc+ca) \leq \frac{9}{8} \Leftrightarrow \sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc \leq \frac{9}{8};\ (5)$$

Applying again AM-GM, we have $\sum_{cyc}a^2b \geq 3abc$, then from (5) it follows that

$$\frac{9}{8} \geq \sum_{cyc}ab^2+\sum_{cyc}a^2b+3abc \geq \sum_{cyc}ab^2+3abc+3abc = \sum_{cyc}ab^2+6abc$$

Hence $\sum_{cyc}ab^2+6abc \leq \frac{9}{8}$. From (1),(2) it follows that $P \geq \frac{8}{(\lambda+1)^{n+1}}$.

So, minimum of expression $P = \sum_{cyc}\frac{a}{b(b+2c)(a+\lambda c)^2}$ is $\frac{8}{(\lambda+1)^{n+1}}$ attained for $a=b=c=\frac{1}{2}$.

Note. For $n=1, \lambda=2$ we get problem SP.304 from Number 22-RMM Autumn Edition 2021, proposed by Hoang Le Nhat Tung-Hanoi-Vietnam.

5. Let $a,b,c>0$ such that $(a+b)(b+c)(c+a)=1$ and $\lambda \geq 2, \mu \geq 2$. Find the minimum value of expression

$$P = \frac{a^n}{b(b+\mu c)(a+\lambda c)^{n+1}} + \frac{b^n}{c(c+\mu a)(b+\lambda a)^{n+1}} + \frac{c^n}{a(a+\mu b)(c+\lambda b)^{n+1}}$$

Marin Chirciu

Solution by proposer

$$P = \sum_{cyc}\frac{a^n}{b(b+\mu c)(a+\lambda c)^{n+1}} = \sum_{cyc}\frac{\left(\frac{a}{a+\lambda c}\right)^{n+1}}{ab(b+\mu c)} \overset{Holder}{\geq} \frac{\left(\sum_{cyc}\frac{a}{a+\lambda c}\right)^{n+1}}{3^{n-1}\sum_{cyc}ab(b+\mu c)} \overset{(1)}{\geq}$$

$$\geq \frac{\left(\frac{3}{\lambda+1}\right)^{n+1}}{3^{n-1}\sum_{cyc}a^2b+3\mu abc} \overset{(2)}{\geq} \frac{\frac{9}{(\lambda+1)^{n+1}}}{\frac{3(\mu+1)}{8}} = \frac{24}{(\mu+1)(\lambda+1)^{n+1}}.$$

Let's proof inequality (1): $\sum_{cyc}\frac{a}{a+\lambda c} \geq \frac{3}{\lambda+1}$

$$\sum_{cyc}\frac{a}{a+\lambda c} = \sum_{cyc}\frac{a^2}{a^2+\lambda ac} \overset{BCS}{\geq} \frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}(a^2+\lambda ac)} = \frac{\sum_{cyc}a^2+2\sum_{cyc}bc}{\sum_{cyc}a^2+\lambda\sum_{cyc}bc} \overset{(3)}{\geq} \frac{3}{4}$$

Where (3) $\Leftrightarrow (\lambda-2)\sum_{cyc} a^2 \geq (\lambda-2)\sum_{cyc} bc \Leftrightarrow (\lambda-2)\sum_{cyc}(b-c)^2 \geq 0$ true for $\lambda \geq 2$ and $\sum_{cyc}(b-c)^2 \geq 0$, with equality when $a=b=c$.

Let's proof inequality (2): $\sum_{cyc} a^2b + 3\mu abc \leq \frac{3(\mu+1)}{8}$

From AM-GM we have: $a+b \geq 2\sqrt{ab}$ and analogously, then $(a+b)(b+c)(c+a) \geq 8abc$ and from $(a+b)(b+c)(c+a)=1$ we get $abc \leq \frac{1}{8}$; (4) with equality for $a=b=c=\frac{1}{2}$. Now,

$$(a+b+c)(ab+bc+ca)=(a+b)(b+c)(c+a)+abc=1+abc \overset{(4)}{\leq} 1+\frac{1}{8}=\frac{9}{8}$$

$$(a+b+c)(ab+bc+ca) \leq \frac{9}{8} \Leftrightarrow \sum_{cyc} ab^2 + \sum_{cyc} a^2b + 3abc \leq \frac{9}{8};\ (5)$$

Applying again AM-GM, we have $\sum_{cyc} a^2b \geq 3abc$, then from (5) it follows that

$$\frac{9}{8} \geq \sum_{cyc} ab^2 + \sum_{cyc} a^2b + 3abc \geq \sum_{cyc} ab^2 + 3abc + 3abc = \sum_{cyc} ab^2 + 6abc$$

Hence $\sum_{cyc} ab^2 + 6abc \leq \frac{9}{8}$; (6)

$\sum_{cyc} a^2b + 3\mu abc \leq \frac{3(\mu+1)}{8}$ true from (6) and (4), for $\mu \geq 2$ we get (2).

From (1),(2) it follows that $P \geq \frac{24}{(\mu+1)(\lambda+1)^{n+1}}$.

So, minimum of expression $P=\sum_{cyc}\frac{a^n}{b(b+\mu c)(a+\lambda c)^{n+1}}$ is $\frac{24}{(\mu+1)(\lambda+1)^{n+1}}$ attained for $a=b=c=\frac{1}{2}$.

Note. For $n=1, \lambda=3, \mu=2$ we get problem SP.304 from RMM Autumn Edition 2021, proposed by Hoang Le Nhat-Hanoi-Vietnam

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

OPPENHEIM'S INEQUALITY REVISITED

By Bogdan Fuștei-Romania

NOTE BY EDITOR: *In his unmistakable style with which the author has accustomed us in the previous articles in this paper is approached the famous Oppenheim inequality in triangle. The author provides an interesting generalization and numerous applications with inequalities and identities that result from it. I notice once again the way in which the author presents his thought process in its entire evolution for obtaining the results-fact that makes the article extremely useful also from a methodical perspective.*

Let M –be any point in euclidian space and $x,y,z \in \mathbb{R}$. In ΔABC holds:

$$\boldsymbol{(x+y+z)(xMA^2+yMB^2+zMC^2) \geq yza^2+zxb^2+xyc^2}$$

Proof: *We have:* $x\overrightarrow{MA}+y\overrightarrow{MB}+z\overrightarrow{MC} \geq 0$ *then:*

$$x^2MA^2+y^2MB^2+z^2MC^2+2xy\overrightarrow{MA}\cdot\overrightarrow{MB}+2yz\overrightarrow{MB}\cdot\overrightarrow{MC}+2zx\overrightarrow{MC}\cdot\overrightarrow{MA} \geq 0$$

$$x^2MA^2+y^2MB^2+z^2MC^2+xy(MA^2+MB^2-AB^2)+yz(MB^2+MC^2-BC^2)$$
$$+zx(MC^2+MA^2-AC^2) \geq 0$$

$$x(x+y+z)MA^2+y(x+y+z)MB^2+z(x+y+z)MC^2\geq yza^2+zxb^2+xyc^2$$
$$(x+y+z)(xMA^2+yMB^2+zMC^2)\geq yza^2+zxb^2+xyc^2$$
If we take $M=0$ then $OA=OB=OC=R$ then we have that:
$$(x+y+z)^2R^2\geq yza^2+zxb^2+xyc^2\ (\boldsymbol{Bottema's\ Inequality})$$
Let be $x,y,z\in\mathbb{R}_+;x=a^2x_1,y=b^2y_1,z=c^2z_1;\ x_1,y_1,z_1\in\mathbb{R}_+$ then follows:
$(a^2x_1+b^2y_1+c^2z_1)^2R^2\geq a^2b^2c^2(x_1y_1+y_1z_1+z_1x_1)$ *and from $abc=4RS$ we get:*
$a^2x_1+b^2y_1+c^2z_1\geq 4S\sqrt{x_1y_1+y_1z_1+z_1x_1}$ ***(Oppenheim-Băndilă-Chiriță)***
In $A_1B_1C_1$ with sides $a_1=\sqrt{a},b_1=\sqrt{b},c_1=\sqrt{c}$ the following relationship holds:
$$ax_1+by_1+cz_1\geq 4S_1\sqrt{x_1y_1+y_1z_1+z_1x_1}$$
$$16S^2=2\sum_{cyc}a^2b^2-(a^4+b^4+c^4)\Rightarrow 16S_1^2=2(ab+bc+ca)-(a^2+b^2+c^2)$$
$ab+bc+ca=s^2+4Rr+r^2;\ a^2+b^2+c^2=2(s^2-4Rr-r^2)$ *it follows*
$$16S_1^2=4r(4R+r)\Rightarrow 4S_1^2=r(4R+r)\Rightarrow S_1=\frac{1}{2}\sqrt{r(4R+r)}$$
So, we get: $ax_1+by_1+cz_1\geq 2\sqrt{r(4R+r)(x_1y_1+y_1z_1+z_1x_1)}$
But $4R+r\geq s\sqrt{4-\frac{2r}{R}}$ *which follows from* $s^2\leq\frac{R(4R+r)^2}{2(2R-r)}$ $(\boldsymbol{Blundon})$.
Therefore,
$$ax_1+by_1+cz_1\geq 2\sqrt{S\sqrt{4-\frac{2r}{R}}(x_1y_1+y_1z_1+z_1x_1)}$$
Let be: $x_1=\frac{x}{a},y_1=\frac{y}{b},z_1=\frac{z}{c};x,y,z>0$ *then follows*
$$x+y+z\geq 2\sqrt{S\sqrt{4-\frac{2r}{R}}\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{xz}{ac}\right)}$$
$$S=\frac{absinC}{2}=\frac{bcsinA}{2}=\frac{acsinB}{2}$$
$$(x+y+z)^2\geq 4S\sqrt{4-\frac{2r}{R}}\cdot\sum_{cyc}\frac{xy}{ab}=\sqrt{4-\frac{2r}{R}}\cdot\sum_{cyc}\frac{xy}{ab}\cdot 2absinC=\sqrt{4-\frac{2r}{R}}\cdot 2\sum_{cyc}xysinC$$
In ΔABC; $x,y,z>0$ the following relationship holds:
$$\boldsymbol{(x+y+z)^2\geq 2\sqrt{4-\frac{2r}{R}}(yzsinA+zxsinB+xysinC)\ (Oppenheim\ generalization)}$$
Proof.
We prove it that: $ax_1+by_1+cz_1\geq 2\sqrt{r(4R+r)(x_1y_1+y_1z_1+z_1x_1)};x_1,y_1,z_1>0$
Let be: $x_1=\frac{x}{a},y_1=\frac{y}{b},z_1=\frac{z}{c};x,y,z>0$ *then*
$$\frac{(x+y+z)^2}{r_a+r_b+r_c}\geq 4r\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ca}\right)$$
But $tan\frac{A}{2}=\frac{r_a}{s}$ $(and\ analogs)$; $r_a+r_b+r_c=4R+r$, *so we get:*
$(x+y+z)^2\geq 4r(R+r)\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ca}\right)$ *hence* $(x+y+z)^2\geq\frac{4S(4R+r)}{s}\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ca}\right)$ *and*

$$(x+y+z)^2 \ge \sum_{cyc} tan\frac{A}{2} \cdot \sum_{cyc} 2absinC \cdot \frac{xy}{ab}$$

Therefore: $(x+y+z)^2 \ge 2(xysinC + xzsinB + yzsinA)\sum_{cyc} tan\frac{A}{2}$

For $x, y, z > 0$ *we have* $ax+by+cz \ge 2\sqrt{r(4R+r)(xy+yz+zx)}$ *and put* $x = \frac{b}{a}$, $y = \frac{c}{b}, z = \frac{a}{c}$ *we get:*

$$a\cdot\frac{b}{a} + b\cdot\frac{c}{b} + c\cdot\frac{a}{c} \ge 2\sqrt{r(4R+r)\left(\frac{b}{a}\cdot\frac{c}{b} + \frac{b}{a}\cdot\frac{a}{c} + \frac{c}{b}\cdot\frac{a}{c}\right)} \Leftrightarrow$$

$$2s \ge 2\sqrt{r(4R+r)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)} \Leftrightarrow s \ge \sqrt{r(4R+r)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$$

But $4R+r \ge s\sqrt{3}$ *then* $s \ge \sqrt{rs\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$ *and hence*

$$s \ge r\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) (\boldsymbol{Ion\ Cristian\ Miu - refinement\ Mitrinovic})$$

How $4R+r \ge s\sqrt{4-\frac{2r}{R}}$ *then we have:* $s \ge r\sqrt{4-\frac{2r}{R}}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$ *and from* $s^2 \ge r(4R+r)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$ *we get:* $\frac{s}{r} \ge \frac{4R+r}{s}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$ *and with* $r_a = s\cdot tan\frac{A}{2}$ *follows that:* $\frac{s}{r} \ge \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(tan\frac{A}{2}+tan\frac{B}{2}+tan\frac{C}{2}\right)$.

From $cot\frac{A}{2}+cot\frac{B}{2}+cot\frac{C}{2} = \frac{s}{r}$ *we get a new inequality:*

$$\boldsymbol{\frac{cot\frac{A}{2}+cot\frac{B}{2}+cot\frac{C}{2}}{tan\frac{A}{2}+tan\frac{B}{2}+tan\frac{C}{2}} \ge \frac{a}{b}+\frac{b}{c}+\frac{c}{a}}$$

From $ax+by+cz \ge 2\sqrt{r(4R+r)(xy+yz+zx)}; x, y, z > 0$ *and for* $x = \frac{c}{a}, y = \frac{a}{b}$, $z = \frac{b}{c}$ *we have:* $a\cdot\frac{c}{a} + b\cdot\frac{a}{b} + c\cdot\frac{b}{c} \ge 2\sqrt{r(4R+r)\left(\frac{c}{a}\cdot\frac{a}{b} + \frac{c}{a}\cdot\frac{b}{c} + \frac{a}{b}\cdot\frac{b}{c}\right)}$ *therefore,*

$s \ge \sqrt{r(4R+r)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)}$ *and from* $4R+r \ge s\sqrt{3}$ *we get a new inequality:*

$$\boldsymbol{s \ge \sqrt{sr\sqrt{3}\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)}}$$

From $s^2 \ge 3r^2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)$ *follows:* $\frac{s^2}{3r^2} \ge \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)$

But $4R+r \ge s\sqrt{4-\frac{2r}{R}}$ *then:* $s \ge \sqrt{4-\frac{2r}{R}}\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)$ *and from* $s \ge r\sqrt{4-\frac{2r}{R}}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$ *it follows:* $\frac{s^2}{r^2} \ge \left(4-\frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)$.

From $ax+by+cz \ge 2\sqrt{r(4R+r)(xy+yz+zx)}; x, y, z > 0$ *and for* $x = \frac{c}{a}, y = \frac{a}{b}$, $z = \frac{b}{c}$ *we get* $s \ge \sqrt{r(4R+r)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$ *and hence*

$$s^2 \ge r(4R+r)\sqrt{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)} \Leftrightarrow \frac{s}{r} \ge \frac{4R+r}{s}\sqrt{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

From $r_a = s \cdot tan\frac{A}{2} (and\ analogs);\ r_a + r_b + r_c = 4R + r;\ \sum_{cyc} cot\frac{A}{2} = \frac{s}{r}$ *we get:*

$$\frac{\boldsymbol{cot\frac{A}{2} + cot\frac{B}{2} + cot\frac{C}{2}}}{\boldsymbol{tan\frac{A}{2} + tan\frac{B}{2} + tan\frac{C}{2}}} \geq \sqrt{\left(\frac{\boldsymbol{a}}{\boldsymbol{b}} + \frac{\boldsymbol{b}}{\boldsymbol{c}} + \frac{\boldsymbol{c}}{\boldsymbol{a}}\right)\left(\frac{\boldsymbol{c}}{\boldsymbol{b}} + \frac{\boldsymbol{b}}{\boldsymbol{a}} + \frac{\boldsymbol{a}}{\boldsymbol{c}}\right)}$$

Similarly:

$$\frac{\boldsymbol{cot\frac{A}{2} + cot\frac{B}{2} + cot\frac{C}{2}}}{\boldsymbol{tan\frac{A}{2} + tan\frac{B}{2} + tan\frac{C}{2}}} \geq \frac{\boldsymbol{c}}{\boldsymbol{b}} + \frac{\boldsymbol{b}}{\boldsymbol{a}} + \frac{\boldsymbol{a}}{\boldsymbol{c}}$$

Now, m_a, m_b, m_c *it can be the lengths sides of on triangle by area* $S_m = \frac{3}{4}S$*, semiperimeter* $s_m = \frac{m_a+m_b+m_c}{2};\ S_m = r_m s_m;\frac{3}{4}S_m = \frac{m_a+m_b+m_c}{2}\cdot r_m \Rightarrow r_m = \frac{3S}{2(m_a+m_b+m_c)}$

We have that: $s \geq r\sqrt{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)$ *and* $s \geq r\sqrt{3}\left(\frac{c}{b} + \frac{b}{a} + \frac{a}{c}\right)$ *so, for* $\Delta m_a m_b m_c$ *we have:*

$$\frac{m_a + m_b + m_c}{2} \geq \frac{3S}{2(m_a + m_b + m_c)}\sqrt{3}\left(\frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a}\right) \Leftrightarrow$$

$$\boldsymbol{(m_a + m_b + m_c)^2 \geq 3\sqrt{3}S\left(\frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a}\right)}$$

Similarly:

$$(m_a + m_b + m_c)^2 \geq 3\sqrt{3}S\left(\frac{m_c}{m_b} + \frac{m_b}{m_a} + \frac{m_a}{m_c}\right)$$

Therefore, we get a new inequality:

$$\boldsymbol{(m_a + m_b + m_c)^4 \geq \left(3\sqrt{3}S\right)^2\left(\frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a}\right)\left(\frac{m_c}{m_b} + \frac{m_b}{m_a} + \frac{m_a}{m_c}\right)}$$

and hence

$$\frac{(m_a + m_b + m_c)^2}{3\sqrt{3}S} \geq \sqrt{\left(\frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a}\right)\left(\frac{m_c}{m_b} + \frac{m_b}{m_a} + \frac{m_a}{m_c}\right)}$$

We know that:

$$s^2 = n_a^2 + 2r_a h_a \Leftrightarrow s^2 - n_a^2 = 2r_a h_a \Leftrightarrow$$

$$(s - n_a)(n + n_a) = 2r_a h_a \Leftrightarrow s - n_a = \frac{2r_a h_a}{s + n_a} \Leftrightarrow \frac{s - n_a}{h_a} = \frac{2r_a}{s + n_a} (and\ analogs)$$

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$$

$$\frac{s}{r} = \sum_{cyc}\frac{n_a}{h_a} + 2\sum_{cyc}\frac{r_a}{s + n_a}\ and\ \frac{s}{r} = \sum_{cyc}\frac{n_a}{r_a} + 2\sum_{cyc}\frac{h_a}{s + n_a}$$

$$\frac{s}{r} \geq \sqrt{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) and \frac{s}{r} \geq \sqrt{3}\left(\frac{c}{b} + \frac{b}{a} + \frac{a}{c}\right)$$

So, it follows that:

$$\sum_{cyc}\frac{n_a}{h_a} \geq \sqrt{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) - 2\sum_{cyc}\frac{r_a}{s + n_a}\ and\ \sum_{cyc}\frac{n_a}{h_a} \geq \sqrt{3}\left(\frac{c}{b} + \frac{b}{a} + \frac{a}{c}\right) - 2\sum_{cyc}\frac{r_a}{s + n_a}$$

$$\sum_{cyc}\frac{n_a}{r_a} \geq \sqrt{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) - 2\sum_{cyc}\frac{h_a}{s + n_a}\ and\ \sum_{cyc}\frac{n_a}{r_a} \geq \sqrt{3}\left(\frac{c}{b} + \frac{b}{a} + \frac{a}{c}\right) - 2\sum_{cyc}\frac{h_a}{s + n_a}$$

$$\boldsymbol{\frac{s}{r} \geq \sqrt{4 - \frac{2r}{R}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)} and \frac{s}{r} \geq \sqrt{4 - \frac{2r}{R}\left(\frac{c}{b} + \frac{b}{a} + \frac{a}{c}\right)}}$$

Therefore,

$$\sum_{cyc}\frac{n_a}{h_a}\ge\sqrt{4-\frac{2r}{R}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}-2\sum_{cyc}\frac{r_a}{s+n_a}\,and\,\sum_{cyc}\frac{n_a}{h_a}\ge\sqrt{4-\frac{2r}{R}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}-2\sum_{cyc}\frac{r_a}{s+n_a}$$

$$\sum_{cyc}\frac{n_a}{r_a}\ge\sqrt{4-\frac{2r}{R}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}-2\sum_{cyc}\frac{h_a}{s+n_a}\,and\,\sum_{cyc}\frac{n_a}{r_a}\ge\sqrt{4-\frac{2r}{R}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}-2\sum_{cyc}\frac{h_a}{s+n_a}$$

Applying Jensen Inequality for function convex $f(x)=tan\frac{x}{4}, x\in\left(0,\frac{\pi}{4}\right)$ we have:

$$tan\frac{A}{4}+tan\frac{B}{4}+tan\frac{C}{4}\ge 3tan\left(\frac{\frac{A}{4}+\frac{B}{4}+\frac{C}{4}}{3}\right)\Leftrightarrow tan\frac{A}{4}+tan\frac{B}{4}+tan\frac{C}{4}\ge 3tan\frac{\pi}{12}$$

$$tan\frac{\pi}{12}=tan\frac{1}{2}\left(\frac{\pi}{6}\right)=2-\sqrt{3}; tan\frac{x}{2}=\frac{1-cosx}{sinx}; cot\frac{x}{2}=\frac{1+cosx}{sinx}$$

$$cos\frac{A}{2}=\frac{s-a}{AI}(and\ analogs); sin\frac{A}{2}=\frac{r}{AI}(and\ analogs)$$

$$tan\frac{A}{4}=\frac{1-cos\frac{A}{2}}{sin\frac{A}{2}}=\frac{1-\frac{s-a}{AI}}{\frac{r}{AI}}=\frac{AI+a-s}{AI}\cdot\frac{AI}{r}=\frac{AI+a-s}{r}$$

So, we have: $tan\frac{A}{4}=\frac{AI+a-s}{r}(and\ analogs)$

$$\boldsymbol{\sum_{cyc}tan\frac{A}{4}=\frac{AI+BI+CI+a+b+c-3s}{r}}$$

$$\sum_{cyc}tan\frac{A}{4}=\frac{AI+BI+CI-s}{r}\Rightarrow\frac{AI+BI+CI-s}{r}\ge\frac{s}{r}+3\left(2-\sqrt{3}\right)$$

$$cos\frac{B-C}{2}=\left(\frac{b+c}{a}\right)sin\frac{A}{2}\ (and\ analogs); cos\frac{B-C}{2}=\frac{h_a}{w_a}(and\ analogs)$$

$$\frac{AI}{r}=\frac{1}{sin\frac{A}{2}}(and\ analogs)\Rightarrow\frac{h_a}{w_a}=\frac{b+c}{a}\cdot\frac{r}{AI}$$

$$\frac{b+c}{a}=\frac{ab+ac}{a^2}; bc=2Rh_a(and\ analogs);\ a^2=2R\cdot\frac{h_bh_c}{h_a}(and\ anaolgs)$$

$$\frac{b+c}{a}=\frac{2R(h_b+h_c)}{2R\cdot\frac{h_bh_c}{h_a}}=h_a\cdot\frac{h_b+h_c}{h_bh_c}(and\ analogs)$$

$$\frac{h_a}{w_a}=h_a\cdot\frac{h_b+h_c}{h_bh_c}\cdot\frac{r}{AI}\Rightarrow\frac{AI}{r}=\frac{w_a}{h_b}+\frac{w_a}{h_c}(and\ analogs)$$

Therefore,

$$\boldsymbol{\frac{AI+BI+CI}{r}=\sum_{cyc}\frac{w_b+w_c}{h_a}\ge\frac{s}{r}+3\left(2-\sqrt{3}\right)}$$

$$\boldsymbol{\frac{s}{r}=\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}+2\sum_{cyc}\frac{r_a}{s+n_a}}$$

So, we get a new inequality:

$$\boldsymbol{\sum_{cyc}\frac{w_b+w_c-n_a}{h_a}\ge 3\left(2-\sqrt{3}\right)+2\sum_{cyc}\frac{r_a}{s+n_a}}$$

Now,

$$\sum_{cyc} tan\frac{A}{4} = \frac{AI + BI + CI - s}{r} = \sum_{cyc}\frac{w_b + w_c - n_a}{h_a} - 2\sum_{cyc}\frac{r_a}{s + n_a}$$

and from

$$\frac{s}{r} \geq \frac{4R + r}{s}\sqrt{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}; 4R + r \geq s\sqrt{4 - \frac{2r}{R}}$$

So, we get a new inequality:

$$\boldsymbol{\frac{s}{r} \geq \sqrt{\left(4 - \frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}}$$

Denote:

$$Q = \sqrt{\left(4 - \frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)} \Rightarrow \sum_{cyc} tan\frac{A}{4} \leq \sum_{cyc}\frac{w_b + w_c}{h_a} - Q$$

$$cot\frac{A}{4} = \frac{1 + cos\frac{A}{2}}{sin\frac{A}{2}} = \frac{1 + \frac{s-a}{AI}}{sin\frac{A}{2}} = \frac{AI + s - a}{AI}\cdot\frac{AI}{r} = \frac{AI + s - a}{r}$$

Hence,

$$\boldsymbol{\sum_{cyc} cot\frac{A}{4} = \frac{AI + BI + CI + s}{r}}$$

So, we have the identity:

$$\boldsymbol{\sum_{cyc} cot\frac{A}{4} = \frac{AI + BI + CI + s}{r} = \sum_{cyc}\frac{w_b + w_c}{h_a} + \frac{s}{r}}$$

Therefore,

$$\sum_{cyc} cot\frac{A}{4} \geq Q + \sum_{cyc}\frac{w_b + w_c}{h_a} + \frac{s}{r} = \sum_{cyc}\frac{n_a}{h_a} + 2\sum_{cyc}\frac{r_a}{s + n_a}$$

$$\sum_{cyc} cot\frac{A}{4} = 2\sum_{cyc}\frac{r_a}{s + n_a} + \sum_{cyc}\frac{w_b + w_c - n_a}{h_a}$$

Denote $\boldsymbol{N_a}$ *–**Nagel point**. Applying **Van Aubel** theorem, we have:*

$$\frac{AN_a}{n_a - AN_a} = \frac{s-c}{s-a} + \frac{s-b}{s-a} = \frac{a}{s-a} \Rightarrow \frac{n_a - AN_a}{AN_a} = \frac{s-a}{a} \Rightarrow \frac{n_a}{AN_a} = \frac{s}{a}(and\ analogs)$$

$$AN_a = \frac{an_a}{s}; 2s = a + b + c \Rightarrow (a + b + c)AN_a = 2an_a \Rightarrow$$

$$1 + \frac{b + c}{a} = \frac{2n_a}{AN_a}(and\ analogs)$$

$$2S = ah_a; 2S = 2sr \Rightarrow ah_a = 2sr \Rightarrow \frac{h_a}{r} = \frac{a + b + c}{a} = 1 + \frac{b + c}{a}$$

Then, we have: $\frac{h_a}{2r} = \frac{n_a}{AN_a}(and\ analogs)$ *and hence*

$$\frac{AN_a + BN_a + CN_a}{2r} = \frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c}$$

$$\frac{AN_a}{a} = \frac{n_a}{s}(and\ analogs) \Rightarrow \sum_{cyc}\frac{AN_a}{a} = \frac{n_a + n_b + n_c}{s}$$

In $\Delta ABC: \frac{MA}{a} + \frac{MB}{b} + \frac{MC}{c} \geq \sqrt{3};\ \forall M \subset (ABC).$

For $M = N_a \Rightarrow \frac{AN_a}{a} + \frac{BN_a}{b} + \frac{CN_a}{c} \ge \sqrt{3} \Rightarrow n_a + n_b + n_c \ge s\sqrt{3}$

Now,

$\frac{b+c}{a} = \frac{r_a+h_a}{r_a} = 1 + \frac{h_a}{r_a}(and\ analogs)$ *and hence* $2 + \frac{h_a}{r_a} = \frac{2n_a}{AN_a}$

So, we have a new inequality:

$$\mathbf{2\sum_{cyc}\frac{n_a}{AN_a} = 6 + \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c} = \frac{h_a + h_b + h_c}{r}}$$

$$bc = 2Rh_a(and\ analogs);\ h_a + h_b + h_c = \frac{ab + bc + ca}{2R};$$

$$ab + bc + ca = s^2 + 4Rr + r^2$$

So, it follows that:

$$\sum_{cyc}\frac{n_a}{AN_a} = \frac{s^2 + 4Rr + r^2}{4Rr};\ s^2 \ge 16Rr - 5r^2(\boldsymbol{Gerretsen})$$

$$\mathbf{\sum_{cyc}\frac{n_a}{AN_a} \ge 5 - \frac{r}{R}}$$

From $\frac{h_a}{2r} = \frac{n_a}{AN_a}(and\ analogs)$ *and from* $h_a \le s_a(and\ analogs)$

we get: $\frac{s_a}{2r} \ge \frac{n_a}{AN_a}(and\ analogs)$ *then*

$$\mathbf{\frac{s_a + s_b + s_c}{2r} \ge \sum_{cyc}\frac{n_a}{AN_a}}$$

Now, we prove it that: $\frac{R}{r} - 1 \ge \frac{n_a}{h_a}(and\ analogs)$

$$\mathbf{\frac{R - r}{r} \ge \frac{AN_a}{2r} \Rightarrow 2(R - r) \ge AN_a(and\ analogs)}$$

Therefore,

$$\mathbf{6(R - r) \ge AN_a + BN_a + CN_a}$$

$$\frac{AN_a}{a} = \frac{n_a}{s}; \frac{AI}{w_a} = \frac{b + c}{2s} \Rightarrow \frac{2AI}{(b + c)w_a} = \frac{1}{s}$$

$$\frac{AN_a}{a} = \frac{2AI}{(b + c)w_a}\cdot n_a \Rightarrow \frac{AN_a}{AI} = \frac{a}{b + c}\cdot\frac{n_a}{w_a}(and\ analogs)$$

$$\frac{1}{sin\frac{A}{2}} = \frac{b + c}{a}\cdot\frac{w_a}{h_a}(and\ analogs) \Rightarrow \frac{2a}{b + c} = 2\frac{w_a}{h_a}sin\frac{A}{2}$$

$$\frac{AN_a}{AI} = 2\frac{n_a}{h_a}sin\frac{A}{2}(and\ analogs) \Rightarrow \frac{AN_a}{AI} \le 2\left(\frac{R}{r} - 1\right)sin\frac{A}{2}(and\ analogs)$$

$$\sum_{cyc}\frac{AN_a}{AI} < 2\left(\frac{R}{r} - 1\right)\sum_{cyc}sin\frac{A}{2}$$

$$sin\frac{A}{2} = \sqrt{\frac{r}{2R}}\cdot\sqrt{\frac{r_a}{h_a}} \Rightarrow 2sin\frac{A}{2} = \sqrt{\frac{2r}{R}}\cdot\sqrt{\frac{r_a}{h_a}} \Rightarrow \mathbf{\frac{AN_a}{AI} = \frac{n_a}{h_a}\sqrt{\frac{2r}{R}}\cdot\sqrt{\frac{r_a}{h_a}}(and\ analogs)}$$

From $\frac{R}{2r} \ge \frac{m_a}{h_a}(\boldsymbol{Panaitopol\ inequality})$ *we get:*

$$\mathbf{\frac{AN_a}{AI} = \frac{n_a}{h_a}\sqrt{\frac{2r}{R}}\cdot\sqrt{\frac{r_a}{h_a}} \le \frac{n_a}{h_a}\sqrt{\frac{h_a}{m_a}}\cdot\sqrt{\frac{r_a}{h_a}} = \frac{n_a}{h_a}\sqrt{\frac{r_a}{m_a}}}$$

So, we have: $\frac{AN_a}{AI} \le \frac{n_a}{h_a}\sqrt{\frac{r_a}{m_a}}$ $(and\ analogs)$ *and summing, it follows that:*

$$\sum_{cyc}\frac{\boldsymbol{AN_a}}{\boldsymbol{AI}} \le \sum_{cyc}\frac{\boldsymbol{n_a}}{\boldsymbol{h_a}}\sqrt{\frac{\boldsymbol{r_a}}{\boldsymbol{m_a}}}$$

