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CONTENT

SPECIAL LIMITS WITH RIEMANN'S SUMS

By Daniel Sitaru – Romania

ABSTRACT. In this paper it is developed a method for calculus of sequences' limits using Riemann's sums.

Main result:

If $\alpha,\beta\in\mathbb{R},\beta\neq 0,\alpha<\beta;f:[\alpha,\alpha+\beta]\to\mathbb{R},f$ continuous then:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}f\left(\alpha+\frac{i\beta}{n}\right)f\left(\alpha+\frac{j\beta}{n}\right)=\frac{1}{2}\left(\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta}f(x)\,dx\right)^2 \quad (1)$$

Proof.

Let be:

$$\Delta_n=\left(\alpha<\alpha+\frac{\beta}{n}<\alpha+\frac{2\beta}{n}<\dots<\alpha+\frac{(n-1)\beta}{n}<\beta\right)$$

Denote $x_n^i=\xi+n^i=\alpha+\frac{i\beta}{n}$

$$||\Delta_n||=x_n^i-x_n^{i-1}=\alpha+\frac{i\beta}{n}-\alpha-\frac{(i-1)\beta}{n}=\frac{\beta}{n}$$

$$\lim_{n\to\infty}||\Delta_n||=\lim_{n\to\infty}\frac{\beta}{n}=0$$

$$x_n^{i-1}\le\xi_n^i\le x_n^i$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f\left(\alpha+\frac{i\beta}{n}\right)=\lim_{n\to\infty}\frac{1}{\beta}\cdot\frac{\beta}{n}\sum_{i=1}^{n}f\left(\alpha+\frac{i\beta}{n}\right)=$$

$$=\frac{1}{\beta}\lim_{n\to\infty}\sum_{i=1}^{n}f\left(\alpha+\frac{i\beta}{n}\right)\left(\alpha+\frac{i\beta}{n}-\alpha-\frac{(i-1)\beta}{n}\right)=\frac{1}{\beta}\lim_{n\to\infty}\sum_{i=1}^{n}f(\xi_n^i)(x_n^i-x_n^{i-1})=$$

$$=\frac{1}{\beta}\lim_{n\to\infty}\sigma_{\Delta_n}(f_1\xi^n)=\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta}f(x)\,dx \qquad (2)$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}f^2\left(\alpha+\frac{i\beta}{n}\right)=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^{n}f^2\left(\alpha+\frac{i\beta}{n}\right)\right)=$$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f^2\left(\alpha+\frac{i\beta}{n}\right)=0\cdot\left(\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta}f(x)\,dx\right)=0$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}f^2\left(\alpha+\frac{i\beta}{n}\right)=0 \qquad (3)$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n} f\left(\alpha+\frac{i\beta}{n}\right)f\left(\alpha+\frac{j\beta}{n}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{2}\left[\left(\frac{1}{n}\sum_{i=1}^{n} f\left(\alpha+\frac{i\beta}{n}\right)\right)^2-\frac{1}{n^2}\sum_{i=1}^{n} f^2\left(\alpha+\frac{i\beta}{n}\right)\right]=$$

$$\overset{(2);(3)}{=}\frac{1}{2}\left(\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} f(x)\,dx\right)^2-0=\frac{1}{2}\left(\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} f(x)\,dx\right)^2$$

Corollary 1: If $\alpha,\beta\in\mathbb{R};\beta\neq 0;\alpha<\beta$ then:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\left(\alpha+\frac{i\beta}{n}\right)\left(\alpha+\frac{j\beta}{n}\right)=\frac{\beta^2+2\alpha\beta}{8\beta^2}$$

Proof: We take in (1): $f(x)=x$.

Corollary 2: If $\alpha,\beta\in\mathbb{R};\alpha\neq 0;\alpha<\beta$ then:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\cos\left(\alpha+\frac{i\beta}{n}\right)\cos\left(\alpha+\frac{j\beta}{n}\right)=\frac{(\sin(\alpha+\beta)-\sin\alpha)^2}{2\beta^2}\quad(2)$$

Proof: We take in (1): $f(x)=\cos x$.

Corollary 3: If $0<\alpha<\beta$ then:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\frac{1}{\left(\alpha+\frac{i\beta}{n}\right)\left(\alpha+\frac{j\beta}{n}\right)}=\frac{1}{2\alpha^2(\alpha+\beta^2)}$$

Proof: We take in (1): $f(x)=\frac{1}{x}$

Corollary 4:

$$\lim_{n\to\infty}\frac{1}{n^4}\sum_{1\le i<j\le n} ij=\frac{1}{8}$$

Proof: We take in (1): $f(x)=x,\alpha=0,\beta=1$.

Corollary 5:

$$\lim_{n\to\infty}\frac{1}{n^6}\sum_{1\le i<j\le n} i^2j^2=\frac{1}{18}$$

Proof: We take in (1): $f(x)=x^2;\alpha=0;\ \beta=1$.

Corollary 6:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\cos i\cdot\cos j=\frac{\sin^2 1}{2}$$

Proof: We take in (2): $\alpha = 0;\ \beta = 1$.

Proposed problems:

1. If $0 < \alpha < \beta$ then find:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\ln\left(\alpha+\frac{i\beta}{n}\right)\ln\left(\alpha+\frac{j\beta}{n}\right)$$

2. If $\alpha,\beta\in\mathbb{R},\beta\neq 0;\alpha<\beta$ then find:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\ln\left(\alpha+\frac{i\beta}{n}\right)\ln\left(\alpha+\frac{j\beta}{n}\right)$$

2. If $\alpha,\beta\in\mathbb{R};\beta\neq 0;\alpha<\beta$ then find:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\sin\left(\alpha+\frac{i\beta}{n}\right)\sin\left(\alpha+\frac{j\beta}{n}\right)$$

3. If $0 < \alpha < \beta$ then find:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\frac{1}{\left(\alpha+\frac{i\beta}{n}\right)^3\left(\alpha+\frac{j\beta}{n}\right)^3}$$

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QUASI-EXACT DIFFERENTIAL EQUATION

By Benny Le Van-Vietnam

Abstract: This article examines a specific variant of exact differential equation whose form is a matrix product of exact partial derivatives and a modifying vector.

1. Introduction

It is widely perceived (see e.g. [1]) that if $P(x,y)$ and $Q(x,y)$ are two bivariable function such that:

$$\frac{\partial[P(x,y)]}{\partial y}=\frac{\partial[Q(x,y)]}{\partial x}\qquad(1)$$

Then the exact first-ordered ordinary differential equation:

$$P(x,y)\,dx+Q(x,y)dy=0\qquad(2)$$

Has solutions as:

$$\begin{cases} U(x,y)=const\\ \left[\begin{array}{l} U(x,y)=\int\limits_{x_0}^{x}P(x,y_0)\,dx+\int\limits_{y_0}^{y}Q(x,y)\,dy+const\\ U(x,y)=\int\limits_{x_0}^{x}P(x,y)\,dx+\int\limits_{y_0}^{y}Q(x_0,y)\,dy+const \end{array}\right. \end{cases}\qquad(3)$$

Of which, $U(x,y)$ is the potential function of (1) where exactness of the above differential equation is determined by the criteria that:

$$\begin{cases} \frac{\partial[U(x,y)]}{\partial x} = P(x,y) \quad \frac{\partial[U(x,y)]}{\partial y} = Q(x,y) \\ \frac{\partial^2[U(x,y)]}{\partial x \partial y} = \frac{\partial[P(x,y)]}{\partial y} = \frac{\partial[Q(x,y)]}{\partial x} \end{cases} \tag{4}$$

Accordingly, we define the quasi-exact differential equation (QDE) as:

$$P(x,y)\,dx + R(x,y)Q(x,y)dy = 0 \tag{5}$$

An alternate form of QDE is $vw = 0$ where $v = [P(x,y)\,dx \quad Q(x,y)dy]$ are exact partial derivatives, $w = [1 \quad R(x,y)]'$ is the modifying vector, and $R(x,y)$ is the modifier.

We shall solve (5) where $R(x,y)$ is (i) a constant in Section 2; and a variable function in Section 3, and Section 4, respectively. Following, Section 5 provides discussion.

2. Constant quasi-exact differential equation

In the case $R(x,y)$ is a constant r ($r \neq 0$ and $r \neq 1$)[1], the constant QDE becomes:

$$P(x,y)\,dx + rQ(x,y)dy = 0 \tag{6}$$

We shall transform (6) to the exact form by multiplying both sides with an integrating factor $S(x,y)$ which is not a constant. Equation (6) becomes:

$$P(x,y)S(x,y)\,dx + rQ(x,y)S(x,y)dy = 0 \tag{7}$$

It is supposed to find $S(x,y)$ such that:

$$\frac{\partial[P(x,y)S(x,y)]}{\partial y} = \frac{\partial[rQ(x,y)S(x,y)]}{\partial x}$$

$$\Leftrightarrow S\frac{\partial P}{\partial y} + P\frac{\partial S}{\partial y} = r\left(S\frac{\partial Q}{\partial x} + Q\frac{\partial S}{\partial x}\right)$$

$$\Leftrightarrow S\left(\frac{\partial P}{\partial y} - r\frac{\partial Q}{\partial x}\right) + P\frac{\partial S}{\partial y} - rQ\frac{\partial S}{\partial x} = 0$$

With $U(x,y)$ as determined under (3) and (4), we could rewrite the above as:

$$S\left(\frac{\partial^2 U}{\partial x \partial y} - r\frac{\partial^2 U}{\partial x \partial y}\right) + \frac{\partial U \partial S}{\partial x \partial y} - r\frac{\partial U \partial S}{\partial x \partial y} = 0$$

$$\Leftrightarrow S(1-r)\frac{\partial^2 U}{\partial x \partial y} + (1-r)\frac{\partial U \partial S}{\partial x \partial y}$$

$$\Leftrightarrow S\partial^2 U + \partial U \partial S = 0$$

Since $S(x,y) \neq const$, we could divide both sides of the above by ∂S^2 and consequently obtain a second-ordered ordinary differential equation:

$$S\frac{\partial^2 U}{\partial S^2} + \frac{\partial U}{\partial S} = 0 \Leftrightarrow S\frac{d^2 U}{dS^2} + \frac{dU}{dS} = 0 \tag{8}$$

In equation (8), replacing $T = dU/dS$ gives:

$$S\frac{dT}{dS} + T = 0 \Leftrightarrow \frac{dT}{T} = -\frac{dS}{S} \Leftrightarrow \ln|T| = -\ln|S| + c \Leftrightarrow T = \frac{k}{S}$$

Henceforth, we obtain:

$$\frac{dU}{dS} = \frac{k}{S} \Leftrightarrow dU = k\frac{dS}{S} \Leftrightarrow U = k\ln|S| + l \Leftrightarrow S = ae^{bU}$$

The finding that $S = ae^{bU}$ gives solutions of (7) are $W(x,y) = const$, of which:

[1] If $r = 0$, solutions are $P(x,y) = 0$ or $x = const$; if $r = 1$, the QDE becomes an exact differential equation.

$$W(x,y) = \int_{x_0}^{x} P(x,y_0)S(x,y_0)\,dx + r\int_{y_0}^{y} Q(x,y)S(x,y)\,dy + const$$
$$= \int_{x_0}^{x} ae^{bU(x,y_0)}\left.\frac{\partial U(x,y)}{\partial x}\right|_{y=y_0} dx + r\int_{y_0}^{y} ae^{bU(x,y)}\frac{\partial U(x,y)}{\partial y}dy + const \qquad (9)$$

Or:

$$W(x,y) = \int_{x_0}^{x} P(x,y)S(x,y)\,dx + r\int_{y_0}^{y} Q(x_0,y)S(x_0,y)\,dy + const$$
$$= \int_{x_0}^{x} ae^{bU(x,y)}\frac{\partial U(x,y)}{\partial x}dx + r\int_{y_0}^{y} ae^{bU(x_0,y)}\left.\frac{\partial U(x,y)}{\partial y}\right|_{x=x_0} dy + const \qquad (10)$$

Example 1

Solve the following differential equation:

$$(ye^x + e^y)\,dx = (xe^y + e^x)dy \qquad (11)$$

Equation (11) is quasi-exact where $P(x,y) = ye^x + e^y$, $Q(x,y) = xe^y + e^x$, and $r = -1$. Besides, formula (3) gives $U(x,y) = xe^y + ye^x$.

It is found that the integrating factor is:

$$S(x,y) = ae^{bU(x,y)} = ae^{b(xe^y+ye^x)}$$

Thus, solution of Example 1 is $W(x,y) = const$, where:

$$W(x,y) = \int_{x_0}^{x} ae^{b(xe^{y_0}+y_0e^x)}(y_0e^x + e^{y_0})\,dx - \int_{y_0}^{y} ae^{b(xe^y+ye^x)}(xe^y + e^x)\,dy + const$$
$$W(x,y) = \frac{a}{b}e^{b(xe^{y_0}+y_0e^x)}\Big|_{x_0}^{x} - \frac{a}{b}e^{b(xe^y+ye^x)}\Big|_{y_0}^{y} + const$$
$$W(x,y) = \frac{2a}{b}e^{b(xe^{y_0}+y_0e^x)} - \frac{a}{b}e^{b(xe^y+ye^x)} + const$$

Simplifying $a = b$, then a solution of (11) is $W(x,y) = const$, where:

$$W(x,y) = 2e^{b(xe^{y_0}+y_0e^x)} - e^{b(xe^y+ye^x)} + const$$

An alternate expression is:

$$W(x,y) = e^{b(xe^y+ye^x)} - 2e^{b(x_0e^y+ye^{x_0})} + const$$

3. Univariable quasi-exact differential equation

We shall solve the QDE (5) when $R(x,y) \neq const$. A unique case is:

$$\frac{\partial(RQ)}{\partial x} = \frac{\partial Q}{\partial x} \Leftrightarrow Q\frac{\partial R}{\partial x} + R\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial x} \Leftrightarrow Q\frac{\partial R}{\partial x} = (1-R)\frac{\partial Q}{\partial x} \Leftrightarrow \frac{\partial R}{1-R} = \frac{\partial Q}{Q}$$

The above case results in a separable differential equation:

$$\frac{dR}{1-R} = \frac{dQ}{Q} \Leftrightarrow \ln|1-R| = \ln|Q| + c \Leftrightarrow 1-R = kQ \Leftrightarrow R = 1-kQ$$

Of which, $k = const$. In this case, (5) becomes exact whose solution is under the form of (3). For $R(x,y) \neq 1 - kQ(x,y)$, it is supposed to find the integrating factor $S(x,y) \neq const$ such that:

$$\begin{cases} P(x,y)S(x,y)\,dx + R(x,y)Q(x,y)S(x,y)dy = 0 \\ \dfrac{\partial[P(x,y)S(x,y)]}{\partial y} = \dfrac{\partial[R(x,y)Q(x,y)S(x,y)]}{\partial x} \end{cases} \qquad (12)$$

Finding $S(x,y)$:

$$S\frac{\partial P}{\partial y}+P\frac{\partial S}{\partial y}=QS\frac{\partial R}{\partial x}+RS\frac{\partial Q}{\partial x}+RQ\frac{\partial S}{\partial x}\Leftrightarrow S\frac{\partial^2 U}{\partial x\partial y}+\frac{\partial U\partial S}{\partial x\partial y}=RS\frac{\partial^2 U}{\partial x\partial y}+R\frac{\partial U\partial S}{\partial x\partial y}+S\frac{\partial R\partial U}{\partial x\partial y} \quad (13)$$

This includes a specific case that $\partial R/\partial x=0\Leftrightarrow R(x,y)=R(y)$ or $\partial R/\partial x=0\Leftrightarrow R(x,y)=R(x)$. In this case, the process of finding $S(x,y)$ turns equivalent to Section 2.

Henceforth, the QDE (5) is comprehensively solvable if $R(x,y)$ is a univariable function. Without loss of generality, we assume $R(x,y)=R(y)$ and (13) becomes:

$$S\frac{\partial^2 U}{\partial x\partial y}+\frac{\partial U\partial S}{\partial x\partial y}=RS\frac{\partial^2 U}{\partial x\partial y}+R\frac{\partial U\partial S}{\partial x\partial y}$$
$$\Leftrightarrow S\partial^2U+\partial U\partial S=RS\partial^2U+R\partial U\partial S$$
$$\Leftrightarrow S(1-R)\partial^2U+(1-R)\partial U\partial S=0$$

Since $R\neq const$ and $S\neq const$, we could transform the above equation as:

$$S\frac{\partial^2U}{\partial S^2}+\frac{\partial U}{\partial S}=0\Leftrightarrow S\frac{d^2U}{dS^2}+\frac{dU}{dS}=0\Leftrightarrow U=k\ln|S|+l\Leftrightarrow S=ae^{bU}$$

Thus, solutions of (5) are $W(x,y)=const$, where:

$$W(x,y)=\int_{x_0}^{x}P(x,y_0)S(x,y_0)\,dx+\int_{y_0}^{y}R(y)Q(x,y)S(x,y)\,dy+const$$
$$=\int_{x_0}^{x}ae^{bU(x,y_0)}\left.\frac{\partial U(x,y)}{\partial x}\right|_{y=y_0}dx+\int_{y_0}^{y}aR(y)e^{bU(x,y)}\frac{\partial U(x,y)}{\partial y}dy+const \quad (14)$$

Or:

$$W(x,y)=\int_{x_0}^{x}P(x,y)S(x,y)\,dx+\int_{y_0}^{y}R(y)Q(x_0,y)S(x_0,y)\,dy+const$$
$$=\int_{x_0}^{x}ae^{bU(x,y)}\frac{\partial U(x,y)}{\partial x}dx+\int_{y_0}^{y}aR(y)e^{bU(x_0,y)}\left.\frac{\partial U(x,y)}{\partial y}\right|_{x=x_0}dy+const \quad (15)$$

Example 2

Solve the QDE:

$$\left(\ln y+\frac{y}{x}\right)dx+(x+y\ln x)\,dy=0 \quad (16)$$

Equation (16) is quasi-exact where:

$$\begin{cases}P(x,y)=\ln y+\dfrac{y}{x}\\ Q(x,y)=\ln x+\dfrac{x}{y}\\ R(x,y)=y\end{cases}$$

The exact differential formula gives:

$$\begin{cases}U(x,y)=x\ln y+y\ln x\\ S(x,y)=ae^{bU(x,y)}=axye^{b(x+y)}\end{cases}$$

Thus, solution of (16) is $W(x,y)=const$, where:

$$W(x,y)=\int_{x_0}^{x} axy_0\left(\ln y_0+\frac{y_0}{x}\right)e^{b(x+y_0)}\,dx+\int_{y_0}^{y} axy(x+y\ln x)e^{b(x+y)}\,dy+const$$

Or:

$$W(x,y)=\int_{x_0}^{x} axy\left(\ln y+\frac{y}{x}\right)e^{b(x+y)}\,dx+\int_{y_0}^{y} ax_0y(x_0+y\ln x_0)e^{b(x_0+y)}\,dy+const$$

4. Bivariable generalized quasi-exact differential equation

This Section considers equation (13) in a generalized case that $\partial R/\partial x\neq 0$ and $\partial R/\partial y\neq 0$. Accordingly, we obtain:

$$S\partial^2U+\partial U\partial S=RS\partial^2U+R\partial U\partial S+S\partial R\partial S$$

$$\Leftrightarrow S\frac{\partial^2U}{\partial S^2}+\frac{\partial U}{\partial S}=RS\frac{\partial^2U}{\partial S^2}+R\frac{\partial U}{\partial S}+S\frac{\partial R}{\partial S}\Leftrightarrow S(1-R)\frac{\partial^2U}{\partial S^2}+(1-R)\frac{\partial U}{\partial S}-S\frac{\partial R}{\partial S}=0$$

Thus, we obtain a second-ordered partial differential equation:

$$\frac{\partial^2U}{\partial S^2}+\left(\frac{1}{S}\right)\frac{\partial U}{\partial S}+\left(\frac{1}{R-1}\right)\frac{\partial R}{\partial S}=0 \qquad (17)$$

Due to coherent characteristics, equation (17) is extremely hard to solve in a generalized manner, especially when $S(x,y)$ is not constant. On the other hand, a feasible direction is assuming that solutions of (17) are $U=U(S,R)$ and somehow transforming (17) to a linear second-ordered partial differential equation (see e.g. [2], [3], [4], and [5]).

5. Discussion

We have examined the quasi-exact differential equation which is obtained by modifying the exact differential form. We have shown that the QDE is solvable when the modifier is either constant or univariable. To solve the QDE comprehensively, we suggest three potential pathways for further researches which include (i) upgrading the order, e.g. second-ordered QDE; (ii) generalizing the number of variables, e.g. QDE in vector space; and (iii) generalizing the solution for the integrating factor as indicated by a complicated partial differential equation, i.e. (17).

Acknowledgement: This article is inspired by Prof. Dr. Jalil Hajimir-Toronto-Canada

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Solve the following partial differential equation:

$$\begin{cases} \dfrac{\partial u(x;t)}{\partial t} = k\dfrac{\partial^2 u(x;t)}{\partial x^2} \\ u(x;0) = 0 \\ u(0;t) = h \end{cases}$$

Of which, $x > 0$, $t > 0$, $k > 0$ and h are constant.The problem is a heat equation on $(0;+\infty)$ with homogeneous initial conditions and constant boundary conditions, and therefore has solutions expressed as:

$$u(x;t) = \int_0^t \frac{x}{\sqrt{4k\pi(t-s)^3}} exp\left[-\frac{x^2}{4k(t-s)}\right] h(s)\,ds = \frac{2h}{\sqrt{\pi}}\int_0^t \frac{x}{4\sqrt{k(t-s)^3}} exp\left[-\left(\frac{x}{2\sqrt{k(t-s)}}\right)^2\right] ds$$

Transforming $y = x/\left[2\sqrt{k(t-s)}\right]$, dy and boundary values are determined as:

$$\begin{cases} \dfrac{dy}{ds} = \dfrac{x}{4\sqrt{k(t-s)^3}} \\ \lim\limits_{s\to t^-} y = +\infty \\ y(s=0) = \dfrac{x}{2\sqrt{kt}} \end{cases}$$

Thus, we could rewrite the solution:

$$u(x;t) = \frac{2h}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{kt}}}^{+\infty} e^{-y^2}\,dy = h\left(\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{kt}}}^{0} e^{-y^2}\,dy + \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-y^2}\,dy\right)$$

$$= h\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{kt}}} e^{-y^2}\,dy\right) = h\left[1 - erf\left(\frac{x}{2\sqrt{kt}}\right)\right] = h \times erfc\left(\frac{x}{2\sqrt{kt}}\right)$$

In the above calculation, we employ the Poisson's integral (given $a > 0$) that:

$$I = \int_0^{+\infty} e^{-ax^2}\,dx = \int_0^{+\infty} e^{-ay^2}\,dy > 0 \Rightarrow I^2 = \iint_0^{+\infty} e^{-a(x^2+y^2)}\,dx\,dy$$

Replacing $x = r\cos\varphi$ and $y = r\sin\varphi$, we get $\{r;\varphi\} \in (0;\pi/2) \times (0;+\infty)$ and:

$$\frac{\partial(x;y)}{\partial(r;\varphi)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} = \cos\varphi & \dfrac{\partial x}{\partial \varphi} = -r\sin\varphi \\ \dfrac{\partial y}{\partial r} = \sin\varphi & \dfrac{\partial y}{\partial \varphi} = r\cos\varphi \end{vmatrix} = r$$

Henceforth, the square of Poisson's integral is:

$$I^2 = \int_0^{\frac{\pi}{2}}\int_0^{+\infty} e^{-ar^2}\, r\, dr\, d\varphi = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{+\infty} re^{-ar^2}\, dr = \frac{\pi}{2}\int_0^{+\infty} re^{-ar^2}\, dr = \frac{\pi}{4a}\int_0^{+\infty} e^{-ar^2}\, d(ar^2)$$

$$= \frac{\pi}{4a}\int_0^{+\infty} e^{-z}\, dz = \frac{\pi}{4a} e^{-z}\Big|_{+\infty}^{0} = \frac{\pi}{4a}$$

Therefore:

$$I = \frac{\sqrt{\pi}}{2\sqrt{a}}$$

Saigon, 11 août 2020

As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality.

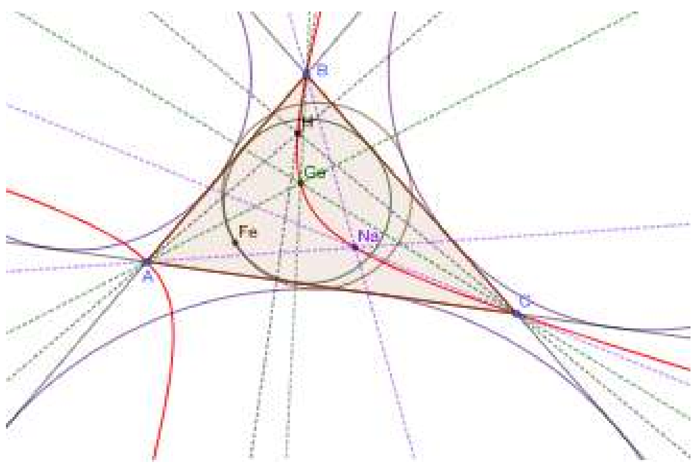
Albert Einstein

God exists since mathematics is consistent, and the Devil exists since we cannot prove it.

Andre Weil

ABOUT NAGEL'S AND GERGONNE'S CEVIANS-(V)

By Bogdan Fuştei-Romania



In ΔABC we show it that $\boldsymbol{s^2 = n_a^2 + 2r_a h_a}$ **(**and analogs)

$$s^2 - n_a^2 = (s + n_a)(s - n_a) = 2r_a h_a \Rightarrow \frac{s - n_a}{h_a} = \frac{2n_a}{s + n_a}$$

$\frac{s}{h_a} = \frac{n_a}{h_a} + \frac{2r_a}{s+n_a}$ (and analogs) $2S = a \cdot h_a = 2sr$ (and analogs)

$$\frac{a}{2r} = \frac{s}{h_a} + \frac{n_a}{s+n_a}$$

$$\boldsymbol{\frac{s}{r} = \sum_{cyc} \frac{n_a}{h_a} + 2\sum_{cyc} \frac{r_a}{n_a + s}}$$

$\frac{s-n_a}{r_a}=\frac{2h_a}{n_a+s}$ (and analogs) and $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$

So, we have: $\frac{s}{r}=\sum_{cyc}\frac{n_a}{h_a}+2\sum_{cyc}\frac{r_a}{n_a+s}$ but $h_a=\frac{2r_br_c}{r_b+r_c}\Rightarrow\frac{1}{h_a}=\frac{1}{2}\left(\frac{1}{r_b}+\frac{1}{r_c}\right)$ hence $\frac{n_a}{h_a}=\frac{1}{2}\left(\frac{n_a}{r_b}+\frac{n_a}{r_c}\right)$ (and analogs), summing, we get:

$$\mathbf{2\sum_{cyc}\frac{n_a}{h_a}=\sum_{cyc}\frac{n_b+n_c}{r_a};\quad (1)}$$

$$\mathbf{\frac{2s}{r}=2\sum_{cyc}\frac{n_a}{h_a}+\sum_{cyc}\frac{4r_a}{n_a+s};\quad (2)}$$

$$\mathbf{\frac{s}{r}=\sum_{cyc}\frac{n_a}{h_a}+\sum_{cyc}\frac{2r_a}{n_a+s};\quad (3)}$$

From (1), (2), (3) it follows that

$$\frac{3s}{r}=\frac{\sum n_a}{r}+2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}$$

We know that $s\ge r\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\Rightarrow\frac{3s}{r}=3\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\Rightarrow$

$$\frac{\sum n_a}{r}+2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge 3\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$$

Hence,

$$2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge 3\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-\frac{\sum n_a}{r}$$

Summing, it follows

$$4\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge 3\sqrt{3}\sum_{cyc}\frac{b+c}{a}-2\frac{\sum n_a}{r}\Leftrightarrow$$

$$\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\frac{3\sqrt{3}}{4}\sum_{cyc}\frac{b+c}{a}-\frac{\sum n_a}{2r}$$

Now,

$\frac{s}{r}\ge\sqrt{4-\frac{2r}{R}}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$ and $\frac{s}{r}\ge\sqrt{4-\frac{2r}{R}}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)$ hence

$$\frac{3s}{r}\ge 3\sqrt{4-\frac{2r}{R}}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\Leftrightarrow\frac{\sum n_a}{r}+2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge 3\sqrt{4-\frac{2r}{R}}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$$

$$2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge 3\sqrt{4-\frac{2r}{R}}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-\frac{\sum n_a}{r}$$

Similarly, we have:

$$2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\sqrt{4-\frac{2r}{R}}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)-\frac{\sum n_a}{r}$$

Adding, we get:

$$4\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge 3\sqrt{4-\frac{2r}{R}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)}-2\frac{\sum n_a}{r}$$

Hence,

$$\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge \frac{3}{4}\sqrt{4-\frac{2r}{R}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)}-\frac{\sum n_a}{2r}$$

$$h_a=\frac{2sr}{a}=\frac{(a+b+c)r}{a}\Rightarrow h_a=\left(1+\frac{b+c}{a}\right)r\Rightarrow\frac{h_a-r}{r}=\frac{b+c}{a}\Rightarrow$$

$$\sum_{cyc}\frac{b+c}{a}=\frac{\sum(h_a-r)}{r}$$

Hence,

$$\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\frac{3\sqrt{3}\sum h_a-2\sum n_a-9\sqrt{3}r}{4r}\Leftrightarrow\frac{9\sqrt{3}}{4}+\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\frac{3\sqrt{3}\sum h_a-2\sum n_a}{4r}$$

Now, we know that

$$\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge \frac{3}{4}\sqrt{4-\frac{2r}{R}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)}-\frac{\sum n_a}{2r}$$

Hence,

$$\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge \frac{3}{4}\sqrt{4-\frac{2r}{R}\cdot\frac{\sum(h_a-r)}{r}}-\frac{\sum n_a}{2r}$$

$$\frac{9}{4}\sqrt{4-\frac{2r}{R}}+\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\frac{3}{4}\sqrt{4-\frac{2r}{R}}\cdot\frac{\sum h_a}{4r}-\frac{\sum n_a}{2r}$$

We know that:

$$\frac{s}{r}\ge\sqrt{\left(4-\frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)},\qquad \frac{3s}{r}=\frac{\sum n_a}{r}+2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}$$

$$\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\sqrt{\left(4-\frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

And we know that:

$$s\ge\sqrt{r(4R+r)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}\Leftrightarrow\frac{s}{r}\ge\sqrt{\left(1+\frac{4R}{r}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$$

Hence,

$$\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\ge\sqrt{\left(1+\frac{4R}{r}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$$

And similarly,

$$\boldsymbol{s\ge\sqrt{r(4R+r)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}}$$

Hence,

$$\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq\sqrt{\left(1+\frac{4R}{r}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

$$\left(\frac{s}{r}\right)^2\geq\sqrt{\left(1+\frac{4R}{r}\right)^2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

$$\frac{s}{r}\geq\sqrt[4]{\left(1+\frac{4R}{r}\right)^2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}=Q$$

Therefore,

$$\boldsymbol{\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}\geq 0}$$

Now, we know that

$\frac{\sum AI}{r}\geq\frac{s}{r}+3(2-\sqrt{3})$ and $\frac{s}{r}=\frac{\sum n_a}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}$ it follows that

$$\frac{\sum(3AI-n_a)}{3r}\geq 3\left(2-\sqrt{3}\right)+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}$$

We known that $n_a+g_a\geq 2m_a$ (and analogs), then $n_a\geq 2m_a-g_a$

Hence,

$$\frac{s}{r}\geq\frac{\sum(2m_a-g_a)}{3r}+\frac{2}{3}\sum_{cyc}\frac{h_a+2r_a}{n_a+s}$$

$$h_a=\left(1+\frac{b+c}{a}\right)r;\ \frac{b+c}{a}=\frac{r_a+h_a}{r_a}=1+\frac{h_a}{r_a}$$

$$h_a=\left(2+\frac{h_a}{r_a}\right)r\Rightarrow r_ah_a=(2r_a+h_a)r$$

$bc=s^2+r_a^2-4Rr_a$, then we have $\frac{r_ah_a}{r}=2r_a+h_a$ (and analogs)

$$\frac{3s}{r}=\frac{\sum n_a}{r}+2\sum_{cyc}\frac{h_a+2r_a}{n_a+s}=\frac{\sum n_a}{r}+\frac{2}{r}\sum_{cyc}\frac{h_ar_a}{n_a+s}$$

Hence,

$$3s=n_a+n_b+n_c+2\sum_{cyc}\frac{h_ar_a}{n_a+s}$$

We know that $n_a+g_a\geq 2m_a\Rightarrow n_a\geq 2m_a-g_a$

$$3s\geq\sum_{cyc}(2m_a-g_a)+\ 2\sum_{cyc}\frac{h_ar_a}{n_a+s}$$

From $n_a+n_b+n_c\geq s\sqrt{3}\Rightarrow(n_a+n_b+n_c)\sqrt{3}\geq 3s=n_a+n_b+n_c+2\sum_{cyc}\frac{h_ar_a}{n_a+s}$

Therefore,

$$\frac{\sqrt{3}-1}{2}(n_a+n_b+n_c)\geq\sum_{cyc}\frac{h_ar_a}{n_a+s}$$

$$|b-c|\geq n_a-g_a\Rightarrow g_a+|b-c|\geq n_a\Rightarrow\frac{1}{n_a}\geq\frac{1}{g_a+|b-c|}\Rightarrow$$

$$3s \geq \frac{b+c}{2} \cdot cos\frac{A}{2} \Rightarrow \frac{m_a}{cos\frac{A}{2}} \geq \frac{b+c}{2}$$

Adding, it follows that

$$\sum_{cyc} \frac{m_a}{cos\frac{A}{2}} \geq \sum_{cyc} \frac{b+c}{2} = a+b+c = 2s$$

Hence,

$$\frac{1}{2}\sum_{cyc} \frac{m_a}{cos\frac{A}{2}} \geq s \Rightarrow \frac{3}{2}\sum_{cyc} \frac{m_a}{cos\frac{A}{2}} \geq 3s$$

$$\frac{3}{2}\sum_{cyc} \frac{m_a}{cos\frac{A}{2}} \geq n_a+n_b+n_c+2\sum_{cyc}\frac{h_a r_a}{n_a+s}$$

We know that $n_a g_a \geq m_a w_a \Rightarrow n_a \geq \frac{m_a w_a}{g_a}$ then

$$\boldsymbol{3s \geq \sum_{cyc} \frac{m_a w_a}{g_a} + 2\sum_{cyc} \frac{h_a r_a}{n_a+s}}, \qquad \boldsymbol{cosAcosBcosC = \frac{s^2-(2R+r)^2}{4R^2}}$$

If ΔABC is obtuse triangle, then $cosAcosBcosC \leq 0$ equality if triangle is right.

$$s^2-(2R+r)^2 \geq 0 \Rightarrow s \geq 2R+r \Rightarrow 3s \geq 3(2R+r)$$

$n_a+n_b+n_c+2\sum_{cyc}\frac{h_a r_a}{n_a+s} \geq 3(2R+2)$ for non-obtuse triangle

$n_a+n_b+n_c < 3(2R+2) - 2\sum_{cyc}\frac{h_a r_a}{n_a+s}$ for obtuse triangle.

We know that

$$\frac{s}{r} = \sum_{cyc}\frac{n_a}{h_a} + 2\sum_{cyc}\frac{r_a}{n_a+s}$$

$$\frac{s}{r} = \sum_{cyc}\frac{n_a}{r_a} + 2\sum_{cyc}\frac{h_a}{n_a+s}$$

So, for non-obtuse triangle we have:

$$\sum_{cyc}\frac{n_a}{h_a} \geq 1 + \frac{2R}{r} - 2\sum_{cyc}\frac{r_a}{n_a+s}$$

$$\sum_{cyc}\frac{n_a}{r_a} \geq 1 + \frac{2R}{r} - 2\sum_{cyc}\frac{h_a}{n_a+s}$$

$Let\ be\ P \in (ABC), A, B, C$ –non-collinear. If $PA = x; PB = y; PC = z$ then $ayz+bxz+cxy \geq abc$ (Cocea-Hayashi inequality)

Let be $P = N_a$ –Nagel's point, then $AN_a = \sqrt{(b-c)^2+4r^2}$ (and analogs).

But we show it that: $\frac{AN_a}{2r} = \frac{n_a}{h_a}$ (and analogs), hence

$$\frac{aBN_a \cdot CN_a}{4r^2} + \frac{bCN_a \cdot AN_a}{4r^2} + \frac{cAN_a \cdot BN_a}{4r^2} \geq \frac{abc}{4r^2} \Leftrightarrow$$

$$\frac{aBN_a \cdot CN_a}{4r^2} + \frac{bCN_a \cdot AN_a}{4r^2} + \frac{cAN_a \cdot BN_a}{4r^2} \geq \frac{R}{r} \cdot s$$

$bc = s^2 + r_a^2 - 4Rr_a$ (and analogs)

$$h_a h_b h_c = \frac{2S}{a} \cdot \frac{2S}{b} \cdot \frac{2S}{c} = \frac{2S^2}{R}$$

$$ah_a = bh_b = ch_c = 2S = 2sr$$

$$2S\sum_{cyc} n_b n_c \geq \frac{R}{r}\cdot s\cdot \frac{2S^2}{R} \Rightarrow 2sr\sum_{cyc} n_b n_c \geq 2s\cdot s^2 r \Rightarrow \sum_{cyc} n_b n_c \geq s^2$$

So, it follows a new inequality

$$\sum_{cyc} n_b n_c \geq s^2 = \sum_{cyc} r_a r_b$$

But $\sum n_a^2 \geq \sum n_a n_b \Rightarrow (\sum n_a)^2 \geq 3\sum n_a n_b \geq 3s^2$, hence

$$\sum n_a \geq s\sqrt{3}$$

Let be G_e –Gergonne's point, then $AA_1 = g_a = AG_a + G_a A_1$

From Van-Aubel's theorem, we have:

$$\frac{AG_e}{A_1 G_e} = \frac{s-a}{s-b} + \frac{s-a}{s-c}$$

$s(s-a) = r_b r_c$ (and analogs)

$$\frac{AG_e}{A_1 G_e} = \frac{s-a}{s-b} + \frac{s-a}{s-c} = \frac{r_b r_c}{r_a r_c} + \frac{r_b r_c}{r_a r_b} = \frac{r_b + r_c}{r_a}$$

Hence,

$\frac{AG_e}{A_1 G_e} = \frac{r_b + r_c}{r_a}$ (and analogs)

$$\frac{A_1 G_e}{AG_e} = \frac{r_a}{r_b + r_c} \Rightarrow \frac{A_1 G_e + AG_e}{AG_e} = \frac{r_a + r_b + r_c}{r_b + r_c};\ r_a + r_b + r_c = 4R + r$$

$AG_e = \frac{g_a(r_b + r_c)}{4R + r}$ (and analogs)

Adding, it follows that

$$\sum_{cyc} AG_e = \frac{1}{4R + r}\sum_{cyc} g_a(r_b + r_c)$$

$\frac{AG_e}{g_a} = \frac{r_b + r_c}{4R + r}$ (and analogs)

$\frac{AG_e}{g_a} + \frac{BG_e}{g_b} + \frac{CG_e}{g_c} = 2$ and $\frac{g_a}{4R + r} = \frac{AG_e}{r_b + r_c}$

Adding, it follows that

$$\sum_{cyc} \frac{AG_e}{r_b + r_c} = \frac{g_a + g_b + g_c}{r_a + r_b + r_c}$$

But $g_a \leq AI + r$ (and analogs), from triangle inequality, hence

$g_a + g_b + g_c \leq 3r + AI + BI + CI$ then

$$\sum_{cyc} \frac{AG_e}{r_b + r_c} \leq \frac{3r + AI + BI + CI}{r_a + r_b + r_c}$$

But $m_a + m_b + m_c \leq r_a + r_b + r_c$ hence,

$$\sum_{cyc} \frac{AG_e}{r_b + r_c} \leq \frac{3r + AI + BI + CI}{m_a + m_b + m_c}$$

$\frac{g_a}{AG_e} = \frac{4R + r}{r_b + r_c}; 2r_b r_c = h_a(r_b + r_c)$ (and analogs)

$$\frac{g_a}{AG_e} = \frac{h_a(r_b + r_c)}{2r_b r_c} \Rightarrow \frac{g_a}{h_a} = \frac{(r_b + r_c)AG_e}{2r_b r_c} = \frac{(4R + r)r_a \cdot AG_e}{2r_a r_b r_c};$$

$r_a r_b r_c = Ss; ah_a = bh_b = ch_c = 2S$

$$\frac{ag_a}{ah_a} = \frac{ag_a}{2S} = \frac{(4R + r)r_a \cdot AG_e}{2Ss} \Rightarrow ag_a = \frac{(4R + r)r_a \cdot AG_e}{s}$$

$$tan\frac{A}{2}=\frac{r_a}{s}; r_a+r_b+r_b=4R+r\Rightarrow ag_a=\left(tan\frac{A}{2}+tan\frac{B}{2}+tan\frac{C}{2}\right)r_aAG_e$$

which follows from Blundon's inequality $\frac{4R+r}{s}\geq\sqrt{4-\frac{2r}{R}}$ then

$$\frac{ag_a}{r_aAG_e}\geq\sqrt{4-\frac{2r}{R}}$$

Adding, it follows a new inequality

$$\frac{1}{3}\sum_{cyc}\frac{ag_a}{r_aAG_e}\geq\sqrt{4-\frac{2r}{R}}$$

From $ag_a=\frac{4R+r}{s}\cdot r_a\cdot AG_e$ we get

$$\sum_{cyc}ag_a=\frac{4R+r}{s}\sum_{cyc}r_aAG_e$$

$$\frac{\sum ag_a}{\sum r_aAG_e}=\frac{4R+r}{s}=\sum tan\frac{A}{2}\geq\sqrt{4-\frac{2r}{R}}$$

Let be $P\in Int(ABC)\Rightarrow\frac{PA}{a}+\frac{PB}{b}+\frac{PC}{c}\geq\sqrt{3}$

$AG_e=\frac{g_a(r_b+r_c)}{4R+r}\Rightarrow\frac{AG_e}{a}=\frac{g_a}{4R+r}\cdot\frac{r_b+r_c}{a}$ (and analogs)

$$a=\sqrt{(r_a-r)(r_b+r_c)};\ sin\frac{A}{2}=\sqrt{\frac{r_a-r}{4R}}; cos\frac{A}{2}=\sqrt{\frac{r_b+r_c}{4R}}$$

Hence,

$$cot\frac{A}{2}=\sqrt{\frac{r_b+r_c}{r_a-r}}=\frac{r_b+r_c}{\sqrt{(r_a-r)(r_b+r_c)}},\qquad cot\frac{A}{2}=\frac{r_b+r_c}{a}$$

$$\frac{AG_e}{a}=\frac{g_a}{4R+r}\cdot cot\frac{A}{2}\Rightarrow\frac{AG_e}{a}+\frac{BG_e}{b}+\frac{CG_e}{c}\geq\sqrt{3}$$