Now, N_a *—Nagel's point*

$$AN_a = \frac{an_a}{s}(and\ analogs);\ n_a^2 = s^2 - 2r_ah_a(and\ analogs)$$

$$AN_a^2 = \frac{(s^2 - 2r_ah_a)a^2}{s^2} = \left(1 - \frac{2r_ah_a}{s^2}\right)a^2$$

$$r_a = \frac{S}{s-a};\ h_a = \frac{2S}{a} \Rightarrow 2r_ah_a = \frac{4S^2}{a(s-a)}$$

$$AN_a^2 = a^2 - \frac{4S^2}{a(s-a)}\cdot\frac{a^2}{s^2};\ S^2 = s(s-a)(s-b)(s-c)(\boldsymbol{Heron})$$

$$AN_a^2 = a^2 - \frac{4s(s-a)(s-b)(s-c)}{s^2}\cdot\frac{a}{s-a}$$

$$AN_a^2 = a\left(a - \frac{4(s-b)(s-c)}{s}\right)(and\ analogs)$$

But $a^2 = (b-c)^2 + b(s-b)(s-c)(and\ analogs)$

$$AN_a^2 = (b-c)^2 + 4(s-b)(s-c) - \frac{4a(s-b)(s-c)}{s}$$

$$AN_a^2 = (b-c)^2 + 4(s-b)(s-c)\left(1 - \frac{a}{s}\right) = (b-c)^2 + \frac{4s(s-a)(s-b)(s-c)}{s^2}$$

$$AN_a^2 = (b-c)^2 + \frac{4S^2}{s^2};\ S = sr$$

So, we get a new identity:

$$\boldsymbol{AN_a^2 = (b-c)^2 + 4r^2(and\ analogs); AN_a^2 = \frac{a^2n_a^2}{s^2}} \Rightarrow$$

$$\boldsymbol{\frac{n_a^2}{s^2} = \frac{(b-c)^2 + 4r^2}{a^2}(and\ analogs)}$$

Using AM-QM Inequality, we have:

$$\sqrt{\frac{(b-c)^2 + 4r^2}{2}} \ge \frac{|b-c| + 2r}{2} \Rightarrow \frac{an_a}{s} \ge \frac{|b-c| + 2r}{\sqrt{2}}$$

So, we get a new inequality:

$$\boldsymbol{\frac{n_a}{s} \ge \frac{|b-c| + 2r}{a\sqrt{2}}(and\ analogs); abc = 4RS = 4Rrs}$$

$$n_an_bn_c \ge \frac{s^3(|b-c| + 2r)(|c-a| + 2r)(|a-b| + 2r)}{2\sqrt{2}\cdot 4Rrs} \Leftrightarrow$$

$$n_an_bn_c \ge \frac{s^2(|b-c| + 2r)(|c-a| + 2r)(|a-b| + 2r)}{8\sqrt{2}\cdot Rr}$$

But

$$n_a \ge \frac{|b-c| + 2r}{a\sqrt{2}}\cdot s; \frac{s}{a} = \frac{h_a}{2r}(and\ analogs); n_a \ge \frac{h_a}{2r}\cdot\frac{|b-c| + 2r}{\sqrt{2}}$$

So, we get inequality:

$$\frac{\boldsymbol{n_a}}{\boldsymbol{h_a}} \geq \frac{|\boldsymbol{b}-\boldsymbol{c}|+\mathbf{2}\boldsymbol{r}}{\mathbf{2}\sqrt{\mathbf{2}}\boldsymbol{r}}\ (\boldsymbol{and\ analogs})$$

$$\frac{\boldsymbol{n_a}}{\boldsymbol{h_a}}+\frac{\boldsymbol{n_b}}{\boldsymbol{h_b}}+\frac{\boldsymbol{n_c}}{\boldsymbol{h_c}} \geq \frac{|\boldsymbol{a}-\boldsymbol{b}|+|\boldsymbol{b}-\boldsymbol{c}|+|\boldsymbol{c}-\boldsymbol{a}|+\mathbf{2}\boldsymbol{r}}{\mathbf{2}\sqrt{\mathbf{2}}\boldsymbol{r}}+\frac{\mathbf{3}\sqrt{\mathbf{3}}}{\mathbf{2}}$$

$$\boldsymbol{a}\sqrt{\mathbf{2}} \geq \frac{\boldsymbol{s}(|\boldsymbol{b}-\boldsymbol{c}|+\mathbf{2}\boldsymbol{r})}{\boldsymbol{n_a}}(\boldsymbol{and\ analogs})$$

We have prove it that:

$$\frac{AN_a}{2r}=\frac{n_a}{h_a}(and\ analogs)\ and\ AN_a^2=(b-c)^2+4r^2(and\ analogs)$$

So, we get:

$$\frac{n_a^2}{h_a^2}=\frac{(b-c)^2+4r^2}{4r^2}=1+\frac{(b-c)^2}{4r^2}(and\ analogs)$$

Now,

$$s^2=n_a^2+2r_ah_a(and\ analogs)$$

$$s^2-n_a^2=2r_ah_a \Leftrightarrow (s-n_a)(s+n_a)=2r_ah_a \Leftrightarrow \frac{s-n_a}{h_a}=\frac{2r_a}{s+n_a}(and\ analogs)$$

$$2S=ah_a=2sr \Rightarrow \frac{s}{h_a}=\frac{a}{2r}(and\ analogs)$$

So, we get:

$$\frac{a}{2r}=\frac{n_a}{h_a}+\frac{2r_a}{s+n_a}(and\ analogs)$$

Now,

$$cot\frac{B}{2}+cot\frac{C}{2}=\sqrt{\frac{r_ar_c}{rr_b}}+\sqrt{\frac{r_ar_b}{rr_c}}=\frac{\sqrt{r_ar_br_c}}{r_b\sqrt{r}}+\frac{\sqrt{r_ar_br_c}}{r_c\sqrt{r}};\ r_ar_br_c=s^2r$$

So, it follows that:

$$cos\frac{B}{2}+cot\frac{C}{2}=\frac{s}{r_b}+\frac{s}{r_c};\ h_a=\frac{2r_br_c}{r_b+r_c}(and\ analogs)$$

$$cot\frac{B}{2}+cot\frac{C}{2}=\frac{2s}{h_a}(and\ analogs)$$

$$cot\frac{B}{2}+cot\frac{C}{2}=\frac{a}{r}(and\ analogs)$$

So, we get a new inequality:

$$cot\frac{B}{2}+cot\frac{C}{2}=\frac{2n_a}{h_a}+\frac{4r_a}{s+n_a}\ (and\ analogs)$$

$$cot\frac{A}{4}=\frac{AI+s-a}{r}=\frac{AI}{r}+\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}+2\sum_{cyc}\frac{r_a}{s+n_a}-\frac{2n_a}{h_a}-\frac{4r_a}{s+n_a}=$$

$$=\frac{AI}{r}+\frac{n_b}{h_b}+\frac{n_c}{h_c}-\frac{n_a}{h_a}+2\left(\frac{r_b}{s+n_b}+\frac{r_c}{s+n_c}-\frac{r_a}{s+n_a}\right)$$

Finally, we get:

$$\boldsymbol{cot}\frac{\boldsymbol{A}}{\mathbf{4}}=\frac{\boldsymbol{AI}}{\boldsymbol{r}}+\frac{\boldsymbol{n_b}}{\boldsymbol{h_b}}+\frac{\boldsymbol{n_c}}{\boldsymbol{h_c}}-\frac{\boldsymbol{n_a}}{\boldsymbol{h_a}}+\mathbf{2}\left(\frac{\boldsymbol{r_b}}{\boldsymbol{s}+\boldsymbol{n_b}}+\frac{\boldsymbol{r_c}}{\boldsymbol{s}+\boldsymbol{n_c}}-\frac{\boldsymbol{r_a}}{\boldsymbol{s}+\boldsymbol{n_a}}\right)$$

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ABOUT AN INEQUALITY BY FLORICĂ ANASTASE-I

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\boldsymbol{9R(bch_a + cah_b + abh_c) \ge 2r^2(s^2 + r^2 + 4Rr)}$$

Proposed by Florică Anastase-Romania

Solution: Inequality can be written as:

$$\sum_{cyc} bch_a \ge \frac{2r^2}{9R}(s^2 + r^2 + 4Rr)$$

Because triplets (h_a, h_b, h_c) and (bc, ca, ab) are same order, from Chebyshev's inequality, we have:

$$\sum_{cyc} bch_a \ge \frac{1}{3}\sum_{cyc} h_a \sum_{cyc} bc \ge \frac{1}{3}\cdot\frac{s^2 + r^2 + 4Rr}{2R}(s^2 + r^2 + 4Rr) \overset{(1)}{\ge}$$

$\ge \frac{2r^2}{9R}(s^2 + r^2 + 4Rr)$, where $(1) \Leftrightarrow 3(s^2 + r^2 + 4Rr) \ge 4r^2$, which it follows from Gerretsen inequality $s^2 \ge 16Rr - 5r^2$ and Euler inequality $R \ge 2r$.

2) In ΔABC the following relationship holds:

$$\boldsymbol{\frac{8r^2}{3R}(4R + r)^2 \le \sum_{cyc} bch_a \le \frac{2R}{3}(4R + r)^2}$$

Marin Chirciu

Solution: **Lemma. 3) In ΔABC the following relationship holds:**

$$\boldsymbol{\sum_{cyc} bch_a = \frac{s^2(s^2 + 2r^2 - 8Rr) + r^2(4R + r)^2}{2R}}$$

Proof. Using $h_a = \frac{2F}{a}$, we get:

$$\sum_{cyc} bch_a = \sum_{cyc} bc\cdot\frac{2F}{a} = 2F\sum_{cyc}\frac{bc}{a} = 2F\cdot\frac{s^2(s^2 + 2r^2 - 8Rr) + r^2(4R + r)^2}{4RF} =$$

$= \frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{2R}$, which follows from

$$\sum_{cyc}\frac{bc}{a} = \frac{s^2(s^2 + 2r^2 - 8Rr) + r^2(4R + r)^2}{4RF}$$

Using Lemma and Blundon Gerretsen: $\frac{r(4R+r)^2}{R+r} \le s^2 \le \frac{R(4R+r)^2}{2(2R-r)}$

Equality holds if and only if triangle is equilateral.

4) In ΔABC the following relationship holds:

$$\boldsymbol{\left(1 + \frac{2}{3}\cdot\frac{r}{R}\right)\cdot r(4R + r)^2 \le \sum_{cyc} bch_a \le \frac{4}{3}\cdot r(4R + r)^2}$$

Marin Chirciu

Solution: **Lemma. 5)In ΔABC the following relationship holds:**

$$\boldsymbol{\sum_{cyc} bcr_a = r[s^2 + (4R + r)^2]}$$

Proof: Using $r_a = \frac{F}{s-a}$ we get:

$$\sum_{cyc} bcr_a = \sum_{cyc} bc \cdot \frac{F}{s-a} = F\sum_{cyc}\frac{bc}{s-a} = sr \cdot \frac{s^2+(4R+r)^2}{s} = r[s^2+(4R+r)^2],$$

which follows from $\sum_{cyc}\frac{bc}{s-a} = \frac{s^2+(4R+r)^2}{s}$.

Using Lemma and Blundon Gerretsen: $\frac{r(4R+r)^2}{R+r} \le s^2 \le \frac{R(4R+r)^2}{2(2R-r)}$.

Equality holds if and only if triangle is equilateral.

6) In ΔABC the following relationship holds:

$$\sum_{cyc} bcr_a \le \frac{R}{2r}\sum_{cyc} bch_a$$

Marin Chirciu

Solution: Using up Lemmas, we get:

$$\sum_{cyc} bch_a = \frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{2R}$$

$$\sum_{cyc} bcr_a = r[s^2+(4R+r)^2]$$

Inequality can be written as:

$$r[s^2+(4R+r)^2] \le \frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{2R} \Leftrightarrow$$

$s^2(s^2-2r^2-8Rr) \ge 3r^2(4R+r)^2$, which folllows from Gerretsen inequality:

$s^2 \ge 16Rr-5r^2$ and Euler inequality $R \ge 2r$.

Remains to prove that:

$$(16Rr-5r^2)(16Rr-5r^2-2r^2-8Rr) \ge 3r^2(4R+r)^2 \Leftrightarrow$$

$$5R^2-11Rr+2r^2 \ge 0 \Leftrightarrow (R-2r)(5R-r) \ge 0$$

Equality holds if and only if triangle is equilateral.

7) In ΔABC the following relationship holds:

$$\left(1+\frac{2}{3}\cdot\frac{r}{R}\right)\cdot r(4R+r)^2 \le \sum_{cyc} bch_a \le \frac{R}{2r}\sum_{cyc} bch_a \le \frac{R^2}{3r}(4R+r)^2$$

Proposed by Marin Chirciu-Romania

Solution. See inequalities 6), 4), 2).

Equality holds if and only if triangle is equilateral.

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ABOUT PROBLEM 12214 FROM AMERICAN MATHEMATICAL MONTHLY

By Marian Dincă-Romania

Let x, y and z be the lengths of the medians of a triangle with area F. Prove:

$$\frac{xyz(x+y+z)}{xy+yz+zx} \ge \sqrt{3}F$$

Proposed by George Apostolopoulos-Messolonghi-Greeece

Solution by Marian Dincă-Romania

$$\frac{3}{4}F(a,b,c)=F(x,y,z)\Rightarrow F(a,b,c)=\frac{4}{3}F(x,y,z)$$

$$\frac{xyz(x+y+z)}{xy+yz+zx}\geq\sqrt{3}F(x,y,z)$$

Let α,β,γ the angles of triangle to sides x,y,z, result:

$$F(x,y,z)=\frac{xysin\alpha}{2}=\frac{xzsin\beta}{2}=\frac{yzsin\gamma}{2}$$

$$xy+yz+zx=\frac{2F(x,y,z)}{sin\alpha}+\frac{2F(x,y,z)}{sin\beta}+\frac{2F(x,y,z)}{sin\gamma}$$

$$xyz(x+y+z)=\frac{2F(x,y,z)}{sin\alpha}\cdot\frac{2F(x,y,z)}{sin\beta}+\frac{2F(x,y,z)}{sin\beta}\cdot\frac{2F(x,y,z)}{sin\gamma}+\frac{2F(x,y,z)}{sin\alpha}\cdot\frac{2F(x,y,z)}{sin\gamma}$$

Now, $\frac{xyz(x+y+z)}{xy+yz+zx}\geq\sqrt{3}F\Leftrightarrow$

$$\frac{\frac{2F(x,y,z)}{sin\alpha}\cdot\frac{2F(x,y,z)}{sin\beta}+\frac{2F(x,y,z)}{sin\beta}\cdot\frac{2F(x,y,z)}{sin\gamma}+\frac{2F(x,y,z)}{sin\alpha}\cdot\frac{2F(x,y,z)}{sin\gamma}}{\frac{2F(x,y,z)}{sin\alpha}+\frac{2F(x,y,z)}{sin\beta}+\frac{2F(x,y,z)}{sin\gamma}}\geq\frac{4}{\sqrt{3}}F(x,y,z)$$

$$\frac{sin\alpha+sin\beta+sin\gamma}{sin\alpha\cdot sin\beta+sin\beta\cdot sin\gamma+sin\gamma\cdot sin\alpha}\geq\frac{2}{\sqrt{3}}$$

Or, $sin\alpha\cdot sin\beta+sin\beta\cdot sin\gamma+sin\gamma\cdot sin\alpha\leq\frac{1}{3}(sin\alpha+sin\beta+sin\gamma)^2$

$$\because xy+yz+zx\leq\frac{1}{3}(x+y+z)^2$$

And $\frac{1}{3}(sin\alpha+sin\beta+sin\gamma)^2\leq(sin\alpha+sin\beta+sin\gamma)\cdot\frac{\sqrt{3}}{2}\Leftrightarrow$

$$sin\alpha+sin\beta+sin\gamma\leq\frac{3\sqrt{3}}{2};(Jensen)$$

Hence,

$$sin\alpha\cdot sin\beta+sin\beta\cdot sin\gamma+sin\gamma\cdot sin\alpha\leq(sin\alpha+sin\beta+sin\gamma)\cdot\frac{\sqrt{3}}{2}$$

Therefore,

$$\frac{sin\alpha+sin\beta+sin\gamma}{sin\alpha\cdot sin\beta+sin\beta\cdot sin\gamma+sin\gamma\cdot sin\alpha}\geq\frac{2}{\sqrt{3}}$$

AN USEFUL TRIGONOMETRIC SUBSTITUTION TO PROVE ALGEBRAIC INEQUALITIES

By Marius Drăgan, Neculai Stanciu – Romania

Abstract: This paper presents a method to solve certain algebraic inequalities using geometric inequalities.

Keywords: algebraic inequalities, geometric inequalities, algebraic identities, geometric inequalities.

MSC: 08A20, 51M16, 26D05

Main results

I. The refinement of the inequality $\prod(x+y)^2\geq 4\prod(x^2+yz)$, (Vasile Cârtoaje).

If $x,y,z\geq 0$, then holds the following inequality;

$$(x+y)^2(y+z)^2(z+x)^2 \geq 4(x^2+yz)(y^2+zx)(z^2+xy)+32x^2y^2z^2$$

Proof. If we denote $u=\frac{x}{\alpha}, v=\frac{y}{\alpha}, w=\frac{z}{\alpha}$, where $\alpha=\sqrt{\sum xy}, \sum uv=1$, then the inequality from the statement is successively equivalent to

$$\prod(x+y)^2 \geq 4\left(2x^2y^2z^2+\sum x^3y^3+xyz\sum x^3\right)+32x^2y^2z^2 \Leftrightarrow$$
$$\Leftrightarrow \prod(x+y)^2 \geq 40x^2y^2z^2+4\sum x^3y^3+4xyz\sum x^3$$
$$\Leftrightarrow \prod(u+v)^2 \geq 40u^2v^2w^2+4\sum u^3v^3+4uvw\sum u^3$$

Since we have $\sum uv=1$ there exists the triangle ABC with $u=\tan\frac{A}{2}, v=\tan\frac{B}{2}, w=\tan\frac{C}{2}$ and also are well-known the identities $\prod\tan\frac{A}{2}=\frac{r}{s}, \prod\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=\frac{4R}{s}$

$$\prod\tan^3\frac{A}{2}=\frac{(4R+r)^3}{s^3}-\frac{12R}{s}, \prod\tan^3\frac{A}{2}\tan^3\frac{B}{2}=1-\frac{12Rr}{s^2}$$

Hence, the last inequality becomes successively

$$\frac{4R^2}{s^2} \geq \frac{10r^2}{s^2}+1-\frac{12Rr}{s^2}+\frac{r}{s}\left(\frac{(4R+r)^3}{s^3}-\frac{12R}{s}\right) \Leftrightarrow \frac{4R^2+24Rr-10r^2}{s^2} \geq 1+\frac{r(4R+r)^3}{s^4}$$

We denote $u=\frac{1}{s^2}, x=\frac{R}{r}$ and we consider the function

$$f:\left[\frac{1}{4R^2+4Rr+3r^2}, \frac{1}{16Rr-5r^2}\right] \to R,$$
$$f(u)=r(4R+r)^3u^2-(4R^2+24Rr-10r^2)u+1$$

The last inequality is equivalent to $f(u) \leq 0, \forall u \in [u_1, u_2]=\left[\frac{1}{4R^2+4Rr+3r^2}, \frac{1}{16Rr-5r^2}\right]$

So, it remains to prove that $f\left(\frac{1}{16Rr-5r^2}\right) \leq 0$ and $f\left(\frac{1}{4R^2+4Rr+3r^2}\right) \leq 0$.

$f\left(\frac{1}{16Rr-5r^2}\right) \leq 0 \Leftrightarrow 5x^2-11x+2 \geq 0 \Leftrightarrow (x-2)(5x-1) \geq 0$, true since $x \geq 2$.

$$f\left(\frac{1}{4R^2+4Rr+3r^2}\right) \leq 0 \Leftrightarrow (4x+1)^3-(4x^2+24x-10)(4x^2+4x+3)+(4x^2+4x+3)^2 \leq 0$$

$\Leftrightarrow 4x^3-5x^2-x-10 \geq 0 \Leftrightarrow (x-2)^2(4x^2+3x+2) \geq 0$, which is true.

II. If $x,y,z \geq 0$, then the following inequality holds:

$$\sum x^4(y+z)+2xyz\sum xy \geq \sum x^3(y^2+z^2)+2xyz\sum x^2$$

Proof. Denoting $u=\frac{x}{\alpha}, v=\frac{y}{\alpha}, w=\frac{z}{\alpha}$, where $\alpha=\sqrt{\sum xy}, \sum uv=1$, then the inequality is successively equivalent to $\sum x^4(y+z)+2xyz\sum xy \geq \sum x^3(y^2+z^2)+2xyz\sum x^2$

$$\Leftrightarrow \sum x^4\left(\sum x-x\right)+2xyz\sum xy \geq \sum x^3\left(\sum x^2-x^2\right)+2xyz\sum x^2 \Leftrightarrow$$
$$\Leftrightarrow \sum x^4\sum x+2xyz\sum xy \geq \sum x^3\sum x^2+2xyz\sum x^2 \Leftrightarrow$$
$$\Leftrightarrow \sum u^4\sum u+2uvw\sum uv \geq \sum u^3\sum u^2+2uvw\sum u^2$$

Since we have $\sum uv=1$ there exists the triangle ABC with $u=\tan\frac{A}{2}, v=\tan\frac{B}{2}, w=\tan\frac{C}{2}$

Using the identities

$$\sum u=\frac{4R+r}{s}, uvw=\frac{r}{s}, \sum x^4=2-\frac{16R(4R+r)}{s^2}+\frac{(4R+r)^4}{s^4}$$
$$\sum x^3=\frac{(4R+r)^3}{s^3}-\frac{12R}{s}, \sum x^2=\frac{(4R+r)^2}{s^2}-2,$$

the last inequality becomes

$$\left(2-\frac{16R(4R+r)}{s^2}+\frac{(4R+r)^4}{s^4}\right)\left(\frac{4R+r}{s}\right)+2\cdot\frac{r}{s}\geq$$
$$\geq\left(\frac{(4R+r)^3}{s^3}-\frac{12R}{s}\right)\left(\frac{(4R+r)^2}{s^2}-2+\frac{2r}{s}\left(\frac{(4R+r)^2}{s^2}-2\right)\right),$$

which after some algebra is equivalent to $p^2\leq\frac{R(4R+r)^2}{4R-2r}$, i.e. Bilcev's inequality.

Remark 1. The inequality can be written as

$$(x-y)^2xy(x+y-z)+(y-z)^2yz(y+z-x)+(z-x)^2zx(z+x-y)\geq 0$$

III. If $x,y,z\geq 0$, then the following inequality holds:

$$\frac{8xyz}{\prod(x+y)}+\frac{(\sum xy)^2}{xyz\sum x}\geq 4$$

(The American Mathematical Monthly).

Proof. We denote $u=\frac{x}{\alpha},v=\frac{y}{\alpha},w=\frac{z}{\alpha}$, where $\alpha=\sqrt{\sum xy},\sum uv=1$. Since we have $\sum uv=1$ we shall consider the triangle ABC with $u=\tan\frac{A}{2},v=\tan\frac{B}{2},w=\tan\frac{C}{2}$.
Using the identities $\sum uv=1,\sum u=\frac{4R+r}{s},uvw=\frac{r}{s},\prod(u+v)=\frac{4R}{s}$, then the inequality to prove is successively equivalent to

$$\frac{8uvw}{\prod(u+v)}+\frac{(\sum uv)^2}{uvw\sum u}\geq 4\Leftrightarrow\frac{8\cdot\frac{r}{s}}{\frac{4R}{s}}+\frac{1}{\frac{r}{s}\cdot\frac{4R+r}{s}}\geq 4\Leftrightarrow\frac{2r}{R}+\frac{s^2}{r(4R+r)}\geq 4\Leftrightarrow$$
$$\Leftrightarrow s^2\geq\frac{(4R-2r)\cdot r\cdot(4R+r)}{R}$$

In [1] it is proved that $s^2\geq\frac{r(16R^2-20Rr+3r^2)}{R-r}$.
So, if we denote $\frac{R}{r}=x$ it suffices to show that

$$\frac{16x^2-20x+3}{x-1}\geq\frac{(4x-2)(4x+1)}{x}\Leftrightarrow$$
$$\Leftrightarrow 16x^3-20x^2+3x\geq(x-1)(16x^2+4x-8x-2)\Leftrightarrow$$
$$\Leftrightarrow 16x^3-20x^2+3x\geq 16x^3-4x^2-2x-16x^2+4x+2\Leftrightarrow x\geq 2\Leftrightarrow R\geq 2r,$$ i.e. Euler's inequality. The proof is complete.

Remark 2. For other inequality solved by this method see for e.g. [2].

References:

[1] M. Drăgan, I.V. Maftei, S. Rădulescu, ***Some consequences of Blundon inequality***, Gazeta Matematică, No. 1, 2010, 3-9
[2] M. Drăgan, I.V. Maftei, S. Rădulescu, ***Inegalități Matematice (Extinderi și generalizări). Tehnici și metode de demonstrație și rafinare***, Editura Didactică și Pedagogică, 2012.

FEW OUTSTANDING LIMITS (II)

By Florică Anastase-Romania

Problem 1.

If $(x_n)_{n\geq 1}$ –sequence of real numbers such that

$$x_{n+1}=x_n+\left(\frac{1}{\pi}\right)^{x_n},x_1\in\mathbb{R}_+.$$ **Find:**

$$\boldsymbol{\Omega = \lim_{n\to\infty} \frac{x_n}{\sqrt[n]{log^2 n}} \cdot \int_0^\pi \frac{log(x+n)}{x^2+n\pi} dx}$$

Solution:

$$x_{n+1} - x_n = \left(\frac{1}{\pi}\right)^{x_n} > 0 \Rightarrow (x_n)_{n\geq 1} \nearrow$$

Suppose that $\exists l \in R$ such that $l = \lim_{n\to\infty} x_n \Rightarrow \left(\frac{1}{\pi}\right)^l = 0 \Rightarrow \lim_{n\to\infty} x_n = +\infty$

Hence,

$$0 \leq x \leq \pi \to \begin{cases} \dfrac{1}{\pi^2+n\pi} \leq \dfrac{1}{x^2+n\pi} \leq \dfrac{1}{n\pi} \\ logn \leq log(x+n) \leq log(\pi+n) \end{cases}$$

$$\frac{logn}{\pi^2+n\pi} \leq \frac{log(x+n)}{x^2+n\pi} \leq \frac{log(\pi+n)}{n\pi}$$

$$\frac{logn}{\pi^2+n\pi}\int_0^\pi dx \leq \int_0^\pi \frac{log(x+n)}{x^2+n\pi}dx \leq \frac{log(\pi+n)}{n\pi}\int_0^\pi dx$$

$$\frac{n}{n+\pi} \leq \frac{n}{logn}\int_0^\pi \frac{log(x+n)}{x^2+n\pi}dx \leq \frac{log(n+\pi)}{logn}$$

$$\frac{n}{n+\pi}\cdot\frac{x_n}{logn} \leq \frac{nx_n}{log^2 n}\int_0^\pi \frac{log(x+n)}{x^2+n\pi}dx \leq \frac{log(n+\pi)}{logn}\cdot\frac{x_n}{logn}$$

$$\frac{n}{n+\pi}\cdot\frac{x_n}{logn} \leq \frac{x_n}{\sqrt[n]{log^2 n}}\cdot\int_0^\pi \frac{log(x+n)}{x^2+n\pi}dx \leq \frac{log(n+\pi)}{logn}\cdot\frac{x_n}{logn}$$

$$\lim_{n\to\infty}\frac{x_n}{logn}\cdot\frac{log(n+\pi)}{logn} = \lim_{n\to\infty}\frac{x_n}{logn}\cdot\frac{log\left(n\left(1+\frac{\pi}{n}\right)\right)}{logn} =$$

$$= \lim_{n\to\infty}\frac{x_n}{logn}\cdot\left(1+\frac{log\left(1+\frac{\pi}{n}\right)}{logn}\right) = \lim_{n\to\infty}\frac{x_n}{logn} =$$

$$= \lim_{n\to\infty}\frac{x_{n+1}-x_n}{log(n+1)-logn} = \lim_{n\to\infty}\frac{\left(\frac{1}{\pi}\right)^{x_n}}{log\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{n\pi^{-x_n}}{log\left(1+\frac{1}{n}\right)^n} =$$

$$= \lim_{n\to\infty}\frac{n}{\pi^{x_n}} = \lim_{n\to\infty}\frac{n+1-n}{\pi^{x_{n+1}}-\pi^{x_n}} = \lim_{n\to\infty}\frac{1}{\pi^{x_n}(\pi^{-x_n}-1)} \overset{\pi^{x_n}=y}{\triangleq} \lim_{y\to 0}\frac{y}{\pi^y-1} = \frac{1}{log\pi}$$

$$\lim_{n\to\infty}\frac{n}{n+\pi}\cdot\frac{x_n}{logn} = \lim_{n\to\infty}\frac{x_n}{logn} = \frac{1}{log\pi}$$

Therefore,

$$\Omega = \lim_{n\to\infty}\frac{x_n}{\sqrt[n]{log^2 n}}\cdot\int_0^\pi \frac{log(x+n)}{x^2+n\pi}dx = \frac{1}{log\pi}$$

Problem 2.

If $(a_n)_{n\geq 1}$, $a_1 = a > 0$, $a_{n+1} = n\cdot\sqrt{a_n}+1$. Find:

$$\Omega = \boldsymbol{\lim_{n\to\infty}\frac{n^3}{log(n)}\int_0^a \frac{log(x+n)}{x^2+na_n}dx}$$

Solution:

$$a_1 = a > 0, a_{n+1} = n\cdot\sqrt{a_n}+1 > 1 \Rightarrow a_n > 1, \forall n \geq 2.$$

$$n\sqrt{a_n} < a_{n+1} < (n+1)\sqrt{a_n}, \forall n \geq 2 \Rightarrow$$
$$logn + \frac{1}{2}loga_n < loga_{n+1} < log(n+1) + \frac{1}{2}loga_n \Leftrightarrow$$
$$2^{n+1}logn + 2^n loga_n < 2^n loga_{n+1} < 2^{n+1}log(n+1) + 2^n loga_n \Leftrightarrow$$
$$2^{n+1}logn < 2^{n+1}loga_{n+1} - 2^n loga_n < 2^{n+1}log(n+1)$$

Let us denote: $b_n = 2^n loga_n \Rightarrow 2^{n+1}logn < b_{n+1} - b_n < 2^{n+1}log(n+1)$ and summing, we get:

$$\sum_{k=3}^{n} 2^k lo\,g(k-1) < b_n - b_2 < \sum_{k=3}^{n} 2^k logk$$
$$\frac{b_2}{2^n} + \frac{1}{2^n}\sum_{k=3}^{n} 2^k log(k-1) - 2logn < log\left(\frac{a_n}{n^2}\right) < \frac{b_2}{2^n} + \frac{1}{2^n}\sum_{k=3}^{n} 2^k logk - log2$$

Let us denote:

$$x_n = \frac{1}{2^n}\left(\sum_{k=3}^{n} 2^k log(k-1) - 2^{n+1}logn\right) = \frac{u_n}{v_n}$$
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{u_{n+1}-u_n}{v_{n+1}-v_n} = \lim_{n\to\infty}\frac{2^{n+1}logn - 2^{n+2}log(n+1) + 2^{n+1}logn}{2^{n+1} - 2^n} = 0$$

Analogously,

$$y_n = \frac{1}{2^n}\left(\sum_{k=3}^{n} 2^k logk - 2^{n+1}logn\right) \to 0$$

How,

$$\frac{b_2}{2^n} + x_n < log\left(\frac{a_n}{n^2}\right) < \frac{b_2}{2^n} + y_n, \forall n \geq 2$$

It follows that: $\lim_{n\to\infty} log\left(\frac{a_n}{n^2}\right) = 0 \Rightarrow \lim_{n\to\infty}\left(\frac{a_n}{n^2}\right) = 1; (1)$

$$0 \leq x \leq a \to \begin{cases} \dfrac{1}{a^2 + na_n} \leq \dfrac{1}{x^2 + na_n} \leq \dfrac{1}{na_n} \\ logn \leq log(x+n) \leq log(a+n) \end{cases}$$
$$\frac{logn}{a^2 + na_n} \leq \frac{log(x+n)}{x^2 + na_n} \leq \frac{log(a+n)}{na_n}, \frac{logn}{a^2 + na_n}\int_0^a dx \leq \int_0^a \frac{log(x+n)}{x^2 + na_n}dx \leq \frac{log(a+n)}{na_n}\int_0^a dx$$
$$\frac{na}{a^2 + na_n} \leq \frac{n}{logn}\cdot\int_0^\pi \frac{log(x+n)}{x^2 + na_n}dx \leq \frac{alog(n+a)}{a_n logn}$$
$$\frac{n^3 a}{a^2 + na_n} \leq \frac{n^3}{logn}\cdot\int_0^\pi \frac{log(x+n)}{x^2 + na_n}dx \leq \frac{an^2 log(n+a)}{a_n logn}$$
$$\frac{a}{\frac{a^2}{n^3} + \frac{a_n}{n^2}} \leq \frac{n^3}{logn}\cdot\int_0^\pi \frac{log(x+n)}{x^2 + na_n}dx \leq \frac{alog(n+a)}{\frac{a_n}{n^2}\cdot logn}; (2)$$

From (1), (2) it follows that: $\Omega = \lim_{n\to\infty}\frac{n^3}{log(n)}\int_0^a \frac{log(x+n)}{x^2+na_n}dx = a$

Problem 3.