Therefore,

$$\boldsymbol{\sum_{cyc}g_a cot\frac{A}{2}\geq(r_a+r_b+r_c)\sqrt{3}}$$

$tan\frac{A}{2}=\frac{r_a}{s}=\frac{1}{cot\frac{A}{2}}\Rightarrow cot\frac{A}{2}=\frac{s}{r_a}$ (and analogs)

$g_a cot\frac{A}{2}=\frac{g_a}{r_a}\cdot s$ (and analogs)

Hence,

$$\frac{g_a}{r_a}+\frac{g_b}{r_b}+\frac{g_c}{r_c}\geq\frac{r_a+r_b+r_c}{s}\sqrt{3}$$

$$4R+r=r_a+r_b+r_c\geq s\sqrt{4-\frac{2r}{R}}\Rightarrow\frac{r_a+r_b+r_c}{s}\geq\sqrt{4-\frac{2r}{R}}$$

Therefore,

$$\frac{g_a}{r_a}+\frac{g_b}{r_b}+\frac{g_c}{r_c}\geq\sqrt{3\left(4-\frac{2r}{R}\right)}$$

But $g_a \le AI + r \Rightarrow \frac{g_a}{h_a} \le \frac{AI}{r_a} + \frac{r}{r_a}; \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$ hence,

$\frac{g_a}{r_a} + \frac{g_b}{r_b} + \frac{g_c}{r_c} \le 1 + \frac{AI}{r_a} + \frac{BI}{r_b} + \frac{CI}{r_c}$ and then

$$\boldsymbol{\frac{AI}{r_a} + \frac{BI}{r_b} + \frac{CI}{r_c} \ge \frac{r_a + r_b + r_c}{s}\sqrt{3} - 1}$$

$$\boldsymbol{\frac{AI}{r_a} + \frac{BI}{r_b} + \frac{CI}{r_c} \ge \sqrt{3\left(4 - \frac{2r}{R}\right)}}$$

For $P = I, I$ –incenter, hence Cocea-Hayashi inequality becomes:

$$a \cdot BI \cdot CI + b \cdot CI \cdot AI + c \cdot AI \cdot BI \ge abc = 3Rrs$$

$$AI = \frac{r}{\sin\frac{A}{2}} \Rightarrow AI \cdot BI \cdot CI = \frac{r^3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\sin\frac{A}{2} = \sqrt{\frac{rr_a}{bc}} \Rightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \sqrt{\frac{r^3 \cdot r_a r_b r_c}{4RS \cdot 4RS}} = \frac{r}{4R}$$

$$\Rightarrow AI \cdot BI \cdot CI = r^3 \cdot \frac{4R}{r} = 4Rr^2$$

$$\frac{a}{AI} + \frac{b}{BI} + \frac{c}{CI} \ge \frac{abc}{4Rr^2} = \frac{s}{r}$$

Now,

$$\frac{s}{r} = \frac{\sum n_a}{3r} + \frac{2}{3}\sum_{cyc}\frac{2r_a + h_a}{s + n_a} \Rightarrow \frac{a}{AI} + \frac{b}{BI} + \frac{c}{CI} \ge \frac{\sum n_a}{3r} + \frac{2}{3}\sum_{cyc}\frac{2r_a + h_a}{s + n_a}$$

But $AI = \sqrt{2R(h_a - 2r)}$, then

$$\sum_{cyc}\frac{a}{\sqrt{h_a - 2r}} \ge \frac{\sqrt{2R}}{3}\left(\frac{\sum n_a}{r} + 2\sum_{cyc}\frac{2r_a + h_a}{s + n_a}\right)$$

But $AI = \sqrt{(r_b - r)(r_c - r)}$, then

$$\sum_{cyc}\frac{a}{\sqrt{(r_b - r)(r_c - r)}} \ge \frac{\sqrt{2R}}{3}\left(\frac{\sum n_a}{r} + 2\sum_{cyc}\frac{2r_a + h_a}{s + n_a}\right)$$

We know that:

$$m_a^2 = r_b r_c + \frac{1}{4}(b - c)^2 \Rightarrow \frac{m_a^2}{r^2} = \frac{r_b r_c}{r^2} + \frac{(b - c)^2}{4r^2}$$

But $\frac{(b-c)^2}{4r^2} = \frac{n_a^2}{h_a^2} - 1$, hence $\frac{m_a^2}{r^2} = \frac{r_b r_c}{r^2} + \frac{n_a^2}{h_a^2} - 1$

Adding, it follows a new identity

$$\frac{\sum m_a^2}{r^2} = \frac{s^2}{r^2} + \sum_{cyc}\frac{n_a^2}{h_a^2} - 3$$

But $\sum\frac{n_a^2}{h_a^2} \ge \sum\frac{n_a n_b}{h_a h_b}$, then it follows that

$$\frac{\sum m_a^2}{r^2} \ge \left(\frac{s}{r}\right)^2 + \sum_{cyc}\frac{n_a n_b}{h_a h_b} - 3$$

But $\frac{s}{r} = \frac{\sum n_a}{3r} + \frac{2}{3}\sum_{cyc}\frac{2r_a + h_a}{s + n_a}$, hence

$$\frac{\sum m_a^2}{r^2} = \left(\frac{\sum n_a}{3r} + \frac{2}{3}\sum_{cyc}\frac{2r_a + h_a}{s + n_a}\right)^2 + \sum_{cyc}\frac{n_a^2}{h_a^2} - 3$$

Now, $\frac{m_a^2}{r^2} = \frac{n_a^2}{h_a^2} + \frac{r_b r_c - r^2}{r^2}$

Using QM-AM inequality $\sqrt{x^2 + y^2} \ge \frac{x+y}{\sqrt{2}}$ for $x^2 = \frac{n_a^2}{h_a^2}; y^2 = \frac{r_b r_c - r^2}{r^2}$, we get:

$$\frac{m_a}{r} \ge \frac{1}{\sqrt{2}}\left(\frac{n_a}{h_a} + \sqrt{\frac{r_b r_c - r^2}{r^2}}\right)$$

Adding, it follows a new inequality:

$$\frac{\sum m_a}{r} \ge \frac{1}{\sqrt{2}}\left(\sum_{cyc}\frac{n_a}{h_a} + \sum_{cyc}\sqrt{\frac{r_b r_c - r^2}{r^2}}\right)$$

But $(\sum m_a)^2 \le 4s^2 - 16Rr + 5r^2$ (Chu&Yang inequality), hence

$$\left(\frac{\sum m_a}{r}\right)^2 \le \frac{4s^2}{r^2} - \frac{16R}{r} + 5$$

$$\frac{s^2}{r^2} + \sum_{cyc}\frac{n_a^2}{h_a^2} - 3 \le \frac{4s^2}{r^2} - \frac{16R}{r} + 5 - 2\sum_{cyc}\frac{m_b m_c}{r^2}$$

Finally, it follows

$$\sum_{cyc}\frac{n_a^2}{h_a^2} + 2\sum_{cyc}\frac{m_b m_c}{r^2} \le \frac{3s^2}{r^2} - \frac{16R}{r} + 8$$

But $\sum\frac{n_a^2}{h_a^2} \ge \sum\frac{n_a n_b}{h_a h_b}$, hence

$$\sum_{cyc}\frac{n_a n_b}{h_a h_b} + 2\sum_{cyc}\frac{m_b m_c}{r^2} \le \frac{3s^2}{r^2} - \frac{16R}{r} + 8$$

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ABOUT AN INEQUALITY BY DAN RADU SECLĂMAN-I

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{m_a^2 + m_b^2 + m_c^2}{r_a + r_b + r_c} \le 2R - r$$

Proposed by Dan Radu Seclăman – Romania

***Solution**: We prove the following lemma:*

Lemma. 2) In ΔABC the following inequality holds:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}=\frac{3(s^2-r^2-4Rr)}{2(4R+r)}$$

Proof. *Using* $\sum m_a^2=\frac{3}{4}\sum a^2$, $\sum a^2$, $\sum a^2=2(s^2-r^2-4Rr)$ *and* $\sum r_a=4R+r$.

Let's get back to the main problem.

Using the Lemma the inequality can be rewritten:

$$\frac{3(s^2-r^2-4Rr)}{2(4R+r)}\le 2R-r\Leftrightarrow 3s^2\le 16R^2+8Rr+r^2$$

which follows from Euler's inequality $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark. *We obtain an inequality having an opposite sense.*

3) In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}\ge\frac{2r}{R}(2R-r)$$

Marin Chirciu

Solution: *Using Lemma, we write the inequality:*

$$\frac{3(s^2-r^2-4Rr)}{2(4R+r)}\ge\frac{2r}{R}(2R-r)\Leftrightarrow 3Rs^2\ge r(44R^2-5Rr-4r^2)$$

which follows from Gerretsen's inequality $s^2\ge 16Rr-5r^2$.

It remains to prove that:

$$3R(16Rr-5r^2)\ge r(44R^2-5Rr-4r^2)\Leftrightarrow 2R^2-5Rr+2r^2\ge 0\Leftrightarrow(R-2r)(2R-r\ge 0,$$

obviously from Euler's inequality $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark. *The double inequality can be written:*

4) In ΔABC the following relationship holds:

$$\frac{2r}{R}(2R-r)\le\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}\le 2R-r$$

Solution *See inequality 1) and inequality 3). Equality holds if and only if the triangle is equilateral.*

Reference:

Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT AN INEQUALITY BY D.M.BĂTINEȚU-GIURGIU-I

By Marin Chirciu – Romania

1) Let $x,y,z>0$ with the property that $xy+yz+zx=xyz$. Prove that:

$$\frac{x^2y^2}{z(1+xy)^2}+\frac{y^2z^2}{y(1+yz)^2}+\frac{z^2x^2}{y(1+zx)^2}\ge\frac{81}{100}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

Solution *We have* $xy+yz+xz=xyz\Leftrightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$. *Denoting* $\frac{1}{x}=a,\frac{1}{y}=b,\frac{1}{z}=c$ *we have* $a+b+c=1$. *We can reformulate the problem:*

2) If $a,b,c>0$ such that $a+b+c=1$ prove that:

$$\frac{a}{(1+bc)^2}+\frac{b}{(1+ca)^2}+\frac{c}{(1+ab)^2}\ge\frac{81}{100}$$

Proof. *Using Bergström's inequality we obtain:*

$$\sum\frac{a}{(1+bc)^2}=\sum\frac{a^2}{a(1+bc)^2}\ge\frac{(a+b+c)^2}{\sum a(1+bc)^2}=\frac{1}{\sum a(1+bc)^2}$$

It suffices to prove that $\frac{1}{\sum a(1+bc)^2}\ge\frac{81}{100}\Leftrightarrow\sum a(1+bc)^2\le\frac{100}{81}$

As $a,b,c>0$ such that $a+b+c=1$ there is an ΔABC with

$a=\tan\frac{B}{2}\tan\frac{C}{2},b=\tan\frac{C}{2}\tan\frac{A}{2},c=\tan\frac{A}{2}\tan\frac{B}{2}$, *and the inequality that we have to prove ca be written:*

$$\sum\tan\frac{B}{2}\tan\frac{C}{2}\left(1+\tan\frac{C}{2}\tan\frac{A}{2}\tan\frac{A}{2}\tan\frac{B}{2}\right)^2\le\frac{100}{81}\Leftrightarrow\sum\tan\frac{B}{2}\tan\frac{C}{2}\left(1+\frac{r}{s}\tan\frac{A}{2}\right)^2\le\frac{100}{81}\Leftrightarrow$$

$$\Leftrightarrow\sum\tan\frac{B}{2}\tan\frac{C}{2}\left(1+2\frac{r}{s}\tan\frac{A}{2}+\frac{r^2}{s^2}\tan^2\frac{A}{2}\right)\le\frac{100}{81}\Leftrightarrow$$

$$\Leftrightarrow\sum\tan\frac{B}{2}\tan\frac{C}{2}+2\frac{r}{s}\sum\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}+\frac{r^2}{s^2}\sum\tan^2\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\le\frac{100}{81}\Leftrightarrow$$

$$\Leftrightarrow\sum\tan\frac{B}{2}\tan\frac{C}{2}+2\frac{r}{s}\cdot3\prod\tan\frac{A}{2}+\frac{r^2}{s^2}\prod\tan\frac{A}{2}\sum\tan\frac{A}{2}\le\frac{100}{81}\Leftrightarrow$$

$$\Leftrightarrow1+2\frac{r}{s}\cdot3\frac{r}{s}+\frac{r^2}{s^2}\cdot\frac{r}{s}\cdot\frac{4R+r}{s}\le\frac{100}{81}\Leftrightarrow1+6\frac{r^2}{s^2}+\frac{r^3}{s^3}\cdot\frac{4R+r}{s}\le\frac{100}{81}\Leftrightarrow$$

$$\Leftrightarrow\frac{6r^2}{s^2}+\frac{r^3(4R+r)}{s^4}\le\frac{19}{81}\Leftrightarrow19s^4\ge486s^2r^2+81r^3(4R+r)\Leftrightarrow$$

$\Leftrightarrow s^2(19s^2-486r^2)\ge81r^3(4R+r)$, *which follows from Gerretsen's inequality*

$s^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$$(16Rr - 5r^2)[19(16Rr - 5r^2) - 486r^2] \geq 81r^3(4R + r) \Leftrightarrow$$

$\Leftrightarrow 1216R^2 - 278Rr + 706r^2 \geq 0 \Leftrightarrow (R - 2r)(1216R - 353r) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Observations.

O1. *Above we have used the known inequalities in triangle:*

$$\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1, \prod \tan\frac{A}{2} = \frac{r}{s} \text{ and } \sum \tan\frac{A}{2} = \frac{4R+r}{s}$$

O2. *Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2$ *follows from the remarkable distance in triangle* $GI^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr)$. *As* $GI^2 \geq 0$ *it follows* $s^2 + 5r^2 - 16Rr \geq 0 \Leftrightarrow$

$\Leftrightarrow s^2 \geq 16Rr - 5r^2$. *Equality holds if* $GI^2 = 0 \Leftrightarrow G \equiv 1 \Leftrightarrow \Delta ABC$ *is equilateral.*

Equality holds if and only if the triangle is equilateral.

Remark. *The problem can be developed.*

3) Let be $x, y, z > 0$ ***having the property that*** $xy + yz + zx = xyz$ ***and*** $n \geq 0$***. Prove that:***

$$\frac{x^2y^2}{z(n + xy)^2} + \frac{y^2z^2}{y(n + yz)^2} + \frac{z^2x^2}{y(n + zx)^2} \geq \frac{81}{(9n + 1)^2}$$

Marin Chirciu

Solution *We have* $xy + yz + zx = xyz \Leftrightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. *Denoting* $\frac{1}{x} = a, \frac{1}{y} = b, \frac{1}{z} = c$ *we have* $a + b + c = 1$. *We can reformulate the problem:*

4) If $a, b, c > 0$ ***such that*** $a + b + c = 1$ ***and*** $n \geq 0$ ***prove that:***

$$\frac{a}{(n + bc)^2} + \frac{b}{(n + ca)^2} + \frac{c}{(n + ab)^2} \geq \frac{81}{(9n + 1)^2}$$

Proof. *Using Bergström's inequality:*

$$\sum \frac{a}{(n + bc)^2} = \sum \frac{a^2}{a(n + bc)^2} \geq \frac{(a + b + c)^2}{\sum a(n + bc)^2} = \frac{1}{\sum a(n + bc)^2}$$

It suffices to prove that: $\frac{1}{\sum a(n+bc)^2} \geq \frac{81}{(9n+1)^2} \Leftrightarrow \sum a(n + bc)^2 \leq \frac{(9n+1)^2}{81}$

As $a, b, c > 0$ *such that* $a + b + c = 1$ *there is* ΔABC *with*

$a = \tan\frac{B}{2}\tan\frac{C}{2}, b = \tan\frac{C}{2}\tan\frac{A}{2}, c = \tan\frac{A}{2}\tan\frac{B}{2}$, *and the inequality that we have to prove can be written:*

$$\sum \tan\frac{B}{2}\tan\frac{C}{2}\left(n + \tan\frac{C}{2}\tan\frac{A}{2} \cdot \tan\frac{A}{2}\tan\frac{B}{2}\right)^2 \leq \frac{(9n + 1)^2}{81} \Leftrightarrow$$

$$\Leftrightarrow \sum \tan\frac{B}{2}\tan\frac{C}{2}\left(n+\frac{r}{s}\tan\frac{A}{2}\right)^2 \le \frac{(9n+1)^2}{81} \Leftrightarrow$$

$$\Leftrightarrow \sum \tan\frac{B}{2}\tan\frac{C}{2}\left(n^2+2n\frac{r}{s}\tan\frac{A}{2}+\frac{r^2}{s^2}\tan^2\frac{A}{2}\right) \le \frac{(9n+1)^2}{81} \Leftrightarrow$$

$$\Leftrightarrow n^2\sum \tan\frac{B}{2}\tan\frac{C}{2}+2n\frac{r}{s}\sum \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}+\frac{r^2}{s^2}\sum \tan^2\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \le \frac{(9n+1)^2}{81}$$

$$\Leftrightarrow n^2\sum \tan\frac{B}{2}\tan\frac{C}{2}+2n\frac{r}{s}\cdot 3\prod \tan\frac{A}{2}+\frac{r^2}{s^2}\prod \tan\frac{A}{2}\sum \tan\frac{A}{2} \le \frac{(9n+1)^2}{81} \Leftrightarrow$$

$$\Leftrightarrow n^2+2n\frac{r}{s}\cdot 3\frac{r}{s}+\frac{r^2}{s^2}\cdot\frac{r}{s}\cdot\frac{4R+r}{s} \le \frac{(9n+1)^2}{81} \Leftrightarrow n^2+6n\frac{r^2}{s^2}+\frac{r^3}{s^3}\cdot\frac{4R+r}{s} \le \frac{(9n+1)^2}{81}$$

$$\Leftrightarrow \frac{6nr^2}{s^2}+\frac{r^3(4R+r)}{s^4} \le \frac{18n+1}{81} \Leftrightarrow (18n+1)s^4 \ge 486ns^2r^2+81r^3(4R+r) \Leftrightarrow$$

$\Leftrightarrow s^2[(18n+1)s^2-486nr^2] \ge 81r^3(4R+r)$, *which follows from Gerretsen's inequality* $s^2 \ge 16Rr-5r^2$. *It remains to prove that:*

$$(16Rr-5r^2)[(18n+1)(16Rr-5r^2)-486nr^2] \ge 81r^3(4R+r) \Leftrightarrow$$

$$\Leftrightarrow 64(18n+1)R^2-(2664n+121)Rr+(720n-14)r^2 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)[64(18n+1)R+(7-360n)r] \ge 0$, *obviously from Euler's inequality* $R \ge 2r$.

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NESBITT'S TYPE INEQUALITIES WITH CONSTRAINT

By Florentin Vișescu-Romania

Let $a, b, c = [0, \infty)$ such that at most one is zero. Prove that:

1. If $8(a^2+b^2+c^2) = 11(ab+bc+ca)$ then:

$$\frac{69}{40} \le \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \le \frac{51}{28}$$

2. If $a^2+b^2+c^2 = 2(ab+bc+ca)$ then:

$$2 \le \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \le \frac{12}{5}$$ **(Lorian Saceanu)**

3. If $5(a^2+b^2+c^2) = 17(ab+bc+ca)$ then:

$$\frac{17}{5} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \le \frac{15}{4}$$

4. If $a^2 + b^2 + c^2 = 4(ab + bc + ca)$ then:

$$4 \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \le \frac{3(3\sqrt{2} + 16)}{4}$$

We will prove that:

Lemma 1:

If $a, b, c \in [0, \infty)$ such that at most one is zero then denoting $a + b + c = p$, $ab + bc + ca = q$ and $abc = r$, we have:
(1) $p^2 \ge 3q$

(2) $p^2 \in [3q, 4q] \Rightarrow r \in \left[\frac{9pq - 2p^3 - 2(p^2-3q)^{\frac{3}{2}}}{27}; \frac{9pq - 2p^3 + 2(p^2-3q)^{\frac{3}{2}}}{27}\right]$

(3) $p \in [4q, \infty) \Rightarrow r \in \left[0, \frac{9pq - 2p^3 + 2(p^2-3q)^{\frac{3}{2}}}{27}\right]$

Proof.