Find:

$$\Omega = \boldsymbol{\lim_{n\to\infty}\left(\frac{n}{logn}\right)^e \cdot e^{\int_0^e log\left(\frac{log(x+e)}{x^2+ne}\right)dx}}$$

Solution:

$$\frac{\beta-\alpha}{\int_{\alpha}^{\beta}\frac{1}{f(x)}dx} \le e^{\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} log(f(x))dx} \le \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(x)dx, \alpha < \beta \ (*)$$ means Integral Inequality

For $x \in [0,e] \Rightarrow ne \le x^2 + ne \le e^2 + ne \Rightarrow \frac{logn}{e^2+ne} \le \frac{log(x+e)}{x^2+ne} \le \frac{log(e+n)}{ne}$

Let $f: [0,e] \to \mathbb{R}, f(x) = \frac{log(x+e)}{x^2+ne}$ we have:

$$\frac{e}{\int_0^e \frac{1}{f(x)}dx} = \frac{e}{\int_0^e \frac{x^2+ne}{log(x+e)}dx} \ge \frac{e}{\int_0^e \frac{e^2+ne}{logn}dx} = \frac{1}{e}\cdot\frac{logn}{e+n}, \qquad (1)$$

$$\frac{1}{e}\int_0^e f(x)dx = \frac{1}{e}\int_0^e \frac{log(x+e)}{x^2+ne}dx \le \frac{1}{e}\int_0^e \frac{log(e+n)}{ne}dx = \frac{1}{e}\cdot\frac{log(e+n)}{n}, \quad (2)$$

From (1), (2) we get:

$$\frac{1}{e}\cdot\frac{logn}{e+n} \le e^{\frac{1}{e}\int_0^e log\left(\frac{log(x+e)}{x^2+ne}\right)dx} \le \frac{1}{e}\cdot\frac{log(e+n)}{n}$$

$$\frac{1}{e}\cdot\frac{n}{e+n} \le \frac{n}{logn}\cdot e^{\frac{1}{e}\int_0^e log\left(\frac{log(x+e)}{x^2+ne}\right)dx} \le \frac{1}{e}\cdot\frac{log(e+n)}{logn}$$

$$\frac{1}{e^e}\cdot\left(\frac{n}{e+n}\right)^e \le \left(\frac{n}{logn}\right)^e\cdot e^{\int_0^e log\left(\frac{log(x+e)}{x^2+ne}\right)dx} \le \frac{1}{e^e}\cdot\left(\frac{log(e+n)}{logn}\right)^e$$

$$\lim_{n\to\infty}\frac{n}{e+n} = \lim_{n\to\infty}\frac{log(e+n)}{logn} = 1$$

So,

$$\Omega = \lim_{n\to\infty}\left(\frac{n}{logn}\right)^e\cdot e^{\int_0^e log\left(\frac{log(x+e)}{x^2+ne}\right)dx} = \frac{1}{e^e}$$

Problem 4.

Let $(a_n)_{n\ge1}, a_n \in (0,\infty)$ be sequence of real numbers such that $a_1 = \sqrt{a}$, $a > 0, a_{n+1}^2 = n\cdot a_n + 1$ then find:

$$\Omega = \lim_{n\to\infty}\frac{a_n}{n^3}\int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx, n \in \mathbb{N}, n \ge 2$$

Solution 1:

$$a_1 = \sqrt{a}, a > 0;\ a_{n+1}^2 = n\cdot a_n + 1$$

Let: $a_{n+1}^2 = x_{n+1} > 0 \Rightarrow x_1 = a > 0, a_{n+1} = \sqrt{x_{n+1}}$;

$$x_{n+1} = n\cdot\sqrt{x_n} + 1$$

How $x_{n+1} = n\cdot\sqrt{x_n} + 1 > 1 \Rightarrow x_n > 1, \forall n \ge 2, n \in \mathbb{N}$ then

$$n\cdot\sqrt{x_n} < x_{n+1} < (n+1)\sqrt{x_n}, \forall n \ge 2 \Leftrightarrow$$

$$log\, n + \frac{1}{2}logx_n < logx_{n+1} < log(n+1) + \frac{1}{2}logx_n \Leftrightarrow$$

$$2^{n+1}log\, n + 2^n logx_n < 2^{n+1}logx_{n+1} < 2^{n+1}log(n+1) + 2^n logx_n$$

$$2^{n+1}log\, n < 2^{n+1}logx_{n+1} - 2^n logx_n < 2^{n+1}log(n+1)$$

Let: $y_n = 2^n logx_n \Rightarrow 2^{n+1}log\, n < y_{n+1} - y_n < 2^{n+1}log(n+1)$

and summing, we get:

$$\sum_{k=3}^{n} 2^k log(k-1) < y_n - y_2 < \sum_{k=3}^{n} 2^k log\, k$$

$$\frac{y_2}{2^n}+\frac{1}{2^n}\sum_{k=3}^{n}2^k log(k-1)-2log\, n < log\left(\frac{x_n}{n^2}\right) < \frac{y_2}{2^n}+\frac{1}{2^n}\sum_{k=3}^{n}2^k log\, k-2log\, n$$

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^{n}2^k log(k-1)-2log\, n\right)=\lim_{n\to\infty}\frac{1}{2^n}\left(\sum_{k=3}^{n}2^k log\, k-2^{n+1}log\, n\right)\overset{L.C-Stolz}{=}$$

$$=\lim_{n\to\infty}\frac{2^{n+1}log\, n-2^{n+2}log(n+1)+2^{n+1}log\, n}{2^{n+1}-2^n}=4\lim_{n\to\infty}log\left(\frac{n+1}{n}\right)=0;(1)$$

Analogously,

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^{n}2^k log\, k-2log\, n\right)=0;(2)$$

From (1),(2) we get:

$$\lim_{n\to\infty}log\left(\frac{x_n}{n^2}\right)=0\Rightarrow\lim_{n\to\infty}\left(\frac{x_n}{n^2}\right)=\lim_{n\to\infty}\left(\frac{a_n^2}{n^2}\right)=1\Rightarrow\lim_{n\to\infty}\left(\frac{a_n}{n}\right)=1$$

Now,

$$\frac{x^{2n}+1}{x^n+1}>\sqrt{2}-1\Leftrightarrow x^{2n}-(\sqrt{2}-1)x^n+2-\sqrt{2}>0; t=x^n>0\Rightarrow$$

$$t^2-(\sqrt{2}-1)t+2-\sqrt{2}>0,\Delta_t=-5+2\sqrt{2}<0\Rightarrow$$

$$\sqrt{2}-1<\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx;\ \ (3)$$

$$\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}=\sqrt[n]{(x^{2n}+1)\cdot\underbrace{1\cdot1\ldots.1}_{(n-2)}\cdot\frac{1}{x^n+1}}\overset{AM-GM}{\leq}\frac{(x^{2n}+1)+n-2+\frac{1}{x^n+1}}{n}\Leftrightarrow$$

$$\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}\leq\frac{1}{n}\left(x^{2n}+n-1+\frac{1}{x^n+1}\right)$$

$$\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx\leq\frac{1}{n}\int_0^1x^{2n}dx+\frac{n-1}{n}+\frac{1}{n}\underbrace{\int_0^1\frac{1}{x^n+1}}_{\leq1}\leq\frac{1}{n(2n+1)}+1;(4)$$

From (3),(4) we have:

$$\sqrt{2}-1<\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx<\frac{1}{n(2n+1)}+1\Leftrightarrow$$

$$(\sqrt{2}-1)\frac{a_n}{n}\cdot\frac{1}{n^2}<\frac{a_n}{n^3}\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx<\frac{a_n}{n}\cdot\frac{1}{n^2}\left(\frac{1}{n(2n+1)}+1\right)$$

$$So,\Omega=\lim_{n\to\infty}\frac{a_n}{n^3}\int_0^1\sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx=0$$

Solution 2:

$$a_1=\sqrt{a}, a>0;\ a_{n+1}^2=n\cdot a_n+1$$

Let: $a_{n+1}^2=x_{n+1}>0\Rightarrow x_1=a>0, a_{n+1}=\sqrt{x_{n+1}};\ x_{n+1}=n\cdot\sqrt{x_n}+1$

How $x_{n+1}=n\cdot\sqrt{x_n}+1>1\Rightarrow x_n>1,\forall n\geq2, n\in\mathbb{N}$ then

$$n\cdot\sqrt{x_n} < x_{n+1} < (n+1)\sqrt{x_n}, \forall n\ge 2 \Leftrightarrow$$
$$log\,n + \frac{1}{2}logx_n < logx_{n+1} < log(n+1) + \frac{1}{2}logx_n \Leftrightarrow$$
$$2^{n+1}log\,n + 2^n logx_n < 2^{n+1}logx_{n+1} < 2^{n+1}log(n+1) + 2^n logx_n$$
$$2^{n+1}log\,n < 2^{n+1}logx_{n+1} - 2^n logx_n < 2^{n+1}log(n+1)$$

Let: $y_n = 2^n logx_n \Rightarrow 2^{n+1}log\,n < y_{n+1} - y_n < 2^{n+1}log(n+1)$ and summing, we get:

$$\sum_{k=3}^{n} 2^k log(k-1) < y_n - y_2 < \sum_{k=3}^{n} 2^k log\,k$$
$$\frac{y_2}{2^n} + \frac{1}{2^n}\sum_{k=3}^{n} 2^k log(k-1) - 2log\,n < log\left(\frac{x_n}{n^2}\right) < \frac{y_2}{2^n} + \frac{1}{2^n}\sum_{k=3}^{n} 2^k log\,k - 2log\,n$$
$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^{n} 2^k log(k-1) - 2log\,n\right) = \lim_{n\to\infty}\frac{1}{2^n}\left(\sum_{k=3}^{n} 2^k log\,k - 2^{n+1}log\,n\right) \overset{L.C-Stolz}{=}$$
$$= \lim_{n\to\infty}\frac{2^{n+1}log\,n - 2^{n+2}log(n+1) + 2^{n+1}log\,n}{2^{n+1} - 2^n} = 4\lim_{n\to\infty} log\left(\frac{n+1}{n}\right) = 0; (1)$$

Analogously,

$$\lim_{n\to\infty}\left(\frac{1}{2^n}\sum_{k=3}^{n} 2^k log\,k - 2log\,n\right) = 0; (2)$$

From (1),(2) we get:

$$\lim_{n\to\infty} log\left(\frac{x_n}{n^2}\right) = 0 \Rightarrow \lim_{n\to\infty}\left(\frac{x_n}{n^2}\right) = \lim_{n\to\infty}\left(\frac{a_n^2}{n^2}\right) = 1 \Rightarrow \lim_{n\to\infty}\left(\frac{a_n}{n}\right) = 1$$

Now, let be the function*:*

$$f:[0,1]\to\mathbb{R}, f(x) = \frac{x^{2n}+1}{x^n+1}; f'(x) = \frac{nx^{n-1}(x^{2n}+2x^n-1)}{(x^n+1)^2}$$

x	0 $\qquad$ $\sqrt[n]{\sqrt{2}-1}$ $\qquad$ 1
$f'(x)$	$- - - - - - - - - 0 + + + + + + +$
$f(x)$	$1 \searrow$ $\qquad$ $2(\sqrt{2}-1)$ $\qquad$ $\nearrow 1$

We have: $2(\sqrt{2}-1) \le f(x) \le 1; \forall x\in[0,1] \Rightarrow \sqrt[n]{2(\sqrt{2}-1)} < \sqrt[n]{\frac{x^{2n}+1}{x^n+1}} < 1; \forall x\in[0,1] \Rightarrow$

$$\frac{a_n}{n}\cdot\frac{1}{n^2}\sqrt[n]{2(\sqrt{2}-1)} < \frac{a_n}{n^3}\cdot\sqrt[n]{\frac{x^{2n}+1}{x^n+1}} < \frac{a_n}{n}\cdot\frac{1}{n^2}; \forall x\in[0,1] \Rightarrow$$
$$\frac{a_n}{n}\cdot\frac{1}{n^2}\sqrt[n]{\frac{2}{\sqrt{2}+1}} < \frac{a_n}{n^3}\int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}} < \frac{a_n}{n}\cdot\frac{1}{n^2}; \forall x\in[0,1]$$
$$So, \Omega = \lim_{n\to\infty}\frac{a_n}{n^3}\int_0^1 \sqrt[n]{\frac{x^{2n}+1}{x^n+1}}dx = 0$$

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ROMANIAN MATHEMATICAL MAGAZINE-INTERACTIVE JOURNAL-www.ssmrmh.ro

PROBLEMS FOR JUNIORS

J.234 If $a, b, c, d \in \mathbb{R}$ then:

$$4(ad-bc)^6+4(ac+bd)^6 \geq (a^2+b^2)^3(c^2+d^2)^3$$

Proposed by Daniel Sitaru,Elena Nicolae – Romania

J.235 If $x, y, z \geq 0$ then:

$$\left(\prod_{cyc}(x+1)+\prod_{cyc}(2x+1)\right)\prod_{cyc}(x+2) \geq 2\prod_{cyc}(3x+2)$$

Proposed by Daniel Sitaru,Luiza Dumitrescu – Romania

J.236 If $x, y \geq 0, x+y \leq \pi$ then:

$$2\left(\cos\frac{2x}{3}+\cos\frac{2y}{3}\right)+1 \geq 4\cos\frac{x}{3}\cos\frac{y}{3}$$

Proposed by Daniel Sitaru,Delia Popescu – Romania

J.237 If $a, b, c > 0, \sqrt{ab}+\sqrt{bc}+\sqrt{ca} = 12$ then:

$$\frac{\left(a+b+\sqrt{ab}\right)^3}{(a+b)^2}+\frac{\left(b+c+\sqrt{ca}\right)^3}{(b+c)^2}+\frac{\left(c+a+\sqrt{ab}\right)^3}{(c+a)^2} \geq 81$$

Proposed by Daniel Sitaru,Liliana Argetoianu – Romania

J.238 Let $a, b, c \in \mathbb{R}_+^* = (0, \infty)$; $m \in \mathbb{R}_+ = [0, \infty), abc = 1$, then:

$$\frac{(a+b)^{2m+2}}{(a+b+2c)^{m+1}}+\frac{(b+c)^{2m+2}}{(2a+b+c)^{m+1}}+\frac{(c+a)^{2m+2}}{(a+2b+c)^{m+1}} \geq 3$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

J.239 If $m, n, x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then in any ΔABC, the following inequality holds:

$$\frac{mx+ny}{z}h_a+\frac{my+nz}{x}h_b+\frac{mz+nx}{z}h_c = \frac{24\sqrt{mn}r^2}{R}$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

J.240 In any ΔABC with the area F the following inequality holds:

$$(a+b)^2h_c^2+(b+c)^2h_a^2+(c+a)^2h_b^2 \geq 48F^2$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

J.241 Let M be an interior point in ΔABC with the area F and $x_A = MA, x_B = MB, x_C = MC$. Prove that:

$$ax_A + bx_B + cx_C \geq 4F$$

Proposed by D.M.Bătinețu-Giurgiu – Romania

J.242 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\tan^2 \frac{A}{2}}{\cot \frac{B}{2}\left(\tan \frac{A}{2} + \tan \frac{C}{2}\right)} \geq \frac{1}{2}$$

Proposed by Marian Ursărescu – Romania

J.243 In ΔABC the following relationship holds:

$$\sqrt{(8R^2 - a^2)(8R^2 - b^2)(8R^2 - c^2)} \geq 2\sqrt{5}R^2(2R + r)$$

Proposed by Marian Ursărescu – Romania

J.244 In ΔABC the following relationship holds:

$$2\sqrt{(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2)} \geq 27r^2(9R - 2r)$$

Proposed by Marian Ursărescu – Romania

J.245 If $x, y, z > 0$ and $\lambda \geq 1$ then:

$$\sum_{cyc} \frac{x}{\lambda x + y + z} \leq \frac{3}{\lambda + 2}$$

Proposed by Marin Chirciu-Romania

J.246 In acute ΔABC the following relationship holds:

$$\sqrt{\frac{3}{2}(1 + cosAcosBcosC)} \geq \frac{27}{4} tan\frac{A}{2} tan\frac{B}{2} tan\frac{C}{2}$$

Proposed by Marin Chirciu-Romania

J.247 In ΔABC the following relationship holds:

$$\sum_{cyc} bcr_a \leq \frac{R}{2r} \sum_{cyc} bch_a$$

Proposed by Marin Chirciu-Romania

J.248 If $a, b, c, d > 0$ then:

$$8(1 - a + a^2)(1 - b + b^2)(1 - c + c^2)(1 - d + d^2) \geq (ab + cd)^2 + (1 + abcd)^2$$

Proposed by Marin Chirciu-Romania

J.249 If $a, b, c > 0$ then:

$$\frac{ab\sqrt{ab}}{(a + b)^3} + \frac{bc\sqrt{bc}}{(b + c)^3} + \frac{ca\sqrt{ca}}{(c + a)^3} + \frac{3(a + b)(b + c)(c + a)}{64abc} \geq \frac{3}{4}$$

Proposed by Marin Chirciu-Romania

J.250 In ΔABC the following relationship holds:

$$\sum_{cyc} sec^3\frac{A}{2}sec\frac{B}{2} \geq \frac{8R\sqrt{3}}{s} \geq \frac{16}{3}$$

Proposed by Marin Chirciu-Romania

J.251 If $a,b,c,d > 0$ then:

$$4(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2) \geq (1+abcd)^2$$

Proposed by Marin Chirciu-Romania

J.252 If $a,b,c > 0$ and $2n \in \mathbb{N}$ then:

$$\frac{a^n b^n}{(a+b)^{2n}} + \frac{b^n c^n}{(b+c)^{2n}} + \frac{c^n a^n}{(c+a)^n} + \frac{2n(a+b)(b+c)(c+a)}{2^{2n+3}abc} \geq \frac{2n+3}{2^{2n}}$$

Proposed by Marin Chirciu-Romania

J.253 $x,y,z \geq 0, z = \max(x,y,z)$

$$A = 9x^3y^3(x^3+y^3) + 6x^2y^2z^2(x^6+y^6+z^6) + 18z^{12}$$
$$B = (5x^6+5y^6+8z^6)(x^3y^3+x^2y^2z^2+z^6)$$

Prove that: $A \geq B$

Proposed by Daniel Sitaru,Mihai Ionescu – Romania

J.254 If $x,y,z > 0$ then:

$$\sqrt{x}(x+2z) + \sqrt{y}(y+2x) + \sqrt{z}(z+2y) \leq (x+y+z)\sqrt{3(x+y+z)}$$

Proposed by Daniel Sitaru,Carina Viespescu – Romania

J.255 If $0 < x,y < \frac{\pi}{2}$ then:

$$\frac{\sin^2 x}{\sin^6 y} + \frac{\sin^2 y}{\cos^6 x} + \frac{\cos^2 x}{\cos^6 y} + \frac{\cos^2 y}{\sin^6 x} \geq 16$$

Proposed by Daniel Sitaru,Eugenia Turcu – Romania

J.256 In any ΔABC the following relationship holds:

$$\left(\sum_{cyc}\frac{m_a^3}{h_b+h_c}\right)\left(\sum_{cyc}\frac{1}{(h_a+h_b)^2}\right) \geq \frac{9}{8}$$

Proposed by D.M.Bătineţu-Giurgiu, Claudia Nănuţi-Romania

J.257 If $m,n \in \mathbb{R}_+ = [0,\infty), x,y,z \in \mathbb{R}_+^* = (0,\infty), m+n = 2$ and ΔABC with F –area,then prove:

$$\frac{xa^m}{(y+z)h_a^n} + \frac{yb^m}{(y+z)h_b^n} + \frac{zc^m}{(x+y)h_c^n} \geq \frac{\sqrt{3}}{2^{n-1}} \cdot F^{1-n}$$

Proposed by D.M.Bătineţu-Giurgiu, Claudia Nănuţi-Romania

J.258 If $m,n \in \mathbb{R}_+ = [0,\infty), m+n = 2$ and ΔABC with F –area,then prove:

$$\frac{a^m}{h_a^n} + \frac{b^m}{h_b^n} + \frac{c^m}{h_c^n} \geq 2^m\sqrt{3} \cdot F^{1-n}$$

Proposed by D.M.Bătineţu-Giurgiu, Claudia Nănuţi-Romania

J.259 If $M \in Int(\Delta ABC), x = MA, y = MB, z = MC$ then prove:

$$\sum_{cyc} \frac{\frac{x^2y^2}{a^2b^2}}{\frac{2xy}{ab} + \frac{3yz}{bc} + \frac{4zx}{ca}} \geq \frac{1}{8}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.260 Solve for real numbers:

$$\frac{x^2 + x}{x^2 + x + 1} + \frac{y^2 + y}{y^2 + y + 1} + \frac{z^2 + z}{z^2 + z + 1} + 1 = 0$$

Proposed by Daniel Sitaru,Roxana Vasile – Romania

J.261 If $x, y, z \in \mathbb{R}, a, b, c > 0$ then:

$$((x-a)^2 + y^2 + z^2)^2 + (x^2 + (y-b)^2 + z^2)^2 + (x^2 + y^2 + (z-c)^2)^2 \geq$$
$$\geq \frac{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}{2(a^2 + b^2 + c^2)}$$

When equality holds?

Proposed by Daniel Sitaru,Luiza Cremeneanu – Romania

J.262 If $a, b > 0$ then:

$$\sqrt[256]{\frac{a^2 + b^2}{2}}\left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\right)^3 \leq 27\left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2} + \sqrt{\frac{a^2 + b^2}{2}}\right)^4$$

Proposed by Daniel Sitaru,Nineta Oprescu – Romania

J.263 If $m, n \in \mathbb{R}_+ = [0, \infty); m + n, x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then:

$$\frac{y+z}{x}(mb + nc)^2 + \frac{z+x}{y}(mc + na)^2 + \frac{x+y}{z}(ma + nb)^2 \geq 8(m+n)^2\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

J.264 If $m, n \in \mathbb{R}_+ = [0, \infty): m + n, x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then:

$$\frac{x+y}{z}(ma + nb)^4 + \frac{y+z}{x}(mb + nc)^4 + \frac{z+x}{y}(mc + na)^4 \geq 32 \cdot (m+n)^4F^2$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

J.265 If $m \geq 0, x, y, z > 0$, then in any ΔABC with the area F the following inequality holds:

$$\frac{x}{(y+z)^m}a^{2(m+1)} + \frac{y}{(z+x)^m}b^{2(m+1)} + \frac{z}{(x+y)^m}c^{2(m+1)} \geq 2^{m+2}(xy + yz + zx)^{\frac{1-m}{2}}F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

J.266 In any ΔABC with the area F the following inequality holds:

$$(h_b + h_c)a^5 + (h_c + h_a)b^5 + (h_a + h_b)c^5 \geq 64F^3$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

J.267 In any ΔABC the following inequality holds:

$$\frac{a^2}{h_b^2 + h_c^2} + \frac{b^2}{h_c^2 + h_a^2} + \frac{c^2}{h_a^2 + h_b^2} \geq 2$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.268 If $x \in \mathbb{R}_+^* = (0,\infty)$ and $m \in \mathbb{R}_+ = [0,\infty)$, and $[x],\{x\}$ are respectively the whole part and the fractionary part of x, then:

$$(x+[x])^{m+1}+(x+\{x\})^{m+1}+3^m([x]^{m+1}+\{x\}^{m+1}) \geq \frac{3^m}{2^{m-2}}x^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.269 If $a,b,x,y,z \in \mathbb{R}_+^* = (0,\infty)$, then:

$$\sum_{cyc}\frac{x^2+y^2}{a(y+z)^2+bxy} \geq \frac{6}{4a+b}\cdot\frac{xy+yz+zx}{x^2+y^2+z^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

J.270 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\frac{n_a}{h_a} \leq \sqrt{\sum_{cyc}\cot^2\frac{A}{2}}$$

Proposed by Bogdan Fuștei – Romania

J.271 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\frac{b^2n_b+c^2n_c-a^2n_a}{bc\sqrt{n_bn_c}} \leq 3$$

Proposed by Bogdan Fuștei – Romania

J.272 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{bm_b+cm_c-am_a}{\sqrt{bcm_bm_c}} \leq 3$$

Proposed by Bogdan Fuștei – Romania

J.273 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\prod_{cyc}\left(h_a+\sqrt[3]{n_a+\sqrt[4]{m_a}}\right) \geq \sqrt[16]{\prod_{cyc}h_am_an_a}$$

Proposed by Bogdan Fuștei – Romania

J.274 In ΔABC the following relationship holds:

$$\sum_{cyc}am_a \leq \frac{\sqrt{3}(r^2+4rR)}{2}+\frac{\sqrt{3}}{4}\sum_{cyc}(n_a^2+g_a^2)$$

Proposed by Bogdan Fuștei – Romania

J.275 In ΔABC the following relationship holds:

$$2\sqrt{2}m_a \geq \frac{h_b+h_c}{2\sin\frac{A}{2}}+|b-c|\sin\frac{A}{2}$$

Proposed by Bogdan Fuștei – Romania

J.276 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne cevian, the following relationship holds:

$$4m_a^2 \geq (b+c)^2\cos^2\frac{A}{2}+(n_a-g_a)^2\sin^2\frac{A}{2}$$

Proposed by Bogdan Fuștei – Romania

J.277 In ΔABC the following relationship holds:

$$\frac{|r_a - r_b| + |r_b - r_c| + |r_c - r_a|}{s} \geq 2\sum_{cyc} \frac{m_a - w_a}{r_a}$$

Proposed by Bogdan Fuștei – Romania

J.278 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{r_a}{n_a + m_a + h_a} \geq \frac{4R - 2r}{3R}$$

Proposed by Bogdan Fuștei – Romania

J.279 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{m_a}{AI} \geq \frac{1}{3R - 2r}\sum_{cyc}(m_a + n_a)$$

Proposed by Bogdan Fuștei – Romania

J.280 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{m_a}{AI} \geq \frac{1}{3R}\sum_{cyc}(m_a + n_a + h_a)$$

Proposed by Bogdan Fuștei – Romania

J.281 Solve for real numbers:

$$\sqrt{x + [x]} + \sqrt{x - [x]} = 2, [*] - GIF$$

Proposed by Jalil Hajimir-Canada

J.282 Solve for real numbers:

$$x[x] + \frac{[x]}{x} = 9, \ \ [*] - GIF$$

Proposed by Jalil Hajimir-Canada

J.283 Solve for real numbers:

$$\big[x[x]\big] + x[x] = 2x^2, \ \ [*] - GIF$$

Proposed by Jalil Hajimir-Canada

J.284 If $x, y, z > 0$ then:

$$(x^3 + y^3 + z^3)^3\left(\frac{x^6}{y^3} + \frac{y^6}{z^3} + \frac{z^6}{x^3}\right) \geq 3(x^4 + y^4 + z^4)^3$$

Proposed by Daniel Sitaru,Roxana Popescu – Romania

J.285 If $x, y \in \mathbb{R}, x^2 + y^2 + 178 \leq 10x + 26y$ then:

$$x^4 + y^4 + 2x^2y^2 + 16384 \leq 320(10x + 26y - 178)$$

Proposed by Daniel Sitaru,Aurelia Petrică – Romania

J.286 If $a, b, c > 0, a + b + c = 3$ then:

$$\frac{a^2}{\sqrt{b^2 + 9bc + 6c^2}} + \frac{b^2}{\sqrt{c^2 + 9ca + 6a^2}} + \frac{c^2}{\sqrt{a^2 + 9ab + 6b^2}} \geq \frac{3}{4}$$

Proposed by Daniel Sitaru,Monica Stanca – Romania

J.287 If $a, b, c, d > 0, abcd = 4$ then:

$$\sum_{cyc} \frac{a^{14}}{a^6 + (bcd)^2} \geq 32$$

Proposed by Daniel Sitaru, Cătălin Nicola– Romania

J.288 In ΔABC the following relationship holds:

$$\sqrt{\frac{g_a r_a}{m_a}}+\sqrt{\frac{g_b r_b}{m_b}}+\sqrt{\frac{g_c r_c}{m_c}}\le\frac{s}{\sqrt{r}}$$

Proposed by Marian Ursărescu-Romania

J.289 In ΔABC the following relationship holds:

$$\sqrt{\frac{a^4+b^4}{2}}+\sqrt{\frac{b^4+c^4}{2}}+\sqrt{\frac{c^4+a^4}{2}}\le 4(3R^2-2Rr+r^2)$$

Proposed by Marian Ursărescu-Romania

J.290 In ΔABC the following relationship holds:

$$\sqrt{\frac{r_a^2+r_b^2}{2}}+\sqrt{\frac{r_b^2+r_c^2}{2}}+\sqrt{\frac{r_c^2+r_a^2}{2}}\le 8R-7r$$

Proposed by Marian Ursărescu-Romania

J.291 Find all functions $f:\mathbb{R}\to\mathbb{R}$ with the property:

$$f(x)=f\left(\frac{x}{\sqrt{1+x^2}}\right),\forall x\in\mathbb{R}$$

Proposed by Marian Ursărescu-Romania

J.292 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\left(\sum_{cyc} n_a n_b\right)\left(\sum_{cyc} m_a w_a\right)\ge s^4$$

Proposed by Bogdan Fuștei-Romania

J.293 In ΔABC, the following relationship holds:

$$\frac{R}{2r}\ge\sqrt{1+\frac{\left(4-\frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2(b^2-c^2)^2}{(a+b+c)^4}}$$

Proposed by Bogdan Fuștei-Romania

J.294 In ΔABC, the following relationship holds:

$$8\prod_{cyc}\frac{r_a}{s+n_a}\ge\prod_{cyc}\left(\frac{n_a+g_a+\sqrt{2r_b r_c}+\sqrt{r r_a}}{4(h_a-r)}-\frac{n_a}{h_a}\right)$$

Proposed by Bogdan Fuștei-Romania

J.295 In ΔABC, the following relationship holds:

$$2m_a\ge\sqrt{(n_a-g_a)^2-(r_b-r_c)^2 tan^2\frac{A}{2}}$$

Proposed by Bogdan Fuștei-Romania

J.296 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian, the following relationship holds:

$$\frac{n_a+n_b+n_c}{r}+2\cdot\sum_{cyc}\frac{2r_a+h_a}{2m_a+s-g_a}\ge 3\cdot\sqrt{\frac{16R}{r}-5+\frac{R-2r}{R-r}}$$

Proposed by Bogdan Fuștei-Romania

J.297 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian, the following relationship holds:

$$2\sum_{cyc}\frac{n_a g_a}{h_a w_a} \geq \sum_{cyc}\frac{m_b + m_c}{m_a}$$

Proposed by Bogdan Fuștei-Romania

J.298 In $\Delta ABC, g_a$ –Gergonne's cevian, the following relationship holds:

$$\sum_{cyc}\frac{h_a}{|m_a + w_b + w_c + \sqrt{3}(g_a - a)|} \geq \frac{g_a + g_b + g_c - s}{2\sqrt{3}r}$$

Proposed by Bogdan Fuștei-Romania

J.299 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian, the following relationship holds:

$$\frac{g_a + g_b + g_c}{r} \geq \sum_{cyc}\left(\frac{n_a}{r_a} + \frac{2h_a}{n_a}\right)$$

Proposed by Bogdan Fuștei-Romania

J.300 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\sum_{cyc}cot^2\frac{A}{2} = \sum_{cyc}\frac{n_a^2}{r_a^2} + 2\sum_{cyc}\frac{h_a}{r_a}$$

Proposed by Bogdan Fuștei-Romania

J.301 In ΔABC, the following relationship holds:

$$\frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \geq \sum_{cyc}\frac{2m_a - g_a}{h_a}$$

Proposed by Bogdan Fuștei-Romania

J.302 If $a, b, c > 0, a + b + c = 3$ then

$$(a^2 + b^2 + c^2)(ab + bc + ca)^5 \leq 729$$

Proposed by Marin Chirciu-Romania

J.303 If $a, b, c > 0, a + b + c \leq 3$ and $n \in \mathbb{N}$ then

$$(a^2 + b^2 + c^2)(ab + bc + ca)^{n+2} \leq 3^{n+3}$$

Proposed by Marin Chirciu-Romania

J.304 In ΔABC the following relationship holds:

$$\frac{32r}{R}\cdot\left(\frac{2r}{R}\right)^{\frac{1}{3}} \leq \sum_{cyc}\frac{(b + c)^2}{m_a^2} \leq 4\left(\frac{R}{r} + 2\right)$$

Proposed by Marin Chirciu-Romania

J.305 If $a, b, c > 0$ such that $a + b + c = 1$ and $n \in \mathbb{N}, n \geq 2, \lambda \geq 0$ then

$$\sum_{cyc}\frac{(3a)^n}{(b + \lambda)(c + \lambda)} \geq \frac{27}{(3\lambda + 1)^2}$$

Proposed by Marin Chirciu-Romania

J.306 If $-\frac{1}{2} \leq x_k \leq \frac{1}{2}, 1 \leq k \leq n$ such that $x_1^3 + x_2^3 + \cdots + x_n^3 = 0$ then

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq \frac{n}{3}$$

Proposed by Marin Chirciu-Romania

J.307 If $a,b,c>0$ such that $abc=1$ and $\lambda\ge 0$ then

$$\frac{a^4}{b^4+\lambda c^2}+\frac{b^4}{c^4+\lambda b^2}+\frac{c^4}{a^4+\lambda b^2}+\frac{a^3+b^3+c^3}{\lambda+1}\ge\frac{6}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

J.308 In $\Delta ABC, m(\sphericalangle BAC)=40^\circ, m(\sphericalangle ABC)=60^\circ$. Prove that:

$$\left(\frac{b}{c}\right)^3+3\left(\frac{b}{c}\right)^2=3$$

Proposed by Mehmet Şahin-Turkey

J.309 If in $\Delta ABC, m(\sphericalangle B)=m(\sphericalangle C)=10^\circ$ then

$$\left(\frac{a}{b}\right)^6-6\left(\frac{a}{b}\right)^4+12\left(\frac{a}{b}\right)^2+4=0$$

Proposed by Mehmet Şahin-Turkey

J.310 Let ΔDEF be the pedal triangle of $P\in Int(\Delta ABC)$ in ΔABC. Prove that:

$$DE^2+EF^2+FD^2\ge 12\sqrt{3}\cdot\sqrt[3]{[PEF]\cdot[PDF]\cdot[PDE]}$$

Proposed by Mehmet Şahin-Turkey

J.311 In ΔABC the following relationship holds:

$$\frac{64}{9R^3}\le\frac{1}{r^3}-\left(\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\right)\le\frac{8}{9r^3}$$

Proposed by Mehmet Şahin-Turkey

J.312 In $\Delta ABC, D\in(BC), E\in(CA), F\in(AB), GD\parallel AB, GE\parallel BC, CF\parallel CA$.