The equation which has a, b, c as roots is
$x^3 - px^2 + qx - r = 0$. We consider the function $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2 - px^2 + qx - r$. In order that $f(x) = 0$ to have three real positive roots it must that: $f'(x) = 3x^2 - 2px + q = 0$ to have real positive roots.
Then $\Delta = 4p^2 - 12q \ge 0 \Rightarrow p^2 \ge 3q$ (1)
Let $x_{1,2} = \frac{2p \pm 2\sqrt{p^2-3q}}{6} = \frac{p \pm \sqrt{p^2-3q}}{3}$ the root of $f'(x) = 0$
Then $f\left(\frac{p-\sqrt{p^2-3q}}{3}\right) = \left(\frac{p-\sqrt{p^2-3q}}{3}\right)^3 - p\left(\frac{p-\sqrt{p^2-3q}}{3}\right)^2 + q\left(\frac{p-\sqrt{p^2-3q}}{3}\right) - r =$
$= \frac{9pq-2p^3+2(p^2-3q)^{\frac{3}{2}}}{27} - r$ and

$$f\left(\frac{p+\sqrt{p^2-3q}}{3}\right) = \left(\frac{p+\sqrt{p^2-3q}}{3}\right)^3 - p\left(\frac{p+\sqrt{p^2-3q}}{3}\right)^2 + q\left(\frac{p+\sqrt{p^2-3q}}{3}\right) + r$$
$$= \frac{9pq - 3p^3 - 2(p^2-3q)^{\frac{3}{2}}}{27} - r$$

Obviously $f(0) = -r$ and $\lim_{x\to\infty} f(x) = \infty$
Then, according to Rolle's sequence and taking into account that:
$\frac{p-\sqrt{p^2-3q}}{3} \le \frac{p+\sqrt{p^2-3q}}{3}$ we obtain:

$$f\left(\frac{p+\sqrt{p^2-3q}}{3}\right) \le 0 \Rightarrow r \ge \frac{9p^2 - 2p^3 - 2(p^2-3q)^{\frac{3}{2}}}{27}$$
$$f\left(\frac{p-\sqrt{p^2-3q}}{3}\right) \ge 0 \Rightarrow r \le \frac{9pq - 2p^3 + 2(p^2-3q)^{\frac{3}{2}}}{27}$$
$$f(0) \le 0 \Rightarrow -r \le 0 \Rightarrow r \ge 0$$

In conclusion $r \in \left[\frac{9pq-2p^3-2(p^2-3q)^{\frac{3}{2}}}{27}; \frac{9pq-2p^3+2(p^2-3q)^{\frac{3}{2}}}{27}\right] \cap [0,\infty)$

Taking into account that $p^2 \geq 3q$ we prove that $9pq - 2p^3 + 2(p^2-3q)^{\frac{3}{2}} \geq 0$, namely $2(p^2-3q)^{\frac{3}{2}} \geq 2p^3 - 9pq$.We prove that $2(p^2-3q)^{\frac{3}{2}} \geq p(2p^2-9q)$

If $2p^2 - 9q \leq 0$ namely $p^2 \leq \frac{9}{2}q$ the inequality is obvious. If $p^2 \geq \frac{9}{2}q$ the inequality is equivalent with $4(p^2-3q)^3 \geq p^2(2p^2-9q)^2 \Leftrightarrow q^2(p^2-4q) \geq 0$ true.

We study the expression's sign $9pq - 2p^3 - 2(p^2-3q)^{\frac{3}{2}}$

We establish the relationship between p^2 and q in order that the expression to be positive.

$$9pq - 2p^3 - 2(p^2-3q)^{\frac{3}{2}} \geq 0 \Leftrightarrow 9pq - 2p^3 \geq 2(p^2-3q)^{\frac{3}{2}} \Leftrightarrow$$
$$\Leftrightarrow p(9q-2p^2) \geq 2(p^2-3q)^{\frac{3}{2}}$$

Obviously for $p^2 > \frac{9}{2}q$ the expression is negative.

For $p^2 \leq \frac{9}{2}q$ (and $p^2 \geq 3q$) the inequality is equivalent with $p^2(9q-2p^2)^2 \geq 4(p^2-3q)^3$

$$\Leftrightarrow q^2(4q-p^2) \geq 0 \Leftrightarrow p^2 \leq 4q$$

Hence for $p^2 \in [3q,4q]$ the expression we study is positive and for $p^2 \leq [4q,\infty)$ the expression is negative.We obtain:

For $p^2 \in [3q,4q] \Rightarrow r \in \left[\frac{9pq-2p^3-2(p^2-3q)^{\frac{3}{2}}}{27}; \frac{9pq-3p^3+2(p^2-3q)^{\frac{3}{2}}}{27}\right]$

For $p^2 \in [4q,\infty) \Rightarrow r \in \left[0, \frac{9pq-2p^3+2(p^2-3q)^{\frac{3}{2}}}{27}\right]$

Lemma 2.

If $a,b,c \in [0,\infty)$ such that at most one is zero, then denoting $a+b+c=p$, $ab+bc+ca=q$ and $abc=r$ we have:

(1) It exist $t \in [0,1)$ such that $\frac{p^2}{q} = \frac{3}{1-t^2}$

(2) If $t \in \left[0,\frac{1}{2}\right] \Rightarrow \frac{r}{p^3} \in \left[\frac{(t+1)^2(1-2t)}{27}; \frac{(t-1)^2(2t+1)}{27}\right]$

(3) If $t \in \left[\frac{1}{2};1\right) \Rightarrow \frac{r}{s^3} \in \left[0, \frac{(t-1)^2(2t+1)}{27}\right]$

Proof.

According to Lemma 1 (1) $\frac{p^2}{q} \geq 3$. We prove that it exists $t \in [0,1)$ such that $\frac{p^2}{q} = \frac{3}{1-t^2}$

$$\Leftrightarrow 1 - t^2 = \frac{3q}{p^2} \Leftrightarrow t^2 = \frac{p^2-3q}{p^2} \Rightarrow t = \pm\frac{\sqrt{p^2-3q}}{p}$$

We choose $t \in \frac{\sqrt{p^2-3q}}{p}$ and we prove $t \in [0,1)$

Obviously $t \geq 0$. We prove $t < 1 \Leftrightarrow \frac{\sqrt{p^2-3q}}{p} < 1 \Leftrightarrow p^2 - 3q < p^2 \Leftrightarrow -3q < 0 \Leftrightarrow q > 0$ true $(a,b,c \in [0,\infty)$ at most one is zero)

So, it exists $t \in [0,1)$ such that $\frac{p^2}{q} = \frac{3}{1-t^2}$ (1), $t \in \frac{\sqrt{p^2-3q}}{p}$

According to Lemma 1

(2) $\frac{p^2}{q} \in [3,4] \Rightarrow \frac{r}{p^3} \in \left[\frac{9\frac{q}{p^2}-2-2\left(1-3\frac{q}{p^2}\right)^{\frac{3}{2}}}{27}; \frac{9\frac{q}{p^2}-2+2\left(1-3\frac{q}{p^2}\right)^{\frac{3}{2}}}{27}\right]$

(3) $\frac{p^2}{q} \in [4,\infty) \Rightarrow \frac{r}{p^3} \in \left[0, \frac{9\frac{q}{p^2}-2+2\left(1-3\frac{q}{p^2}\right)^{\frac{3}{2}}}{27}\right]$

We replace in these relationships $\frac{p^2}{q} = \frac{3}{1-t^2}$ and we obtain:

(2) $\frac{3}{1-t^2} \in [3,4] \Rightarrow \frac{3}{p^3} \in \left[\frac{9\frac{1-t^2}{3}-2-2\left(1-3\frac{1-t^2}{3}\right)^{\frac{3}{2}}}{27}; \frac{9\frac{1-t^2}{3}-2+2\left(1-3\frac{1-t^2}{3}\right)^{\frac{3}{2}}}{27}\right]$

(3) $\frac{3}{1-t^2} \in [4,\infty) \Rightarrow \frac{r}{p^3} \in \left[0, \frac{9\frac{1-t^2}{3}-2+2\left(1-3\frac{1-t^2}{3}\right)^{\frac{3}{2}}}{27}\right]$

After we make the calculus the relationship becomes:

(2) $t \in \left[0, \frac{1}{2}\right] \Rightarrow \frac{r}{p^3} \in \left[\frac{-2t^3-3t^2+1}{27}; \frac{2t^3-3t^2+1}{27}\right]$

(3) $t \in \left[\frac{1}{2}, 1\right) \Rightarrow \frac{r}{s^3} \in \left[0, \frac{2t^3-3t^2+1}{27}\right]$

or

(2) $t \in \left[0, \frac{1}{2}\right] \Rightarrow \frac{r}{p^3} \in \left[\frac{(t+1)^2(1-2t)}{27}; \frac{(t-1)^2(2t+1)}{27}\right] = D_1$

(3) $t \in \left[\frac{1}{2}, 1\right) \Rightarrow \frac{r}{p^3} \in \left[0, \frac{(t-1)^2(2t+1)}{27}\right] = D_2$

NESBITT WITH CONSTRAINT

Theorem: Let $a, b, c \in [0,\infty)$ such that at most one is zero. Let $a+b+c=p$, $ab+bc+ca=q, abc=r$ and $t \in [0,1)$ such that $\frac{p^2}{q} = \frac{3}{1-t^2}$

(1) If $t \in \left[0, \frac{1}{2}\right]$ then:

$$\frac{3}{2}\frac{2t^2+t+2}{(t+1)(2-t)} \le \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \le \frac{3}{2}\frac{2t^2-t+2}{(1-t)(t+2)}$$

(2) If $t \in \left[\frac{1}{2}, 1\right)$ then:

$$\frac{1+2t^2}{1-t^2} \le \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \le \frac{3}{2}\frac{2t^2-t+2}{(1-t)(t+2)}$$

Proof.

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} = \frac{a}{p-a}+\frac{b}{p-b}+\frac{c}{p-c} =$$

$$= \frac{a(p-b)(p-c)+b(p-a)(p-c)+c(p-a)(p-b)}{(p-a)(p-b)(p-c)} =$$

$$= \frac{p^3-2pq+3r}{pq-r} = \frac{1-2\frac{q}{p^2}+3\frac{r}{p^3}}{\frac{q}{p^2}-\frac{r}{p^3}} =$$

$$=\frac{1-2\frac{1-t^2}{3}+3\frac{r}{p^3}}{\frac{1-t^2}{3}-\frac{r}{p^3}}=\frac{\frac{3-2+2t^2}{3}+3\frac{r}{p^3}}{\frac{1-t^2}{3}-\frac{r}{p^3}}=\frac{\frac{1+t^2}{3}+3\frac{r}{p^3}}{\frac{1-t^2}{3}-\frac{r}{p^3}}$$

We denote $\frac{r}{p^3}=x$ and we consider the function:

$f(x)=\frac{\frac{1+2t^2}{3}+3x}{\frac{1-t^2}{3}-x}$; $f: D_1\to\mathbb{R}$ if $t\in\left[0,\frac{1}{2}\right]$ or $f: D_2\to$ if $t\in\left[\frac{1}{2},1\right)$

$$f'(x)=\frac{3\left(\frac{1-t^2}{3}-x\right)+\left(\frac{1+2t^2}{3}+3x\right)}{\left(\frac{1-t^2}{3}-x\right)^2}=\frac{1-t^2-3x+\frac{1+2t^2}{3}+3x}{\left(\frac{1-t^2}{3}-x\right)^2}$$

$$=\frac{3-3t^2+1+2t^2}{3\left(\frac{1-t^2}{3}-x\right)^2}=\frac{4-t^2}{3\left(\frac{1-t^2}{3}-x\right)^2}>0$$

So, f strictly increasing on D_1 and D_2

Then

(1) for $t\in\left[0,\frac{1}{2}\right]$

$$f\left(\frac{(t+1)^2(1-2t)}{27}\right)\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le f\left(\frac{(t-1)^2(2t+1)}{27}\right)$$

(2) for $t\in\left[\frac{1}{2},1\right)$

$$f(0)\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le f\left(\frac{(t-1)^2(2t+1)}{27}\right)$$

namely, after calculus

(1) For $t\in\left[0,\frac{1}{2}\right]$: $\frac{3}{2}\frac{2t^2+t+2}{(t+1)(2-t)}\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{3}{2}\frac{2t^2-t+2}{(1-t)(2+t)}$

(2) For $t\in\left[\frac{1}{2},1\right)$: $\frac{1+2t^2}{1-t^2}\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\le\frac{3}{2}\frac{2t^2-t+2}{(1-t)(2+t)}$

Observation. For $t=\frac{1}{3}$ we obtain ex 1), $t=\frac{1}{2}$ we obtain ex 2), $t=\frac{2}{3}$ we obtain ex 3) and for $t=\frac{\sqrt{2}}{2}$ we obtain ex 4).

Refference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-I

By Marin Chirciu – Romania

1) Let be ΔABC, AD, BE and CF the internal bisectors. Prove that:

$$AF\cdot BC+BD\cdot AC+CE\cdot AB\ge 18r^2$$

Proposed by Marian Ursărescu – Romania

Solution: *We prove the following lemma:*

Lemma:

2) Let be ΔABC, AD, BE and CF the internal bisectors. Prove that:

$$AF\cdot BC+BD\cdot AC+CE\cdot AB=2Rr\cdot\frac{5s^2+r^2+4Rr}{s^2+r^2+2Rr}$$

Solution: *Using the bisector's theorem in* ΔABC *the following relationship holds:*

$BD=\frac{ac}{b+c}, CE=\frac{ab}{a+c}, AF=\frac{bc}{a+b}$. *It follows that:*

$$AF\cdot BC+BD\cdot AC+CE\cdot AB=\frac{bc}{a+b}\cdot a+\frac{ab}{a+c}\cdot c+\frac{ab}{a+c}\cdot c=abc\sum\frac{1}{b+c}=$$

$=4Rrs\cdot\frac{5s^2+r^2+4Rr}{2s(s^2+r^2+2Rr)}=2Rr\cdot\frac{5s^2+r^2+4Rr}{s^2+r^2+2Rr}$, *which follows from* $\sum\frac{1}{b+c}=\frac{5s^2+r^2+4Rr}{2s(s^2+r^2+2Rr)}$.

Let's get back to the main problem.

Using Lemma the inequality from enunciation can be written:

$2Rr\cdot\frac{5s^2+r^2+4Rr}{s^2+r^2+2Rr}\ge 18r^2\Leftrightarrow s^2(5R-9r)\ge r(9r^2+17Rr-4R^2)$, *which follows from Gerretsen's inequality* $s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$(16Rr-5r^2)(5R-9r)\ge r(9r^2+17Rr-4R^2)\Leftrightarrow 14R^2-31Rr+6r^2\ge 0\Leftrightarrow$

$(R-2r)(14R-3r)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark: *The inequality can be strengthened:*

3) Let be $\mathbf{\Delta ABC}$, $\mathbf{AD}$, $\mathbf{BE}$ ***and*** $\mathbf{CF}$ ***be the internal bisectors. Prove that:***

$$AF\cdot BC+BD\cdot AC+CE\cdot AB\ge 9Rr$$

Marin Chirciu

Solution: *Using the Lemma the inequality can be written:*

$2Rr\cdot\frac{5s^2+r^2+4Rr}{s^2+r^2+2Rr}\ge 9Rr\Leftrightarrow s^2\ge 10Rr+7r^2$ *which follows from Gerretsen's inequality* $s^2\ge 16Rr-5r^2$. *It remains to prove that:* $16Rr-5r^2\ge 10Rr+7r^2\Leftrightarrow R\ge 2r$ *(Euler's inequality). Equality holds if and only if the triangle is equilateral.*

Remark: *Inequality 3) is stronger than inequality 1):*

4) Let be $\mathbf{\Delta ABC}$, $\mathbf{AD}$, $\mathbf{BE}$ ***and*** $\mathbf{CF}$ ***be the internal bisectors. Prove that:***

$$AF\cdot BC+BD\cdot AC+CE\cdot AB\ge 9Rr\ge 18r^2$$

Solution: *See inequalities 1), 3) and* $9Rr\ge 18r^2\Leftrightarrow R\ge 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark: *Let's find an inequality having an opposite sense:*

5) Let be $\mathbf{\Delta ABC}$, $\mathbf{AD}$, $\mathbf{BE}$ ***and*** $\mathbf{CF}$ ***the internal bisectors. Prove that:***

$$AF\cdot BC+BD\cdot AC+CE\cdot AB\le\frac{9R^2}{4}$$

Marin Chirciu

Solution: *Using Lemma the inequality can be written:*

$$2Rr\cdot\frac{5s^2+r^2+4Rr}{s^2+r^2+2Rr}\le\frac{9R^2}{2}\Leftrightarrow s^2(9R-20r)+r(18R^2-7Rr-4r^2)\ge 0$$

We distinguish the following cases:

Case 1). If $(9R-20r)\ge 0$, *the inequality is obvious.*

Case 2). If $(9R-20r)<0$, *the inequality can be rewritten:*

$r(18R^2-7Rr-4r^2)\ge s^2(20r-9R)$, *which follows from Gerretsen's inequality* $s^2\le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$r(18R^2-7Rr-4r^2)\ge(4R^2+4Rr+3r^2)(20r-9R)\Leftrightarrow$$

$$\Leftrightarrow 18R^3-13R^2r-30Rr^2-32r^3\ge 0\Leftrightarrow(R-2r)(18R^2+23Rr+16r^2)\ge 0,$$

obviously from Euler's inequality $R\ge 2r$. *Equality holds if and only if the triangle is equilateral.*

Remark: *We can write the double inequality:*

6) Let be $\Delta ABC, AD, BE$ and CF the internal bisectors. Prove that:

$$9Rr \leq AF \cdot BC + BD \cdot AC + CE \cdot AB \leq \frac{9R^2}{2}$$

Marin Chirciu

Solution: *See inequalities 3) and 5). Equality holds if and only if the triangle is equilateral.*

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-I

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{1}{R} \leq \frac{1}{r_a + r_b} + \frac{1}{r_b + r_c} + \frac{1}{r_c + r_a} \leq \frac{R}{4r^2}$$

Proposed by George Apostolopoulos – Messolonghi-Greece

Solution: *We prove the following identity:*

Lemma:

2) In ΔABC the following relationship holds:

$$\frac{1}{r_a + r_b} + \frac{1}{r_b + r_c} + \frac{1}{r_c + r_a} = \frac{s^2 + (4R + r)^2}{4Rs^2}$$

Proof:

Using the formula $r_a = \frac{S}{s-a}$ we obtain:

$\sum \frac{1}{r_b + r_c} = \sum \frac{1}{\frac{S}{s-b} + \frac{S}{s-c}} = \frac{1}{S} \sum \frac{(s-b)(s-c)}{a} = \frac{1}{rs} \cdot \frac{r[s^2 + (4R+r)^2]}{4Rs} = \frac{s^2 + (4R+r)^2}{4Rrs^2}$, *which follows from the known inequality in triangle:* $\sum \frac{(s-b)(s-c)}{a} = \frac{r[s^2 + (4R+r)^2]}{4Rs}$

Let's get back to the main problem.LHS inequality.
Using the Lemma can be written:

$\frac{s^2 + (4R+r)^2}{4Rrs^2} \geq \frac{1}{R} \Leftrightarrow (4R + r)^2 \geq 3s^2$, *(**Doucet's inequality**), which follows from Gerretsen's inequality* $s^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*
$(4R + r)^2 \geq 3(4R^2 + 4Rr + 3r^2) \Leftrightarrow R^2 - Rr - 2r^2 \geq 0 \Leftrightarrow (R - 2r)(R + r) \geq 0$, *obviously from Euler's inequality $R \geq 2r$. Equality holds if and only if the ΔABC is equilateral. RHS inequality. Using the Lemma the inequality holds:*

$\frac{s^2 + (4R+r)^2}{4Rs^2} \leq \frac{R}{4r^2} \Leftrightarrow s^2(R^2 - r^2) \geq r^2(4R + r)^2$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:* $\frac{r(4R+r)^2}{R+r}(R^2 - r^2) \geq r^2(4R + r)^2$
$\Leftrightarrow R \geq 2r$. *(Euler's inequality). Equality holds if and only if the triangle is equilateral.*

Remark: *Inequality 1) can be strenghtened:*

3) In ΔABC the following relationship holds:

$$\frac{1}{4R}\left(5 - \frac{2r}{R}\right) \leq \frac{1}{r_a + r_b} + \frac{1}{r_b + r_c} + \frac{1}{r_c + r_a} \leq \frac{1}{2r}$$

Marin Chirciu

Solution: *Using the Lemma the inequality can be written:*

$$\frac{s^2+(4R+r)^2}{4Rs^2} \ge \frac{1}{4R}\left(5-\frac{2r}{R}\right) \Leftrightarrow s^2 \le \frac{R(4R+r)^2}{2(2R-r)}$$ *(Blundon-Gerretsen's inequality)*

Equality holds if and only if the triangle is equilateral. RHS inequality.

Using Lemma the inequality can be written:

$\frac{s^2+(4R+r)^2}{4Rs^2} \le \frac{1}{2r} \Leftrightarrow s^2 \ge \frac{r(4R+r)^2}{2R-r}$, *which follows from Gerretsen's inequality* $s^2 \ge 16Rr-5r^2 \ge \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:* $\frac{r(4R+r)^2}{R+r} \ge \frac{r(4R+r)^2}{2R-r} \Leftrightarrow R \ge 2r$ *(Euler's inequality). Equality holds if and only if the triangle is equilateral.*

Remark: *Inequality 3) is stronger than inequality 1).*

4) In ΔABC the following inequality holds:

$$\frac{1}{R} \le \frac{1}{4R}\left(5-\frac{2r}{R}\right) \le \frac{1}{r_a+r_b}+\frac{1}{r_b+r_c}+\frac{1}{r_c+r_a} \le \frac{1}{2r} \le \frac{R}{4r^2}$$

Solution:*See 3) and Euler's inequality $R \ge 2r$. Equality holds if and only the ΔABC is equilateral.*

Remark: *If we replace r_a with h_a we propose:*

5) In ΔABC the following relationship holds:

$$\frac{1}{R} \le \frac{1}{h_a+h_b}+\frac{1}{h_b+h_c}+\frac{1}{h_c+h_a} \le \frac{1}{2r}$$

Proposed by Marin Chirciu – Romania

Solution:*We prove the followign lemma:*

Lemma:

6) In ΔABC the following relationship holds:

$$\frac{1}{h_a+h_b}+\frac{1}{h_b+h_c}+\frac{1}{h_c+h_a} = \frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{4rs^2(s^2+2r^2+2Rr)}$$

Proof:

Using the formula $h_a = \frac{2S}{a}$ we obtain:

$$\sum\frac{1}{h_b+h_c} = \sum\frac{1}{\frac{2S}{b}+\frac{2S}{c}} = \frac{1}{2S}\sum\frac{bc}{b+c} = \frac{1}{2rs}\cdot\frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{2s(s^2+2r^2+2Rr)} =$$

$= \frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{4rs^2(s^2+2r^2+2Rr)}$, *which follows from the known identity in triangle:*

$$\sum\frac{bc}{b+c} = \frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{2s(s^2+2r^2+2Rr)}$$

Let's get back to the main problem. ***LHS:****Using the Lemma we write the inequality:*

$$\frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{4rs^2(s^2+2r^2+2Rr)} \ge \frac{1}{R} \Leftrightarrow$$

$$\Leftrightarrow s^2[s^2(R-4r)+r(16R^2-6Rr-4r^2)]+Rr^2(4R+r)^2 \ge 0$$

We distinguish the following cases:

Case 1). If $[s^2(R-4r)+r(16R^2-6Rr-4r^2)] \ge 0$, the inequality is obviously.

Case 2). If $[s^2(R-4r)+r(16R^2-6Rr-4r^2)] < 0$, we write the inequality:

$Rr^2(4R+r)^2 \ge s^2[s^2(4r-R)-r(16R^2-6Rr-4r^2)]$, *which follows from Blundon – Gerretsen's inequality* $s^2 \le \frac{R(4R+r)^2}{2(2R-r)} \le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$Rr^2(4R+r)^2 \ge \frac{R(4R+r)^2}{2(2R-r)}\left[(4R^2+4Rr+3r^2)(4r-R)-r(16R^2-6Rr-4r^2)\right] \Leftrightarrow$$

$\Leftrightarrow 4R^3 + 4R^2r - 15Rr^2 - 18r^3 \geq 0 \Leftrightarrow (R-2r)(2R+r)^2 \geq 0$, *obviously from Euler's inequality* $R \geq 2r$. *Equality holds if and only if* ΔABC *is equilateral.*

RHS *inequality. Using the Lemma the inequality holds:*

$\frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{4rs^2(s^2+2r^2+2Rr)} \leq \frac{1}{2r} \Leftrightarrow s^2(s^2-12Rr) \geq r^2(4R+r)^2$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:*

$\frac{r(4R+r)^2}{R+r}(16Rr - 5r^2 - 12Rr) \geq r^2(4R+r)^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality)*

Equality holds if and only if ΔABC *is equilateral.*

Remark: *Between the sums* $\sum\frac{1}{r_b+r_c}$ *and* $\sum\frac{1}{h_b+h_c}$ *the following inequality holds:*

7) In ΔABC ***the following relationship holds:***

$$\sum\frac{1}{r_b+r_c} \leq \sum\frac{1}{h_b+h_c}$$

Marin Chirciu

Solution: *Using the following Lemma the above inequality holds:*

$$\frac{s^2+(4R+r)^2}{4Rs^2} \leq \frac{s^4+s^2(16Rr+2r^2)+r^2(4R+r)^2}{4rs^2(s^2+2r^2+2Rr)} \Leftrightarrow$$

$\Leftrightarrow s^2[s^2(R-r) - 2r^2(4R+r)] \geq r^2(4R+r)^2(R+r)$, *which follows from Gerretsen's inequality* $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$. *It remains to prove that:*

$$\frac{r(4R+r)^2}{R+r}[(16Rr-5r^2)(R-r) - 2r^2(4R+r)] \geq r^2(4R+r)^2(R+r) \Leftrightarrow$$

$\Leftrightarrow 15R^2 - 31Rr + 2r^2 \geq 0 \Leftrightarrow (R-2r)(15R-r) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$. *Equality holds if and only if the triangle is equilateral.*

Remark: *We can write the inequalities:*

8) In ΔABC ***the following relationship holds:***

$$\frac{1}{R} \leq \sum\frac{1}{r_b+r_c} \leq \sum\frac{1}{h_b+h_c} \leq \frac{1}{2r}$$

Solution: *See inequalities 1), 7) and 5). Equality holds if and only if* ABC *triangle is equilateral.*

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY VASILE MIRCEA POPA-I

By Marin Chirciu – Romania

1) In acute angled triangle ABC **the following relationship holds:**

$$\tan A + \tan B + \tan C + 3\sqrt{3} \geq 2\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right)$$

Proposed by Vasile Mircea Popa – Romania

Solution: *Using Popoviciu's inequality: If* $f:[a,b] \to \mathbb{R}$ *is a convexe function on* $[a,b]$, *then the following inequality holds:*

$$\frac{f(x)+f(y)+f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right],$$

$\forall x,y,z\in[a,b]$. *We consider the convexe function* $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=\tan x$, *for which we apply Popoviciu's inequality. We obtain:*

$$\frac{\tan A+\tan B+\tan C}{3}+\tan\left(\frac{A+B+C}{3}\right)\geq\frac{2}{3}\left[\tan\left(\frac{A+B}{2}\right)+f\left(\frac{B+C}{2}\right)+f\left(\frac{C+A}{2}\right)\right], \text{ wherefrom}$$

$$\frac{\tan A+\tan B+\tan C}{3}+\tan\left(\frac{\pi}{3}\right)\geq\frac{2}{3}\left[\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)+\tan\left(\frac{\pi}{2}-\frac{A}{2}\right)+\tan\left(\frac{\pi}{2}-\frac{B}{2}\right)\right]\Leftrightarrow$$

$$\Leftrightarrow\frac{\tan A+\tan B+\tan C}{3}+\sqrt{3}\geq\frac{2}{3}\left[\cot\frac{C}{2}+\cot\frac{A}{2}+\cot\frac{B}{2}\right]\Leftrightarrow$$

$$\Leftrightarrow\tan A+\tan B+\tan C+3\sqrt{3}\geq 2\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)$$

Equality holds if and only if the triangle is equilateral.

Remark: *In the same way we can propose:*

2) In acute triangle ABC the following relationship holds:

$$\cot A+\cot B+\cot C+\sqrt{3}\geq 2\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

Marin Chirciu

Solution: *We apply Popoviciu's inequality for the convexe function*

$$f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=\cot x.$$

3) In ΔABC the following relationship holds:

$$\frac{1}{\sin^2 A}+\frac{1}{\sin^2 B}+\frac{1}{\sin^2 C}+4\geq 2\left(\frac{1}{\cos^2\frac{A}{2}}+\frac{1}{\cos^2\frac{B}{2}}+\frac{1}{\cos^2\frac{C}{2}}\right)$$

Marin Chirciu

Solution: *We apply Popoviciu's inequality for the convexe function*

$$f:(0,\pi)\to\mathbb{R}, f(x)=\frac{1}{\sin^2 x}$$

4) In ΔABC the following inequality holds:

$$\frac{r_a}{r_a-r}+\frac{r_b}{r_b-r}+\frac{r_c}{r_c-r}+\frac{9}{2}\geq\frac{4s^2}{s^2+r_ar_b}+\frac{4s^2}{s^2+r_br_c}+\frac{4s^2}{s^2+r_cr_a}$$

Marin Chirciu

Solution: *We apply Popoviciu's inequality for the convexe function* $f:(0,1)\to\mathbb{R}$, $f(x)=\frac{1}{1-x}$, *for* $x=\frac{r}{r_a}, y=\frac{r}{r_b}, z=\frac{r}{r_c}$. *Using* $\sum\frac{r}{r_a}=1, \sum r_br_c=s^2$ *and* $\prod r_a=rs^2$.

5) If $x,y,z>0$ prove that:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{9}{x+y+z}\geq\frac{4}{x+y}+\frac{4}{y+z}+\frac{4}{z+x}$$

Marin Chirciu

Solution: *We apply Popoviciu's inequality for the convexe function* $:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{x}$.

6) If $x,y,z>0$ prove that:

$$\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}+\frac{9}{\sqrt{x+y+z}}\geq\frac{4}{\sqrt{x+y}}+\frac{4}{\sqrt{y+z}}+\frac{4}{\sqrt{z+x}}$$

Marin Chirciu

Solution: *We apply Popoviciu's inequality for the convexe function* $:(0,\infty)\to\mathbb{R}, f(x)=\frac{1}{\sqrt{x}}$.

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

SOME INTERESTING PROPERTIES OF COMPLEX NUMBERS WITH THE SAME MODULUS

By Dana Heuberger-Romania

Abstract: *In this paper we will see some less known theorems concerning complex numbers in Geometry, which allow us to find easier proofs, if we use the affixes of some points of the unit circle.*

Keywords: *complex number, incircle, circumcircle, projection, collinearity, concurrency.*

THEORETICAL APPROACH

In this paper we will see some interesting properties of complex numbers in Geometry which, together with a convenient choice of the axes of the complex plane, allow us to find easier proofs for difficult problems. If we consider the affixes of some points which lie on the unit circle, the proofs may become shorter and nicer.

We will use the notations and the assertions presented in [1].. If M is a point of the complex plane, we denote by m its affix. In this paper we consider that the unit circle is centered at the origin $O(0)$ of the complex plane.

Theorem 1. ***If $A(a)$, $B(b)$ are two points of the complex plane, the equation of the line which contains the point $M(m)$ and is perpendicular to the line AB is:***

$$z(\overline{b}-\overline{a})+\overline{z}(b-a)=m(\overline{b}-\overline{a})+\overline{m}(b-a).$$

Proof. The equation of the line is:

$$\frac{z-m}{a-b}=\frac{\overline{z}-\overline{m}}{\overline{b}-\overline{a}} \Leftrightarrow z(\overline{b}-\overline{a})+\overline{z}(b-a)=m(\overline{b}-\overline{a})+\overline{m}(b-a).$$

Theorem 2. ***We consider a triangle ABC, the center $O(o)$ of its circumscribed circle, its orthocenter $H(h)$, its center of gravity $G(g)$ and a point M of the complex plane.***

a) $o=\frac{|a|^2(b-c)+|b|^2(c-a)+|c|^2(a-b)}{\overline{a}(b-c)+\overline{b}(c-a)+\overline{c}(a-b)}$.

b) $h=\frac{\sum a^2(\overline{c}-\overline{b})+\sum|a|^2(c-b)}{\sum\overline{a}(b-c)}$.

c) $h+2o=3g=a+b+c$.

d) ***AM is the bisector line of $\widehat{BAC}$ iff*** $\frac{(m-a)^2}{(b-a)(c-a)}\in\mathbb{R}_+$.

e) ***AN is the outer bisector line of $\widehat{BAC}$ iff*** $\frac{(n-a)^2}{(b-a)(c-a)}\in\mathbb{R}_-$.

Proof. a) By replacing $m = \frac{a+b}{2}$ in Theorem 1, we deduce that the mediator line of the segment AB has the equation:

$$z(\bar{b} - \bar{a}) + \bar{z}(b - a) + |a|^2 - |b|^2 = 0.$$

Similarly, the mediator line of the segment AC has the equation:

$$z(\bar{c} - \bar{a}) + \bar{z}(c - a) + |a|^2 - |c|^2 = 0.$$

By multiplying the first equation with $(c - a)$, the second one with $(b - a)$ and then by substracting the equations thus obtained, the conclusion follows.

b) By using Theorem 1, we find the equations of the heights from A and B of the triangle:

$$h_A: z(\bar{c} - \bar{b}) + \bar{z}(c - b) = a(\bar{c} - \bar{b}) + \bar{a}(c - b),$$

$$h_B: z(\bar{a} - \bar{c}) + \bar{z}(a - c) = b(\bar{a} - \bar{c}) + \bar{b}(a - c).$$

By multiplying the first equation with $(a - c)$, the second one with $(b - c)$ and then by adding the equations thus obtained, we find the conclusion.

c) The relation is a transcription of the fact that O, G and H lie on Euler's line of the triangle ABC.

d) AM is the bisector line of $\widehat{BAC} \Leftrightarrow arg\left(\frac{m-a}{b-a}\right) = arg\left(\frac{c-a}{m-a}\right) \Leftrightarrow \frac{\frac{m-a}{b-a}}{\frac{c-a}{m-a}} \in \mathbb{R}_+ \Leftrightarrow$

$\frac{(m-a)^2}{(b-a)(c-a)} \in \mathbb{R}_+$.

e) Let $D \in AC$ be such that $A \in (CD)$. Then, $k \in \mathbb{R}_+$ exists, such that $d - a = k(a - c)$.

AN is the outer bisector line of $\widehat{BAC}$ iff $arg\left(\frac{b-a}{n-a}\right) = arg\left(\frac{n-a}{d-a}\right) \Leftrightarrow \frac{(n-a)^2}{(b-a)(d-a)} \in \mathbb{R}_+ \Leftrightarrow$

$$-\frac{(n-a)^2}{k(b-a)(c-a)} \in \mathbb{R}_+ \Leftrightarrow \frac{(n-a)^2}{(b-a)(c-a)} \in \mathbb{R}_-.$$

Remark.

i) **If O is the origin of the complex plane, then from any of the assertions b) and c) it follows that $h = a + b + c$.**

ii) **If the point C is the origin of the complex plane, then**

$$o = \frac{ab(\bar{a} - \bar{b})}{\bar{a}b - a\bar{b}}, h = \frac{(\bar{a}b + a\bar{b})(a - b)}{a\bar{b} - \bar{a}b}.$$

Theorem 3. ***The points A and B lie on the unit circle, and the pints C and Z are arbitrary. We have:***

a) $Z \in AB \Leftrightarrow a + ab\bar{z} = a + b.$

b) ***Z lies on the tangent line at A to the circle*** $\Leftrightarrow \boldsymbol{z+a^2\bar{z}=2a}$.

c) ***Z lies on the perpendicular line through C on AB*** $\Leftrightarrow \boldsymbol{z-ab\bar{z}=c-ab\bar{c}}$.

d) ***Z lies on the line through C which is orthogonal to the tangent line at A to the circle***$\Leftrightarrow \boldsymbol{z-a^2\bar{z}=c-a^2\bar{c}}$.

Proof.

a) $Z\in AB\Leftrightarrow \frac{z-a}{b-a}\in\mathbb{R}\Leftrightarrow \frac{z-a}{b-a}=\frac{\bar{z}-\bar{a}}{\bar{b}-\bar{a}}\Leftrightarrow z\bar{b}-z\bar{a}-a\bar{b}=b\bar{z}-a\bar{z}-\bar{a}b$.

By multiplying the previous equality with ab and using $a\bar{a}=b\bar{b}=1$, we deduce:

$$(a-b)(z+ab\bar{z}-a-b)=0\Leftrightarrow z+ab\bar{z}=a+b.$$

b) $\frac{z-a}{0-a}\in i\mathbb{R}\Leftrightarrow \frac{a-z}{a}=\frac{\bar{z}-\bar{a}}{\bar{a}}\Leftrightarrow |a|^2-\bar{a}z=a\bar{z}-|a|^2\Leftrightarrow \bar{a}z+a\bar{z}=2\overset{|\cdot a}{\Leftrightarrow} z+a^2\bar{z}=2a$.

c) $\frac{z-c}{a-b}\in i\mathbb{R}\Leftrightarrow \frac{z-c}{a-b}=\frac{\bar{z}-\bar{c}}{\bar{b}-\bar{a}}\Leftrightarrow (\bar{b}-\bar{a})z-\bar{b}c+\bar{a}c=(a-b)\bar{z}-a\bar{c}+b\bar{c}\overset{|\cdot ab}{\Longleftrightarrow}$

$$(a-b)z-c(a-b)=ab(a-b)\bar{z}-ab\bar{c}(a-b)\Leftrightarrow z-ab\bar{z}=c-ab\bar{c}.$$

d) $ZC\parallel OA\Leftrightarrow \frac{z-c}{a-0}\in\mathbb{R}\Leftrightarrow \frac{z-c}{a}=\frac{\bar{z}-\bar{c}}{\bar{a}}\Leftrightarrow \bar{a}z-\bar{a}c=a\bar{z}-a\bar{c}\overset{|\cdot a}{\Leftrightarrow} z-a^2\bar{z}=c-a^2\bar{c}$.

Theorem 4. ***The points A, B, C and D lie on the unit circle and Z is an arbitrary point. Then:***

a) $\boldsymbol{\frac{a-b}{\bar{a}-\bar{b}}=-ab}$.

b) ***The affix of the projection of the point Z on the line AB is*** $\boldsymbol{m=\frac{a+b+z-ab\bar{z}}{2}}$.

c) ***The affix of the intersection point of the chords AB and CD is*** $\boldsymbol{p=\frac{ab(c+d)-cd(a+b)}{ab-cd}}$.

d) ***The affix of the intersection point of the tangent lines at A and B to the circle is*** $\boldsymbol{q=\frac{2ab}{a+b}}$

Proof. a) $\frac{a-b}{\bar{a}-\bar{b}}=\frac{ab(a-b)}{a\bar{a}b-b\bar{b}a}=\frac{ab(a-b)}{b-a}=-ab$.

b) We consider $M(m)=pr_{AB}(Z)$. As $M\in AB$, from Theorem 3.a) follows:

$$m+ab\bar{m}=a+b;\quad (1.1).$$

Because M lies on the line through Z which is orthogonal to AB, we deduce, using Theorem 3.c): $m-ab\bar{m}=z-ab\bar{z}$; (1.2).

By adding the equalities (1.1) and (1.2) it results $m=\frac{a+b+z-ab\bar{z}}{2}$.

c) We consider $\{P\}=AB\cap CD$. Theorem 1.a) leads to: $p+cd\bar{p}=c+d$ and $p+ab\bar{p}=a+b$.

By multiplying the first equality with ab, the second one with cd and by substracting the equations thus obtained, we deduce: $(ab-cd)p=ab(c+d)-cd(a+b)$.

d) Let $Q(q)$ be the intersection point of these tangent lines. Because Q lies on the tangent line at A to the circle, from Theorem 3.b) we obtain: $q + b^2\overline{q} = 2a$.

But Q also lies on the tangent line at B to the circle, therefore, from Theorem 3.b) we deduce: $q + b^2\overline{q} = 2b$.

By multiplying the first equality with b^2, the second one with a^2 and by substracting the equations thus obtained, it result: $(b^2 - a^2)q = 2ab(b - a) \Leftrightarrow q = \frac{2ab}{a+b}$.

Theorem 5. ***If the incircle of the triangle ABC is the unit circle and its intersection with the lines BC, CA and AB are P, Q, R then:***

a) $\boldsymbol{a = \frac{2qr}{q+r}, b = \frac{2rp}{r+p}, c = \frac{2pq}{p+q}}$.

b) $\boldsymbol{o = \frac{2pqr(p+q+r)}{(p+q)(q+r)(r+p)}}$.

c) $\boldsymbol{h = \frac{2\left(p^2q^2+q^2r^2+r^2p^2+pqr(p+q+r)\right)}{(p+q)(q+r)(r+p)}}$.

Proof. a) It result from Theorem 4.d).

b) From Theorem 2.a), we have: $o = \frac{\sum |a|^2(b-c)}{\sum \overline{a}(b-c)}$.

$$\sum |a|^2(b-c) \overset{a)}{=} \sum \frac{4}{(q+r)(\overline{q}+\overline{r})}\left(\frac{2rp}{r+p} - \frac{2pq}{p+q}\right) =$$

$$= \sum \frac{4qr}{(q+r)(q\overline{q}r + qr\overline{r})} \cdot \frac{2p^2(r-q)}{(r+p)(p+q)} = \frac{8pqr}{(p+q)(q+r)(r+p)} \sum \frac{p(r-q)}{q+r} =$$

$$= \frac{8pqr}{(p+q)^2(q+r)^2(r+p)^2} \sum p^3(r-q).$$

Because $\sum p^3(r-q) = (p+q+r)\sum p^2(r-q)$, we obtain:

$$\sum |a|^2(b-c) = \frac{8pqr(p+q+r)}{(p+q)^2(q+r)^2(r+p)^2} \sum p^2(r-q); \quad (1.3)$$

$$\sum \overline{a}(b-c) = \sum \frac{2\overline{qr}}{\overline{q}+\overline{r}} \cdot \frac{2p^2(r-q)}{(p+q)(r+p)} = \frac{4}{(p+q)(q+r)(r+p)} \sum p^2(r-q); \ (1.4)$$

By dividing the equalities (1.3) and (1.4), we obtain the conclusion.

c) From $h + 2o = a + b + c$, using a) and b), the conclusion follows.

Theorem 6. ***We consider a triangle ABC, inscribed in the unit circle.***

a) $\boldsymbol{u, v, w \in \mathbb{C}}$ ***exist, with*** $\boldsymbol{a = u^2, b = v^2, c = w^2}$ ***such that*** $\boldsymbol{-uv, -uw, -vw}$ ***are the affixes of the midpoints of the arcs AB, AC and A respectively.***

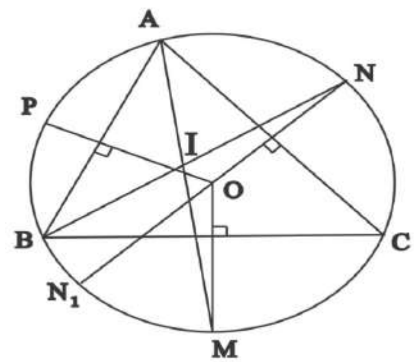
b) The affix of the incenter I of the triangle ABC is $\boldsymbol{i = -(uv + uw + vw)}$.

Proof. a) Let M, N, P be the midpoint of the arcs BC, AC and AB of the unit circle which don't pass through A, B and C respectively.

The numbers $u-1$, $u_2, v_1, v_2, w_1, w_2 \in \mathbb{C}$ exist and are unique, such that

$u_1^2 = u_2^2 = a, v_1^2 = v_2^2 = b, w_1^2 = w_2^2 = c.$

From $\Delta BOM \equiv \Delta MOC$, we find: $\frac{m}{b} = \frac{c}{m}$, therefore,



$$m^2 = bc = (v_k w_t)^2, \forall s, t \in \{1,2\}.$$

Similarly, it follows: $n^2 = (u_s w_t)^2, \forall s, f \in \{1,2\}$.