Let ρ be the circumradii of ΔDEF, G –centroid. Prove that:

$$\rho\ge\frac{a+b+c}{9}$$

Proposed by Mehmet Şahin-Turkey

J.313 If in $\Delta ABC, a+b+c=1$ then:

$$\frac{7R}{2r}+5\ge\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{ab+bc+ca}\ge 12$$

Proposed by Alex Szoros-Romania

J.314 If $x,y,z>0, xyz=1$ then

$$\frac{1}{x^5(y+z)}+\frac{1}{y^5(z+x)}+\frac{1}{z^5(x+y)}\ge\frac{3}{2}$$

Proposed by Rajeev Rastogi-India

J.315 p –prime number, fixed. Find the number of ordered pairs (x,y) such that:

$$x^3+y^3+8=p+xy$$

Proposed by Rajeev Rastogi-India

J.316 Find $n\in\mathbb{N}$ such that: $\frac{n^2+2n+2}{n^2-2n+2}\in\mathbb{N}$

Proposed by Rajeev Rastogi-India

J.317 Determine the number of 6-tuples $(e_1,e_2,\dots,e_6)$ such that $e_1,e_2,\dots,e_6\in\{-1,1\}$ and

$$e_1+3e_2+5e_3+6e_4+7e_5+11e_6\equiv 0(mod\ 3)$$

Proposed by Rajeev Rastogi-India

J.318 Find:

$$\Omega = \underbrace{min}_{x\in\mathbb{R}}\left(\underbrace{max}_{y\in[0,1]}(|y^2 - xy|)\right)$$

Proposed by Rajeev Rastogi-India

J.319 $a_0, a_1 > 0, a_{n+1} = \frac{max(a_n,5)}{min\left(\frac{a_{n-1}}{5},1\right)}, n \ge 1, a_{1010} = 1010$. Find:

$$\Omega = \frac{a_{2020} + a_{2021} + a_{2022} + a_{2023} + a_{2024} + a_{2025}}{a_{2017} + a_{2018} + a_{2019}}$$

Proposed by Rajeev Rastogi-India

J.320 $x^9 + \frac{15}{8}x^6 + \frac{75}{64}x^3 - x + \frac{445}{512} = 0$. If S −sum of all real roots then find $[S]$, $[*] - GIF$.

Proposed by Rajeev Rastogi-India

J.321 Find $x, y \in \mathbb{Z}_+$ such that:

$$4x^3 - 8x^2y - 11xy^2 - 3y^3 - 8x^2 + 20xy + 12y^2 + 3x - 9y = 0$$

Proposed by Rajeev Rastogi-India

J.322 Determine the number of unordered triplets (x, y, z) of positive integers satisfying the equation $x + y + z = 102$

Proposed by Rajeev Rastogi-India

J.323 Determine the number of 6-tuples $(e_1, e_2, \dots, e_6)$ of positive integers satisfying the equation $2e_1 + 2e_2 + 2e_3 + 5e_4 + 5e_5 + 6e_6 = 32$

Proposed by Rajeev Rastogi-India

J.324 In $\Delta ABC, I$ −incenter, R_a, R_b, R_c −circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\frac{R_a^2}{r_a(sinB + sinC)} + \frac{R_b^2}{r_b(sinC + sinA)} + \frac{R_c^2}{r_c(sinA + sinB)} \ge \frac{6R^2}{a + b + c}$$

Proposed by Ertan Yildirim-Turkey

J.325 In ΔABC the following relationship holds:

$$\sum_{cyc} h_a^2(sinB - sinC) = (ab + bc + ca)\prod_{cyc}\frac{b - c}{2R}$$

Proposed by Ertan Yildirim-Turkey

J.326 In ΔABC the following relationship holds:

$$\frac{9r}{4R^2} \le \sum_{cyc}\frac{r_a}{bc} \cdot cos^2\frac{A}{2} \le \frac{9}{16r}$$

Proposed by Ertan Yildirim-Turkey

J.327 In ΔABC the following relationship holds:

$$\frac{r_a(r_b^3 + r_c^3)}{a} + \frac{r_b(r_c^3 + r_a^3)}{b} + \frac{r_c(r_a^3 + r_b^3)}{c} \ge s^3$$

Proposed by Ertan Yildirim-Turkey

J.328 In ΔABC the following relationship holds:

$$\frac{\left(1 - \frac{r}{r_a}\right)\left(1 - \frac{r}{r_b}\right)\left(1 - \frac{r}{r_c}\right)}{(1 - cosA)(1 - cosB)(1 - cosC)} = \frac{8R^3r}{F^2}$$

Proposed by Ertan Yildirim-Turkey

J.329 In $\Delta ABC, H$ –orthocenter, I –incenter, the following relationship holds:

$$\frac{a|AH|+b|BH|+c|CH|}{abc}\le\frac{3}{AI+BI+CI}$$

Proposed by Ertan Yildirim-Turkey

J.330 In $\Delta ABC, I$ –incenter, I_a, I_b, I_c –excenters, the following relationship holds:

$$\frac{AI_a}{\cos\frac{A}{2}}+\frac{BI_b}{\cos\frac{B}{2}}+\frac{CI_c}{\cos\frac{C}{2}}\ge 2(a+b+c)$$

Proposed by Ertan Yildirim-Turkey

J.331 In ΔABC the following relationship holds:

$$\frac{3r}{R}\le\frac{\sin^2A}{\sin^2B+\sin^2C}+\frac{\sin^2B}{\sin^2C+\sin^2A}+\frac{\sin^2C}{\sin^2A+\sin^2B}\le\frac{R}{r}-\frac{1}{2}$$

Proposed by Ertan Yildirim-Turkey

J.332 In ΔABC the following relationship holds:

$$\frac{m_b^2+m_c^2}{h_a}+\frac{m_c^2+m_a^2}{h_b}+\frac{m_a^2+m_b^2}{h_c}\ge 9R$$

Proposed by Ertan Yildirim-Turkey

J.333 In ΔABC the following relationship holds:

$$9r\le\frac{a(m_b+m_c)}{b+c}+\frac{b(m_c+m_a)}{c+a}+\frac{c(m_a+m_b)}{a+b}\le\frac{9R^2}{4r}$$

Proposed by Ertan Yildirim-Turkey

J.334 In $\Delta ABC, T$ –Toricelli's point, the following relationship holds:

$$12r^2\le AT^2+BT^2+CT^2\le 4R^2-4r^2$$

Proposed by Eldeniz Hesenov-Georgia

J.335 In ΔABC, the following relationship holds:

$$\frac{18r}{R}\le(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\le\left(\frac{3R}{2r}\right)^2$$

Proposed by Eldeniz Hesenov-Georgia

J.336 In ΔABC, the following relationship holds:

$$\left(\sum_{cyc}(b+c)^{-2}\right)\left(\sum_{cyc}m_aw_a\right)\ge\frac{9}{4R^2}\left(\prod_{cyc}r_a\right)\left(\prod_{cyc}r_a\right)^{-1}$$

Proposed by Eldeniz Hesenov-Georgia

J.337 In $\Delta ABC, I$ –incenter, R_a, R_b, R_c –circumradii in $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\sqrt[3]{\left(\frac{a}{R_a}\right)^2}+\sqrt[3]{\left(\frac{b}{R_b}\right)^2}+\sqrt[3]{\left(\frac{c}{R_c}\right)^2}\le 5$$

Proposed by Eldeniz Hesenov-Georgia

J.338 In ΔABC, the following relationship holds:

$$\left(\frac{\cos^2A}{a}\right)^2+\left(\frac{\cos^2B}{b}\right)^2+\left(\frac{COS^2c}{C}\right)^2\ge\left(\frac{(r-R)(r+R)}{3R^2}\right)^2$$

Proposed by Eldeniz Hesenov-Georgia

J.339 Let $a,b,c>0$ such that: $a+b+c=6, ab+bc+ca=9, a<b<c$. Prove that:

$$a\in(0,1), b\in(1,3), c\in(3,4).$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.340 If $x,y,z>0$ such that $xyz=1$, then:

a. $(x^{2021}+y^{2021})(y^{2021}+z^{2021})(z^{2021}+x^{2021})\ge 2\sqrt{2\left(2+\sum_{cyc}x^{2021}y^{2021}+\sum_{cyc}x^{2021}\right)}$

b. $\frac{x^2}{(y+z)^2}+\frac{y^2}{(x+z)^2}+\frac{z^2}{(x+y)^2}+\frac{10}{(x+y)(y+z)(z+x)}\ge 2$

c. $\left(\frac{x}{y}\right)^{2020}+\left(\frac{y}{z}\right)^{2019}+\left(\frac{z}{x}\right)^{2018}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge 6$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.341 If $x,y,z>0, x(x+y+z)=3yz$ then:

$$(x+y)^3+(x+z)^3+3(x+y)(y+z)(z+x)\le 5(y+z)^3$$

When equality holds?

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.342 Find all positive real numbers α such that:

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\le\frac{\alpha(a^2+b^2+c^2)}{a+b+c}, \forall a,b,c>0$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.343 In any triangle ABC the following relationship holds:

$$\sqrt{m_a+h_a+g_a+\sqrt{3}a}+\sqrt{m_b+h_b+w_b+\sqrt{3}b}+\sqrt{m_c+h_c+s_c+\sqrt{3}c}\le 2\sqrt{12R+3r}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.344 If $a,b,c>0$ such that $abc=1$, then:

1. $(a+b+c)^3\ge\frac{27}{4}(ab^2+bc^2+ca^2+abc)$
2. $ab^2+bc^2+ca^2+\frac{1}{\sqrt[5]{81(a^3+b^3+c^3)}}\ge\frac{10}{3}$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.345 If $a,,b,c>0$ such that $a+b+c=1$, then:

$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b}+\sqrt[3]{abc}\ge\frac{10}{9(a^2+b^2+c^2)}$$

When does equality holds?

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.346 Find all positive real numbers α such that:

$$\sqrt[3]{abc}+\frac{|a-b|+|b-c|+|c-a|}{\alpha}\ge\frac{a+b+c}{3}, \forall a,b,c>0$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.347 If a,b,c are positive real numbers then:

$$\frac{abc}{(a+b)(b+c)(c+a)}\le\frac{(a+b)(a+b+2c)}{(3a+3b+2c)^2}\le\frac{1}{8}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.348 If x, y, z are positive real numbers such that $x+y+z=3$ and $\alpha \geq 2$ then:

$$\min\left\{\sum \frac{1}{1+\alpha x y^2}, \sum \frac{1}{1+\alpha y x^2}\right\} \geq \frac{3}{1+\alpha}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.349 If $a, b \geq 0$ such that $a+b \leq \frac{4}{5}$ then:

$$\sqrt{\frac{1-a}{1+a}}+\sqrt{\frac{1-b}{1+b}} \leq \sqrt{\frac{1-a-b}{1+a+b}}+1$$

When does equality holds?

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.350 In any triangle ABC the following relationship holds:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq \frac{3p(p^2-3r^2-6Rr)}{p^2-r^2-4Rr}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.351 In $\Delta ABC, \Delta A'B'C'$ the following relationship holds:

$$\sum_{cyc}(a^2+a'^2)+2\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a'^2\right)} \geq 36(r+r')^2$$

Proposed by Daniel Sitaru,Aurel Chiriță – Romania

J.352 If $a, b > 0$ then (ϕ - golden ratio):

$$\left((1+\phi)^{\sqrt{ab}}+\sqrt{\phi^{a+b}}\right)\left((1+\pi)^{\sqrt{ab}}+\sqrt{\pi^{a+b}}\right)$$
$$\leq \left(\phi^{\sqrt{ab}}+\sqrt{(1+\phi)^{a+b}}\right)\left(\pi^{\sqrt{ab}}+\sqrt{(1+\pi)^{a+b}}\right)$$

Proposed by Daniel Sitaru,Nicolae Oprea – Romania

J.353 Solve for real numbers:

$$15\cos x \cdot \cos y \cdot \cos z+4\sum_{cyc} \cos 5x \cdot \cos y \cdot \cos z=0, -\frac{\pi}{2}<x, y, z<\frac{\pi}{2}$$

Proposed by Daniel Sitaru,Iulia Selea – Romania

J.354 Solve for real numbers:

$$\begin{cases} x^{x^2+6x} \cdot y^{y^2+6y} \cdot 3^{9+2xy}=\left(\dfrac{x+y}{3}+1\right)^{(x+y+z)^3} \\ x+3^y=29+\log_3\left(\dfrac{1}{x}\right) \end{cases}$$

Proposed by Daniel Sitaru,Ionuț Ivănescu – Romania

J.355 If $a,b,c,d\in\mathbb{R}$ then:

$$(ad-bc)^8(a^2+b^2)(c^2+d^2)+(ac+bd)^{10}\le(a^2+b^2)^5(c^2+d^2)^5$$

Proposed by Daniel Sitaru,Amelia Curcă Năstăselu – Romania

J.356 In ΔABC the following relationship holds:

$$\left(m_am_b+\frac{b^2+c^2}{2ca}\right)\left(\frac{1}{h_ah_b}+\frac{c^2+a^2}{2bc}\right)\ge\left(m_a(c^2+a^2)+\frac{b^2+c^2}{h_b}\right)\left(\frac{m_b}{2bc}+\frac{1}{2cah_a}\right)$$

Proposed by Daniel Sitaru,Mirea Mihaela Mioara – Romania

J.357 If M is an interior point in ΔABC with the area F and $x=MA,y=MB,z=MC$, then: $x^2h_a^2+y^2h_b^2+z^2h_c^2\ge 4F^2$

Proposed by D.M.Bătineţu-Giurgiu– Romania

J.358 Prove that in any ΔABC with the area F the following inequality holds:

$$(a+b)h_c+(b+c)h_a+(c+a)h_b\ge 12F$$

Proposed by D.M.Bătineţu-Giurgiu– Romania

J.359 If $m\in\mathbb{N}$ and $a,b,c,x,y\in\mathbb{R}_+^*=(0,\infty)$, $abc=1$, then:

$$\frac{(ax+by)^{2m+2}}{(a+b+2c)^{m+1}}+\frac{(bx+cy)^{2m+2}}{(2a+b+c)^{m+1}}+\frac{(cx+ay)^{2m+2}}{(a+2b+c)^{m+1}}\ge\frac{3(x+y)^{2(m+1)}}{4^{m+1}}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.360 In any ΔABC the following inequality holds:

$$\frac{m_a+m_b+m_c}{\frac{h_a+m_b+m_c}{h_a}+\frac{h_b+m_c+m_a}{h_b}+\frac{h_c+m_a+m_b}{h_c}}\ge r$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.361 In any ΔABC the following inequality holds:

$$\sum_{cyc}\frac{w_a+m_b+h_c}{h_a}\ge\frac{1}{r}(h_a+h_b+h_c)\ge 9$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.362 In any ΔABC the following inequality holds:

$$\sum_{cyc}\frac{s_a+s_b+h_c}{h_b}\le\frac{1}{r}(s_a+s_b+s_c)$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.363 In any ΔABC the following inequality holds:

$$\sum_{cyc}\frac{h_a+h_b+m_c}{h_c}\le\frac{1}{r}(m_a+m_b+m_c)$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.364 In any ΔABC the following inequality holds:

$$\sum_{cyc}\frac{w_a+w_b+h_c}{h_a}\le\frac{1}{r}(w_a+w_b+w_c)$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.365 If $m,n\in\mathbb{R}_+=[0,\infty)$; $m+n=2$ then in any ΔABC with the area F the following inequality holds: $\frac{(a^2+b^2)w_c^m}{c^n}+\frac{(b^2+c^2)w_a^m}{a^n}+\frac{(c^2+a^2)w_b^m}{b^n}\ge 3\cdot 2^{m+1}F^m$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.366 Let $m, n, t \in \mathbb{R}_+ = [0, \infty), m + n = 4$ and $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ the in ΔABC with the area F the following inequality holds:

$$\frac{y+z+6t}{x+3t} \cdot \frac{a^m}{h_a^n} + \frac{z+x+6t}{y+3t} \cdot \frac{b^m}{h_b^n} + \frac{x+y+6t}{z+3t} \cdot \frac{c^m}{h_c^n} \ge 2^{n-1} F^{2-n}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.367 Let $m, n, t \in \mathbb{R}_+ = [0, \infty), m + n = 4$ and $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in ΔABC with the area F the following inequality holds:

$$\frac{y+z+6t}{x+3t} \cdot \frac{a^m}{h_a^n} + \frac{z+x+6t}{y+3t} \cdot \frac{b^m}{h_b^n} + \frac{x+y+6t}{z+3t} \cdot \frac{c^m}{h_c^n} \ge 2^{n-1} F^{2-n}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.368 Let $m \in \mathbb{R}_+ = [0, \infty)$ and $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\left(\sum_{cyc} \frac{\sin^{m+1} x}{(\cos y + \cos z)^m}\right)\left(\sum_{cyc} \frac{\cos^{m+1} x}{(\sin y + \sin z)^m}\right) \ge \frac{1}{2^{2m}}\left(\sum_{cyc} \sqrt{\sin x \cdot \cos x}\right)^z$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.369 If $m, n \in \mathbb{R}_+, m + n = 2, x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then in any ΔABC with the area F the following inequality holds:

$$\frac{(x+y) \cdot a^m}{z h_a^n} + \frac{(y+z) \cdot b^m}{x h_b^n} + \frac{(z+x) c^m}{y \cdot h_c^n} \ge 2^{3-n}\sqrt{3} \cdot F^{1-n}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.370 Let $\Delta A_1B_1C_1$ be the circumcevian triangle of I – incenter in ΔABC. Prove that:

$$IA_1^2 + IB_1^2 + IC_1^2 \ge 12r^2$$

Proposed by Marian Ursărescu – Romania

J.371 $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$, different in pairs, $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$.
If $z_1z_2(z_1 + z_2) + z_2z_3(z_2 + z_3) + z_3z_1(z_3 + z_1) + 2z_1z_2z_3 = 0$ then:

$$\prod_{cyc}(AB^2 + AC^2 - BC^2) = 0$$

Proposed by Marian Ursărescu – Romania

J.372 Solve for real numbers:

$$5^x + 4^{\frac{1}{x}} + 25^x \cdot 16^{\frac{1}{x}} = 2527$$

Proposed by Marian Ursărescu – Romania

J.373 In $\Delta ABC, AA_1, BB_1, CC_1$ – internal bisectors, $\{A_2\} = AA_1 \cap B_1C_1$, $\{B_2\} = BB_1 \cap C_1A_1, \{C_2\} = CC_1 \cap A_1B_1, I$ – incenter. Prove that:

$$\frac{AA_2}{IA_2} + \frac{BB_2}{IB_2} + \frac{CC_2}{IC_2} \le 3\left(1 + \frac{R}{r}\right)$$

Proposed by Marian Ursărescu – Romania

J.374 $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$, different in pairs, $A(z_1), B(z_2), C(z_3)$

$$|z_A| = |z_B| = |z_C|, \sum_{cyc} ab(a-b)(z_A - z_B) = 0$$

Prove that: $AB = BC = CA$.

Proposed by Marian Ursărescu – Romania

J.375 Solve for real numbers:

$$x + 9^{\log_x 27} + x \cdot 9^{\log_x 27} = 279$$

Proposed by Marian Ursărescu – Romania

J.376 In ΔABC the following relationship holds:

$$\frac{r_a}{\sqrt[3]{r_a + 2r_b}} + \frac{r_b}{\sqrt[3]{r_b + 2r_c}} + \frac{r_c}{\sqrt[3]{r_c + 2r_a}} \ge 3\sqrt[3]{\frac{3Rr}{2}}$$

Proposed by Marian Ursărescu – Romania

J.377 In ΔABC the following relationship holds:

$$\left(\frac{m_a m_b}{m_a + m_b}\right)^2 + \left(\frac{m_b m_c}{m_b + m_c}\right)^2 + \left(\frac{m_c m_a}{m_c + m_a}\right)^2 \ge \frac{27r^2}{4}$$

Proposed by Marian Ursărescu – Romania

J.378 In $\Delta ABC, I$ – incenter, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$.

Prove that:

$$R_a + R_b + R_c \ge \frac{5R}{2} + r$$

Proposed by Marian Ursărescu – Romania

J.379 $z_1, z_2, z_3, z_4 \in \mathbb{C} - \{0\}$ – different in pairs, $|z_1| = |z_2| = |z_3| = |z_4| = 1$

$$|z_1z_2 + z_1z_3 + z_2z_3|^2 + |z_1z_2 + z_1z_4 + z_2z_4|^2 + |z_1z_3 + z_1z_4 + z_3z_4|^2 +$$
$$+|z_2z_3 + z_2z_4 + z_3z_4|^2 = 4$$

Find:

$$\Omega = \frac{1}{z_1^{2021}} + \frac{1}{z_2^{2021}} + \frac{1}{z_3^{2021}} + \frac{1}{z_4^{2021}}$$

Proposed by Marian Ursărescu – Romania

J.380 $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$, different in pairs, $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$

$$\sum_{cyc} \big| |z_1 - z_2|z_1 - z_3| \; + (z_1 - z_3)|z_1 - z_2| \big| = 3\sum_{cyc} |z_1 - z_2| \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

J.381 In $\Delta ABC, K$ – Lemoine's point, the following relationship holds:

$$\frac{48r^4}{R^3} \le AK + BK + CK \le \frac{3R^2}{2r}$$

Proposed by Marian Ursărescu – Romania

J.382 In ΔABC the following relationship holds:

$$\frac{3}{2} \le \sum_{cyc} \frac{h_b + h_c}{b + c} tan \frac{A}{2} \le \frac{3R}{4r}$$

Proposed by Marin Chirciu-Romania

J.383 In ΔABC the following relationship holds:

$$\frac{w_a^{n+1}}{w_a^n}+\frac{w_b^{n+1}}{w_b^n}+\frac{w_c^{n+1}}{w_c^n}\ge 2r\left(5-\frac{r}{R}\right), n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

J.384 In ΔABC the following relationship holds:

$$\sum_{cyc} cot^3\frac{A}{2}cot\frac{B}{2}\ge\frac{s^2}{r^2}\ge\frac{16R}{r}-5$$

Proposed by Marin Chirciu-Romania

J.385 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(r_b^{n+1}+r_c^{n+1})^2}{r_b^n+r_c^n}\le\frac{27R^3}{4r}, n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

J.386 Let $\lambda>0$. Solve for real numbers:

$$\begin{cases} y^3-3\lambda x^2+3\lambda^2x-\lambda^3=0\\ z^3-3\lambda y^2+3\lambda^2y-\lambda^3=0\\ x^3-3\lambda z^2+3\lambda^2z-\lambda^3=0\end{cases}$$

Proposed by Marin Chirciu-Romania

J.387 In $\Delta ABC, I$ –incenter, R_a, R_b, R_c –circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\frac{4}{3}\le\sum_{cyc}\frac{R_a^2}{r_a^2}\le\frac{2R}{3r}$$

Proposed by Marin Chirciu-Romania

J.388 In ΔABC the following relationship holds:

$$w_a^2+w_b^2+w_c^2+n\cdot OI^2\ge s^2, n\ge 2$$

Proposed by Marin Chirciu-Romania

J.389 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_a(h_b^2+h_c^2)}{a}\le\sum_{cyc}\frac{r_a(r_b^2+r_c^2)}{a}$$

Proposed by Marin Chirciu-Romania

J.390 In ΔABC the following relationship holds:

$$\frac{h_a^{n+1}}{h_a^n}+\frac{h_b^{n+1}}{h_b^n}+\frac{h_c^{n+1}}{h_c^n}\ge 2r\left(5-\frac{r}{R}\right), n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

J.391 In ΔABC the following relationship holds:

$$\sum_{cyc} csc^3\frac{A}{2}csc\frac{B}{2}\ge\frac{24R}{r}$$

Proposed by Marin Chirciu-Romania

J.392 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(s_b^{n+1}+s_c^{n+1})^2}{s_b^n+s_c^n}\le\frac{27}{2}R^2, n\in\mathbb{N}$$

Proposed by Marin Chirciu-Romania

J.393 In ΔABC the following relationship holds:

$$\left(\frac{m_a}{m_b}\right)^3+\left(\frac{m_b}{m_c}\right)^3+\left(\frac{m_c}{m_a}\right)^3\ge 2+\frac{3(a^2+b^2+c^2)}{2(a^2+b^2+c^2)+ab+bc+ca}$$

Proposed by Daniel Sitaru,Claudiu Ciulcu – Romania

J.394 In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\frac{1}{a^2\cdot AI^4}+\frac{1}{b^2\cdot BI^4}+\frac{1}{c^2\cdot CI^4}\ge\frac{9}{64r^4(r^2+2R^2)}$$

Proposed by Daniel Sitaru,Lavinia Trincu– Romania

J.395 If $m,n,x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then in any triangle ABC the following relationship holds:

$$\frac{mx+ny}{az}+\frac{my+nz}{bx}+\frac{mz+nx}{cy}\ge\frac{2\sqrt{mn}\cdot\sqrt{3}}{R}$$

Proposed by D.M.Bătineţu-Giurgiu, Claudia Nănuţi-Romania

J.396 If $x,y,z\in\mathbb{R}_+^*=(0,\infty), t\in\mathbb{R}_+=[0,\infty)$ and ΔABC with F –area, then prove:

$$\frac{y+z+6t}{x+3t}h_a+\frac{z+x+6t}{y+3t}h_b+\frac{x+y+6t}{z+3t}h_c\ge\frac{4\sqrt{3}\cdot F}{R}$$

Proposed by D.M.Bătineţu-Giurgiu, Claudia Nănuţi-Romania

J.397 If $m\in\mathbb{N}, M\in Int(\Delta ABC)$ and F –area of $\Delta ABC, x_A=MA, x_B=MB, x_C=MC$ then prove:

$$3m+\left(\frac{x_A}{h_a}\right)^{m+1}+\left(\frac{x_B}{h_b}\right)^{m+1}+\left(\frac{x_C}{h_c}\right)^{m+1}\ge 2(m+1)$$

Proposed by D.M.Bătineţu-Giurgiu, Claudia Nănuţi-Romania

J.398 If $m,n\in\mathbb{R}_+=[0,\infty); m+n,x,y,z>0$ and F –area of ΔABC then prove:

$$\sum_{cyc}\frac{m(x+y)+n(x+z)}{(ny+mz)h_a}\cdot a^3\ge 16F$$

Proposed by D.M.Bătineţu-Giurgiu, Claudia Nănuţi-Romania

J.399 If $a, b, c, d, e > 0, a + b + c + d + e = 41$ then:

$$\frac{a^2}{\sqrt[3]{4}} + \frac{b^2 + c^2 + d^2}{\sqrt[3]{2}} + e \geq 41\left(8\sqrt[3]{4} - \sqrt[3]{2} - 5\right)$$

When equality holds?

Proposed by Daniel Sitaru,Mihaela Nascu – Romania

J.400 If $a, b, c, d > 0, a + b + c + d = \log_{10}^2 98$ then:

$$\left(\frac{1}{a} + \frac{1}{c}\right)\log_{10}^4 9 + \left(\frac{1}{b} + \frac{1}{d}\right)\log_{10}^4 11 > 4$$

Proposed by Daniel Sitaru,Corina Ionescu – Romania

J.401 Solve for real numbers:

$$\sqrt[5]{5\sqrt{5} + x} - \sqrt[5]{5\sqrt{5} - x} = \sqrt[5]{2}$$

Proposed by Daniel Sitaru,Laura Zaharia – Romania

J.402 Solve for real numbers:

$$\begin{cases} x, y, z, t > 1 \\ \log_y x^{\frac{2}{x+y}} + \log_z y^{\frac{2}{y+z}} + \log_t z^{\frac{2}{z+t}} + \log_x t^{\frac{2}{t+x}} = \dfrac{16}{x + y + z + t} \\ (2x + 3y + 4z + 100)^{10} = 10^{12}x^2y^3z^4 \end{cases}$$

Proposed by Daniel Sitaru,Delia Schneider – Romania

J.403 If $m, n \in \mathbb{R}_+ = [0, \infty); m + n, x, y, z > 0$ then in ΔABC with the area F the following inequality holds:

$$\sum_{cyc} \frac{m(x + y) + n(x + z)}{(ny + mz)h_a} a^3 \geq 16F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.404 If $a, b, x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then:

$$\sum_{cyc} \frac{x^2 + y^2}{a(y + z)^2 + bxy} \geq \frac{6}{4a + b} \cdot \frac{xy + yz + zx}{x^2 + y^2 + z^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

J.405 If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\left(\sum_{cyc} \frac{\sin^2 z}{\cos y + \cos z}\right)\left(\sum_{cyc} \frac{\cos^2 z}{\sin y + \sin z}\right) \geq \frac{1}{4}\left(\sum_{cyc} \sqrt{\sin x \cdot \cos x}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu,Gabriel Tică– Romania

J.406 If $m,n,u,v\in\mathbb{R}_+^*=(0,\infty)$ and $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then:

$$\left(\sum_{cyc}\frac{\sin^2 x}{u\cos y+v\cos t}\right)\left(\sum_{cyc}\frac{\cos^2 x}{m\sin y+n\sin z}\right)\geq\frac{1}{(m+n)(u+v)}\left(\sum_{cyc}\sqrt{\sin x\cdot\cos x}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu,Dan Nănuți– Romania

J.407 If $x\in\mathbb{R}_+^*=(0,\infty)$ and $m\in\mathbb{R}_+=[0,\infty)$ then $2^m\left(([x])^{m+1}+(\{x\})^{m+1}\right)\geq x^{m+1}$

where $[x]$ is the whole part of x and $\{x\}$ is the fractionary part of x

Proposed by D.M. Bătinețu-Giurgiu,Dan Nănuți– Romania

J.408 If $m\in\mathbb{N}^*$, then in any triangle the following inequality holds:

$$(2m+1)R(s^2-r^2-4Rr)\geq(mR+r)(s^2+r^2+4Rr)$$

where s is triangle's semiperimeter.