We choose $u \in \{u_1, u_2\}, v \in \{v_1, v_2\}, w \in \{w_1, w_2\}$ such that $m = -vw$ and $n = -uw$.

Let $N_1(uw)$ be the diametrically opposite point of N.

From $\mu\left(\widehat{N_1OP}\right) = A = \mu\left(\widehat{MOB}\right)$, it results that $\frac{n_1}{p} = \frac{m}{b}$,

therefore $p = \frac{bn_1}{m} = \frac{v^2uw}{-uw} = -uv$.

b) Because $\{I\} = AM \cap BN$, from Theorem 4.c) we deduce that:

$$i = \frac{am(b+n) - bn(a+m)}{am - bn} = \frac{-u^2vw(v^2 - uw) + v^2uw(u^2 - vw)}{-u^2vw + v^2uw}$$
$$= -(uv + uw + vw)$$

2. PROBLEMS

P1. ***The incircle of the triangle ABC is the unit circle. We denote its center by I. Let D, E, F be the touchpoints of the incircle with BC, AC and AB respectively. We denote: $\{M\} = AI \cap DE, \{N\} = BI \cap EF, \{P\} = CI \cap FD, \{Q\} = AI \cap DF$.***

Prove that the following assertions are true:

- ***a)*** $IM \cdot IN \cdot IP = 1$.
- ***b)*** I, E, C, Q ***are concyclic.***
- ***c)*** $BM = IM \cdot EC$.

Dana Heuberger

Solution. a) $ID = IE = IF = 1$ and using Theorem 4.d) it results:

$$a = \frac{2ef}{e+f}, b = \frac{2fd}{f+d}, c = \frac{2de}{d+e}.$$

We obtain: $\bar{a} = \frac{2}{e+f}$, $\bar{b} = \frac{2}{f+d}$ and $\bar{c} = \frac{2}{d+e}$. As $M \in DE$, we have:

$$m + de\overline{m} = d + e; \quad (2.1)$$

A, I, M are collinear, therefore $\frac{m}{a} = \frac{\bar{m}}{\bar{a}} \Leftrightarrow \bar{m} = m \cdot \frac{\bar{a}}{a} = \frac{m}{ef}$.

By replacing in (2.1), it follows :

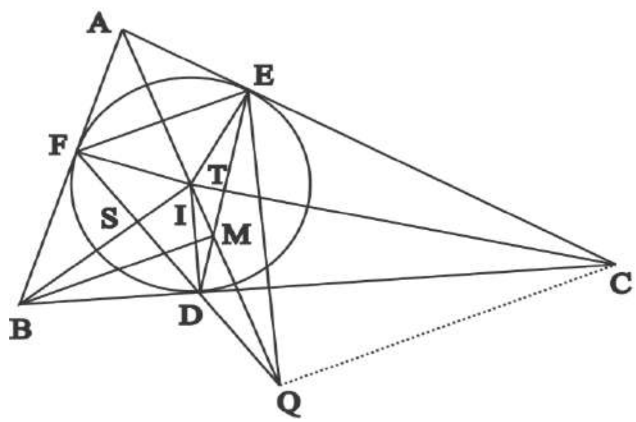
$m + \frac{dm}{f} = d + e \Leftrightarrow m = \frac{(d+e)\cdot f}{d+f}$; (2.2)

As $Q \in DF$, we have: $q + fd\bar{q} = f + d$; (2.3)

A, I, Q are collinear, so $\bar{q} = q \cdot \frac{\bar{a}}{a} = \frac{q}{ef}$.

By replacing in (2.3), we deduce:

$$q = \frac{(d+f)\cdot e}{d+e}; \quad (2.4)$$



Similarly, we find:

$$n = \frac{(e+f)\cdot d}{e+d}; \quad (2.5)$$

$$p = \frac{(f+d)\cdot e}{f+e}; \quad (2.6)$$

By multiplying the equalities (2.2), (2.5) and (2.6), we obtain:

$$mnp = def \Rightarrow IM \cdot IN \cdot IP = 1.$$

b) We denote: $\{S\} = BI \cap DF, \{T\} = CI \cap DE$. From (2.2) it follows: $m = \frac{tf}{s}$, and using moduli, we obtain $IM = \frac{IT}{IS}$.

From (2.4), it follows: $q = \frac{es}{t}$, and using moduli, we obtain $IQ = \frac{IS}{IT} = \frac{1}{IM}$.

Then, as $\frac{IQ}{IE} = \frac{IE}{IM}$ and $\widehat{QIE} \equiv \widehat{EIM}$, the triangles QIE and EIM are similar, therefore $\mu(\widehat{IQE}) = \mu(\widehat{IEM}) = \frac{C}{2}$, which means that I, E, C, Q are concyclic.

c) $m = \frac{d+e}{2de} \cdot \frac{2def}{d+f} = \frac{be}{c}$, therefore $\frac{m}{b} = \frac{e}{c} \Leftrightarrow \Delta BIM \sim \Delta CIE$. We obtain:

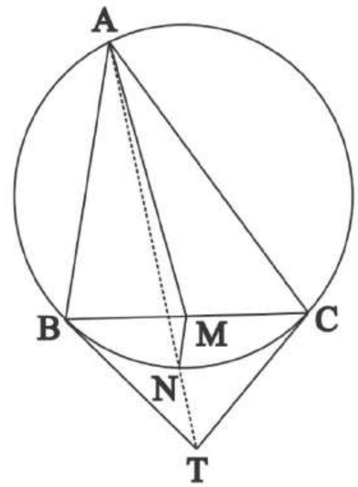
$$BM = \frac{IM \cdot EC}{IE} = IM \cdot CE.$$

P2. *ABC is an arbitrary triangle and T is the intersection point of the tangents in B and C at its circumscribed circle. Let M be the midpoint of the segment BC and N the second intersection point of the line AT with the circle. Prove that $BM^2 = MN \cdot AM$.*

Dana Heuberger

Solution. We consider that the circumscribed circle of the triangle ABC is the unit circle.

Using Theorem 4.d) we obtain: $t = \frac{2bc}{b+c}$ thus $\bar{t} = \frac{2}{b+c}$.



Using Theorem 3.a) we deduce: $t + an\bar{t} = a + n \Leftrightarrow$

$$n = \frac{a-t}{a\bar{t}-1}.$$

Replacing, we find: $n = \frac{ab+ac-2bc}{2a-b-c}$.

Then,

$$n - m = n - \frac{b+c}{2} = \frac{(b-c)^2}{4\left(a - \frac{b+c}{2}\right)}.$$

Therefore, $MN = \frac{BC^2}{4AM}$, and the conclusion follows.

P3. *The quadrilateral $ABCD$ is inscribed in $\mathcal{C}(O,1)$, such that*

$$\mu(\widehat{AOB}) + \mu(\widehat{COD}) \neq \pi$$

We denote by M and P the projections of the points A and C on the line BD. The points N and Q are the projections of B and D on the line AC.
Prove that $MNPQ$ is a parallelogram iff $ABCD$ is a rectangle.

Dana Heuberger

Solution. " $\Leftarrow$ " The assertion is obvious. Moreover, $MNPQ$ is a rectangle.
$\Rightarrow$ We choose the Cartesian coordinate system with the origin in the point O, and such that:
$a = cost_1 + i \cdot sint_1, b = cost_2 + i \cdot sint_2, c = cost_3 + i \cdot sint_3,$
$d = cost_4 + i \cdot sint_4$, with $0 < t_1 < t_2 < t_3 < t_4 < 2\pi$. We prove that $ac + bd \neq 0$.
Suppose that $ac + bd = 0$. It follows:

$$cos(t_1 + t_3) + cos(t_2 + t_4) + i\big(sin(t_1 + t_3) + sin(t_2 + t_4)\big) \Leftrightarrow$$

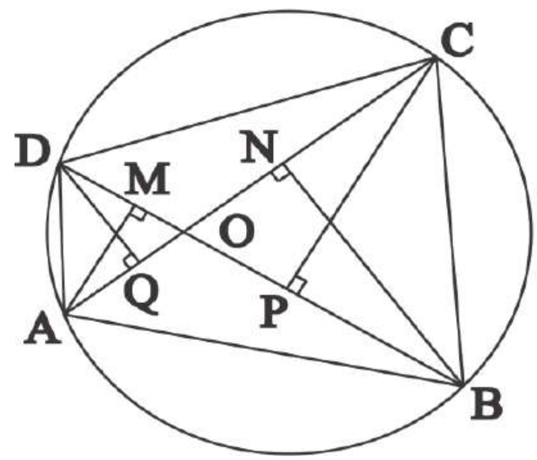
$$2cos\frac{t_2 + t_4 - t_1 - t_3}{2}\left(cos\frac{t_1 + t_2 + t_3 + t_4}{2} + i \cdot sin\frac{t_1 + t_2 + t_3 + t_4}{2}\right) = 0 \Leftrightarrow$$

$$cos\frac{t_2 + t_4 - t_1 - t_3}{2} = 0 \Leftrightarrow (t_2 - t_1) + (t_4 - t_3) = \pi \Leftrightarrow$$

$\mu(\widehat{AOB}) + \mu(\widehat{COD}) = \pi$, false.

Therefore, $ac + bd \neq 0$.

Using Theorem 4.b) we obtain:



$$m = \frac{b + d + a - bd\bar{a}}{2} = \frac{a + b + d}{2} - \frac{bd}{2a},$$

$$p = \frac{b + c + d - bd\bar{c}}{2} = \frac{b + c + d}{2} - \frac{bd}{2c},$$

$$n = \frac{a + b + c - ac\bar{b}}{2} = \frac{a + b + c}{2} - \frac{ac}{2b},$$

$$q = \frac{a + c + d - ac\bar{d}}{2} = \frac{a + c + d}{2} - \frac{ac}{2d}.$$

$$m+p=n+q\Leftrightarrow$$

$$\frac{a+b+d}{2}-\frac{bd}{2a}+\frac{b+c+d}{2}-\frac{bd}{2c}=\frac{a+b+c}{2}-\frac{ac}{2b}+\frac{a+c+d}{2}-\frac{ac}{2d}$$

It follows: $m+p=n+q\Leftrightarrow\frac{(b+d)(ac+bd)}{bd}=\frac{(a+c)(ac+bd)}{ac}$.

As $ac+bd\neq 0$, we deduce that: $\frac{b+d}{bd}=\frac{a+c}{ac}\Leftrightarrow ab(c-d)=cd(b-a)$.

Therefore, $|a|\cdot|b|\cdot CD=|c|\cdot|d|\cdot AB$, so $ab=CD$.

Similarly, $\frac{bd}{bd}=\frac{a+c}{ac}\Leftrightarrow bc(a-d)=ad(b-c)$, therefore $AD=BC$. In the end, $ABCD$ is an parallelogram, namely $ABCD$ is a rectangle.

P4. ***The point H is the orthocenter of the triangle ABC. The points D, E and F lie on the circumcircle of the triangle ABC such that $AD\parallel BE\parallel CF$. The points S, T and U are the respective reflections of D, E and F across the lines BC, CA and AB. Prove that the points S, T, U and H are concyclic.***

([2], 4, p.7)

Solution. We consider that the circumscribed circle of the triangle ABC is the unit circle, with the center O. Then, $h=a+b+c$.

$AD\parallel BE\Leftrightarrow\frac{a-d}{b-e}=\frac{\bar{a}-\bar{d}}{\bar{b}-\bar{e}}\Leftrightarrow\frac{a-d}{b-e}=\frac{a-d}{e-b}\cdot\frac{be}{ad}\Leftrightarrow be=ad$.

Similarly, $AD\parallel CF\Leftrightarrow ad=cf$.

Theorem 4.b) leads to $m=\frac{b+c+d-bc\bar{d}}{2}$.

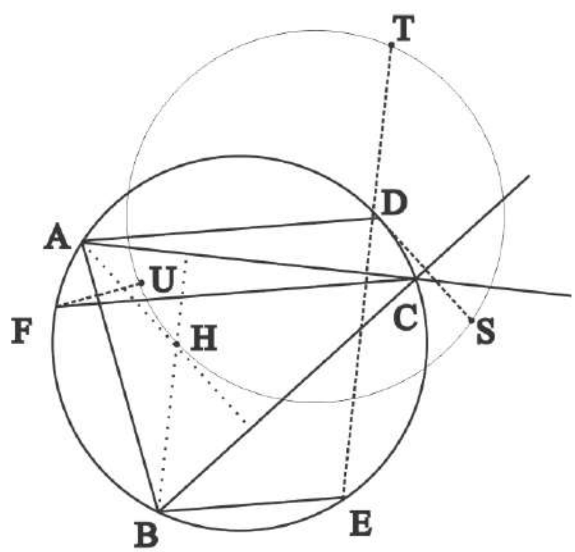
Because $s=2m-d$, we obtain

$$s=b+c-bc\bar{d}.$$

Similarly, we deduce:

$$t=a+c-ac\bar{e}$$

$$u=a+b-ab\bar{f}.$$



U,H,S,T are cyclic iff $\alpha=\frac{u-h}{t-h}\cdot\frac{t-s}{u-s}\in\mathbb{R}$.

We have: $\alpha=\frac{-c-\frac{ab}{f}}{-b-\frac{ac}{e}}\cdot\frac{a-b-\frac{ac}{e}+\frac{bc}{d}}{a-c-\frac{ab}{f}+\frac{bc}{d}}$,

therefore, $\alpha=\frac{(ab+cf)(ae-be)}{(ac+be)(af-fc)}=\frac{(ab+ad)(ae-ad)}{(ac+ad)(af-ad)}=\frac{(b+d)(e-d)}{(c+d)(f-d)}$.

From here,

$$\bar{\alpha} = \frac{\frac{1}{b}+\frac{1}{d}}{\frac{1}{c}+\frac{1}{d}} \cdot \frac{\frac{1}{e}-\frac{1}{d}}{\frac{1}{f}+\frac{1}{d}} = \frac{b+d}{bd} \cdot \frac{d-e}{de} \cdot \frac{cd}{c+d} \cdot \frac{df}{d-f} = \alpha \cdot \frac{cf}{be} = \alpha,$$

which leads to $\alpha \in \mathbb{R}$. Therefore, the points U, H, S, T are concyclic.

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[2] *https://www.scribd.com/document/295267839/TJUSAMO-2013-2014-Complex* ***Numbers in Geometry***

[3] https://www.yisun.io/teaching.html

FEW OUTSTANDING LIMITS

By Florică Anastase-Romania

Problem 1.

If $n, k \in \mathbb{N}, n \geq k$ and $f_k: \mathbb{R} \to \left[0, \frac{n}{n+k-1}\right]$ continuous function. Prove that:

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} f(k) \cdot \sqrt[n+k-1]{1 - f(k) + \frac{1-k}{n} f(k)}\right) < \log 2$$

Solution.

$$\frac{n}{n+k} = \frac{(n+k-1)f(k) + n - (n+k-1)f(k)}{n+k} \overset{AM-GM}{\geq}$$

$$\geq \sqrt[n+k]{f^{n+k-1}(k)\big(n - (n+k-1)f(k)\big)} \Leftrightarrow$$

$$f^{n+k-1}(k)\big(n-(n+k-1)f(k)\big) \leq \left(\frac{n}{n+k}\right)^{n+k-1} \cdot \frac{n}{n+k} \Leftrightarrow$$

$$f(k)\sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)} \leq \frac{n}{n+k}$$

$$\frac{f(k)}{n} \cdot \sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)} \leq \frac{1}{n} \cdot \frac{n}{n+k} \Leftrightarrow$$

$$\frac{1}{n}\sum_{k=1}^{n} f(k) \cdot \sqrt[n+k-1]{1 - f(n) + \frac{1-k}{n} f(n)} \leq$$

$$\leq \frac{1}{n}\sum_{k=1}^{n} f(k) \cdot \sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)} \leq \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{n}{n+k} =$$

$$\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{n}{n+k}=\int_0^1\frac{1}{x+1}dx=\log 2$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}f(k)\cdot\sqrt[n+k-1]{1-f(k)+\frac{1-k}{n}f(k)}\right)<\log 2$$

Problem 2.

If $(a_n)_{n\geq 1}$ is sequence of real numbers such that

$0<a_k<\frac{n^2}{n^2+k^2-1}\,\forall n,k\in\mathbb{N}$ then prove:

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\cdot\sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)}\right)<\frac{\pi}{4}$$

Solution.

$$\frac{n^2}{n^2+k^2}=\frac{(n^2+k^2-1)a_k+n^2-(n^2+k^2-1)a_k}{n^2+k^2}\overset{AM-GM}{\geq}$$

$$\geq\sqrt[n^2+k^2]{a_k^{n^2+k^2-1}(n^2-(n^2+k^2-1)a_k)}\Rightarrow$$

$$a_k^{n^2+k^2-1}(n^2-(n^2+k^2-1)a_k)\leq\left(\frac{n^2}{n^2+k^2}\right)^{n^2+k^2-1}\cdot\frac{n^2}{n^2+k^2}\Leftrightarrow$$

$$a_k\sqrt[n^2+k^2-1]{\frac{(n^2-(n^2+k^2-1)a_k)(n^2+k^2)}{n^2}}\leq\frac{n^2}{n^2+k^2}\Leftrightarrow$$

$$\frac{a_k}{n}\cdot\sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)}\leq\frac{1}{n}\cdot\frac{n^2}{n^2+k^2}\Leftrightarrow$$

$$\frac{1}{n}\sum_{k=1}^{n}a_k\cdot\sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)}\leq\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{n^2}{n^2+k^2}=\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\cdot\sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)}\right)<$$

$$<\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}=\int_0^1\frac{1}{1+x^2}dx=\frac{\pi}{4}$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\cdot\sqrt[n^2+k^2-1]{\left(1+\frac{k^2}{n^2}\right)\left(1-\left(1-\frac{1}{n^2}+\frac{k^2}{n^2}\right)a_k\right)}\right)<\frac{\pi}{4}$$

Problem 3.

Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\int_{k-1}^{k}\tan^{-1}\left(\frac{n(x-k)}{kx+n^2}\right)dx\right)$$

Solution.

$$\sum_{k=1}^{n}\int_{k-1}^{k} tan^{-1}\left(\frac{n(x-k)}{kx+n^2}\right)dx = \sum_{k=1}^{n}\int_{k-1}^{k} tan^{-1}\left(\frac{x-k}{\frac{k}{n}x+n}\right)dx \overset{x\to nx}{=}$$

$$= n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} tan^{-1}\left(\frac{x-\frac{k}{n}}{1+\frac{k}{n}x}\right) = n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(tan^{-1}x - tan^{-1}\left(\frac{k}{n}\right)\right)dx =$$

$$= n\left(\int_0^1 tan^{-1}x - \frac{1}{n}\sum_{k=1}^{n} tan^{-1}\left(\frac{k}{n}\right)\right)$$

Let $f:\mathbb{R}\to\mathbb{R}, f(x) = tan^{-1}x, f''(x) < 0, \forall x\in\mathbb{R} \Rightarrow f' \searrow$ and let $x\in\left[\frac{k-1}{n},\frac{k}{n}\right)\xrightarrow{MVT}$

$\exists c\in\left(\frac{k-1}{n},\frac{k}{n}\right)$ such that $\frac{f(x)-f\left(\frac{k}{n}\right)}{x-\frac{k}{n}} = f'(c) \Rightarrow$

$$f'\left(\frac{k-1}{n}\right) > \frac{f(x)-f\left(\frac{k}{n}\right)}{x-\frac{k}{n}} > f'\left(\frac{k}{n}\right) \,|\cdot\left(x-\frac{k}{n}\right) < 0 \to$$

$$\left(x-\frac{k}{n}\right)f'\left(\frac{k-1}{n}\right) < f(x) - f\left(\frac{k}{n}\right) < \left(x-\frac{k}{n}\right)f'\left(\frac{k}{n}\right)$$

$$-\frac{1}{2n^2}\cdot f'\left(\frac{k-1}{n}\right) \le \int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(tan^{-1}x - tan^{-1}\left(\frac{k}{n}\right)\right) \le -\frac{1}{2n^2}f'\left(\frac{k}{n}\right), n\ge 2, k\in\{1,2,\dots,n\}$$

Thus,

$$-\frac{1}{2n}\cdot\sum_{k=1}^{n} f'\left(\frac{k-1}{n}\right) \le n\left(\int_0^1 tan^{-1}x - \frac{1}{n}\sum_{k=1}^{n} tan^{-1}\left(\frac{k}{n}\right)\right) \le -\frac{1}{2n}\cdot\sum_{k=1}^{n} f'\left(\frac{k}{n}\right)$$

$$\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n} f'\left(\frac{k-1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n} f'\left(\frac{k}{n}\right) = \int_0^1 f'(x)\,dx = \frac{\pi}{4}$$

Therefore,

$$\Omega = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\int_{k-1}^{k} tan^{-1}\left(\frac{n(x-k)}{kx+n^2}\right)dx\right) = \lim_{n\to\infty} n\left(\int_0^1 tan^{-1}x - \frac{1}{n}\sum_{k=1}^{n} tan^{-1}\left(\frac{k}{n}\right)\right) = -\frac{\pi}{8}$$

Reference:

ROMANIAN MATEMATICAL MAGAZINE-www.ssmrmh.ro

PROBLEMS FOR JUNIORS

J.01 If $a, b, c, d \in \mathbb{R}$ then:

$$(2a + 3b + 4c + 5d)^2 \geq 8(3ab + 5ad + 6bc + 10cd)$$

Proposed by Marian Ursărescu – Romania

J.02 If $a, b, c > 0, n \geq 0, a^3 + b^3 + c^3 = 3$ then:

$$\frac{1}{n + ab(a + b)} + \frac{1}{n + bc(b + c)} + \frac{1}{n + ca(c + a)} \geq \frac{3}{n + 2}$$

Proposed by Marin Chirciu – Romania

J.03 If $a, b, c, d, e, f > 0, a + d = b + e = c + f = 5$ then:

$$(a + b + c)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) \leq 3\left(\frac{a}{d} + \frac{b}{e} + \frac{c}{f}\right)$$

Proposed by Daniel Sitaru – Romania

J.04 If $a, b, c > 0, n \geq 0$ then:

$$\frac{a^3}{n + ab^2} + \frac{b^3}{n + bc^2} + \frac{c^3}{n + ca^2} \geq \frac{3abc}{n + abc}$$

Proposed by Marin Chirciu – Romania

J.05 If $a, b, c, d > 0, a + b + c + d = 1, 0 \leq n \leq 1$ then:

$$\frac{a^2}{na + bcd} + \frac{b^2}{nb + cda} + \frac{c^2}{nc + dab} + \frac{d^2}{nd + abc} \geq \frac{16}{n + 16}$$

Proposed by Marin Chirciu, Octavian Stroe – Romania

J.06 In $\Delta ABC, M, E \in (AB), N, F \in (AC)$ such that $\overrightarrow{AE} = m\overrightarrow{EB}, \overrightarrow{AF} = n\overrightarrow{FC}, \overrightarrow{MO} = p\overrightarrow{ON}$ and

$$\frac{MB}{MA} = \frac{NA}{NC} = \lambda, m, n, p\lambda \in \mathbb{R}^*; p \neq -1, \lambda \neq 1; m \cdot p = 1.$$

Prove that: E, O, F are collinear if and only if $p = n$.