Proposed by D.M. Bătinețu-Giurgiu,Neculai Stanciu– Romania

J.409 If $m\in\mathbb{R}_+=[0,\infty)$ then in any ΔABC with the area F the following inequality holds:

$$\sum_{cyc}\frac{a^{m+2}}{(b+c-a)^m}\geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu,Neculai Stanciu– Romania

J.410 If M is an interior point in ΔABC and $x=MA, y=MB, z=MC$, then:

$$\left(\sum_{cyc}\frac{x}{a}\right)^2\left(\sum_{cyc}\left(\frac{xy}{ab}\right)^4\right)\geq\frac{3x^2y^2z^2}{a^2b^2c^2}$$

Proposed by D.M. Bătinețu-Giurgiu,Neculai Stanciu– Romania

J.411 If $t\in(0,\infty)$, M is an interior point in ΔABC and $x=MA, y=MB, z=MC$, then:

$$\left((1+t)\left(\frac{xy}{ab}+\frac{yz}{bc}\right)+\frac{zx}{ca\cdot t}\right)\left(\frac{\frac{xy}{ab}+\frac{yz}{bc}}{1+t}+\frac{zxt}{ca}\right)\geq 1$$

Proposed by D.M. Bătinețu-Giurgiu,Neculai Stanciu– Romania

J.412 In ΔABC the following relationship holds:

$$5+2\sum_{cyc}\frac{n_a n_b}{h_a h_b}\leq\frac{4R}{r}+\sum_{cyc}\frac{n_a^2}{h_a^2}$$

Proposed by Bogdan Fuștei – Romania

J.413 In ΔABC, n_a – Nagel's cevian, g_a – Gergonne cevian, the following relationship holds:

$$4\sqrt{3}\sum_{cyc}\cos\frac{A}{2}\leq 9+\sum_{cyc}\frac{3(b+c)-n_a-g_a-\sqrt{2r_b r_c}}{a}$$

Proposed by Bogdan Fuștei – Romania

J.414 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne cevian, the following relationship holds:

$$\sum_{cyc}\frac{g_a}{n_a}\le 1+\frac{r}{R}\left(1+\frac{(4R+r)^2}{s^2}\right)$$

Proposed by Bogdan Fuștei – Romania

J.415 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{h_a}{r}|r_b-r_c|\ge 2\cos\frac{A}{2}\sqrt{n_a^2-h_a^2}$$

Proposed by Bogdan Fuștei – Romania

J.416 In ΔABC the following relationship holds:

$$\frac{1}{R}\ge\sum_{cyc}\frac{\frac{h_c}{h_b}+\frac{h_c}{h_b}}{m_a+r_b+r_c-h_a}$$

Proposed by Bogdan Fuștei – Romania

J.417 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$6\sqrt{3}\sum_{cyc}\frac{a\cdot n_a}{bc}\le\sum_{cyc}\frac{h_a}{n_a}\cdot\sum_{cyc}\cot^2\frac{A}{2}$$

Proposed by Bogdan Fuștei – Romania

J.418 In ΔABC the following relationship holds:

$$2+\frac{\prod\cos\frac{A}{2}}{\sum\cos\frac{A}{2}}\ge\sum_{cyc}\cos\frac{A}{2}\cdot\sqrt{1-\frac{\prod\cos\frac{A}{2}}{\sum\cos\frac{A}{2}}}$$

Proposed by Bogdan Fuștei – Romania

J.419 In ΔABC the following relationship holds:

$$\frac{2bcm_bm_c}{\sqrt{2a^2b^2+2a^2c^2+8b^2c^2+a^4-2b^4-2c^4}}\ge\sqrt{\frac{\prod_{cyc}(am_a+bm_b-cm_c)}{am_a+bm_b+cm_c}}$$

Proposed by Bogdan Fuștei – Romania

J.420 Solve for real numbers:

$$\begin{cases}logx=[y]\\ log(y+1)=[x]-1\end{cases},\quad [*]-GIF$$

Proposed by Jalil Hajimir-Canada

J.421 Solve for real numbers:

$$\sqrt[4]{(1+2^x)(1+2^{3x})}>\frac{1+6^x}{\sqrt{1+3^{2x}}}$$

Proposed by Jalil Hajimir-Canada

J.422 In ΔABC the following relationship holds:

$$\frac{\sec^4 A}{a^5\cdot w_a}+\frac{\sec^4 B}{b^5\cdot w_b}+\frac{\sec^4 C}{c^5\cdot w_c}\ge\frac{8\sqrt{2}}{3sR^4\cdot\sqrt{Rr}}$$

Proposed by Daniel Sitaru,Elena Grigore – Romania

J.423 Solve for real numbers:

$$\begin{cases} \sqrt[3]{(x+3)^2} + 6\cdot\sqrt[3]{(z+3)^2} = 5\cdot\sqrt[3]{(x-3)(y+3)} \\ x,y,z > 0 \\ \dfrac{2x^2}{yz(y+z)} + \dfrac{2y^2}{zx(z+x)} + \dfrac{2z^2}{xy(x+y)} = \dfrac{9}{x+y+z} \end{cases}$$

Proposed by Daniel Sitaru,Dan Grigorie – Romania

J.424 In ΔABC the following relationship holds:

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \le \frac{\sqrt{3}}{72r^3}$$

Proposed by Daniel Sitaru,Elena Alexie – Romania

J.425 Solve for real numbers:

$$\begin{cases} x,y,z > 0 \\ 32(x^5+y^5+1) = (x+y)^5 + (x+1)^5 + (y+1)^5 \\ x+z = \sqrt{xz} + \sqrt{\dfrac{x^2+z^2}{2}} \end{cases}$$

Proposed by Daniel Sitaru,Daniela Beldea – Romania

J.426 $A(a), B(b), C(c), a,b,c \in \mathbb{C}^*$ –different in pairs, $|a| = |b| = |c| = 1$. Prove that:

$$\sum_{cyc}\left(|a^2+ab+bc+ca| + |a-b|^2\right) = 12 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

J.427 $z_1, z_2, z_3 \in \mathbb{C}^*$, distinct, $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), c(z_3)$.

$$\sum_{cyc}\left||z_1-z_3|(z_1-z_2) + |z_1-z_2|(z_1-z_3)\right|^{-2} = 3\left(\sum_{cyc}|z_1-z_2|\right)^{-2} \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

J.428 $A(a), B(b), C(c), a,b,c \in \mathbb{C}^*$ –diffrerent in pairs, $|a| = |b| = |c| = 1$. Prove that:

$$\sum_{cyc}|2a+b+c|^2 = 3 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

J.429 $z_1, z_2, z_3 \in \mathbb{C}^*$, distinct, $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$

$$\sum_{cyc}\frac{|2z_1-z_2-z_3|}{\left||z_1-z_3|(z_1-z_2) + |z_1-z_2|(z_1-z_3)\right|} = 9\left(\sum_{cyc}|z_1-z_2|\right)^{-1} \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

J.430 Let $\Delta A'B'C'$ be the circumcevian triangle of centroid in ΔABC. Prove that:

$$\frac{[A'B'C']}{[ABC]} \geq \left(\frac{2r}{R}\right)^2$$

Proposed by Marian Ursărescu-Romania

J.431 In ΔABC the following relationship holds:

$$m_a r_a + m_b r_b + m_c r_c \leq \frac{3R}{2r}(2R^2 + r^2)$$

Proposed by Marian Ursărescu-Romania

J.432 In $\Delta ABC, G$ –centroid, K –Lemoine's point the following relationship holds:

$$AG \cdot AK + BG \cdot BK + CG \cdot CK \geq \frac{24r^3}{R}$$

Proposed by Marian Ursărescu-Romania

J.433 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\left(\frac{n_a}{r_a}\right)^2 + \left(\frac{n_b}{r_b}\right)^2 + \left(\frac{n_c}{r_c}\right)^2 \geq 13 + \frac{4R - 2(h_a + h_b + h_c)}{r}$$

Proposed by Bogdan Fuştei-Romania

J.434 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian, the following relationship holds:

$$min\left\{\sum_{cyc} n_a g_a \cdot \left(\sum_{cyc} h_a h_b\right)^{-1}, \sum_{cyc} m_a w_a \cdot \left(\sum_{cyc} h_a h_b\right)^{-1}\right\} \geq \frac{R}{2r}$$

Proposed by Bogdan Fuştei-Romania

J.435 In ΔABC, the following relationship holds:

$$\frac{n_a + g_a}{b + c} + \frac{n_b + g_b}{c + a} + \frac{n_c + g_c}{a + b} \geq 4\prod_{cyc} cos\left(\frac{\pi - A}{4}\right)$$

Proposed by Bogdan Fuştei-Romania

J.436 In $\Delta ABC, g_a$ –Gergonne's cevian, the following relationship holds:

$$\frac{R}{r} \geq 1 + \sqrt[3]{\frac{m_a m_b m_c w_a w_b w_c}{h_a h_b h_c g_a g_b g_c}} \geq 2$$

Proposed by Bogdan Fuştei-Romania

J.437 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\sum_{cyc} \frac{n_a g_a}{w_a} \geq \sqrt{\sum_{cyc} h_a h_b} \cdot \sum_{cyc} tan\frac{A}{2}$$

Proposed by Bogdan Fuştei-Romania

J.438 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian, the following relationship holds:

$$sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\le\sum_{cyc}\sqrt{\frac{b^2+c^2-2n_ag_a}{2bc}}$$

Proposed by Bogdan Fuștei-Romania

J.439 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\sqrt[4]{\frac{m_am_bm_cw_aw_bw_c}{r^6}}\ge\sum_{cyc}\left(\frac{n_a}{h_a}+\frac{2r_a}{s+n_a}\right)$$

Proposed by Bogdan Fuștei-Romania

J.440 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne cevian, the following relationship holds:

$$\sum_{cyc}\frac{a}{r_a-r}\ge\sum_{cyc}\left(\frac{2m_a-g_a}{r_a}+\frac{2h_a}{s+n_a}\right)$$

Proposed by Bogdan Fuștei-Romania

J.441 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian, the following relationship holds:

$$\prod_{cyc}\left(\frac{n_a+g_a}{s_a}-\frac{h_c}{h_b}\right)\ge 1$$

Proposed by Bogdan Fuștei-Romania

J.442 In ΔABC the following relationship holds:

$$\frac{9Rs}{2}\le\sum_{cyc}\frac{r_a(r_b^2+r_c^2)}{a}\le\frac{2s}{R}(2R-r)^2$$

Proposed by Marin Chirciu-Romania

J.443 In ΔABC the following relationship holds:

$$3\sqrt[6]{\frac{4r}{R^2s^2}}\le\sum_{cyc}\sqrt{\frac{r_a}{m_bm_c}}\le\frac{4R+r}{s}\sqrt{\frac{1}{r}}$$

Proposed by Marin Chirciu-Romania

J.444 In ΔABC the following relationship holds:

$$4\left(\frac{2r}{R}\right)^{\frac{2}{3}}\le\sum_{cyc}\frac{a^2}{m_a^2}\le 4\left(\frac{R}{r}-1\right)$$

Proposed by Marin Chirciu-Romania

J.445 In ΔABC the following relationship holds:

$$\sum_{cyc}r_a^3r_b\ge 3r^2(4R+r)^2$$

Proposed by Marin Chirciu-Romania

J.446 In ΔABC the following relationship holds:

$$2R^2r \le \sum_{cyc}\left(\frac{AH \cdot AI}{\sqrt{r_b + r_c}}\right)^2 \le R(4R^2 - 3Rr - 6r^2)$$

Proposed by Marin Chirciu-Romania

J.447 In ΔABC the following relationship holds:

$$\frac{2R}{r} - 1 \le \sum_{cyc}\frac{R}{r_a - r} \le \left(\frac{R}{r}\right)^2 - \frac{R}{r} + 1$$

Proposed by Marin Chirciu-Romania

J.448 If $a, b, c > 0$ such that $abc = 1$ and $n \in \mathbb{N}^*$ then:

$$\sum_{cyc}\frac{a^{2n+2} + a^{2n}b^2}{(ac + b^2)^2} \ge \frac{3}{2}$$

Proposed by Marin Chirciu-Romania

J.449 Let $\lambda \ge 2$ be positive real numbers. Solve for real numbers:

$$\frac{(\lambda + 1 + x)(\lambda + x)(\lambda - 1 + x)}{(\lambda + 1 - x)(\lambda - x)(\lambda - 1 - x)} = \frac{3(9\lambda^2 - 1)}{1 - \lambda^2}$$

Proposed by Marin Chirciu-Romania

J.450 In ΔABC the following relationship holds:

$$\frac{m_a^3}{m_a^2} + \frac{m_b^3}{m_b^2} + \frac{m_c^3}{m_c^2} \ge 6r\left(2 - \frac{r}{R}\right)$$

Proposed by Marin Chirciu-Romania

J.451 O −circumcenter, Ω −first Brocard point in $\Delta ABC, O_a, O_b, O_c$ −circumcenters of $\Delta B\Omega C, \Delta C\Omega A, \Delta A\Omega B, d_a = OO_a, d_b = OO_b, d_c = OO_c$. Prove that:

$$[ABC] \le [O_aO_bO_c] \le \frac{3\sqrt{3} \cdot d_ad_bd_c}{8r}$$

Proposed by Mehmet Şahin-Turkey

JP.452 O −circumcenter, Ω −first Brocard point in $\Delta ABC, O_a, O_b, O_c$ −circumcenters of $\Delta B\Omega C, \Delta C\Omega A, \Delta A\Omega B, d_a = OO_a, d_b = OO_b, d_c = OO_c$. Prove that:

$d_ad_bd_c = R^3, R$ −circumradii of ΔABC

Proposed by Mehmet Şahin-Turkey

J.453 In $\Delta ABC, G$ −centroid, $D, E \in BC, F, K \in CA, M, N \in AB, \overline{DGK}, \overline{EGN}, \overline{MGF}$ −antiparallels, P_1, P_2, P_3 −perimeters of $\Delta GDE, \Delta GFK, \Delta GNM$. Prove that:

$$\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{3(2R - r)}{2F}$$

Proposed by Mehmet Şahin-Turkey

J.454 In $\Delta ABC, G$ −centroid, $M, F \in BC, D, K \in AB, E, N \in AC, \overline{MGN}, \overline{DGE}$,

$\overline{KGF}$ −antiparallels. Prove that: $GM \cdot GE \cdot GK = \frac{abc}{27}$.

Proposed by Mehmet Şahin-Turkey

J.455 In $\Delta ABC, G$ −centroid, $M, N \in BC, P, Q \in CA, R, S \in AB, \overline{MGQ}, \overline{NGR}$,

$\overline{SGP}$ −antiparallels, $\varphi_a, \varphi_b, \varphi_c$ −circumradii in $\Delta GMN, \Delta GPQ, \Delta GRS$. Prove that:

$$\varphi_a\varphi_b\varphi_c = \frac{R^3}{27}$$

Proposed by Mehmet Şahin-Turkey

J.456 O −circumcenter in acute $\Delta ABC, F, K \in (AB), M, L \in (BC), E, N \in (CA)$

$\overline{FOE}, \overline{MON}, \overline{LOK}$ are the antiparallels. Let $\varphi_a, \varphi_b, \varphi_c$ be circumradii of

$\Delta AFE, \Delta BLK, \Delta CMN$. Prove that:

$$\varphi_a + \varphi_b + \varphi_c = \frac{R^2}{r}$$

Proposed by Mehmet Şahin-Turkey

J.457 In ΔABC the following relationship holds:

$$4(R+r)^3 \ge \frac{1}{16}\prod_{cyc}\left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_br_c}}\right) \ge 27R^2r$$

Proposed by Alex Szoros-Romania

J.458 Having 10 numbers with in interval [10,90), there are 3 of them which are the lengths of the sides of a triangle

Proposed by Alex Szoros-Romania

J.459 Prove that $F \le \frac{abc(b+c)h_a}{(a+b)(b+c)(b+c-a)}$ if and only if $(r_a - r_b)(r_a - r_c) \ge 0$

Proposed by Alex Szoros-Romania

J.460 In ΔABC if $F = 1$, then

$$R \ge \frac{\frac{cosA}{a} + \frac{cosB}{b} + \frac{cosC}{c}}{1 + cosAcosBcosC} \ge 2r$$

Proposed by Alex Szoros-Romania

J.461 In acute triangle ABC the following relationship holds:

$$(a^2+1)^n + (b^2+1)^n + (c^2+1)^n \ge 3 + 4n(R+r)^2, n \in \mathbb{N}$$

Proposed by Alex Szoros-Romania

J.462 In ΔABC the following relationship holds:

$$\frac{4R}{r} \ge \left(\frac{h_a}{r} - 1\right)\left(\frac{h_b}{r} - 1\right)\left(\frac{h_c}{r} - 1\right) \ge 4 + \frac{8}{3}\sum_{cyc}\frac{a}{b+c}$$

Proposed by Alex Szoros-Romania

J.463 In ΔABC the following relationship holds:

$$\frac{4R}{r} \geq \left(\frac{h_a}{r} - 1\right)\left(\frac{h_b}{r} - 1\right)\left(\frac{h_c}{r} - 1\right) \geq 4 + \frac{8}{3}\sum_{cyc} \frac{a}{b+c}$$

Proposed by Alex Szoros-Romania

J.464 If $ab + bc + ca = abc$, then $4R \geq \frac{a+b}{l_c} + \frac{b+c}{l_a} + \frac{c+a}{l_a} \geq 8r$

Proposed by Alex Szoros-Romania

J.465 In ΔABC the following relationship holds:

$$\frac{h_a + h_b + h_c}{3r} \geq 1 + \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}$$

Proposed by Alex Szoros-Romania

J.466 In$\Delta ABC, \omega$ –Brocard's point, the following relationship holds:

$$\sqrt{\frac{R}{r}\left(\frac{r_b}{r_c} + \frac{r_c}{r_b}\right)} \geq \frac{\sin(A+\omega)}{\sin\omega}$$

Proposed by Alex Szoros-Romania

J.467 In acute ΔABC the following relationship holds:

$$8\sum_{cyc} \cos A \leq \sum_{cyc}\left(\frac{r + r_a}{R}\right)^2 \leq 12$$

Proposed by Alex Szoros-Romania

J.468 If the equation has the roots: $\frac{x^2\sqrt{\phi}+1}{x^2\sqrt{\phi}-1} + \frac{x\phi-1}{x\phi+1} + \phi = 0, \alpha, \beta, \gamma$ then prove that:

$\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \phi^{\frac{7}{2}}, \phi$ –Golden ratio.

Proposed by Srinivasa Raghava-AIRMC-India

J.469 Let $\begin{cases} x + y + z = \phi + 1 \\ x^2 + y^2 + z^2 = \phi^2 + 2 \\ x^3 + y^3 + z^3 = \phi^3 + 3 \end{cases}$, then prove that:

$$x^5 + y^5 + z^5 - (x^4 + y^4 + z^4) = xyz + \frac{7\phi^2}{2}$$

Proposed by Srinivasa Raghava-AIRMC-India

J.470 Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(f(x) + y) = 3x + f(f(y) - 2x), \forall x, y \in \mathbb{R}$$

Proposed by Rajeev Rastogi-India

J.471 If $ABCD$ −cyclic quadrilateral, O −circumcenter, $AB = \sqrt{2}, m(\sphericalangle AOB) = 90^\circ$ then $[ABCD] \le 2$

Proposed by Rajeev Rastogi-India

J.472 In ΔABC the following relationship holds:

$$\frac{m_a^2}{r_b r_c} + \frac{m_b^2}{r_c r_a} + \frac{m_c^2}{r_a r_b} \le 1 + \frac{R}{r}$$

Proposed by Adil Abdullayev-Azerbaijan

J.473 In ΔABC the following relationship holds:

$$\frac{9 r_a r_b r_c}{g_a g_b g_c} + \frac{2r}{R} \ge 10$$

Proposed by Adil Abdullayev-Azerbaijan

J.474 If in $\Delta ABC, S_p$ −Spieker point, N_a −Nagel's point then:

$$[AS_pN_a] + [CS_pN_a] \ge [BS_pN_a]$$

Proposed by Adil Abdullayev-Azerbaijan

J.475 In ΔABC the following relationship holds:

$$\frac{m_a^2 + m_b^2 + m_c^2}{3\sqrt{3}F} \le \frac{1}{9} + \frac{8}{9}\left(\frac{R}{2r}\right)^4$$

Proposed by Adil Abdullayev-Azerbaijan

J.476 In ΔABC the following relationship holds:

$$1 + \sqrt{1 + \sqrt{\sum_{cyc} r_a^2 \cdot \sum_{cyc} r_a^{-2}}} \le \sqrt{1 + \frac{4R}{r}}$$

Proposed by Adil Abdullayev-Azerbaijan

J.477 In ΔABC the following relationship holds:

$$\sqrt{(m_a + m_b + m_c)\left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}\right)} \le 3\left(\frac{R}{2r}\right)^2$$

Proposed by Adil Abdullayev-Azerbaijan

J.478 In ΔABC the following relationship holds:

$$\frac{w_a}{w_b + w_c} + \frac{w_b}{w_c + w_a} + \frac{w_c}{w_a + w_b} \le \frac{s}{2\sqrt{3}r}$$

Proposed by Adil Abdullayev-Azerbaijan

J.479 In ΔABC the following relationship holds:

$$\frac{s_a}{m_a + s_a} + \frac{s_b}{m_b + s_b} + \frac{s_c}{m_c + s_c} + \frac{r_a r_b r_c}{w_a w_b w_c} > \frac{5}{2}$$

Proposed by Adil Abdullayev-Azerbaijan

J.480 In ΔABC the following relationship holds:

$$\frac{3}{2}+\sum_{cyc}\frac{w_a}{w_b+w_c}\le\frac{3R}{2r}$$

Proposed by Adil Abdullayev-Azerbaijan

J.481 In ΔABC the following relationship holds:

$$\frac{2}{R}\le\frac{m_a}{m_bm_c}+\frac{m_b}{m_cm_a}+\frac{m_c}{m_am_b}\le\frac{1}{r}$$

Proposed by Adil Abdullayev-Azerbaijan

J.482 In ΔABC the following relationship holds:

$$\frac{m_a}{s_a}+\frac{m_b}{s_b}+\frac{m_c}{s_c}+\frac{8abc}{(a+b)(b+c)(c+a)}\ge 4$$

Proposed by Adil Abdullayev-Azerbaijan

J.483 In ΔABC the following relationship holds:

$$\left(\frac{m_am_b}{ab}\right)^2+\left(\frac{m_bm_c}{bc}\right)^2+\left(\frac{m_cm_a}{ca}\right)^2\ge\frac{27}{16}$$

Proposed by Adil Abdullayev-Azerbaijan

J.484 If $x,y,z>0$ then:

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+\frac{8xyz}{(x+y)(y+z)(z+x)}\ge 10$$

Proposed by Adil Abdullayev-Azerbaijan

J.485 In ΔABC the following relationship holds:

$$\sum_{cyc}cos(A-B)\le 2\sum_{cyc}\frac{ab}{a^2+b^2}$$

Proposed by Adil Abdullayev-Azerbaijan

J.486 In ΔABC the following relationship holds:

$$\frac{r_a^3}{cos^6\frac{A}{2}}+\frac{r_b^3}{cos^6\frac{B}{2}}+\frac{r_c^3}{cos^6\frac{C}{2}}\ge 24R^3$$

Proposed by Adil Abdullayev-Azerbaijan

J.487 In ΔABC the following relationship holds:

$$\frac{2r_ar_br_c}{w_aw_bw_c}\ge\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{1}{2}$$

Proposed by Adil Abdullayev-Azerbaijan

J.488 In ΔABC the following relationship holds:

$$cos\frac{A-B}{2}cos\frac{B-C}{2}cos\frac{C-A}{2}\le\frac{k^4}{8}+\frac{3k^2}{8}+\frac{1}{2},k=\frac{m_a+m_b+m_c}{r_a+r_b+r_c}$$

Proposed by Adil Abdullayev-Azerbaijan

J.489 In ΔABC the following relationship holds:

$$\frac{cosA\cdot cosB}{ab}+\frac{cosB\cdot cosC}{bc}+\frac{cosC\cdot cosA}{ca}=\frac{1}{4R^2}$$

Proposed by Ertan Yildirim-Turkey

J.490 In ΔABC the following relationship holds:

$$\frac{a^2}{tanB+tanC}+\frac{b^2}{tanC+tanA}+\frac{c^2}{tanA+tanB}=2F$$

Proposed by Ertan Yildirim-Turkey

J.491 In ΔABC the following relationship holds:

$$\frac{a+b+c}{R}\le\sum_{cyc}\sqrt{\frac{asinA+bsinB}{R}}\le 3\sqrt{3}$$

Proposed by Ertan Yildirim-Turkey

J.492 In ΔABC the following relationship holds:

$$\frac{1+cosA}{r_a}+\frac{1+cosB}{r_b}+\frac{1+cosC}{r_c}=\frac{2}{r}-\frac{1}{R}$$

Proposed by Ertan Yildirim-Turkey

J.493 In ΔABC the following relationship holds:

$$\frac{4}{3}\left(4+\frac{r}{R}\right)\le\sum_{cyc}\frac{r_a}{m_a}\left(\frac{b}{c}+\frac{c}{b}\right)\le\frac{4R}{r}-2$$

Proposed by Ertan Yildirim-Turkey

J.494 In ΔABC the following relationship holds:

$$\frac{r_b+r_c}{b+c-a}+\frac{r_c+r_a}{c+a-b}+\frac{r_a+r_b}{a+b-c}=\frac{F}{r^2}$$

Proposed by Ertan Yildirim-Turkey

J.495 Let ΔDEF be the orthic triangle of acute ΔABC. Prove that:

$$\sum_{cyc}\frac{EF}{AH}\cdot\sum_{cyc}\frac{EF}{BE\cdot CF}=\frac{1}{r}+\frac{1}{R}$$

Proposed by Ertan Yildirim-Turkey

J.496 In ΔABC the following relationship holds:

$$\frac{asinA}{m_a}+\frac{bsinB}{m_b}+\frac{csinC}{m_c}\ge\frac{6r}{R}$$

Proposed by Ertan Yildirim-Turkey

J.497 In ΔABC the following relationship holds:

$$\frac{cosA \cdot cosB}{ab} + \frac{cosB \cdot cosC}{bc} + \frac{cosC \cdot cosA}{ca} = \frac{1}{4R^2}$$

Proposed by Ertan Yildirim-Turkey

J.498 In ΔABC the following relationship holds:

$$\frac{1}{cotA - cot2A} + \frac{1}{cotB - cot2B} + \frac{1}{cotC - cot2C} = \frac{2F}{R^2}$$

Proposed by Ertan Yildirim-Turkey

J.499 In $\Delta ABC, I$ −incenter, K, L, M −intersection points of AI, BI, CI with incircle. Prove that:

$$AK^2 + BL^2 + CM^2 \geq 3r^2$$

Proposed by Eldeniz Hesenov-Georgia

J.500 In ΔABC, the following relationship holds:

$$\sum_{cyc} \frac{m_a w_a}{\sqrt{r_b + r_c}} \leq \frac{9R\sqrt{3R}}{4}$$

Proposed by Eldeniz Hesenov-Georgia

J.501 In $\Delta ABC, I$ −incenter, I_a, I_b, I_c −excenters, R_a, R_b, R_c −circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\sum_{cyc} \frac{[AIB]}{I_a I_b} = (R_a + R_b + R_c) sin\frac{A}{2} sin\frac{B}{2} sin\frac{C}{2}$$

Proposed by Eldeniz Hesenov-Georgia

J.502 In ΔABC, the following relationship holds:

$$\frac{w_a^2 + w_b^2 + w_c^2}{\left(\frac{1}{w_a}\right)^2 + \left(\frac{1}{w_b}\right)^2 + \left(\frac{1}{w_b}\right)^2} \leq \left(\frac{3R(R + r)}{2}\right)^2$$

Proposed by Eldeniz Hesenov-Georgia

J.503 In ΔABC, the following relationship holds:

$$\frac{m_a^2}{r_b^2 + r_c^2} + \frac{m_b^2}{r_c^2 + r_a^2} + \frac{m_c^2}{r_a^2 + r_b^2} \leq \frac{3}{2}$$

Proposed by Eldeniz Hesenov-Georgia

J.504 In $\Delta ABC, I$ −incenter, H −orthocenter, I_a, I_b, I_c −excenters, the following relationship holds:

$$\sum_{cyc} \left(\frac{AI \cdot AH \cdot AI_a}{h_a m_a}\right)^2 \leq \left(\frac{4R^2}{\sqrt{3}r}\right)^2 - (8r)^2$$

Proposed by Eldeniz Hesenov-Georgia

J.505 In any ΔABC the following relationship holds:

$$1.\ w_a^2 + w_b^2 + w_c^2 \geq g_a^2 + g_b^2 + g_c^2 + \frac{r}{6R}(R^2 - 4r^2)$$

$$2.\ \min\{h_a, w_b, g_c\} \leq \frac{4R+r}{3}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.506 In any triangle ABC the following relationship holds:

$$1.\ n_a^2 + n_b^2 + n_c^2 + \frac{3r}{R}(R^2 - 4r^2) \geq m_a^2 + m_b^2 + m_c^2$$

$$2.\ m_a^2 + m_b^2 + m_c^2 \geq p^2 + r(R - 2r)$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.507 In ΔABC the following relationship holds:

$$1.\ \frac{h_a}{h_b} + \frac{h_b}{h_c} + \frac{h_c}{h_a} \geq \frac{p^2 + r^2 + 4Rr}{6Rr}$$

$$2.\ \frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} \geq \frac{4R+r}{3r}$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.508 In any triangle ABC the following relationship holds:

$$w_a^2 + w_b^2 + w_c^2 \geq g_a^2 + g_b^2 + g_c^2 + \frac{2019r}{2020(R + 3r)}(R^2 - 4r^2)$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

J.509 In ΔABC the following relationship holds:

$$\frac{2}{R} \leq \frac{\cos A + \cos B}{r_a} + \frac{\cos C + \cos A}{r_b} + \frac{\cos B + \cos C}{r_c} < \frac{3}{2r}$$

Proposed by Radu Diaconu – Romania

J.510 In ΔABC the following relationship holds:

$$\min\left(\frac{\mu(A)\cos A}{g_a}, \frac{\mu(B)\cos B}{g_b}, \frac{\mu(C)\cos C}{g_c}\right) \leq \frac{\pi}{18r}$$

Proposed by Radu Diaconu – Romania

J.511 If in $\Delta ABC, \mu(A) = 90°, 2h_a^2 - 9ah_a + 2a^2 = 0$ then:

$$\frac{\pi^2}{6\sqrt{6}} \leq \left(\mu^2(B) + \mu^2(C)\right) \cdot \cos^2\left(\frac{B - C}{2}\right) \leq \frac{\pi^2}{4}$$

Proposed by Radu Diaconu – Romania

J.512 If in $\Delta ABC, a + b + c = 5R$ then:

$$5R^2 \leq a(s - m_a) + b(s - m_b) + c(s - m_c) < \frac{27R^2}{2}$$

Proposed by Radu Diaconu – Romania

J.513 In $\Delta ABC, m(\sphericalangle A) = 90°, g_a$ – Gergonne's cevian, the following relationship holds:

$$\frac{9r}{R^2} \le \left(\frac{g_a^2}{h_a} + \frac{g_b^2}{h_b} + \frac{g_c^2}{h_c}\right)\left(\frac{1}{b^2} + \frac{1}{c^2}\right) < \frac{23R^2}{48r^3}$$

Proposed by Radu Diaconu – Romania

J.514 In ΔABC the following relationship holds:

$$\left(1 + \frac{1}{a}\tan\frac{A}{2}\right)\left(1 + \frac{1}{b}\tan\frac{B}{2}\right)\left(1 + \frac{1}{c}\tan\frac{C}{2}\right) \ge \left(1 + \frac{9}{2}\cdot\frac{r}{s^2}\right)^3$$

Proposed by Florică Anastase – Romania

J.515 In ΔABC the following relationship holds:

$$\left(1 + \frac{s^2 + r^2\cot^2\frac{A}{2}}{\cot\frac{A}{2}}\right)\left(1 + \frac{s^2 + r^2\cot^2\frac{B}{2}}{\cot\frac{B}{2}}\right)\left(1 + \frac{s^2 + r^2\cot^2\frac{C}{2}}{\cot\frac{C}{2}}\right) \ge \left(1 + \frac{10rs}{3}\right)^3$$

Proposed by Florică Anastase – Romania

J.516 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{w_a^n}{h_a^n + m_a^n} \le \frac{3}{2}\cdot\left(\frac{R}{2r}\right)^{\frac{2n}{3}}, n \in \mathbb{N}$$

Proposed by Florică Anastase – Romania

J.517 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sqrt{s_a}}{\sqrt{h_b} + \sqrt{h_c}}(w_a^2 + m_b m_c) + \sum_{cyc} h_a^2 \ge \frac{4rs^2}{R}$$

Proposed by Florică Anastase – Romania

J.518 In ΔABC the following relationship holds:

$$\left(\sum_{cyc}\frac{1}{m_a m_b}\right)\left(4s + \sum_{cyc}\frac{bc}{a}\right) \ge \frac{54}{s}$$

Proposed by Florică Anastase – Romania

J.519 In ΔABC the following relationship holds:

$$\sqrt[3]{\prod_{cyc}\left(1 + \frac{1}{m_a} + \frac{1}{m_b}\right)} \ge \frac{\sum(a+2)\sin\frac{A}{2}}{\sum a\sin\frac{A}{2}} > 1 + \frac{2}{s}$$

Proposed by Florică Anastase – Romania

J.520 If in $\Delta ABC, m(\sphericalangle B) = 28°, m(\sphericalangle C) = 56°$ then:

$$27s(2b^2 + ba + a^2) + 54abc > 52s^3$$

Proposed by Daniel Sitaru,Elena Iacob Meda – Romania

J.521 The orthic triangle of obtuse ΔABC is similar with ΔABC. Prove that:

$$s^4+2r^2s^2+r^2(4R+r)^2=2R^2(a^2+b^2+c^2)$$

Proposed by Daniel Sitaru,Gilena Dobrică – Romania

J.522 If $a,b,c>0$ then:

$$27(a^4+b^4+c^4)^3(a^6+b^6+c^6)^2\geq(a^3+b^3+c^3)^8$$

Proposed by Daniel Sitaru,Mihaela Dăianu – Romania

J.523 Solve for real numbers:

$$\begin{cases} x,y,z\geq 1\\ xyz=3\\ \log_3^2 x^2\cdot\log_3(yz)+\log_3^2 y^2\cdot\log_3(zx)+\log_3^2 z^2\cdot\log_3(xy)=1\end{cases}$$

Proposed by Daniel Sitaru,Mihaela Pupăză – Romania

J.524 If $x,y,z>0$ then:

$$\sum_{cyc}\frac{xy+(x+y+z)(2x+2y+z)}{(x+y+z)(3x+3y+2z)}\leq 2$$

Proposed by Daniel Sitaru,Simona Radu – Romania

J.525 In ΔABC the following relationship holds:

$$\frac{s^4-2s^2(4Rr-r^2)+r^2(4R+r)^2}{a^2b^2c^2}\geq\frac{81(s^2-r^2-4Rr)}{8s^4}$$

Proposed by Daniel Sitaru,Mădălina Giurgescu – Romania

J.526 a,b,c,d – sides, r – inradii in a bicentric quadrilateral. Prove that:

$$\frac{a^4}{b}+\frac{b^4}{c}+\frac{c^4}{d}+\frac{d^4}{a}\geq 32r^3$$

Proposed by Daniel Sitaru,Ileana Duma – Romania

J.527 If $0<\alpha\leq x,y,z\leq\beta$ then:

$$\left(\sqrt[3]{xyz}+\frac{x+y+z}{3}+\sqrt{\frac{x^2+y^2+z^2}{3}}\right)\left(\frac{1}{\sqrt[3]{xyz}}+\frac{3}{x+y+z}+\sqrt{\frac{3}{x^2+y^2+z^2}}\right)\leq\frac{9(\alpha+\beta)^2}{4\alpha\beta}$$

Proposed by Daniel Sitaru,Dan Mitricoiu – Romania

J.528 If $0<\alpha\leq x,y,z,t\leq\beta$ then:

$$(m_h+m_g+m_a+m_q)\left(\frac{1}{m_h}+\frac{1}{m_g}+\frac{1}{m_a}+\frac{1}{m_q}\right)\leq\frac{4(\alpha+\beta)^2}{\alpha\beta}$$

$$m_h=\frac{4xyzt}{xyz+xyt+xzt+yzt};\ m_g=\sqrt[4]{xyzt}$$

$$m_a=\frac{x+y+z+t}{4};\ m_q=\sqrt{\frac{x^2+y^2+z^2+t^2}{4}}$$

Proposed by Daniel Sitaru,Tatiana Cristea – Romania

J.529 If in $\Delta ABC, \mu(A) = 2\mu(B)$ then:

$$\frac{2(b^3 + c^3 + ab^2 + abc)}{abc} + \frac{9s^2}{4(2b^2 + c^2 + bc)} > 33$$

Proposed by Daniel Sitaru,Anicuța Patricia Bețiu – Romania

J.530 If $a, b, c > 0, \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = 4$ then:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{9}{a + b + c} \geq 16$$

Proposed by Daniel Sitaru,Maria Lavinia Popa – Romania

J.531 In ΔABC the following relationship holds:

$$R^2 + \frac{4}{a^2b^2c^2}\left(\sum_{cyc} bc(s-a)^2\right)^2 + 11r^2 \geq 21Rr$$

Proposed by Daniel Sitaru,Sorin Pîrlea – Romania

J.532 Solve for real numbers:

$$\begin{cases} 0 < x, y < \frac{\pi}{2}, x + y = \frac{5\pi}{6} \\ 4\sin^2 x \sin^2 y\,(\sin^2 x + \sin^2 y) + 4\cos^2 x \cos^2 y\,(\cos^2 x + \cos^2 y) = \sin^2 2x + \sin^2 2y \end{cases}$$

Proposed by Daniel Sitaru,Dorina Goiceanu – Romania

J.533 In acute ΔABC the following relationship holds:

$$\frac{\sqrt{\sin 2A} + \sqrt{\sin 2B} + \sqrt{\sin 2C}}{\sqrt{\tan A} + \sqrt{\tan B} + \sqrt{\tan C}} \geq \frac{\sqrt{2r^2 + 4Rr - 2R^2}}{R}$$

Proposed by Marian Ursărescu – Romania

J.534 Let $\Delta A'B'C'$ be the circumcevian triangle of H – orthocenter in acute ΔABC.

Prove that:

$$\frac{[A'B'C']}{[ABC]} \geq 4\left(\left(\frac{r}{R}\right)^2 + \frac{2r}{R} - 1\right)$$

Proposed by Marian Ursărescu – Romania

J.535 $z_1, z_2, z_3 \in \mathbb{C}^*$ - different in pairs, $|z_1| = |z_2| = |z_3|, A(z_1), B(z_2), C(z_3)$

Prove that:

$$\sum_{cyc}(|z_1 - z_2|^2 + |z_1 - z_3|^2)\, z_2z_3 = 2\sum_{cyc}|z_2 - z_3|^2 z_2z_3 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

J.536 $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$, different in pairs,

$$|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3). \text{ If } z_1^2 + z_2z_3 - z_1z_2 - z_1z_3 = 0$$

then:

$$AB^2 + AC^2 = BC^2$$

Proposed by Marian Ursărescu – Romania

J.537 Let $\Delta A'B'C'$ be the circumcevian triangle of orthocenter in acute ΔABC. Prove that:

$$\frac{AA'}{BA' \cdot CA'} + \frac{BB'}{CB' \cdot AC'} + \frac{CC'}{AC \cdot BC'} \geq \frac{1}{R}\left(2 + \frac{R^2}{r^2}\right)$$

Proposed by Marian Ursărescu – Romania

J.538 In ΔABC the following relationship holds:

$$(s_a + s_b)c^3 + (s_b + s_c)a^3 + (s_c + s_a)b^3 \geq 16\sqrt{3}F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.539 If $x, y, z, t, u > 0, n \in \mathbb{N}^*$ then:

$$x^n \cdot t^{n+1} + uyz \cdot \sqrt[n]{u} \geq tu(n+1) \cdot \sqrt[n+1]{\left(\frac{xyz}{n}\right)^n}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.540 If $a, b, cm > 0, abc = 1$ then:

$$\frac{a^m b^{2m+1}}{a^{3m+1} + b + c} + \frac{b^m c^{2m+1}}{b^{3m+1} + c + a} + \frac{c^m a^{2m+1}}{c^{3m+1} + a + b} \geq 1$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.541 If $0 < x \leq y \leq z$ then:

$$\frac{x}{z + \sqrt{xy}} + \frac{y}{x + \sqrt{yz}} + \frac{z}{y + \sqrt{zx}} \geq \frac{3}{2}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.542 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{1}{h_a^2} \cdot \sum_{cyc} \frac{1}{(a+b)^2} \geq \frac{9}{16F^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.543 If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{x + 3y}{3x + y + 4z} \cdot a^2 + \frac{y + 3z}{3y + z + 4x} \cdot b^2 + \frac{z + 3y}{3z + x + 4y} \cdot c^2 \geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.544 If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{x+y}{z}a^2b^2+\frac{z}{x+y}c^4\right)^2\geq\frac{1600F^4}{3}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.545 In ΔABC the following relationship holds:

$$a^6\tan^2\frac{A}{2}+b^6\tan^2\frac{B}{2}+c^6\tan^2\frac{C}{2}\geq 64r^2F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.546 If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{y+z}{x}bc+\frac{x}{y+z}a^2\right)^2\geq 100F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.547 In any triangle ABC the following relationship holds:

$$1.\ \sqrt{\frac{m_am_b}{h_c}}+\sqrt{\frac{m_bm_c}{h_a}}+\sqrt{\frac{m_cm_a}{h_b}}\leq\frac{4R+r}{\sqrt{3r}}$$

$$2.\ n_a^2+n_b^2+n_c^2\geq\max\left\{m_a^2+m_b^2+m_c^2, w_a^2+w_b^2+w_c^2+\frac{r}{2R}(R^2-4r^2)\right\}+\frac{3r}{8R}\cdot\frac{h_ah_bh_c}{w_aw_bw_c}(R^2-4r^2)$$

Proposed by Nguyen Van Canh - Vietnam

J.548 In any ΔABC:

$$3\leq\frac{1}{2}\left(\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}\right)\leq\frac{b+c-a}{c+a-b}+\frac{c+a-b}{a+b-c}+\frac{a+b-c}{b+c-a}$$

When does equality holds?

Proposed by Nguyen Van Canh - Vietnam

J.549 In any ΔABC:

$$\cos^2\left(\frac{A-B}{2}\right)\cot\frac{C}{2}+\cos^2\left(\frac{B-C}{2}\right)\cot\frac{A}{2}+\cos^2\left(\frac{C-A}{2}\right)\cot\frac{B}{2}=\frac{p(p^2+r^2-6Rr)}{4rR^2}$$

Proposed by Nguyen Van Canh - Vietnam

J.550 If $a, b, c > 0$ then:

$$5(a+b+c)^6\geq 729\,abc(a^3+b^3+c^3+2abc)$$

Proposed by Nguyen Van Canh - Vietnam

J.551 In any triangle ABC:

$$2\left(\sin^3\frac{A}{2}+\sin^3\frac{B}{2}+\sin^3\frac{C}{2}\right)-\left(\sin^2\frac{A}{2}\sin\frac{B}{2}+\sin^2\frac{B}{2}\sin\frac{C}{2}+\sin^2\frac{C}{2}\sin\frac{A}{2}\right)<3$$

Proposed by Nguyen Van Canh - Vietnam

J.552 In any triangle ABC:

$$\frac{a}{12(b+c)+a}+\frac{b}{12(a+c)+b}+\frac{c}{12(a+b)+c}\geq\frac{3}{25}$$

When does equality holds?

Proposed by Nguyen Van Canh - Vietnam

J.553 In any triangle ABC the following relationship holds:

$$\frac{a^4+b^4+c^4}{ab+bc+ca}+\frac{3abc}{a+b+c}\geq\frac{8(m_a^2+m_b^2+m_c^2)}{9}$$

Proposed by Nguyen Van Canh - Vietnam

J.554 In any triangle ABC:

$$6\leq\frac{a}{\sqrt{m_a}}+\frac{b}{\sqrt{m_b}}+\frac{c}{\sqrt{m_c}}+\frac{\sqrt{m_a}}{a}+\frac{\sqrt{m_b}}{b}+\frac{\sqrt{m_c}}{c}\leq 3\sqrt{\frac{R}{r}}\left(\sqrt{R}+\frac{1}{2\sqrt{2r}}\right)$$

Proposed by Nguyen Van Canh - Vietnam

J.555 In triangle ABC the following relationship holds:

$$\frac{h_a+h_b+h_c}{a^3+b^3+c^3}\leq\frac{\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}}{ab^2+bc^2+ca^2}\leq\frac{1}{8\sqrt{3}r^2}$$

Proposed by Nguyen Van Canh - Vietnam

J.556 In any triangle ABC:

$$w_a^2+w_b^2+w_c^2+8r(R-2r)\geq p^2$$

Proposed by Nguyen Van Canh - Vietnam

J.557 In any ΔABC (acute), n_a – Nagel's cevian, the following relationship holds:

$$\frac{9}{4}\leq\frac{r}{2R}+\frac{R}{r}\leq\frac{n_a^2}{bc}+\frac{n_b^2}{ac}+\frac{n_c^2}{ab}<2\left(\frac{m_a^2+w_a^2}{bc}+\frac{m_b^2+w_b^2}{ac}+\frac{m_c^2+w_c^2}{ab}\right)$$

Proposed by Nguyen Van Canh - Vietnam

J.558 In any ΔABC the following relationship holds:

$$\frac{m_aw_a+m_bw_b+m_cw_c}{\sqrt{m_ar_a}+\sqrt{m_br_b}+\sqrt{m_cr_c}}\leq\frac{(4R+r)^2}{9r}$$

Proposed by Nguyen Van Canh - Vietnam

J.559 If $a,b,c>0$ then:

$$3\left(\sqrt[5]{a}+\sqrt[5]{b}+\sqrt[5]{c}+1\right)+a^2+b^2+c^2\geq 5(ab+bc+ca)$$

Proposed by Nguyen Van Canh - Vietnam

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.107 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{2^{n-1}\cdot(n-1)!}\prod_{k=2}^{n}\left(k\cdot(k+1)^{\frac{1}{k}}+(k-1)\cdot k^{\frac{1}{1-k}}\right)\right)$$

Proposed by Daniel Sitaru,Iulia Sanda – Romania

S.108 $A,B\in M_n(\mathbb{R}),\left(4\sqrt{10+2\sqrt{5}}\right)(A^2+B^2)=\left(\sqrt{5}-1\right)(AB-BA)$

If $\det(AB-BA)\neq 0$, then n is divisible with 20.

Proposed by Marian Ursărescu – Romania

S.109 $p\geq 2, p\in\mathbb{N}, 0<x_1<1, x_{n+1}=x_n(1-x_n)^p, y_{n+1}=y_n+\frac{1}{y_n^p}, n\geq 1$

Find:

$$\Omega=\lim_{p\to\infty}\left(\lim_{n\to\infty}x_n\cdot y_n^p\right)^p$$

Proposed by Marian Ursărescu – Romania

S.110 $x_0\in(0,1), y_0>0, x_{n+1}=\frac{2}{\pi}x_n\cdot\cos^{-1}x_n, y_{n+1}=y_n+\frac{1}{y_n^{p-1}}, p\geq 2, p\in\mathbb{N}$

Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[p]{x_n}\cdot y_n\right)$$

Proposed by Marian Ursărescu – Romania

S.111 If $A\in M_5(\mathbb{R}), \det A\neq 0, Tr(A^{-1})^*=1$ then:

$$\det(A^2+I_5)\geq(Tr\,A^*-1)^2$$

Proposed by Marian Ursărescu – Romania

S.112 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^2}\cdot\left(\sum_{i+j+k+l=n}ijkl\right)\cdot\left(\sum_{i+j+k=n}ijk\right)^{-1}$$

Proposed by Daniel Sitaru,Nicolae Radu – Romania

S.113 Prove that for any acute triangle ABC the following inequality holds:

$$\frac{\sqrt{\tan A}+\sqrt{\tan B}+\sqrt{\tan C}}{\tan A+\tan B+\tan C}\leq\frac{1}{\sqrt[4]{3}}$$

Proposed by Vasile Mircea Popa – Romania

S.114 Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{n^6+k^2}\cos\frac{2k\pi}{n}$$

Proposed by Vasile Mircea Popa – Romania

S.115 Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\sum_{i=1}^{k} i(i+1)(i+2)^2\right)^{-1}$$

Proposed by Vasile Mircea Popa – Romania

S.116 In acute ΔABC the following relationship holds:

$$\frac{\sqrt{\cot A}+\sqrt{\cot B}+\sqrt{\cot C}}{\cot A+\cot B+\cot C}\le\sqrt[4]{3}$$

Proposed by Vasile Mircea Popa – Romania

S.117 Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\left(\sum_{k=1}^{n}k^3\binom{n}{k}^2\right)\left(\sum_{k=1}^{n}k^2\binom{n}{k}^2\right)^{-1}}$$

Proposed by Daniel Sitaru,Mihaela Stăncele – Romania

S.118 Prove without softs:

$$2020^{2019}\cdot 2016^{4048} > 2018\cdot 2019\cdot 2017^{8065}$$

Proposed by Daniel Sitaru,Daniela Stoian – Romania

S.119 Find:

$$\Omega = \lim_{n\to\infty}\left((1-i)\prod_{k=1}^{n}\frac{k^2+k+1+i}{\sqrt{(k^2+1)(k^2+2k+2)}}\right)$$

Proposed by Daniel Sitaru,Gigi Zaharia – Romania

S.120 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{8^n}{n(2n+1)^2}\cdot\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)\cdot\prod_{k=1}^{n}\sin^2\left(\frac{k\pi}{2n+1}\right)\cdot\prod_{k=n+1}^{2n}\sin\left(\frac{k\pi}{2n+1}\right)\right)$$

Proposed by Daniel Sitaru,Alecu Orlando – Romania

S.121 Find:

$$\Omega = \lim_{n\to\infty}\left(n-\sum_{k=1}^{n}\sqrt[n]{10k-9}\right)\left(n-\sum_{k=1}^{n}\sqrt[n]{10k-4}\right)^{-1}$$

Proposed by Daniel Sitaru,Ileana Stanciu – Romania

S.122 Solve for natural numbers:

$$\begin{cases} \left(\frac{x+y}{2}\right)^{\frac{1}{xy}} \cdot \left(\frac{z+x}{2}\right)^{\frac{1}{xy}} = x^{\frac{1}{x(x+y)}} \cdot y^{\frac{1}{y(y+z)}} \\ \left(\frac{y+z}{2}\right)^{\frac{1}{yz}} \cdot \left(\frac{x+y}{2}\right)^{\frac{1}{xy}} = y^{\frac{1}{y(y+z)}} \cdot z^{\frac{1}{z(z+x)}} \\ \left(\frac{z+x}{2}\right)^{\frac{1}{xy}} \cdot \left(\frac{y+z}{2}\right)^{\frac{1}{yz}} = z^{\frac{1}{z(z+x)}} \cdot x^{\frac{1}{x(x+y)}} \\ (x+1)(y+2)(z+3) = 336 \end{cases}$$

Proposed by Daniel Sitaru,Cristian Moanță – Romania

S.123 In acute ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\tan A}{\sqrt[3]{\tan A + \tan B + \tan C}} \left(1 + \frac{\tan B}{\sqrt[3]{\tan A + \tan B + \tan C}}\right) \geq 6$$

Proposed by Daniel Sitaru,Lucian Tuțescu – Romania

S.124 Find:

$$\Omega = \lim_{n\to\infty} \frac{\sqrt{(1+n!)^{n!}}}{n \cdot (n!)!}$$

Proposed by Daniel Sitaru,Claudiu Coandă – Romania

S.125 In $\Delta ABC, M_n \in (BC), N_n \in (CA), P_n \in (AB), n \in \mathbb{N}, n \geq 2$

$\frac{BM_n}{CM_n} = \frac{CN_n}{AN_n} = \frac{AP_n}{BP_n} = n$. Find:

$$\Omega = \frac{1}{a^2+b^2+c^2} \cdot \lim_{n\to\infty} \left(AM_n^2 + BN_n^2 + CP_n^2\right)$$

Proposed by Daniel Sitaru – Romania

S.126 Let $m \in \mathbb{R}_+ = [0,\infty); u, v \in \mathbb{R}_+^* = (0,\infty)$ and $x, y, z \in \left(0, \frac{\pi}{2}\right)$, then:

$$\left(\sum_{cyc} \frac{\sin^{m+1} x}{(u\cos y + v\cos z)^m}\right)\left(\sum_{cyc} \frac{\cos^{m+1} x}{(u\sin y + v\sin z)^m}\right) \geq$$
$$\geq \frac{1}{(u+v)^{2m}} \left(\sum_{cyc} \sqrt{\sin x \cdot \cos x}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu,Dan Nănuți– Romania

S.127 Prove:

$$4tan^{-1}\left(\frac{1}{2}\right) + 2tan^{-1}\left(\frac{1}{3}\right) + tan^{-1}\left(\frac{1}{4}\right) < \frac{11}{4}$$

Proposed by Jalil Hajimir-Canada

S.128 If $0 < a, b, c < \frac{\pi}{3}$ then:

$$\sum_{cyc} a \cdot \csc(\csc(\csc(a))) \geq (a+b+c) \cdot \csc\left(\csc\left(\csc\left(\frac{a^2+b^2+c^2}{a+b+c}\right)\right)\right)$$

Proposed by Daniel Sitaru – Romania

S.129 If $x, y, z > 0, x + y + z = 3s$ then ΔABC holds:

$$\frac{x^2}{(m_a + m_b + m_c)^2} + \frac{y^2 + z^2}{(w_a + w_b + w_c)^2} \geq 1$$

Proposed by Daniel Sitaru – Romania

S.130 If $x, y, z \geq 0$ then:

$$\sqrt[18]{\prod_{cyc}(1+x^2)} \cdot \sum_{cyc} \cos\left(\frac{\tan^{-1} x}{8}\right) \geq 3$$

Proposed by Daniel Sitaru – Romania

S.131 Prove without any software:

$$\int_{\frac{1}{e}}^{e} \frac{e^{x^2}}{x^2} dx > 2\int_0^1 \cosh x \cdot e^{\operatorname{sech}^2 x} dx$$

Proposed by Daniel Sitaru – Romania

S.132 If $0 < a \leq b$ then:

$$25(e^{4b} - e^{4a})(e^{6b} - e^{6a}) \leq 6\left(\sqrt{e^{5(a-b)}} + \sqrt{e^{5(b-a)}}\right)(e^{5b} - e^{5a})^2$$

Proposed by Daniel Sitaru – Romania

S.133 In ΔABC the following relationship holds:

$$\begin{vmatrix} 9R^2 & a^2 & b^2 & c^2 \\ a^2 & 9R^2 & c^2 & b^2 \\ b^2 & c^2 & 9R^2 & a^2 \\ c^2 & b^2 & a^2 & 9R^2 \end{vmatrix} > 0$$

Proposed by Daniel Sitaru – Romania

S.134 If $a, b, c, d, e, f, g, h, i > 0$,

$$a^2 + b^2 + c^2 = d^2 + e^2 + f^2 = g^2 + h^2 + i^2 = \sqrt[3]{2} \text{ then:}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \cdot \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \leq 2$$

Proposed by Daniel Sitaru – Romania

S.135 If $A,B\in M_3(\mathbb{R}), det((AB-BA)^2+AB-BA+I_3)=0$ then find:

$$\Omega=det(AB-BA)$$

Proposed by Marian Ursărescu-Romania

S.136 $x_0\in(0,1), y_0>0, x_{n+1}=x_n\cdot\sqrt[2020]{1-x_n}, y_{n+1}=y_n+\frac{1}{y_n^{2019}}, n\in\mathbb{N}$.

$$\text{Find: }\Omega=\lim_{n\to\infty}\left(y_n\cdot\sqrt[2020]{x_n}\right)$$

Proposed by Marian Ursărescu-Romania

S.137 If $A\in M_4(\mathbb{R}), detA=-1$ then: $det(A^2+I_4)\ge(TrA^{-1})^2$.

Proposed by Marian Ursărescu-Romania

S.138 Find:

$$\lim_{n\to\infty}\left(\lim_{x\to0}\frac{2\sqrt[3]{1+x}+2\sqrt[3]{8+x}+\cdots+2\sqrt[3]{n^3+x}-n(n+1)}{x}\right)$$

Proposed by Marin Chirciu-Romania

S.139 Solve for real numbers:

$$2\cdot\frac{1+x^2}{1-x^2}=\sqrt{1+x}+\sqrt{1-x}$$

Proposed by Marin Chirciu-Romania

S.140 $k,n\in\mathbb{N}^*$ fixed. Solve for real numbers:

$$x^{2k}-2x^k+3=\sqrt[2n]{x}+\sqrt[2n]{2-x}$$

Proposed by Marin Chirciu-Romania

S.141 Find the number $\alpha\in\left[\frac{\pi}{2},\infty\right)$ so that the double inequality:

$\left(\frac{R}{r}\right)^\alpha\ge\frac{\alpha R}{r}+\left(\frac{b}{c}\right)^\alpha+\left(\frac{c}{b}\right)^\alpha\ge\left(\frac{b}{c}+\frac{c}{b}\right)^\alpha$ holds in any ABC triangle.

Proposed by Alex Szoros-Romania

S.142 Find the largest value of $\lambda>0$ such that the inequality

$\left(\frac{R}{r}\right)^3\ge\frac{\lambda R}{r}+\frac{1}{\lambda}\left(\frac{a^3+b^3}{c^3}+\frac{b^3+c^3}{a^3}+\frac{c^3+a^3}{b^3}\right)$ is true in any ΔABC.

Proposed by Alex Szoros-Romania

S.143 If the equation $\frac{5x^3+x+\phi}{5x^3-\sqrt{5}x}+\frac{\sqrt{5}x^3+x+\phi}{\sqrt{5}x^3-5x}=0$ has roots $\alpha,\beta,\gamma,\delta,\epsilon$ then prove:

$$\alpha^3+\beta^3+\gamma^3+\delta^3+\epsilon^3+3\alpha\beta\gamma\delta\epsilon=0$$

ϕ –Golden Ratio.

Proposed by Srinivasa Raghava-AIRMC-India

S.144

$$f(n)=\frac{2}{n}\sqrt{\sum_{1\le i<j\le n}\cos^2\left(\frac{(i-j)\pi}{n}\right)},\ g(n)=min\left(\sum_{\substack{0\le i,j\le 2n\\ i+j=odd}}\frac{sin^i\alpha\cdot cos^j\alpha}{sin\alpha+cos\alpha}\right)$$

Find:

$$\Omega=\lim_{n\to\infty}\left(f(n)-g(n)\right),\alpha\in\left(0,\frac{\pi}{2}\right),\alpha-\text{fixed}$$

Proposed by Rajeev Rastogi-India

S.145 $a_n=\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}},\ b_n=\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}, n$ –radicals. Find:

$$\Omega=\lim_{n\to\infty}\left(1+\frac{a_n}{2}\right)^{\frac{1}{3^n b_n}}$$

Proposed by Rajeev Rastogi-India

S.146 Let α be the largest real root of the equation $log^2x-[logx]-6=0, [*]-GIF$.

Find:

$$\Omega=\lim_{x\to log\alpha}\left(4sin\left(log\frac{x}{3}\right)-(x-logx)^{\frac{2019}{2}}\cdot cos\left(\frac{1}{x-logx}\right)\right)^{x-logx}$$

Proposed by Rajeev Rastogi-India

S.147 $x^2+2y-2=0, y^2+6z+17=0, z^2-4x-1=0. t=x+y+z$. Find:

$$\Omega=\lim_{x\to-1}\left(\lim_{n\to\infty}\left(\frac{tan(\pi x^{2020})+\frac{(x-t)^n sin(\pi x^{2019})}{1-x^{2019}}}{1+(x-t)^n+x^{2020}}\right)\right)$$

Proposed by Rajeev Rastogi-India

S.148 $(x^ne^x)^{(p)}=e^x\sum_{k=0}^{p}u_{t-k}x^{n-k}, (*)^p-p^{th}$ derivative. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{2020\cdot u_{t-2020}}{n\cdot(p-2019)\cdot u_{t-2019}}\right)^n$$

Proposed by Costel Florea-Romania

S.149 $a_1=1, a_{n+1}=\frac{a_n}{1+2a_n}, n\ge 1$. Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k^2(k+1)^2a_{k+1}}\right)^{\pi n^2}$$

Proposed by Costel Florea-Romania

S.150 Solve for real numbers:

$$\sum_{k=1}^{n}\frac{x^4+2x^3+(2k+3)x^2+k^2+k+1}{x^4+2x^3+(2k+2)x^2+(2k+1)x+k^2+k}=n,n\in\mathbb{N},n\geq 1$$

Proposed by Costel Florea-Romania

S.151 Solve for natural numbers:

$$(2n-3)\cdot\prod_{k=3}^{n}\frac{8k^3-12k^2-26k+15}{8k^3+12k^2-26k-15}=10n-65$$

Proposed by Costel Florea-Romania

S.152 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{\sum_{k=1}^{n}k^2\cdot 2^k}{n(\sum_{k=1}^{n}k\cdot 2^k-2)}\right)^n$$

Proposed by Costel Florea-Romania

S.153 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{32}{5}\lim_{x\to 0}\frac{1-cos\frac{x\sqrt{2}}{1\cdot 3}\cdot cos\frac{x\sqrt{3}}{2\cdot 4}\cdot\ldots\cdot cos\frac{x\sqrt{n+1}}{n(n+1)}}{x^2}\right)^{\frac{5}{8}n^2}$$

Proposed by Costel Florea-Romania

S.154 Find:

$$\Omega_1=\lim_{x\to 0}\frac{\prod_{k=4}^{n}sin\left(\frac{k^3+6k^2+11k+6}{k^3-6k^2+11k-6}\cdot x\right)}{x^{n-3}}$$

$$\Omega=\lim_{n\to\infty}\left((12n^2-36n+24)\cdot A_{n-1}^{8}\cdot A_6^5\cdot A_5^4\cdot\Omega_1(n)\right)^n$$

Proposed by Costel Florea-Romania

S.155 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{3n^2}\prod_{k=3}^{n}\frac{k^3-k^2-4}{k^3+k^2+4}\right)^{\frac{7n}{2}\prod_{k=3}^{n}\frac{k^3-8}{k^3+8}}$$

Proposed by Costel Florea-Romania

S.156 $\Omega_1(n)=\lim_{x\to 0}\left(x\cdot\frac{\sum_{k=1}^{n}sin(k(k+1)x)}{\sum_{k=1}^{n}tan((sin^{-1}(kx)^2))}\right)$. Find:

$$\Omega=\lim_{n\to\infty}\sqrt[3]{(\Omega_1(n))^{2n}}$$

Proposed by Costel Florea-Romania

S.157

$$\Omega=\sum_{i=1}^{n}\int_{k}^{k+1}\frac{dx}{x^2+2(2i-1)x+4i(i-1)}$$

Find sum of roots of the equation: $\left|k^2\lim_{k\to\infty}\Omega\right|=2020$

Proposed by Costel Florea-Romania

S.158 Find:

$$\Omega=\lim_{x\to 1}\left(cosx+\frac{1}{2!}-\frac{1}{4!}+\frac{1}{6!}-\frac{1}{8!}+\cdots\right)^{\frac{1}{x-1}}$$

Proposed by Adil Abdullayev-Azerbaijan

S.159 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $\forall n\in\mathbb{Z}$, $[*]$ –GIF:

$$f(f(n))+(n^2+1)!+\left[\sqrt[3]{n}\right]=f(n)+4(f^2(n)+1)!+4\left[\sqrt[3]{f(n)}\right]+3$$

Proposed by Adil Abdullayev-Azerbaijan

S.160 Find all positive real numbers α such that:

$$\cos(2020\alpha x)\ge 1-\frac{x^2}{2},\forall x\ge 0$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.161 Find all functions $\varphi:[-2020,2020]\to\mathbb{R}$ such that:

$$\left(\varphi(x-y)\right)^3=5\varphi(x+2y)-x^3y^3,\forall x\in[-2020;2020]$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.162 Find all positive real numbers α,β such that:

$$(1+e^x)^\alpha\ge 1+\beta x,\forall x\ge 0$$

Proposed by Nguyen Van Canh – BenTre – Vietnam

S.163 In acute ΔABC holds:

$$\min\left(\frac{\sqrt{\mu(A)}}{\sin A},\frac{\sqrt{\mu(B)}}{\sin B},\frac{\sqrt{\mu(C)}}{\sin C}\right)\le\frac{2s\sqrt{3\pi}}{27r}$$

Proposed by Radu Diaconu – Romania

S.164 In acute ΔABC holds:

$$\mu(A)\mu(B)\sin A+\mu(C)\mu(A)\sin B+\mu(B)\mu(C)\sin C>\frac{s(s^2+r^2+4Rr)}{12R^3}$$

Proposed by Radu Diaconu – Romania

S.165 If in $\Delta ABC,\mu(A)=90°$ then:

$$\sqrt{\prod_{cyc}\frac{g_a^2+g_b^2}{\mu^2(A)+\mu^2(B)}}\ge\frac{54R^3\sin^2 2B}{\pi^3}$$

Proposed by Radu Diaconu – Romania

S.166 In acute $\Delta ABC, r_a>1, I$ – incenter, holds:

$$\frac{4r\sqrt{6}}{9R}<\sum_{cyc}\frac{AI}{r_a\sqrt{\cos A}}<\frac{3r}{\min\left\{\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}\right\}\cdot\min\left\{\sqrt{\cos A},\sqrt{\cos B},\sqrt{\cos C}\right\}}$$

Proposed by Radu Diaconu – Romania

S.167 If in ΔABC holds:

$$\frac{r_a \cdot \mu(A)}{\mu(B) + \mu(C)} + \frac{r_b \cdot \mu(B)}{\mu(C) + \mu(A)} + \frac{r_c \cdot \mu(C)}{\mu(A) + \mu(B)} = \frac{9r}{2}$$

then ΔABC is an equilateral one.