Proposed by Florică Anastase – Romania

J.07 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in any ΔABC the following inequality holds:

$$\frac{y + z}{x}h_a + \frac{z + x}{y}h_b + \frac{x + y}{z}h_c \geq \frac{36r^2}{R}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.08 If $m, n \in \mathbb{R}_+, m + n \in \mathbb{R}_+^*$ and M an interior point in ΔABC, and x, y, z the distances from point M to the apices ABC and u, v, w the distances from M to the sides $[BC], [CA], [AB]$. Prove that:

$$\frac{(mx + ny)^2}{u^2 + 2vw} + \frac{(my + nz)^2}{v^2 + 2wu} + \frac{(mz + nx)^2}{w^2 + 2uv} \geq 4(m + n)^2$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.09 If $m \geq 1, n, p \geq 0, n + p = 2m, x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in ΔABC with the area F the following inequality holds:

$$\sum_{cyc} \frac{(x+y)^m a^{2m} \cdot b^n}{x^m \cdot h_b^p} \geq \frac{2^{5m-p}}{3^{m-1}} \cdot F^n$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.10 If $m \geq 1, n, p \geq 0, n + p = 2m, x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in ΔABC with the area F the following inequality holds:

$$\sum_{cyc} \frac{(x+y)^m a^{2m} \cdot b^n}{x^m \cdot h_b^p} \geq \frac{2^{5m-p}}{3^{m-1}} \cdot F^n$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.11 In any ΔABC with the area F the following inequality holds:

$$(h_a^3 + h_b^3)c^5 + (h_b^3 + h_c^3)a^5 + (h_c^3 + h_a^3)b^5 \geq 64\sqrt{3}F^4$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.12 In any ΔABC the following inequality holds:

$$(h_b + h_c)a^3 + (h_c + h_a)b^3 + (h_a + h_b)c^3 \geq 16\sqrt{3}F^2$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.13 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in any ΔABC the following inequality holds:

$$\frac{y+z}{x} h_a + \frac{z+x}{y} h_b + \frac{x+y}{z} h_c \geq \frac{36r^2}{R}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu – Romania

J.14 If $m, n \in \mathbb{R}_+ = [0, \infty), m + n \in \mathbb{R}_+^* = (0, \infty)$ then in any ΔABC with the area F the following inequality holds:

$$(ma + nb)h_c + (mb + nc)h_a + (mc + na)h_c \geq 6(m + n)F$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți – Romania

J.15 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in any ΔABC with the area F the following inequality holds:

$$\frac{(yh_b^2 + zh_c^2)a^4}{x} + \frac{(zh_c^2 + xh_a^2)^4 b}{y} + \frac{(xh_a^2 + yh_b^2)^4 c}{z} \geq 32\sqrt{3}F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.15 In any ΔABC with the area F the following inequality holds:

$$(a^2 + b^2)h_c^2 + (b^2 + c^2)h_a^2 + (c^2 + a^2)h_b^2 \geq 24F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.16 If $x, y, z > 0$, then in any ΔABC the following inequality holds:

$$\frac{y+z}{x} m_a + \frac{z+x}{y} m_b + \frac{x+y}{z} m_c \geq \frac{4\sqrt{3}F}{R}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.17 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in ΔABC with the area F the following inequality holds:

$$\frac{x \cdot a^2}{(y+z)h_a^2} + \frac{y \cdot b^2}{(z+x)h_b^2} + \frac{z \cdot c^2}{(x+y)h_c^2} \geq 2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.18 If $m, n \in \mathbb{R}_+ = [0, \infty); m + n \in \mathbb{R}_+^* = (0, \infty)$ and M an interior point in ΔABC and x, y, z the distances from M to the apices A, B, C and u, v, w the distances from M to the sides $[BC], [CA], [AB]$, then:

$$\frac{(mx+ny)^2}{uv} + \frac{(my+nz)^2}{vw} + \frac{(mz+nz)^2}{wu} \geq 12(m+n)^2$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.19 Let M be an interior point in ΔABC and d_a, d_b, d_c the distances from M to the sides BC, CA, AB, then:

$$\frac{a}{d_a} + \frac{b}{d_b} + \frac{c}{d_c} \geq 6\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.20 Let $m, n \in \mathbb{R}_+ = [0, \infty), m + n = 4, M$ an interior point in ΔABC having the area F and x, y, z the distances from M to the apices A, B, C and d_a, d_b, d_c the distances from M to the sides BC, CA, AB. Prove that:

$$\frac{x^2 a^m}{d_a(d_b+d_c)h_a^n} + \frac{y^2 b^m}{d_c(d_c+d_a)h_b^n} + \frac{z^2 \cdot c^m}{d_c(d_a+d_b)h_c^n} \geq 2^m F^{2-n}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.21 If $m, n \in \mathbb{R}_+ = [0, \infty);\ m + n = 3$ and $x \in \mathbb{R}$ then in any ΔABC with the area F the following inequality holds:

$$\sum_{cyc} \frac{a^m}{(a\cos^2 x + b\sin^2 x)h_a^n} \geq 2^{m-1}\sqrt{3}F^{1-n}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.22 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then in any ΔABC with F –area the following relationship holds:

$$\frac{xa}{(y+z)h_a^3} + \frac{yb}{(z+x)h_b^3} + \frac{zc}{(x+y)h_c^3} \geq \frac{1}{F}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.23 If $x, y, z > 0$ then in any ΔABC with F –area the following relationship holds:

$$\frac{x}{y+z}m_a + \frac{y}{z+x}m_b + \frac{z}{x+y}m_c \geq \frac{\sqrt{3}F}{R}$$

Proposed by D.M.Bătinețu-Giurgiu,Neculai Stanciu-Romania

J.24 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then in any ΔABC with F –area the following relationship holds:

$$\frac{y+z}{xh_b}bc^2 + \frac{z+x}{yh_c}ca^2 + \frac{x+y}{zh_a}ab^2 \geq 16F$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.25 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then in any ΔABC with F –area the following relationship holds:

$$\frac{y+z}{xh_bh_c}bc+\frac{z+x}{yh_ch_a}ca+\frac{x+y}{zh_ah_b}ab\ge 8$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.26 If $m\ge 0; x,y,z>0$ then in any ΔABC with F –area the following relationship holds:

$$\frac{x^{m+2}}{(y+z)^m}a^{4m+4}+\frac{y^{m+2}}{(z+x)^m}b^{4m+4}+\frac{z^{m+2}}{(x+y)^m}c^{4m+4}\ge\frac{2^{3m+4}}{3^{m+1}}(xy+yz+zx)F^{2(m+1)}$$

Proposed by D.M.Bătinețu-Giurgiu,Dan Nănuți-Romania

J.27 If $x,y,z>0$ then in any ΔABC with F –area the following relationship holds:

$$\frac{y+z}{x}m_a^2+\frac{z+x}{y}m_b^2+\frac{x+y}{z}m_c^2\ge\frac{24F^2}{R^2}$$

Proposed by D.M.Bătinețu-Giurgiu, Dan Nănuți-Romania

J.28 If $\Delta A'B'C'$ is the circumcevian triangle of K – Lemoine's point in ΔABC then:

$$\frac{KA'}{KA}+\frac{KB'}{KB}+\frac{KC'}{KC}\ge\frac{(a+b+c)^2}{a^2+b^2+c^2}$$

Proposed by Marian Ursărescu – Romania

J.29 If in $\Delta ABC, K$ – Lemoine's point then:

$$AK^2+BK^2+CK^2\ge\frac{abc(a+b+c)}{a^2+b^2+c^2}$$

Proposed by Marian Ursărescu – Romania

J.30 In $\Delta ABC, G$ – centroid, $E\in(AB), F\in(AC), E,F,G$ – collinears, $S_1=[AEF], S_2=[EBCF]$. It's possible that:

$$\frac{S_1}{S_2}<\frac{4}{5}?$$

Proposed by Marian Ursărescu – Romania

J.31 In acute ΔABC the following relationship holds:

$$x+\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c}\le\frac{R(x+3)(r_a+r_b+r_c)}{2r(h_a+h_b+h_c)}, x\ge-\frac{9}{5}$$

Proposed by Marin Chirciu – Romania

J.32 If $0\le x\le\frac{\sqrt{6}}{3}$ then:

$$\sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2}+\sqrt{2x^2-(\sqrt{2}+\sqrt{6})x+2}\ge 2$$

Proposed by Daniel Sitaru,Elena Nicolae – Romania

J.33 In ΔABC the following relationship holds:

$$h_a^3h_b+h_b^3h_c+h_c^3h_a\ge\frac{54r^4(5R-r)}{R}$$

Proposed by Marian Ursărescu – Romania

J.34 In ΔABC the following relationship holds:

$$m_a^3 m_b + m_b^3 m_c + m_c^3 m_a \geq 81r^3(2R - r)$$

Proposed by Marian Ursărescu – Romania

J.35 In ΔABC the following relationship holds:

$$\frac{r_a^3}{r_b^2} + \frac{r_b^3}{r_c^2} + \frac{r_c^3}{r_a^2} \geq s\sqrt{3}$$

Proposed by Marian Ursărescu – Romania

J.36 If $x, y, z, n > 0, x + y + z = 3n^2$ then:

$$\frac{1}{2n - \sqrt{x}} + \frac{1}{2n - \sqrt{y}} + \frac{1}{2n - \sqrt{z}} \geq \frac{3}{n}$$

Proposed by Marin Chirciu – Romania

J.37 Let ΔDEF be the pedal triangle of $P \in Int\ (\Delta ABC)$ in ΔABC. Prove that:

$$\varphi = \frac{AP \cdot BP \cdot CP}{2(R^2 - OP^2)}$$

φ – circumradii of ΔDEF, R – circumradii of ΔABC

Proposed by Mehmet Șahin – Ankara – Turkey

J.38 If in $\Delta ABC, m(\sphericalangle ABC) = 20°, m(\sphericalangle ACB) = 60°$ then:

$$b(a + b + c) = c^2$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.39 Find the real solution of the equation

$$x^2 - \frac{1}{x^2} + \sqrt{x} - \frac{1}{\sqrt{x}} = \frac{1 + \sqrt{5}}{2}$$

Proposed by Srinivasa Raghava-AIRMC-India

J.40 In ΔABC the following relationship holds:

$$h_a + h_b + h_c \leq \sqrt{2Rr + 5r^2}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)$$

Proposed by Alex Szoros – Romania

J.41 In ΔABC the following relationship holds:

$$\left(\frac{R}{r} + \frac{1}{2}\right)^{\lambda} \geq 1 + \frac{\lambda}{2}\left(\frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab}\right), \lambda \geq 1$$

Proposed by Alex Szoros – Romania

J.42 If $a, b, c > 0, n \geq 1$ then:

$$\frac{a}{na + b + c} + \frac{b}{nb + c + a} + \frac{c}{nc + a + b} \geq \frac{27}{(n + 2)(a + b + c)(ab + bc + ca)}$$

Proposed by Marin Chirciu – Romania

J.43 If $a, b, c > 0$ then:

$$\frac{ab\sqrt{ab}}{(a+b)^3}+\frac{bc\sqrt{bc}}{(b+c)^3}+\frac{ca\sqrt{ca}}{(c+a)^3}+\frac{3(a+b)(b+c)(c+a)}{64abc}\geq\frac{3}{4}$$

Proposed by Marin Chirciu – Romania

J.44 If $a, b, c > 0$ then:

$$\frac{a^2b^2}{(a+b)^4}+\frac{b^2c^2}{(b+c)^4}+\frac{c^2a^2}{(c+a)^4}+\frac{(a+b)(b+c)(c+a)}{32abc}\geq\frac{7}{16}$$

Proposed by Marin Chirciu – Romania

J.45 If $a, b, c > 0$ and $2n \in \mathbb{N}$ then:

$$\frac{a^nb^n}{(a+b)^{2n}}+\frac{b^nc^n}{(b+c)^{2n}}+\frac{c^na^n}{(c+a)^{2n}}+\frac{2n(a+b)(b+c)(c+a)}{2^{2n+3}abc}\geq\frac{2n+3}{2^{2n}}$$

Proposed by Marin Chirciu – Romania

J.46 If $a, b, c > 0$ and $n \in \mathbb{N}, n \geq 2$ then:

$$\sqrt[n]{\frac{a}{b+c}+\frac{b}{c+a}}+\sqrt[n]{\frac{b}{c+a}+\frac{c}{a+b}}+\sqrt[n]{\frac{c}{a+b}+\frac{a}{b+c}}\geq 3$$

Proposed by Marin Chirciu – Romania

J.47 If $a_1, a_2, \dots, a_n > 0, a_1+a_2+\cdots+a_n = n, p, k \in \mathbb{N}$ then:

$$\sum_{i=1}^{n}\left(\frac{a_i^{p+1}+1}{a_i^p+1}\right)^k\geq n$$

Proposed by Marin Chirciu – Romania

J.48 In ΔABC the following relationship holds:

$$m_a\cos A+m_b\cos B+m_c\cos C\leq\frac{1}{9R}\left(\frac{2R^2+r^2}{r}\right)^2$$

Proposed by Marian Ursărescu – Romania

J.49 If $a, b > 0$ then:

$$\left(\frac{5a}{2}+\frac{5b}{2}+3\sqrt{ab}+\frac{4ab}{a+b}\right)\left(a+b+3\sqrt{ab}+\frac{10ab}{a+b}\right)\leq 25(a+b)^2$$

Proposed by Daniel Sitaru,Luiza Dumitrescu – Romania

J.50 Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ \dfrac{xy}{\sqrt{(1+x^2)(1+y^2)}}+\dfrac{yz}{\sqrt{(1+y^2)(1+z^2)}}+\dfrac{zx}{\sqrt{(1+z^2)(1+x^2)}}=\dfrac{3}{4} \\ xy+yz+zx=3 \end{cases}$$

Proposed by Daniel Sitaru,Delia Popescu – Romania

J.51 If $x \in \left(0,\frac{\pi}{2}\right), n \in \mathbb{N}, L_n$ – Lucas numbers, F_n – Fibonacci numbers then:

$$\frac{\sin^2 x\cdot L_n^2}{\sin^2 x+L_n^2}+\frac{\cos^2 x\cdot L_{n+1}^2}{\cos^2 x+L_{n+1}^2}<\frac{5F_{2n+1}}{1+5F_{2n+1}}$$

Proposed by Daniel Sitaru,Liliana Argetoianu – Romania

J.52 If $n, m \in \mathbb{N}, L_n$ – Lucas numbers, F_n – Fibonacci numbers then:

$$\sqrt[3]{F_n F_m^2 L_m L_n^2} + \sqrt[3]{F_m F_n^2 L_n L_m^2} \le 2F_{m+n}$$

Proposed by Daniel Sitaru,Eugenia Turcu – Romania

J.53 If F_n – Fibonacci numbers then:

$$\frac{4}{2 + F_{2n}F_{2n+2}} + \frac{4}{2 + F_{2n+2}F_{2n+4}} + \frac{4}{3 + F_{2n+6}^2} < 1 + F_{2n+1}^{-2} + F_{2n+3}^{-2} + F_{2n+6}^{-2}, n \in \mathbb{N}$$

Proposed by Daniel Sitaru,Roxana Vasile – Romania

J.54 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sum_{cyc} \frac{n_a + r_a}{m_a + w_b + w_c} \ge 2$$

Proposed by Bogdan Fuștei – Romania

J.55 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne cevian, the following relationship holds:

$$\sum_{cyc} \frac{m_a}{\sin A} \ge \sum_{cyc} \frac{n_a^2 + g_a^2 + 2rr_a}{b + c} \sqrt{\frac{b + c - a}{a}}$$

Proposed by Bogdan Fuștei – Romania

J.56 In $\Delta ABC, g_a$ – Gergonne cevian, the following relationship holds:

$$\sum_{cyc} \frac{g_a}{h_a} \le \sqrt{5 + \frac{2R}{r}}$$

Proposed by Bogdan Fuștei – Romania

J.57 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne cevian, ω – Brocard's point the following relationship holds:

$$4\left(1 + \frac{1}{2\sin\omega}\right) \le \sum_{cyc} \frac{n_a^2 + g_a^2 + 2m_a w_a}{h_a^2}$$

Proposed by Bogdan Fuștei – Romania

J.58 In $\Delta ABC, g_a$ – Gergonne cevian, the following relationship holds:

$$\frac{4R}{r} \ge 5 + \sum_{cyc} \frac{m_a^2 w_a^2}{g_a^2 h_a r_a}$$

Proposed by Bogdan Fuștei – Romania

J.59 In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne cevian, the following relationship holds:

$$\frac{R}{r} \ge \frac{5R - r + \sum \frac{n_a g_a}{h_a}}{h_a + h_b + h_c} \ge \frac{5R - r + \sum \frac{m_a w_a}{h_a}}{h_a + h_b + h_c} \ge \frac{9R}{h_a + h_b + h_c} \ge \frac{2(r_a + r_b + r_c)}{h_a + h_b + h_c} \ge 2$$

Proposed by Bogdan Fuștei – Romania

J.60 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$2\left(\frac{R}{r}-1\right)\sum_{cyc} h_a = 4R + r + \sum_{cyc}\frac{n_a^2}{r_a}$$

Proposed by Bogdan Fuștei – Romania

J.61 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{m_a}{w_a h_b h_c}} \geq \frac{1}{r}$$

Proposed by Bogdan Fuștei – Romania

J.62 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\prod_{cyc}\left(\sqrt{\frac{2r_a(n_a+h_a)}{w_a^2}}-1\right) \leq \frac{R}{2r}$$

Proposed by Bogdan Fuștei – Romania

J.63 In ΔABC the following relationship holds:

$$\sqrt{m_b m_c}sinA + \sqrt{m_c m_a}sinB + \sqrt{m_a m_b}sinC \leq \frac{9\sqrt{3}R}{4}$$

Proposed by Ionuț Florin Voinea-Romania

J.64 Solve for real numbers:

$$\sqrt[5]{x^7 - 6x^4} = \sqrt[7]{x^5 + 6x^4}$$

Proposed by Ionuț Florin Voinea-Romania

J.65 $ABCD$ –tangential quadrilateral, O –incenter, $A_1, B_1 C_1, D_1$ –intersection points of incircle with AO, BO, CO, DO. Prove that:

$$AA_1 + BB_1 + CC_1 + DD_1 \geq 4(\sqrt{2}-1)R$$

Proposed by Ionuț Florin Voinea-Romania

J.66 If $x, y, z, t \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{1}{\sqrt{sinx \cdot siny \cdot sinz \cdot sint}} + \frac{1}{\sqrt{cosx \cdot cosy \cdot cosz \cdot cost}} \geq 4$$

Proposed by Ionuț Florin Voinea-Romania

J.67 In ΔABC the following relationship holds:

$$\frac{cosA}{\sqrt{r_b r_c}} + \frac{cosB}{\sqrt{r_c r_a}} + \frac{cosC}{\sqrt{r_a r_b}} \leq \frac{1}{2r}$$

Proposed by Ionuț Florin Voinea-Romania

J.68 If $x, y, z \in \mathbb{R} - \mathbb{Z}$ then exists $m, n \in \mathbb{Z} \cap [-9,9], m^2 + n^2 \neq 0$ such that:

$$|m\{x\} + n\{y\}| < \frac{1}{10}, \quad \{*\} = * - [*], \quad [*] - \text{GIF}.$$

Proposed by Ionuț Florin Voinea-Romania

J.69 If $x > 0$ then:

$$2^{\{x\}} + 2^{\left\{\frac{1}{x}\right\}} < \frac{7}{2}, \quad \{x\} = x - [x], [*] - \text{GIF}.$$

Proposed by Ionuț Florin Voinea-Romania

J.70 If $x, y, z, t > 0$ then:

$$(2 + x^2)(2 + y^2)(2 + z^2)(2 + t^2) > 18\left(\sqrt{xy} + \sqrt{zt}\right)^2$$

Proposed by Ionuț Florin Voinea-Romania

J.71 If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{1}{\sqrt{sinxsiny}} + \frac{1}{\sqrt{cosxcosy}} \geq 2\sqrt{2}$$

Proposed by Ionuț Florin Voinea-Romania

J.72 Let ABC be a triangle. Let P, Q are the points on BC such that $BP = PQ = QC$. Let R be the mid point of AC. Let BR intersect AP and AQ in M and N respectively.

$$\text{Find } \Omega = \left(\frac{ar(\Delta AMR)}{ar(quad.\ MRCP)} + \frac{ar(\Delta AMN)}{ar(\Delta ANR)}\right)$$

Proposed by Rajeev Rastogi – India

J.73 Find the value of K such that

$$1 \times 5 \times 9 \times 13 \times 17 \times \dots \times 2021 \equiv K(mod\ 1000)$$

Proposed by Rajeev Rastogi – India

J.74 Let $S(n)$ denotes the sum of digits of a natural number n, then find the number of all 3 – digit numbers n satisfying the inequality:

$$S^4(n) - 40S^3(n) + 401S^2(n) - 40S(n) + 400 \leq 0$$

Proposed by Rajeev Rastogi – India

J.75 Find the reminder, when the number

$$2019^{2020^{2021}} \text{ is divided by } 7$$

Proposed by Rajeev Rastogi – India

J.76 Find number of irreducible fractions which can be written simultaneously in the forms

$$\frac{7a+2}{5a+1} \text{ and } \frac{4b-1}{3b+2} \text{ for some integers } a \text{ and } b.$$

Proposed by Rajeev Rastogi – India

J.77 Prove that 2020 cannot be expressed as a sum of six odd perfect square numbers.