Proposed by Radu Diaconu – Romania

S.168 If in $\Delta ABC, b^4 + c^4 \le 32, b + c \ge 4, \sin^2 A = \sin B \sin C$, then ΔABC is an equilateral one.

Proposed by Radu Diaconu – Romania

S.169 In ΔABC the following relationship holds:

$$\frac{2\pi}{3R} \le \frac{\mu(A)}{g_a} + \frac{\mu(B)}{g_b} + \frac{\mu(C)}{g_c} < \frac{\pi}{2r}$$

Proposed by Radu Diaconu – Romania

S.170 In acute ΔABC the following relationship holds:

$$\sqrt{bc} \cdot \sin^2 A + \sqrt{cas} \cdot \sin^2 B + \sqrt{ab} \cdot \sin^2 C < \sqrt{3}\pi R$$

Proposed by Radu Diaconu – Romania

S.171 Find a closed form:

$$\Omega(a,b) = \int_a^b \log\left(\frac{\left(\frac{x}{a}\right)^{\frac{1}{x}\arctan\frac{b}{x}}}{\left(\frac{b}{x}\right)^{\frac{1}{x}\arctan\frac{x}{a}}}\right)dx, 1 < a < b$$

Proposed by Daniel Sitaru – Romania

S.172 Find a closed form:

$$\Omega = \int_2^3 \frac{\sin^{-1}(2x-5) \cdot \log x}{x}dx + 2\int_2^3 \frac{\tan^{-1}\left(\sqrt{\frac{3-x}{x-2}}\right) \cdot \log x}{x}dx$$

Proposed by Daniel Sitaru – Romania

S.173 Prove without any software:

$$\int_0^1 \frac{x^3}{1 + x^3 + x^{12}}dx + \int_0^1 \frac{x^5}{1 + x^5 + x^{10}}dx + \int_0^1 \frac{x^7}{1 + x^7 + x^8}dx < 1$$

Proposed by Daniel Sitaru – Romania

S.174 If $0 \le a < 1$ then:

$$\int_0^a \frac{\cos x}{(1 + x\sin x)\sqrt{1-x^2}}dx + \int_0^a \frac{\sqrt{1-x^2}}{1 + x\sin x}dx \ge \sin^{-1}(2\sin a)$$

Proposed by Daniel Sitaru – Romania

S.175 If $0 < a \leq b$ then:

$$\left(\int_a^b e^{x^2}\,dx\right)^4 \leq \left(\int_a^b x^3 e^{x^2}\,dx\right)\left(\int_a^b \frac{e^{x^2}}{x}\,dx\right)^3$$

Proposed by Daniel Sitaru – Romania

S.176 Find:

$$\Omega = \lim_{n\to\infty}\left(n\int_0^{\frac{1}{n}} \frac{x^2}{(\cos x + x\sin x)^2}\,dx\right)$$

Proposed by Daniel Sitaru – Romania

S.177 Solve for real numbers:

$$\int_x^{2x} \log(1+\tan 3x\cdot\tan t)\,dt\cdot\int_{2x}^{3x}\log(1+\tan 5x\cdot\tan t)\,dt = 0$$

Proposed by Daniel Sitaru – Romania

S.178 Find without any software:

$$\Omega = \int_{15\cos(2\arctan 3)}^{20\sin(\arctan 3)} \frac{\sin^3 x + \sin^5 x}{1+\cos^2 x + \cos^4 x}\,dx$$

Proposed by Daniel Sitaru – Romania

S.179 Find without any software:

$$\Omega = \int \frac{(3 - 3\sinh^2 x - \sinh^4 x + \sinh^{10} x)\cosh x}{(\cosh^2 x - 2)^2}\,dx$$

Proposed by Daniel Sitaru – Romania

S.180 Find:

$$\int_1^5 \frac{x + sin(x-1)(x-2)(x-3)(x-4)(x-5)}{2x^2 - 12x + 26}\,dx$$

Proposed by Jalil Hajimir-Canada

S.181 Prove without any software:

$$0.75 < \int_0^{\frac{\pi}{2}} \frac{|log(sinx)|}{sinx + cosx}\,dx < 1.1$$

Proposed by Jalil Hajimir-Canada

S.182 Find without any software:

$$\Omega = \int_0^{\frac{\pi}{2}} \frac{sinx}{sin^3x + cos^3x} dx$$

Proposed by Jalil Hajimir-Canada

S.183 Find without any software:

$$\Omega = \int \sqrt{\frac{2^x - 1}{2^x + 1}} dx$$

Proposed by Jalil Hajimir-Canada

S.184 Prove without any software:

$$\frac{1}{2} < \frac{1}{e-1}\int_1^e logxdx < log\left(\frac{1+e}{2}\right)$$

Proposed by Jalil Hajimir-Canada

S.185 If $0 \leq x < \frac{3\pi}{2}$ then:

$$\int_0^x \frac{\sin 2t \cdot e^{\sin t}}{(1+\sin t)^2} dt \geq \frac{2\sin x}{1+x}$$

Proposed by Daniel Sitaru – Romania

S.186 Find without any software:

$$\Omega = \int \frac{\left(\tan 2x + \tan\left(2x + \frac{\pi}{3}\right) + \tan\left(2x + \frac{2\pi}{3}\right)\right)\cos^2(6x)}{\tan x + \tan\left(x + \frac{\pi}{3}\right) + \tan\left(x + \frac{2\pi}{3}\right)} dx$$

Proposed by Daniel Sitaru – Romania

S.187 Prove without any software:

$$20\int_0^1 (x+1)(x+2)\cdot \ldots \cdot (x+19)dx < \sum_{k=0}^{19} 10^k \cdot 11^{19-k}$$

Proposed by Daniel Sitaru – Romania

S.188 Prove that:

$$\frac{x}{2}\left(1 + e^{-x^2}\right) < \int_0^x e^{-t^2} dt < \frac{1}{\sqrt{2}}, \forall x \in \left(0, \frac{1}{\sqrt{2}}\right)$$

Proposed by Rajeev Rastogi-India

S.189 Find:

$$\Omega = \lim_{n\to\infty} \int_0^1 \frac{n \cdot x^n}{2 + x^n} dx$$

Proposed by Rajeev Rastogi-India

S.190 Prove without softs:

$$\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 + sin^{2020}x}} < \frac{\pi}{6}$$

Proposed by Rajeev Rastogi-India

S.191 If $x_i, i \in \{1, . . ,4\}$ are roots of the equation $x^4 - E_1x^3 + E_2x^2 - E_3x + E_4 = 0$, with $x_1 < x_2 < x_3 < x_4$ and $E_1 = 2(2n + 2k + 3)$,

$$E_2 = 6n^2 + 6(2k + 3)n + 6k^2 + 18k + 11$$

$$E_3 = 4n^3 + 6(2k + 3)n^2 + 2(6k^2 + 18k + 11)n + 4k^3 + 18k^2 + 22k + 6$$

$$E_4 = n^4 + 2(2k + 3)n^3 + (6k^2 + 18k + 11)n^2 + 2(2k^3 + 9k^2 + 11k + 3)n + k^4 + 6k^3 + 11k^2 + 6k, n, k \in N$$

Prove that: $\Omega = x_4^2 - x_3^2 - x_2^2 + x_1^2$ is perfect square.

Proposed by Costel Florea-Romania

S.192 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{2n \cdot \int_0^{log2}(x^2e^x)^{(n+1)}dx \cdot \int_0^2\left(\frac{x^2}{x-1}\right)^{(n+1)}dx}{\int_0^2\left(\frac{x^2+1}{x-1}\right)^{(n+1)}dx \cdot \int_0^{log2}(x^3e^x)^{(n+1)}dx}\right)^n$$

$(*)^{\mathrm{n}} - \mathrm{n}^{\mathrm{th}}$ derivative.

Proposed by Costel Florea-Romania

S.193

$$a = \int_{k+1}^{k+2} \frac{x + 1}{x^4 + 4x^3 + 6x^2 + 4x}dx + \int_{k+1}^{k+2} \frac{x + 1}{x^4 + 4x^3 + 10x^2 + 12x + 8}dx + \cdots +$$

$$+ \int_{k+1}^{k+2} \frac{x + 1}{x^4 + 4x^3 + (4n + 2)x^2 + (8n - 4)x + 4n^2 - 4n}dx$$

$$\Omega = \lim_{x\to\infty}(ax^3)$$

Solve for natural numbers: $\binom{\Omega}{6} = \binom{n}{2014}$

Proposed by Costel Florea-Romania

S.194

$$\Omega(n) = \sum_{k=0}^{n}\int_0^1 \frac{x^2 + (2k + 7)x + k^2 + 7k + 14}{x^4 + 2(2k + 7)x^3 + 3(2k^2 + 14k + 25)x^2 + 2(2k^3 + 21k^2 + 75k + 91)x + k^4 + 14k^3 + 75k^2 + 182k + 168}dx$$

Prove that product of roots of the equation: $|\Omega(n)| = 1$ is positive number.

Proposed by Costel Florea-Romania

S.195 Find:

$$\Omega = \lim_{n\to\infty} \int_0^1 \frac{sin(nx)dx}{sin(nx) + cos(nx)}$$

Proposed by Costel Florea-Romania

S.196 Find:

$$\Omega = \lim_{n\to\infty} \left((2n-1) \int_1^2 \frac{dx}{x + x^{n+1}} + (3n-1) \int_1^2 \frac{dx}{x + x^{n+2}} \right)$$

Proposed by Costel Florea-Romania

S.197 Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{3\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\int_{i+2}^{i+3}\frac{dx}{x^2-(1+2i)x+i^2+i}\right)}{(1+n^3)log^2\frac{4}{3}} \right)^{\frac{2n\pi}{3}}$$

Proposed by Costel Florea-Romania

S.198 Find:

$$\Omega = \lim_{n\to\infty} \left(4 \cdot \lim_{x\to 0} \frac{\sum_{k=1}^{\infty} sin^2\left(\frac{x\sqrt{2k+3}}{k^2+3k+2}\right)}{\sum_{k=1}^{n} x^k} \right)^{\frac{\pi n^2}{4}}$$

Proposed by Costel Florea-Romania

S.199

$$\text{If } a = \int_1^e \left(\sum_{k=0}^{n} (k+1)^2 x^k \right) logx\, dx, \text{find:}$$

$$\Omega = \lim_{n\to\infty} \frac{a}{ne^{n+2}}$$

Proposed by Costel Florea-Romania

S.200 Prove that:

$$\lim_{n\to\infty} \sum_{k=0}^{n} tan^{-1}\left(\frac{n}{n^2+k^2+k}\right) = \frac{\pi}{4}$$

Proposed by Mohammed Bouras-Morocco

S.201 Find:

$$\Omega = \int_0^1 \sqrt{x} \cdot \sqrt{1-\sqrt{x}} \cdot \sqrt{1-\sqrt{1-\sqrt{x}}}dx$$

Proposed by Mohammed Bouras-Morocco

S.202 Find:

$$\Omega=\lim_{n\to\infty} n^3\cdot\int_0^1\left(\frac{1}{n+cos(\pi x)}+\frac{1}{n}sin(nlogx)\right)dx$$

Proposed by Mohammed Bouras-Morocco

S.203

$$\phi_n=\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n-times},a_n=\frac{2m+1}{2(m^2+m)}$$

Prove that:

$$f_{m,n}=\int_{\frac{1}{2}\phi_n}^1\left(\frac{1}{m+1-x^2}-\frac{1}{m+x^2}\right)\left(\frac{1}{m+1-x^2}+\frac{1}{m+x^2}-a_m\right)\frac{dx}{\sqrt{1-x^2}}$$

$$f_{m,n}=8a_m^2\cdot\frac{\sqrt{2-\phi_{n-2}}}{(4m+2)^2-2-\phi_{n-2}}$$

Proposed by Mohammed Bouras-Morocco

S.204

$$\phi_+(n)=\sqrt{n+\sqrt{(n-1)+\cdots+\sqrt{3+\sqrt{2+\sqrt{1}}}}},\phi_+(n)=\sqrt{n-\sqrt{(n-1)-\cdots-\sqrt{3-\sqrt{2-\sqrt{1}}}}}$$

Prove that:

$$\begin{cases}\phi_+(n^2+n)+\phi_-(n^2+n)\le 2n+1; n\in\mathbb{N}\\ \phi_+(n^2)+\phi_-(n^2)\ge 2n\end{cases}$$

Proposed by Mohammed Bouras-Morocco

S.205 Find without softs:

$$\Omega=\int_0^{\frac{\pi}{2}}\frac{\pi x+\sin^4 x-\cos^4 x}{\sin^4 x+\cos^4 x}dx$$

Proposed by Radu Diaconu – Romania

S.206 If $(a_n)_{n\ge 0}$ is sequence of real numbers such that $a_0=a_1=0$ and

$a_{n+2}-2a_{n+1}+a_n=n+1$ then find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^n\log\left(1-\frac{k^2(k^2-1)}{18a_ka_{k+1}}\right)$$

Proposed by Florică Anastase – Romania

S.207 In ΔABC the following relationship holds:

$$a^R\left(\frac{1}{(s-a)^r}-\frac{2^r}{a^r}\right)+b^R\left(\frac{1}{(s-b)^r}-\frac{2^r}{b^r}\right)+c^R\left(\frac{1}{(s-c)^r}-\frac{2^r}{c^r}\right)\geq 0$$

Proposed by Daniel Sitaru – Romania

S.208 If $a,b\in\mathbb{R}, a\leq b$ then:

$$23\cdot a^{22}(b-a)^3\leq\frac{b^{25}-a^{25}}{25}+(a-b)\left(\frac{a+b}{2}\right)^{24}\leq 23\cdot b^{22}(b-a)^3$$

Proposed by Daniel Sitaru – Romania

S.209 Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[5]{n+1}\cdot H_{n+1}-\sqrt[5]{n}\cdot H_n\right)$$

Proposed by Daniel Sitaru – Romania

S.210 If in $\Delta ABC, \mu(B)=2\mu(A), \mu(C)=2\mu(A)+\mu(B)$ then:

$$\frac{F}{a^6}+\frac{F}{b^6}+\frac{F}{c^6}<\frac{17\sqrt{7}}{448r^4}$$

Proposed by Daniel Sitaru – Romania

S.211 If $a,b,x,y>0$ then:

$$\left(a\sqrt{\frac{ax+by}{a+b}}+b\sqrt[3]{\frac{ax+by}{a+b}}\right)^{a+b}\geq\left(a\sqrt{x}+b\sqrt[3]{x}\right)^a\cdot\left(a\sqrt{y}+b\sqrt[3]{y}\right)^b$$

Proposed by Daniel Sitaru – Romania

S.212 Find:

$$\Omega(n)=\lim_{n\to 0}\left(\frac{5^x-1}{x^{n+1}}-\frac{\ln 5}{x^n}-\frac{\ln^2 5}{2x^{n-1}}-\cdots-\frac{\ln^n 5}{n!\cdot x}\right), n\in\mathbb{N}, n\geq 2$$

Proposed by Daniel Sitaru – Romania

S.213 In acute ΔABC the following relationship holds:

$$\prod_{cyc}(\tan A+\cot A)(\cos A+\sec A)\geq\prod_{cyc}(\tan A+\cot B)(\cos A+\sec B)$$

Proposed by Daniel Sitaru – Romania

S.214 If $0<a\leq b<\frac{\pi}{2}$ then:

$$\cos 2a-\cos 2b+4\int_a^b\frac{\cos x}{(\sin x)^{\sin x}}dx\leq 8(\sin b-\sin a)$$

Proposed by Daniel Sitaru – Romania

S.215 If $0 < x, y < \frac{\pi}{2}$ then:

$$3 \cdot (\sin x)^{2\sin^2 x} \cdot (\sin y \cos x)^{2\sin^2 y \cos^2 x} \cdot (\cos y \cos x)^{2\cos^2 y \cos^2 x} \ge 1$$

When equality holds?

Proposed by Daniel Sitaru – Romania

S.216 If $f: \mathbb{R} \to [1, \infty), f$ – continuous, $a \le b, a, b \in \mathbb{R}$ then:

$$4\left(\int_a^b f(x)\,dx\right)^4 + 4(b-a)^3 \ge \int_a^b \left(1 + f(x)\right)^3 dx$$

Proposed by Daniel Sitaru – Romania

S.217 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{2^n} \cdot \int_1^2 \sqrt[n-1]{\left(\frac{1 + x + x^2 + \cdots + x^{n-1}}{n}\right)^n}\,dx\right)$$

Proposed by Daniel Sitaru – Romania

S.218 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{e^n}\int_0^n \frac{dx}{e^x\left(e^{x+2} + 2e - \frac{1}{e}\right)}\right)$$

Proposed by Daniel Sitaru – Romania

S.219 If $a, b, c \in [0,1)$ then:

$$\frac{1}{a^4 - 1} + \frac{1}{b^4 - 1} + \frac{1}{c^2 - 1} \le \frac{2\sqrt{2}}{abc - 1}$$

Proposed by Daniel Sitaru – Romania

S.220 If $0 < a \le b$ then:

$$\frac{(b-a)^2}{2b^2} + \ln\left(\frac{b}{a}\right) \le \frac{b-a}{a}$$

Proposed by Daniel Sitaru – Romania

S.221 If $a, b, c \in (0,1), a + b + c = 1$ then:

$$\int_0^{\sqrt[4]{a}} \left(\frac{x^3 + x^2 + 1}{x - 1}\right)^2 dx + \int_0^{\sqrt[4]{b}} \left(\frac{x^3 + x^2 + 1}{x - 1}\right)^2 dx + \int_0^{\sqrt[4]{c}} \left(\frac{x^3 + x^2 + 1}{x - 1}\right)^2 dx > 1$$

Proposed by Daniel Sitaru – Romania

S.222 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{3^n}\sqrt[n]{\frac{1}{n!}\prod_{i=1}^{n}\left(\sum_{k=1}^{i} k\binom{2i}{2k}\right)}\right)$$

Proposed by Daniel Sitaru – Romania

S.223

$$a,b,c>0, abc=1, \Omega(a)=\int_{-1}^{a}\left(\frac{e^{3x^2}}{1+e^x}+6xe^{3x^2}\log(1+e^x)\right)dx$$

Prove that:

$$\Omega(a)+\Omega(b)+\Omega(c)\geq 3e^{a^2+b^2+c^2}$$

Proposed by Daniel Sitaru – Romania

S.224 If $0<a,b,c\leq 1$ then:

$$\int_a^1 x^x\,dx+\int_b^1 x^x\,dx+\int_c^1 x^x\,dx\geq \log\big((2-a)(2-b)(2-c)\big)$$

Proposed by Daniel Sitaru – Romania

S.225 If $a,b\geq 0, n\in\mathbb{N}, n\geq 2, A\in M_n(\mathbb{R}), A^2=O_n$ then:

$$\det\left(\sqrt{3}(a+b)A+(a^2+ab+b^2)I_n\right)\geq 0$$

Proposed by Daniel Sitaru – Romania

S.226 Find:

$$\Omega=\lim_{n\to\infty}\left(n^2\sum_{k=1}^{n}\frac{1}{k^4-k^2+1}\right)$$

Proposed by Daniel Sitaru – Romania

S.227 If $0<a\leq b\leq c\leq d\leq 4$ then:

$$48d+3(ab^2+bc^2+cd^2)<48a+3(b^3+c^3+d^3)+128$$

Proposed by Daniel Sitaru – Romania

S.228 $A,B\in M_{2019}(\mathbb{R}), p\in\mathbb{R}-\{0\}, I_{2019}+2p(A+B)+2p^2(A^2+B^2)=O_{2019}$

Find: $\Omega=\det(AB-BA)$

Proposed by Marian Ursărescu – Romania

S.229 If $f\in C([0,1]), f(0)=f(1), n\in\mathbb{N}^*$ then: $\exists c_1,c_2,\dots,c_n$ – different in pairs such that:

$$2f'(c_1)+6f'(c_2)+\dots+n(n+1)f'(c_n)=0$$

Proposed by Marian Ursărescu – Romania

S.230 Prove that the following equation has natural roots for $n\in\mathbb{N}, n\geq 5$

$$\frac{1}{x_1x_2x_3}+\frac{1}{x_2x_3x_4}+\dots+\frac{1}{x_{n-2}x_{n-1}x_n}=1$$

Proposed by Marian Ursărescu – Romania

S.231 Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k} i\left(k-i+\frac{1}{2}\right)^2\right]^{-1}$$

Proposed by Vasile Mircea Popa – Romania

S.232 Find:

$$\Omega = \lim_{x\to 0}\frac{(1+2x)^{\frac{3}{x}}-(1+3x)^{\frac{2}{x}}}{x}$$

Proposed by Vasile Mircea Popa – Romania

S.233 Find:

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\sqrt{n^4+k}\cdot\cos\frac{2k\pi}{n}\right)$$

Proposed by Vasile Mircea Popa – Romania

S.234 Find:

$$\Omega = \lim_{x\to\infty}\left(\int_{x}^{x+\frac{2}{x}} t^2\arctan\left(\frac{1}{t}\right)dt\right)$$

Proposed by Vasile Mircea Popa – Romania

S.235

$$x,y\in(-1,1),\ x*y=\frac{x+y}{1+xy}$$

Solve for natural numbers:

$$\frac{1}{2}*\frac{1}{3}*\frac{1}{4}*\dots*\frac{1}{n}=\frac{7}{8}$$

Proposed by Vasile Mircea Popa – Romania

S.236

$$a_n,b_n,r,s>0,n\ge 1,n\in\mathbb{N},\ \lim_{n\to\infty}\frac{a_{n+1}}{n^r\cdot a_n}=a>0,\ \lim_{n\to\infty}\frac{b_{n+1}}{n^s\cdot b_n}=b>0$$

$L_0=2, L_1=1, L_{n+2}=L_{n+1}+L_n$ – Lucas sequence. Find:

$$\Omega(a,b)=\lim_{n\to\infty}\left(\left(\frac{\sqrt[n]{a_n}\cdot\sqrt[n+1]{L_{n+1}}}{n^{r+s}}-\frac{\sqrt[n+1]{a_{n+1}}\cdot\sqrt[n]{L_n}}{(n+1)^{r+s}}\right)\sqrt[n]{b_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.237 Find:

$$\Omega = \lim_{n\to\infty}\left(e^{-2H_n}\sum_{k=1}^{n}\left[\left(\sqrt{k}+\sqrt{k+1}\right)^2\right]\right), [*]\text{- great integer function}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.238 If $a, b > 0, a, b$ – fixed then find:

$$\Omega = \lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}}\cdot\sum_{k=1}^{n}\sqrt{\frac{1}{b^2}+\frac{1}{(a+bk)^2}+\frac{1}{(a+b+bk)^2}}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.239 Let the positive real sequence $(a_n)_{n\geq 1}$, such that $\lim_{n\to\infty}\frac{a_{n+1}}{\sqrt[n]{(2n-1)!!}} = a \in \mathbb{R}_+^*$.

Compute:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.240 If $a, b > 0$, then prove that:

$$2\sqrt{ab}\cdot\frac{\sin x}{x}+\frac{a+b}{2}\cdot\frac{\tan x}{x} > 3\sqrt{ab}, \forall x \in \left(0,\frac{\pi}{2}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.241 Let the positive real sequence $(a_n)_{n\geq 1}$, such that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}} = a \in \mathbb{R}_+^*$.

Compute:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

S.242 Let $\alpha, \beta \geq 0$. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$30f(\alpha x^2-\beta y^2) = \left(f(\alpha\beta y)\right)^{11}+x^{\alpha+1}y^{\beta+2}, \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh - Vietnam

S.243 Let $\alpha \geq \beta > 0$. Find all functions $\varphi: \mathbb{R} \to \mathbb{R}$ such that:

$$\varphi(x) \text{ – continuous at } x = 0 \text{ and } \varphi(\alpha x) = \varphi(\beta x)+\frac{\alpha}{\beta}x^4, \forall x \in \mathbb{R}$$

Proposed by Nguyen Van Canh - Vietnam

S.244 If $x \geq 0$ then:

$$(x^4+4x^3+1)^{x^2+3x+3} \geq 15x^4+12x^3+1$$

Proposed by Nguyen Van Canh - Vietnam

S.245 Let $\alpha, \beta > 0$. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$\alpha \cdot f\left(x^{\alpha} + y^{\beta}\right) = \beta \cdot f^{2}\left(x^{\beta} + y^{\alpha}\right) + \alpha\beta(xy)^{2020}, \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh - Vietnam

S.246 Find all real numbers $\alpha \geq \beta \geq 0$ such that:

$$\ln(1 + \alpha x) \leq \beta x, \forall x \geq 0$$

Proposed by Nguyen Van Canh - Vietnam

S.247 Find all real numbers $\alpha \geq 0$ such that:

$$\sin^{2}\left(\sqrt{\alpha} \cdot x^{2020}\right) + \cos^{2}\left((\alpha^{2} - 1) \cdot x^{2020}\right) = 1, \forall x \in \mathbb{R}$$

Proposed by Nguyen Van Canh - Vietnam

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.71 If $a, b \geq 0$ then:

$$\int_{a}^{b}\int_{a}^{b} |ay - ab + bx|\, dy\, dx \leq a^{2}b^{2}$$

Proposed by Daniel Sitaru – Romania

U.72 If $a, b > 0$ then:

$$4\int_{a}^{b}\int_{a}^{b} \left(\frac{2}{5}\sqrt[60]{x} + \frac{3}{5}\sqrt[40]{y}\right)^{100} dx\, dy \geq (b - a)^{4}$$

Proposed by Daniel Sitaru – Romania

U.73 If $a, b, c > 1, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 18$ then:

$$e^{9 + \frac{\Gamma'(a)}{\Gamma(a)} + \frac{\Gamma'(b)}{\Gamma(b)} + \frac{\Gamma'(c)}{\Gamma(c)}} < abc$$

Proposed by Daniel Sitaru – Romania

U.74 Find:

$$\Omega = \lim_{n\to\infty} n^2\left(e^{H_n-\gamma} - n\right)\left(e^{\frac{3}{n+5}} - e^{\frac{3}{n+7}}\right)$$

Proposed by Daniel Sitaru – Romania

U.75 If $0 < a \le b$ then:

$$\int_a^b\int_a^b\int_a^b \frac{x^2 + y^2 + z^2}{x + y + z}\,dx\,dy\,dz \le \log\left(\frac{b}{a}\right)^{\frac{(b-a)(b^3-a^3)}{3}}$$

Proposed by Daniel Sitaru – Romania

U.76 Find:

$$\Omega = \lim_{n\to\infty}\left(H_{2n}\cdot\int_{-1}^{1} x^{2n+1}\log(1+\gamma^x)\,dx\right)$$

Proposed by Daniel Sitaru – Romania

U.77

Find:

$$\Omega(a) = \int_0^a\left(\int_0^a \log(1+ax)^{\frac{1}{1+x^2}}\,dx\right)da, 0 < a < 1$$

Proposed by Daniel Sitaru – Romania

U.78 Find:

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_{\varepsilon}^{1}\frac{x\log x}{1+x^2+x^4}\,dx\right)$$

Proposed by Vasile Mircea Popa – Romania

U.79 Find:

$$\Omega = \int_0^{\infty}\frac{x^2\ln(1+x)}{x^4+1}\,dx$$

Proposed by Vasile Mircea Popa – Romania

U.80 Find without any software:

$$\Omega = \left\lfloor \sum_{k=1}^{556}\frac{1}{\sqrt[3]{k}}\right\rfloor$$

where $\lfloor x\rfloor$ is the greatest integer function (floor function).

Proposed by Vasile Mircea Popa – Romania

U.81 Find a closed form:

$$\Omega(a)=\int_0^\infty \frac{x^2}{(x^4+1)(1+ax)}dx\,,a>0$$

Proposed by Vasile Mircea Popa – Romania

U.82 Find a closed form:

$$\Omega(a)=\int_0^\infty \frac{x^3}{(x^4+1)(1+a^2x^2)}dx\,,a\neq 0$$

Proposed by Vasile Mircea Popa – Romania

U.83 Find a closed form:

$$\Omega(a)=\int_0^\infty \frac{x^3}{(1+x^2+x^4)(1+a^2x^2)}dx,a\neq 0$$

Proposed by Vasile Mircea Popa – Romania

U.84 If $f,f':(0,\infty)\to(0,\infty), f$ – differentiable, $0<a\le b$ then:

$$\int_a^b\int_a^b \frac{\big(f(x)+f(y)\big)f'(x)f'(y)}{\sqrt{1+f(x)f(y)}}dx\,dy\le \log\left(\frac{f(b)+\sqrt{1+f^2(b)}}{f(a)+\sqrt{1+f^2(a)}}\right)^{f^2(b)-f^2(a)}$$

Proposed by Daniel Sitaru – Romania

U.85 If $0<a\le b<\frac{\pi}{2}$ then:

$$2\int_a^b\int_a^b \cos^2 x\cos^2 y\,(1+\tan x\tan y)|\tan x-\tan y|\,dx\,dy\le (b-a)^2$$

Proposed by Daniel Sitaru – Romania

U.86 Let x be a positive real number. Prove:

$$log\Gamma(2x+1)-2log\Gamma\left(x+\frac{1}{2}\right)\ge log([2x]!)-2log([x]!),[*]-GIF$$

Proposed by Jalil Hajimir-Canada

U.87 Find:

$$\int_0^\infty \frac{tan^{-1}(\sqrt{3}x)}{x(1+3x^2)}dx$$

Proposed by Jalil Hajimir-Canada

U.88 Find:

$$\int_{-\infty}^0 \frac{xe^x}{x+log(1-e^x)}dx$$

Proposed by Jalil Hajimir-Canada

U.89 Find closed forms:

$$\Omega_1 = \int_0^\infty \frac{sin^4(3x)}{x^2}dx; \;\; \Omega_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$$

Proposed by Jalil Hajimir-Canada

U.90 Find closed forms:

$$\Omega_1 = \sum_{n=1}^{\infty} \frac{16n^2+1}{(16n^2-1)^2}, \qquad \Omega_2 = \int_0^{2\pi} \frac{cos^2 3x}{9-8cos^2 x}dx$$

Proposed by Jalil Hajimir-Canada

U.91 Find:

$$\Omega(m,n) = \int_0^\infty \left(\frac{1}{e^n}\right)^{\sqrt[m]{x}} dx, \qquad m,n > 0$$

Proposed by Jalil Hajimir-Canada

U.92 Find the general solution of:

$$\frac{dy}{dx} = \frac{2^{-x}-2^x}{4y+3}$$

Proposed by Jalil Hajimir-Canada

U.93 Find the general solution of:

$$\frac{dy}{dx} = \frac{ye^x + e^y}{xe^y + e^x}$$

Proposed by Jalil Hajimir-Canada

U.94 Solve for real numbers:

$$2x^2 - \pi x + 2\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{n^2+n-1}{n^4+2n^3+4n^2+3n+2}\right)\tan^{-1}\left(\frac{n^4+2n^3+2n^2+n+2}{2n^3+4n^2+3n+1}\right) = 0$$

Proposed by Daniel Sitaru – Romania

U.95 For $n > 1$, we have:

$$\int_0^\infty \int_0^\infty \frac{xsin(tx)}{cosh\left(\frac{\pi x}{n}\right)}dtdx = \frac{n}{2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.96 For $m \geq 1, n > 2$, let $\Psi(m,n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2+xy+y^2)e^{-m(nxy+x^2+y^2)}dydx$ then:

$$\frac{\partial^k}{\partial m^k}\Psi(m,n) = \left(-\frac{1}{m}\right)^k \Gamma(k+2)\Psi(m,n)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.97 Prove the following:

$$\int_1^\infty\int_1^\infty \frac{x+y}{(\sqrt{xy}+1)(1+xy(1+xy))}dydx=\frac{\pi}{3\sqrt{3}}+log3$$

$$\int_1^\infty\int_1^\infty \frac{\sqrt{x}+\sqrt{y}}{(\sqrt{xy}+1)(1+xy(1+xy))}dydx=\frac{4}{3}(3log3-4log2)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.98 Prove the summation:

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\right)^2}{(1+2+3+4+\cdots+n)^2}=4\big(7\zeta(4)-6\zeta(3)\big)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.99 Prove the relation:

$$\sum_{n=1}^{\infty}\frac{\big((3n^3+2n^2-n)mod5\big)}{3n^3+2n^2-n}=\int_0^1\frac{t^{\frac{2}{3}}+3t^{\frac{2}{5}}+4t^{\frac{4}{15}}-4t^{\frac{7}{15}}-16\sqrt[15]{t}+12}{20(1-t)t^{\frac{13}{15}}}dt=$$

$$=\frac{4}{5}\psi^{(0)}\left(\frac{1}{5}\right)+\frac{1}{5}\psi^{(0)}\left(\frac{3}{5}\right)-\frac{3}{5}\psi^{(0)}\left(\frac{2}{15}\right)-\frac{1}{5}\psi^{(0)}\left(\frac{2}{5}\right)-\frac{1}{20}\psi^{(0)}\left(\frac{4}{5}\right)-\frac{3}{20}\psi^{(0)}\left(\frac{8}{15}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.100 If we have

$$\int_0^\infty\left(\frac{sin(\pi x)}{x^3}-\frac{cos(\pi x)}{x^2}-\frac{sinh(\pi x)}{x^3}+\frac{cosh(\pi x)}{x^2}\right)\frac{e^{-\pi x}}{\sqrt{x}}dx=\pi^2\alpha+\pi^3\beta$$

Then find the values of α and β.

Proposed by Srinivasa Raghava-AIRMC-India

U.101 For $\mathcal{R}(n)>0$, prove that:

$$\int_0^\pi\frac{\sqrt[n]{tan2x}}{tanx}dx=i\pi e^{-\frac{i\pi}{2n}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.102 Prove the integral:

$$\int_0^1 xLi_2\left(\frac{(1-x)^4}{(1+x)^4}\right)dx=\frac{\pi^2}{12}+2\pi+4-16log2$$

Here $Li_2(x)$ –is Poly-log function.

Proposed by Srinivasa Raghava-AIRMC-India

U.103 If we have the equation $y''(x)+y(x)=tan2x$, with $y\left(\frac{\pi}{2}\right)=1, y'\left(\frac{\pi}{2}\right)=0$ then:

$$\int_0^\pi\big(y(x)+y(2x)\big)dx=2+\frac{i\pi}{2\sqrt{2}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.104 Compute the sum in a closed form:

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}\right)^3}{(1+2+3+4+\cdots+n)^3}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.105 Prove the sum relation:

$$\sum_{n=0}^{\infty}\frac{(\phi^{2n-1}-1)(\phi^{4n-1}-1)}{(2n-1)\phi^{12n-6}}=log\left(\frac{3^{-\frac{\phi}{3}}5^{\frac{\phi}{4}}\sqrt{27-4\phi}(\phi-1)^{-\frac{\phi}{2}}}{\sqrt{22}}\right)$$

Where ϕ –is Golden Ratio.

Proposed by Srinivasa Raghava-AIRMC-India

U.106 Compute the integral in a closed form:

$$\int_{-\infty}^{\infty}\frac{u^{11}+35u^9+249u^7+120u^6+249u^5+35u^3+u}{(u^4+3u^2+1)(u^4+11u^2+1)(u^4+27u^2+1)u}du$$

Proposed by Srinivasa Raghava-AIRMC-India

U.107 Let the function $\eta(a)$ defined for $0\le a\le 1$

$$\eta(a)=\int_a^1\int_a^1\frac{1}{(x+y)(1+xy)}dydx$$

Then show that: $\int_0^1\eta(a)da=\frac{1}{2}log^2 2$, $\int_0^1\eta(\sqrt{a})da=\frac{1}{8}(\pi-4)^2$

Proposed by Srinivasa Raghava-AIRMC-India

U.108 For $m,n\ge 0$, prove that:

$$\int_{-\infty}^{\infty}\frac{x^2-n}{x^2+n}sin(mx)\frac{dx}{x}=\pi\left(2e^{-m\sqrt{n}}-1\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.109 If $\frac{\partial^2}{\partial x^2}f(x)+\frac{\partial}{\partial x}f(x)=f(x)+e^{-x}$ with $f(0)=1$, $\frac{\partial(x)}{\partial x}|x\to 0=0$ then show that:

$$f(i\pi\sqrt{5})f(-i\pi\sqrt{5})=1$$

Proposed by Srinivasa Raghava-AIRMC-India

U.110 If $a(n+2)+a(n)=\frac{1}{6}n(n+1)(2n+1)$, with $a(0)=0, a(1)=1$ then prove:

$$\int_0^1 a(n)dn=\frac{2}{\pi}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.111

$$\int_0^{\pi} sin^{-1}\left(sec\left(tan^{-1}(sinx)\right)\right)sinxdx = \pi + i\left(\frac{2\sqrt{2}\pi^{\frac{3}{2}}}{\Gamma\left(\frac{1}{4}\right)^2} - \frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{2\pi}}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.112 For any $n > 1$, we have:

$$\int_0^{\infty} log^n\left(coth\left(\frac{x}{\sqrt{n}}\right)\right)dx = n\int_0^{\infty} log^n\left(coth\left(x\sqrt{n}\right)\right)dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.113 Prove that:

$$\int_0^{\infty}\int_0^{\infty} log(1 + e^{-\pi x})cos(\pi xy)dydx = log\sqrt{2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.114

$$R(k) = \int_0^{\infty}\frac{x^{-ik}}{x^2 - x + 1}dx$$

Prove that: $\int_0^{\infty} R(k)dk = \pi$

Proposed by Srinivasa Raghava-AIRMC-India

U.115 If, for $u > 0$, $Q(u) = \int_0^{\infty}\frac{tan^{-1}(ux)}{1+x}\frac{dx}{\sqrt{x}}$ then prove:

$$\int_{-i}^{i}\frac{Q(u)}{u}du = \frac{\pi^2}{4(\pi - 2)}\int_{-i}^{i} Q(u)du$$

Proposed by Srinivasa Raghava-AIRMC-India

U.116 If $\int_0^{\infty}\frac{tan^{-1}(\sqrt[4]{x})}{1+x^2}\frac{dx}{x} = \frac{\pi}{4}log(\alpha)$ then prove that:

$$\alpha^4 - 136\alpha^3 + 2584\alpha^2 - 544\alpha + 16 = 0$$

Proposed by Srinivasa Raghava-AIRMC-India

U.117 Prove this sharp inequality:

$$\int_0^{\frac{\pi}{3}}\int_0^{\frac{\pi}{3}}\frac{dydx}{sin^2x + cos^2y} > \frac{4}{\pi}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.118 If $\prod_{k=1}^{\infty}\left(1 + \frac{1}{(\pi k)^2} + \frac{1}{(\pi k)^6}\right) = sinasinbsinhc$ then we have:

$$a^4 + b^4 + c^4 = 1$$

Proposed by Srinivasa Raghava-AIRMC-India

U.119

$$\int_0^1 \frac{x^4+6x^2+1}{(x^2+1)^3} log\left(log\left(\frac{1}{x}\right)\right) dx = \frac{3}{8}\pi log\left(\frac{4\pi^3}{\Gamma\left(\frac{1}{4}\right)^4}\right) - \frac{G}{\pi}$$

G –Catalan constant.

Proposed by Srinivasa Raghava-AIRMC-India

U.120 Solve the equation:

$$cos(\pi x) + e^{\frac{i\pi x}{3}} sin(\pi x) = e^{\frac{i\pi}{3}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.121 For any positive integer $n \ge 1$

$$\int_0^1 \frac{x^{-x} sin(\pi x)}{(1-x)^{1-x}} \frac{dx}{1+nx} = \frac{\pi}{n^{n-1}\sqrt[n]{n}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.122

$$\int_{\frac{1}{\phi}}^{\phi} \frac{dx}{(x^\phi+\phi)^\phi} = \left(\left(\frac{1}{\phi}\right)^{-\phi} + \phi\right)^{-\frac{1}{\phi}} - \frac{2^{\phi-1}(2\phi+2\phi^{-\phi})^{-\frac{1}{\phi}}}{\phi^2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.123 Let $F(x) = \frac{x^2\phi+x+\frac{1}{\phi}}{x^2\phi-x+\frac{1}{\phi}}$ and if $p(x)\frac{\partial F(x)}{\partial x} + F(x)q(x) = 0$ then:

$$p\left(\sqrt{\sqrt{5}}\right) = \phi^2 q\left(\sqrt{\sqrt{5}}\right), \phi \text{ –Golden ratio}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.124

$$\int_{-\infty}^{\infty} e^{-x^2+x}\left(\mathcal{F}_t\left(e^{t-t^2}\right)(x)\right) dx = e^{\frac{2i}{5}} \int_{-\infty}^{\infty} e^{-x^2-x}\left(\mathcal{F}_t\left(e^{t-t^2}\right)(x)\right) dx$$

$\mathcal{F}_t[*](x)$ –Fourier Transform.

Proposed by Srinivasa Raghava-AIRMC-India

U.125 Prove that for $m > 0$:

$$\pi = \int_{-\infty}^{\infty} \frac{x sin(mx)}{x^2+\left(\frac{2i\pi}{m}\right)^2} dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.126 Simplify this monster: $log\left(\sqrt[3]{\frac{33554432}{225\pi^4}\frac{\Gamma\left(\frac{1}{12}\right)^2\Gamma\left(\frac{17}{12}\right)^2\Gamma\left(\frac{7}{4}\right)^8}{\Gamma\left(\frac{7}{12}\right)^2\Gamma\left(\frac{11}{12}\right)^2}}\right)$

Proposed by Srinivasa Raghava-AIRMC-India

U.127 Compute in a closed form:

$$\int_0^1 \frac{\left(x-\sqrt[3]{x}\right)log\left(x+\sqrt[3]{x}\right)}{x+\sqrt[3]{x}}dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.128 Find:

$$\Omega = \lim_{n\to\infty} n\left(\int_0^1 \frac{1}{x}log(1+x+x^2+x^3)dx - \sum_{k=0}^{n}\frac{1}{(2k+1)^2}\right)$$

Proposed by Mohammed Bouras-Morocco

U.129 Show that:

$$\lim_{n\to\infty}\left(log2-\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+..-\frac{(-1)^n}{n}\right)\right)^n = \sqrt{e}$$

Proposed by Mohammed Bouras-Morocco

U.130 Find without softs:

$$\Omega = \int_0^{\frac{\pi}{2}} \frac{x+\sin^2 x+\cos x+\log(\tan x)}{(\sin x+\cos x)^2}dx$$

Proposed by Radu Diaconu – Romania

U.131 If $f:[a,b]\to(0,\infty), f$ – continuous then:

$$2\int_a^b\int_a^b\int_a^b \frac{f^3(x)}{f^2(x)+f(y)f(z)}dxdydz \ge (b-a)^2\int_a^b f(x)\,dx$$

Proposed by Daniel Sitaru – Romania

U.132 If $n\in\mathbb{N}, n\ge 1$ then:

$$\frac{1}{n!}\le \log^n\left(1+\prod_{k=1}^{n}\left(e^{\frac{1}{k}}-1\right)^{\frac{1}{n}}\right)\le\left(\frac{H_n}{n}\right)^n$$

Proposed by Daniel Sitaru – Romania

U.133

$$\Omega(p) = \lim_{n\to\infty}\left(\frac{1}{p}\sum_{k=1}^{p}\sqrt[n]{k}\right)^{np}, p\in\mathbb{N}-\{0\}$$

Find:

$$\Omega = \sum_{p=1}^{\infty}\frac{1}{\Omega(p)}$$

Proposed by Daniel Sitaru – Romania

U.134 Find:

$$\Omega=\prod_{k=1}^{\infty}\left(\lim_{n\to\infty}\left(\frac{\sqrt[n]{k+1}+\sqrt[n]{k+3}}{\sqrt[n]{k+2}+\sqrt[n]{k+4}}\right)^{n}\right)^{2}$$

Proposed by Daniel Sitaru – Romania

U.135 Find:

$$\Omega=\int_0^{\infty}\frac{x\sin x}{(x^2+1)(2-\cos x)}dx$$

Proposed by Vasile Mircea Popa – Romania

U.136 Find a closed form:

$$\Omega(a)=\int_0^{\infty}\frac{x^3}{(x^4+1)(1+ax)}dx\,,a>0$$

Proposed by Vasile Mircea Popa – Romania

U.137 Find:

$$\Omega=\int_0^{\infty}\frac{x^2\ln(1+x^2)}{1+x^2+x^4}dx$$

Proposed by Vasile Mircea Popa – Romania

U.138 Find:

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\varepsilon}^{\infty}\frac{\ln(x)}{x^4-x^2+1}dx$$

Proposed by Vasile Mircea Popa – Romania

U.139 Prove that:

$$\Psi\left(\frac{5}{8}\right)-\Psi\left(\frac{1}{8}\right)=\pi\sqrt{2}+2\sqrt{2}\ln\left(1+\sqrt{2}\right)$$

where $\Psi(x)$ is the diagamma function.

Proposed by Vasile Mircea Popa – Romania

U.140 Find a closed form:

$$\Omega(a)=\int_0^{\infty}\frac{x^3}{(x^3-x^2+1)(ax+1)}dx,a>0$$

Proposed by Vasile Mircea Popa – Romania

U.141 Find:

$$\Omega=\int_0^{\infty}\frac{dx}{(1+x^2)(4+x^2)(2-\cos x)}$$

Proposed by Vasile Mircea Popa – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-SUMMER 2022

PROBLEMS FOR JUNIORS

JP.361 Find $x, y, z > 0$ such that:

$$\begin{cases} x^3y + y^3z + z^3x = xyz(x+y+z) \\ 2x + 3y + 5z = 10 \end{cases}$$

Proposed by Daniel Sitaru-Romania

JP.362. Let ABC be a triangle with inradius r, and circumradius R. Equilateral triangles with AB, BC and BC, are drawn externally to triangle ABC. Let K, L and M be the centroid's of the equilateral triangles, respectively. Prove that:

$$\frac{2r}{R} \le \frac{[KLM]}{[ABC]} \le \left(\frac{R}{2r}\right)^2$$

Proposed by George Apostolopoulos-Messolonghi-Greece

JP.363. Let a, b, c be positive real numbers with $a^2 + b^2 + c^2 = 12$. Prove that:

$$\frac{a^4}{\sqrt{a^3+1}} + \frac{b^4}{\sqrt{b^3+1}} + \frac{c^4}{\sqrt{c^3+1}} \ge 16$$

Proposed by George Apostolopoulos-Messolonghi-Greece

JP.364. If $x, y, z > 0$ then:

$$\sqrt{x^2 - xy\sqrt{3} + y^2} + \sqrt{y^2 - yz\sqrt{2} + z^2} = \sqrt{x^2 + z^2 - \frac{xz(\sqrt{6}-\sqrt{2})}{2}} \Leftrightarrow \frac{2\sqrt{2}}{x} + \frac{2}{z} = \frac{\sqrt{2}+\sqrt{6}}{y}$$

Proposed by Daniel Sitaru – Romania

JP365. If in ΔABC exists relation $\frac{4}{w_a} = \frac{1}{r} + \frac{1}{r_a}$ then prove that $AH \ge 2r$

Proposed by Marian Ursărescu-Romania

JP.366. In acute ΔABC the following relationship holds:

$$cosA + \sqrt{cosAcosB} + \sqrt[3]{cosAcosBcosC} < 2$$

Proposed by Marian Ursărescu-Romania

JP.367. $a,b,c\in\mathbb{C}^*$ –different in pairs, $A(a),B(b),C(c)$; $|a|=|b|=|c|=1$. If

$(ab)^3+(bc)^3+(ca)^3=3(abc)^2$, then ΔABC is equilateral.

Proposed by Marian Ursărescu-Romania

JP.367. $a,b,c\in\mathbb{C}^*$ –different in pairs, $A(a),B(b),C(c)$; $|a|=|b|=|c|=1$. If

$|a-b|\left(\frac{1}{a}+\frac{1}{b}\right)+|b-c|\left(\frac{1}{b}+\frac{1}{c}\right)+|c-a|\left(\frac{1}{c}+\frac{1}{a}\right)=0$, then ΔABC is equilateral.

Proposed by Marian Ursărescu-Romania

JP.369. Find $x,y,z\in\left(0,\frac{\pi}{2}\right)$ such that:

$$\frac{cos(5x)}{cosx}+\frac{cos(5y)}{cosy}+\frac{cos(5z)}{cosz}+\frac{15}{4}=0$$

Proposed by Daniel Sitaru-Romania

JP.370. In ΔABC the following relationship holds:

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}+\sqrt{\frac{a^2+b^2}{2}}+\sqrt{\frac{b^2+c^2}{2}}+\sqrt{\frac{c^2+a^2}{2}}\le 6\sqrt{3}R$$

Proposed by Daniel Sitaru-Romania

JP.371 Solve in real positive numbers the equation

$$x^{logx}+x^{log4}+x^{log5}=x^{log6}$$

Proposed by D.M. Bătinețu-Giurgiu and Neculai Stanciu-Romania

JP.372. If in ΔABC: $a^2+b^2=2c^2$ then:

$$2am_a+m_c^2\cdot\sqrt{\frac{ab}{m_am_b}}\le\frac{\sqrt{3}}{2}(a^2+b^2+c^2)$$

Proposed by Daniel Sitaru – Romania

JP.373. If $a,b,c<0$; $a+b+c=3$; F_n – Fibonacci numbers; L_n – Lucas numbers; P_n – Pell numbers; $n\in\mathbb{N}$; $n\ge 2$ then:

$$\frac{a^2(P_n-F_n)(P_n-L_n)}{F_nL_n}+\frac{b^2(F_n-L_n)(F_n-P_n)}{L_nP_n}+\frac{c^2(L_n-P_n)(L_n-F_n)}{P_nF_n}\ge 9$$

Proposed by Daniel Sitaru – Romania

JP.374. Solve for complex numbers:

$$57x^6-180x^5+234x^4-159x^3+60x^2-12x+1=0$$

Proposed by Daniel Sitaru – Romania

JP.375. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that:

$$9(a^2+b^2+c^2)-2(a^3+b^3+c^3)\geq 21$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEMS FOR SENIORS

SP.361. Let $(x_n)_{n\geq 1},\ x_1=1$ such that

$$n^2[2(x_{n+1}-x_n-1)-n^2]+2x_n=n[3(n^2+x_n)-x_{n+1}]$$

Find:

$$\lim_{n\to\infty}\left(1+\frac{1}{2^n}\sum_{k=0}^{n}\frac{\binom{n}{k}}{2k+1}\right)^{\frac{x_n}{n^2}}$$

Proposed by Florică Anastase-Romania

SP.362 Let m_a,m_b,m_c be the lengths of the medians of a triangle ΔABC. Prove

$$\frac{4\sqrt{3}}{3R}\leq\frac{cscA}{m_a}+\frac{cscB}{m_b}+\frac{cscC}{m_c}\leq\frac{\sqrt{3}R}{3r^2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

SP.363. Triangle ABC has $|BC|=a,|CA|=b,|AB|=c$,inradius r and circumradius R.

Equilateral triangles A_1BC,B_1CA and C_1AB with centroids K,L and M respectively, are drawn externally to triangle ABC. Prove that:

$3\sqrt{3}\leq[ALM]+[BMK]+[CKL]\leq\frac{3\sqrt{3}}{4}R^2$ - $[XYZ]$ represents the area of triangle XYZ

Proposed by George Apostolopoulos-Messolonghi-Greece

SP.364. Let ABC be a non-right triangle with circumradius R. Squares with sides AB,BC,CA and centroids K,L,M respectively, are drawn externally to triangle ABC. Let α,β,γ be the distance from the vertices A,B,C to the segments $\overline{KM},\overline{KL},\overline{LM}$, respectively. Prove that:

$$\left(\frac{cotA}{\alpha}\right)^2+\left(\frac{cotB}{\beta}\right)^2+\left(\frac{cotC}{\gamma}\right)^2\geq\frac{8\cdot tan75^\circ}{3R^2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

SP.365. Let $f,F:[a,b]\to\mathbb{R}$,such that $F(x)=-f(x)+cosf(x)$. If F —is Riemann integrable, prove that f —is Riemann integrable.

Proposed by Cristian Miu-Romania

SP.366. Let m_a,m_b,m_c be the medians, r_a,r_b,r_c the exradii, r inradius and R circumradius of a triangle ABC. Prove that:

$$\frac{3}{2}\left(\frac{R}{2r}\right)^{-2}\le\frac{r_a^2}{m_b^2+m_c^2}+\frac{r_b^2}{m_c^2+m_a^2}+\frac{r_c^2}{m_a^2+m_b^2}\le 2\left(\frac{R}{2r}\right)^2-\frac{1}{2}\left(\frac{R}{2r}\right)$$

Proposed by George Apostolopoulos-Messolonghi-Greece

SP.367. Let m_a,m_b,m_c be the medians, r_a,r_b,r_c the exradii, r the inradius and R the circumradius of a triangle ABC. Prove that:

$$\frac{8r}{R^2}<\frac{r_a+r_b}{m_am_b}+\frac{r_b+r_c}{m_bm_c}+\frac{r_c+r_a}{m_cm_a}\le\frac{1}{r}\left(3\left(\frac{R}{2r}\right)^4-1\right)$$

Proposed by George Apostolopoulos-Messolonghi-Greece

SP.368. If $0<a<b<1$, then prove:

$$\frac{a(2b-a)}{b\sqrt{a^2+b^2}}<\int_a^b\frac{dx}{x\sqrt{(x^2+a^2)^3}}+\frac{\sqrt{2}}{2}<\frac{a}{\sqrt{a^2+b^2}}+\frac{b-a}{a\sqrt{2}}$$

Proposed by Florică Anastase-Romania

SP.369 Let $(L_n)_{n\ge0},L_0=2,L_1=1,L_{n+2}=L_{n+1}+L_n,\forall n\in\mathbb{N}$, be the Lucas' sequences, and $a,b,c\in\mathbb{R}_+^*$ such that $abc=1$. Prove that:

$$\frac{1}{a^6(bL_n+cL_{n+1})^2}+\frac{1}{b^6(cL_n+aL_{n+1})^2}+\frac{1}{c^6(aL_n+bL_{n+1})^2}\ge\frac{3}{L_{n+2}^2},\forall n\in\mathbb{N}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu-Romania

SP.370 If ABC is a triangle with inradius r and circumradius R, then for any point M in the plane of triangle, $M\notin\{A,B,C\}$, holds the inequality

$$\frac{MA}{MB+MC}+\frac{MB}{MC+MA}+\frac{MC}{MA+MB}\ge\frac{R+r}{R}\ge\frac{3r}{R}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

SP.371. Let $ABCD$ be a tetrahedron, and let M be a point in space, $M\notin\{A,B,C\}$. Prove that:

$$\frac{MA}{MB+MC+MD}+\frac{MB}{MC+MD+MA}+\frac{MC}{MD+MA+MB}+\frac{MD}{MA+MB+MC}\ge\frac{R+r}{R}\ge\frac{4r}{R}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

SP.372. If $f: \mathbb{R}_+^* \to \mathbb{R}_+^*$ with $\lim_{n\to\infty} \frac{f(x)}{x} = a \in \mathbb{R}_+^*$, $(b_n)_{n\geq 1}$ is an arithmetic progression with $b_1, r \in \mathbb{R}_+^*$ and $u, v \in \mathbb{R}$ satisfy $u + v = 1$, then compute:

$$\lim_{n\to\infty}\left((n+1)^{u}\sqrt[n+1]{\left(f(b_1)f(b_2)\dots f(b_n)f(b_{n+1})\right)^{v}} - n^{u}\sqrt[n]{\left(f(b_1)f(b_2)\dots f(b_n)\right)^{v}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

SP.373. If $f: \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a function such that $\lim_{x\to\infty} \frac{f(x)}{x} = c \in \mathbb{R}_+^*$ and $(a_n)_{n\geq 1}$ is a positive sequence such that $\lim_{n\to\infty}(a_{n+1} - a_n) = a \in \mathbb{R}_+^*$, then compute:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{f(a_1)f(a_2)\dots f(a_n)f(a_{n+1})}} - \frac{n^2}{\sqrt[n]{f(a_1)f(a_2)\dots f(a_n)}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

SP.374. Let m_a, m_b, m_c be the lengths of the medians of a triangle with circumradius R and area F. Prove that:

$$\frac{4}{9R^2} \leq \frac{1}{m_a(m_b+2m_c)} + \frac{1}{m_b(m_c+2m_a)} + \frac{1}{m_c(m_a+2m_b)} \leq \frac{\sqrt{3}}{3F}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

SP.375. If $x, y, z \in (0,1)$, then in any ABC triangle with the area F the following inequality holds:

$$\frac{xa^4}{(y+z)^2(1-x^2)} + \frac{yb^4}{(z+x)^2(1-y^2)} + \frac{zc^4}{(x+y)^2(1-z)} \geq 6\sqrt{3}F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

UNDERGRADUATE PROBLEMS

UP.361. Prove that:

$$\int_0^1 \frac{tan^{-1}x}{x\sqrt{1-x^2}}dx = log_2(\sqrt{2}-1)\int_0^{\frac{\pi}{2}} log(sinx)\,dx$$

Proposed by Florică Anastase-Romania

UP.362.

$$\omega_n = \sum_{k=1}^{2n} cot\frac{k\pi}{2n+1}\cdot\left(sin\frac{2k\pi}{2n+1} + icos\frac{2k\pi}{2n+1}\right)$$

Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^{\omega_n+2-(k+n)}}$$

Proposed by Florică Anastase-Romania

UP.363. Let be $(a_n)_{n\geq 1}; (b_n)_{n\geq 1} \subset (0,\infty)$ such that:

$$\lim_{n\to\infty}(a_{n+1}-a_n) = a \in (0,\infty); b_n = \left(\prod_{k=1}^{n} a_{2k-1}\right)^{\frac{1}{k}}$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{a_{n+1}\cdot b_{n+1}}{n+1} - \frac{a_n\cdot b_n}{n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.364. In ΔABC the following relationship holds:

$$\left(\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}\frac{1}{a^2}\right)\cdot\ldots\cdot\left(\sum_{cyc}\frac{1}{a^n}\right) \geq \frac{3^m}{\sqrt{\left(R\sqrt{3}\right)^{n^2+n}}};\ n\in\mathbb{N}^*$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.365. Let be $a_n = \sum_{k=1}^{n} tan^{-1}\left(\frac{1}{k^2+k+1}\right); n\geq 1$. Find:

$$\Omega = \lim_{n\to\infty} n^2\left(e^{a_{n+1}} - e^{a_n}\right)$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.366. If $s_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n}; n\geq 1$ find:

$$\Omega = \lim_{n\to\infty}\left(1 + e^{s_{n+1}} - s^{s_n}\right)^{n\sqrt{n}}$$

Proposed by D.M.Bătinețu-Giurgiu and Daniel Sitaru-Romania

UP.367. Find: $\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{((2n)!!)^n}{(2an)!!}}; a \in \mathbb{N}$

$$0!! = 1, (2k)!! = 2 \cdot 4 \cdot \ldots \cdot (2k), k \in \mathbb{N}^*$$

Proposed by D.M. Bătinețu-Giurgiu and Daniel Sitaru-Romania

UP.368. Let be $\Omega_n = \int_n^{n+1} \frac{x^n}{e^x + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}} dx\,; n \in \mathbb{N}^*$

Prove that: $\Omega_n < n!$

Proposed by D.M. Bătinețu-Giurgiu and Daniel Sitaru-Romania

UP.369. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function; $a, b > 0$; $a < b$; $a + b = s$;

$f(s - x) + f(x) = c$; $\forall x \in \mathbb{R}$; $c > 0$. Find:

$$\Omega = \int_a^b (x^2 - sx + s^2) f(x) dx$$

Proposed by D.M. Bătinețu-Giurgiu and Daniel Sitaru-Romania

UP.370. If $a, b > 0$, then

$$\int_0^{\frac{\pi}{4}} \frac{dx}{(x+1)(a^2 cos^2 x + b^2 sin^2 x)} < \frac{1}{ab(\pi + 4)} \left(\pi \frac{b}{a} + 4tan^{-1}\left(\frac{b}{a}\right)\right)$$

Proposed by Florică Anastase-Romania

UP.371. Let be $(a_n)_{n\geq 1}$; $a_n = \prod_{k=1}^n ((2k-1)!!)^{\frac{1}{k}}$. Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru-Romania

UP.372. $(x)_{n\geq 1}$ –be a positive sequence of real numbers such that $x_1 > 0$ and

$$x_{n+1} = \frac{x_n^2}{2x_n - log(1 + x_n)}$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{(2n)!!}{(2n-1)!!} \cdot \frac{nx_n}{\sqrt{2n+1}}$$

Proposed by Florică Anastase-Romania

UP.373. $k \in \mathbb{N}, k > 0$ and $x_1 > k, x_{n+1} = \frac{x_n^2}{x_n - k}; n \in \mathbb{N}^*$

Find:

$$\Omega = \lim_{n\to\infty} \frac{n^2}{log(logn)} \cdot \frac{\sqrt[n]{H_n} - 1}{x_n}$$

Proposed by Florică Anastase-Romania

UP.374. Calculate the integral:

$$\int_0^1 \frac{x \ln x}{x^3 + x\sqrt{x} + 1} dx$$

Proposed by Vasile Mircea Popa – Romania

UP.375. In any convex polygon $A_1A_2 \dots A_n, n \geq 3$ with the area F and the sides lengths

$A_kA_{k+1} = a_k, k = \overline{1,n}, A_{n+1} = A_1$ the following inequality holds:

$$\sum_{k=1}^{n} \left(a_k - \sqrt{a_k a_{k+1}} + a_{k+1}\right)^2 \geq 4F \cdot \tan\frac{\pi}{n}$$

Proposed by D.M. Bătinețu-Giurgiu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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