Proposed by Rajeev Rastogi – India

J.78 Suppose x is a non-zero real number such that $\left(2020x^2 - \frac{2019}{x}\right)$ and x^{2021} are both rational numbers. Also $x = p + \frac{r}{x^2}$ (where $p, r \in Q$). Prove that x is also a rational number.

Proposed by Rajeev Rastogi – India

J.79 Given x be the least prime divisor of the number $\left|\underbrace{000\ldots0}_{2018\ times}\right|$ also

$$(2x)^{(2x)^{(2x)}} \equiv K(mod\ 100), \text{ then find } K$$

Proposed by Rajeev Rastogi – India

J.80 In ΔABC the following relationship holds:

$$2r(R-2r) + s^2 \le m_a^2 + m_b^2 + m_c^2 \le \frac{27R^2}{4}$$

$$h_a^2 + h_b^2 + h_c^2 + 3r(R-2r) \le s^2$$

Proposed by Nguyen Van Canh-Vietnam

J.81 In ΔABC the following relationship holds:

$$h_a^2 + h_b^2 + h_c^2 + 3r(R-2r) \le w_a^2 + w_b^2 + w_c^2$$

$$h_a^2 + h_b^2 + h_c^2 + 5r(R-2r) \le s^2$$

Proposed by Nguyen Van Canh-Vietnam

J.82 If $x, y, z > 0$ then:

$$\sqrt{\frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z}} + 2\cdot\sqrt{\frac{xy+yz+zx}{x^2+y^2+z^2}} \ge \sqrt{6} + 2$$

Proposed by Nguyen Van Canh-Vietnam

J.83 In ΔABC the following relationship holds:

$$\sqrt{\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2}} \le \frac{m_a + m_b + m_c}{3F} \le \frac{3R}{2F}$$

Proposed by Nguyen Van Canh-Vietnam

J.84 In ΔABC the following relationship holds:

$$w_a^2 + w_b^2 + w_c^2 + \frac{2rm_a(R-2r)}{h_a} \ge s^2$$

Proposed by Nguyen Van Canh-Vietnam

J.85 In $\Delta ABC, I$ –incenter, R, r –its circumradius and inradius. If R_a, R_b, R_c –are the circumradii of the triangles IBC, ICA, IAB, then prove that:

$$2r \le \sqrt[3]{R_a \cdot R_b \cdot R_c} \le R$$

Proposed by Dorin Mărghidanu-Romania

J.86 If $x \geq i, i = \overline{1,n}$ such that $\frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{n}{x_n} = n - 1$, then

$$\sqrt{x_1 + x_2 + \cdots + x_n} \geq \sqrt{x_1 - 1} + \sqrt{x_2 - 2} + \cdots + \sqrt{x_n - n}$$

Proposed by Dorin Mărghidanu-Romania

J.87 If x, y, z, m, n are strictly positive real numbers, prove that:

$$2mn(x + y + z) + \frac{(mx + ny)(nx + my)}{x + y} + \frac{(my + nz)(ny + mz)}{y + z} + \frac{(mz + nx)(nz + mx)}{z + x} \leq \frac{(m + n)^2(x + y + z)}{2}$$

Proposed by Dorin Mărghidanu-Romania

J.88 If x_i, a_i are real numbers such that $x_i \geq a_i > 0, i = \overline{1,n}$ then prove that

$$\frac{a_1}{x_1} + \frac{a_2}{x_2} + \cdots + \frac{a_n}{x_n} + \frac{\left(\sqrt{x_1 - a_1} + \sqrt{x_2 - a_2} + \cdots + \sqrt{x_n - a_n}\right)^2}{x_1 + x_2 + \cdots + x_n} \leq n$$

Proposed by Dorin Mărghidanu-Romania

J.89 If $a_1, a_2, \dots, a_n > 0$ and $a_1 + a_2 + \cdots + a_n = S$, then

$$\left(\frac{S}{a_1} - 1\right)^{a_1} \cdot \left(\frac{S}{a_2} - 1\right)^{a_2} \cdot \ldots \cdot \left(\frac{S}{a_n} - 1\right)^{a_n} \leq (n - 1)^S$$

Proposed by Dorin Mărghidanu-Romania

J.90 If a, b are real numbers, prove that

$$\frac{\sum_{k=0}^{2n}(-1)^k a^{2n-k} b^k}{\sum_{k=0}^{2n} a^{2n-k} b^k} \geq \frac{1}{2n + 1}$$

Proposed by Dorin Mărghidanu-Romania

J.91 If $a_1, a_2, \dots, a_n > 0$ and $a_1 + a_2 + \cdots + a_n = S$, then

$$a_1 \cdot \sqrt{\frac{a_1}{S - a_1}} + a_2 \cdot \sqrt{\frac{a_2}{S - a_2}} + \cdots + a_n \cdot \sqrt{\frac{a_n}{S - a_n}} \geq \frac{S}{\sqrt{n - 1}}$$

Proposed by Dorin Mărghidanu-Romania

J.92 If $a_1, a_2, b_1, b_2 \in \mathbb{R}$ then prove

$$(1 + a_1^4)(1 + a_2^4)(1 + b_1^4)(1 + b_2^4) \geq (a_1 b_1 + a_2 b_2)^4$$

Proposed by Dorin Mărghidanu-Romania

J.93 In ΔABC the following relationship holds:

$$\frac{1}{R^2} \leq \frac{a}{b^3} + \frac{b}{c^3} + \frac{c}{a^3} \leq \frac{1}{2r}\sqrt{\left(\frac{a}{b^2}\right)^2 + \left(\frac{b}{c^2}\right)^2 + \left(\frac{c}{a^2}\right)^2}$$

Proposed by Florică Anastase – Romania

J.94 If $a, b, c, d > 1$ then:

$$\frac{\log b}{\log^2(a^2 b)} + \frac{\log c}{\log^2(a^2 b^2 c)} + \frac{\log d}{\log^2(a^2 b^2 c^2 d)} < \frac{\log \sqrt[4]{bcd}}{\log a \cdot \log(abcd)}$$

Proposed by Florică Anastase – Romania

J.95 In ΔABC the following relationship holds:

$$\frac{s}{R^2} \le \sum_{cyc} \frac{\sin A \cdot \sin B}{\sin A + \sin B} \le \frac{1}{2R} + \frac{s(2R-r)}{3}$$

Proposed by Florică Anastase – Romania

J.96 If $a_i, b_i \in (1,\infty), \forall i = \overline{1,n}$ such that: $\frac{a_1^2}{a_1^2+\frac{1}{b_1^2}} + \frac{a_2^2}{a_2^2+\frac{1}{b_2^2}} + \cdots + \frac{a_n^2}{a_n^2+\frac{1}{b_n^2}} = n-1$ then prove:

$$\prod_{i=1}^{n} \log_{a_i}\left(a_i^{b_i} - a_i\right) \cdot \log_{bi}\left(b_i^{a_i} - b_i\right) \cdot \log_{a_i}\left(a_i^{b_i} + a_i\right) \cdot \log_{b_i}\left(b_i^{a_i} + b_i\right) \le (n-1)^n$$

Proposed by Florică Anastase – Romania

J.97 Let M an interior point in ΔABC with the area F and $x_A = MA, x_B = MB, x_C = MC$.

Prove that:

$$a^2x_A^2 + b^2x_B^2 + c^2x_C^2 \ge \frac{16}{3}F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

J.98 If $x, y, z \in \mathbb{R}_+^* = (0,\infty)$, then in any ΔABC with the area F the following inequality holds:

$$\frac{(yh_b^2 + zh_c^2)a^6}{x} + \frac{(zh_c^2 + xh_a^2)b^6}{y} + \frac{(xh_a^2 + yh_b^2)c^6}{z} \ge 128F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

J.99 If $m, n \ge 0, m + n = 3, x \in \mathbb{R}$ then in any ΔABC with the area F the following inequality holds:

$$\sum_{cyc} \frac{a^m}{(a\sin^2 x + b\cos^2 x)h_a^n} \ge 2^{m-1}\sqrt{3}F^{1-n}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

J.100 Let $n \in \mathbb{N}, n \ge 2; a, b, x_k, y \in \mathbb{R}_+^* = (0,\infty), k = \overline{1,n}, X_n = \sum_{k=1}^{n} x_k, c \in \mathbb{R}_+ = [0,\infty)$ such that $aX_n > b \cdot \max_{1\le k\le n} x_k$, then:

$$\sum_{k=1}^{n} \frac{x_k}{aX_n - bx_k + cy} \ge \frac{nX_n}{(an-b)X_n + cny}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

J.101 In any ΔABC the following inequality holds:

$\frac{a^3}{h_b+h_c} + \frac{b^3}{h_c+h_a} + \frac{c^3}{h_a+h_b} \ge 4F$ where F is the triangle's area.

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

J.102 If $m,n\geq 0, m+n,x,y,z\in\mathbb{R}_+^* = (0,\infty)$ and ΔABC with the area F, then:

$$\sum_{cyc}\frac{m(x+y)+n(y+z)}{nx+mz}a^4\geq 32F^2$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.103 If $m\geq 0$, then in any ΔABC the following inequality holds:

$$\frac{m(x+y)+x+z}{y+mz}a^2+\frac{m(y+z)+y+x}{z+mx}b^2+\frac{m(z+x)+z+y}{x+my}\geq 8\sqrt{3}F$$

where F is the area of ΔABC, $\forall x,y,z\in\mathbb{R}_+ = [0,\infty)$.

Proposed by D.M. Bătinețu – Giurgiu – Romania

J.104 If $m\geq 0, x,y,z\in\mathbb{R}_+^* = (0,\infty)$ and ΔABC with the area F, then:

$$\sum_{cyc}\frac{m(x+y)+y+z}{x+mz}a^4\geq 32F^2$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

J.105 Let M be an interior point in ΔABC, x,y,z the distances from point M to the apices A,B,C and u,v,w the distances from M to the sides BC,CA,AB. Prove that:

$$\frac{x^4}{(vw)^2}+\frac{y^4}{(wu)^2}+\frac{z^4}{(uv)^2}\geq 48$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.106 Let $m,n\in\mathbb{R}_+ = [0,\infty); m+n\in\mathbb{R}_+^* = (0,\infty)$. So in ΔABC triangle with the area F the following inequality holds:

$$\frac{y+z}{x}(mb+nc)^2+\frac{z+x}{y}(mc+na)^2+\frac{x+y}{z}(m_a+n_b)^2\geq t\cdot F$$

$\forall x,y,z\in\mathbb{R}_+^*$, then:

$$\frac{v+w}{u}(mb+mc)^4+\frac{w+u}{v}(mc+na)^4+\frac{u+v}{w}(ma+nb)^4\geq\frac{t^2F^2}{6}$$

$\forall u,v,w\in\mathbb{R}_+^*$.

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

J.107 If $m,n\in\mathbb{R}_+ = [0,\infty);\ m+n=2$ then in any ΔABC with the area F the following inequality holds:

$$\frac{(a^2+b^2)w_c^m}{c^n}+\frac{(b^2+c^2)w_a^m}{a^n}+\frac{(c^2+a^2)w_b^m}{b^n}\geq 3\cdot 2^{m+1}F^m$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți – Romania

J.108 If $m,n,p,t,u\in\mathbb{R}_+^*=(0,\infty)$ then in any ΔABC with F –area the following relationship holds:

$$\sum_{cyc}\frac{(ma+nb)^4}{pa+tb+nc}\ge\frac{12(m+n)^4RF}{p+t+u}$$

Proposed by D.M.Bătinețu-Giurgiu,Gabriel Tică-Romania

J.109 If $t\ge 0;x,y,z>0$ then in any ΔABC with F –area the following relationship holds:

$$\frac{x+t}{y+z+2t}m_a^2+\frac{y+t}{z+x+2t}m_b^2+\frac{z+t}{x+y+2t}m_c^2\ge\frac{3\sqrt{3}F}{2R}$$

Proposed by D.M.Bătinețu-Giurgiu-Romania

J.110 If $x,y,z>0$ then in any ΔABC with F –area the following relationship holds:

$$\frac{y^2+z^2}{x^2}h_a^2+\frac{z^2+x^2}{y^2}h_b^2+\frac{x^2+y^2}{z^2}h_c^2\ge 4\left(\frac{F}{R}\right)^2$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.111 If $m\ge 0;x,y,z>0$ then in any ΔABC with F –area the following relationship holds:

$$\frac{x^{m+1}+y^{m+1}}{z^{m+1}a^2}h_a^{m-1}+\frac{y^{m+1}+z^{m+1}}{x^{m+1}b^2}h_b^{m-1}+\frac{z^{m+1}+x^{m+1}}{y^{m+1}c^2}h_c^{m-1}\ge 2^m\left(\sqrt{3}\right)^{m+1}\frac{F^{m-1}}{R^{m+1}}$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

J.112 If $x,y,z>0$ then in any ΔABC with F –area the following relationship holds:

$$\frac{y+z}{x}a+\frac{z+x}{y}b+\frac{x+y}{z}c\ge 4\sqrt[4]{27}F$$

Proposed by D.M.Bătinețu-Giurgiu,Claudia Nănuți-Romania

J.113 In any triangle ABC with s –semiprimeter, the following relationship holds:

$$\frac{m_a}{tb+uc}+\frac{m_b}{tc+ua}+\frac{m_c}{ta+ub}\ge\frac{s}{(t+u)R},\forall t,u\ge 0,t+u>0$$

Proposed by D.M.Bătinețu-Giurgiu,Neculai Stanciu-Romania

J.114 If $m\in\mathbb{N},M\in Int(\Delta ABC),F$ –area of $\Delta ABC;x_A=MA,x_B=MB,x_C=MC$ then prove:

$$3m+(ax_A)^{m+1}+(bx_B)^{m+1}+(cx_C)^{m+1}\ge 4(m+1)F$$

Proposed by D.M.Bătinețu-Giurgiu,Neculai Stanciu-Romania

J.115 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then in any ΔABC with F –area the following relationship holds:

$$\frac{(y^2+z^2)a^4b^3}{x^2h_b}+\frac{(z^2+x^2)b^4c^3}{y^2h_c}+\frac{(x^2+y^2)c^4a^3}{z^2h_a}\ge\frac{256}{3}F^3$$

Proposed by D.M.Bătinețu-Giurgiu,Neculai Stanciu-Romania

J.116 If $m,n\in\mathbb{R}_+=[0,\infty), m+n=4, M\in Int(\Delta ABC), F$ –area of $\Delta ABC; x=d(M,A)$, $y=d(M,B), z=d(M,C); u=d(M,BC), v=d(M,CA), w=d(M,AB)$ then prove:

$$\frac{x^2a^m}{(uv+vw)h_a^n}+\frac{y^2b^m}{(vw+wu)h_b^n}+\frac{z^2c^m}{(wu+uv)h_c^n}\geq 2^{4-n}F^{2-n}=2^mF^{m-2}$$

Proposed by D.M.Bătinețu-Giurgiu, Gabriel Tică-Romania

J.117 If $\Delta A'B'C'$ is the circumcevian triangle of incenter of ΔABC then:

$$\frac{A'B'\cdot B'C'\cdot C'A'}{AB\cdot BC\cdot CA}\leq\sqrt{\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right)}$$

Proposed by Marian Ursărescu – Romania

J.118 Find all functions $f:\mathbb{N}^*\to\mathbb{N}^*$ such that:

$$f(1)-f(2)+f(3)+\cdots+(-1)^{n-1}f(n)=$$
$$=(n^2+n-1)\left(\sqrt{f(1)}-\sqrt{f(2)}+\cdots+(-1)^{n-1}\sqrt{f(n)}\right)$$

Proposed by Marian Ursărescu – Romania

J.119 In acute ΔABC the following relationship holds:

$$\cos A\sin(\sin A)+\cos B\sin(\sin B)+\cos C\sin(\sin C)\leq\frac{3}{2}\sin\left(\frac{\sqrt{3}R}{4r}\right)$$

Proposed by Marian Ursărescu – Romania

J.120 If $a,b,c>0, n,k\in\mathbb{N}, a^k+b^k+c^k=3$ then:

$$\frac{1}{(a^{k+1}+k)^n}+\frac{1}{(b^{k+1}+k)^n}+\frac{1}{(c^{k+1}+k)^n}\geq\frac{3}{(k+1)^n}$$

Proposed by Marin Chirciu – Romania

J.121 Solve for real numbers:

$$2^{x^2+(n-1)x}+2^{-x^2+(n+1)x}=2^{1+nx}, n\in\mathbb{R}$$

Proposed by Marin Chirciu – Romania

J.122 In ΔABC the following relationship holds:

$$2(m_a+m_b+m_c)\leq\frac{a^2}{h_a-r}+\frac{b^2}{h_b-r}+\frac{c^2}{h_c-r}$$

Proposed by Marin Chirciu – Romania

J.123 In ΔABC the following relationship holds:

$$\sqrt[6]{\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a^4\right)\left(\sum_{cyc}a^5\right)}\geq 4S$$

Proposed by Marin Chirciu – Romania

J.124 In ΔABC the following relationship holds:

$$\frac{r_b r_c}{r_a^2}+\frac{r_c r_a}{r_b^2}+\frac{r_a r_b}{r_c^2}+\frac{2nr}{R}\geq n+3, n\leq 8$$

Proposed by Marin Chirciu – Romania

J.125 If $x,y\in\mathbb{R}, n\in\mathbb{N}^*$ then:

$$2^{(n+1)x-ny}+n\cdot 2^y\geq (n+1)\cdot 2^x$$

Proposed by Marin Chirciu – Romania

J.126 If in $\Delta ABC, m(\sphericalangle B)=2m(\sphericalangle A), m(\sphericalangle C)=4m(\sphericalangle A)$ then:

$$h_a^2+h_b^2+h_c^2>7\sqrt{21}R^2$$

Proposed by Daniel Sitaru,Mihai Ionescu – Romania

J.127 In ΔABC, o_a – circumcevian, the following relationship holds:

$$\frac{h_a h_b h_c}{o_a o_b o_c}\leq\left(\frac{s^2+r^2+2Rr}{8R^2}\right)^4$$

Proposed by Daniel Sitaru,Carina Viespescu – Romania

J.128 $z_1,z_2,z_3\in\mathbb{C}-\{0\}$, different in pairs, $A(z_1),B(z_2),C(z_3)$,

$|z_1|=|z_2|=|z_3|=1$. If:

$$\frac{z_1}{z_2+z_3-z_1}+\frac{z_2}{z_3+z_2-z_2}+\frac{z_3}{z_1+z_2-z_3}+\frac{3}{2}=0$$

then: $AB=BC=CA$

Proposed by Marian Ursărescu – Romania

J.129 Let ω be the inradii of circumcevian triangle of incenter in acute ΔABC.

Prove that:

$$\omega\geq\sqrt{\frac{Rr}{2}}$$

Proposed by Marian Ursărescu – Romania

J.130 In acute $\Delta ABC, GD, GE, GF$ – bisectors of $\sphericalangle BGC, \sphericalangle CGA, \sphericalangle AGB, G$ – centroid.

If $D\in(BC), E\in(AC), F\in(BA)$ then:

$$\frac{[DEF]}{[ABC]}\geq 2\left(\frac{r}{R}\right)^3$$

Proposed by Marian Ursărescu – Romania

J.131 In $\Delta ABC, H$ – orthocenter the following relationship holds:

$$AH\cdot CH^3+BH\cdot AH^3+CH\cdot BH^3\leq\frac{16}{3}(4R^2-13r^2)^2$$

Proposed by Marian Ursărescu – Romania

J.132 If in $\Delta ABC, I$ – incenter then:

$$[AIB]\cdot[AIC]+[BIC]\cdot[BIA]+[CIA]\cdot[CIB]\leq r^2(R+r)^2$$

Proposed by Marian Ursărescu – Romania

J.133 Let $\Delta A'B'C'$ be the circumcevian triangle of Nagel's point of ΔABC. Prove that:

$$\frac{[A'B'C']}{[ABC]} \le \left(\frac{R}{r} - 1\right)^3$$

Proposed by Marian Ursărescu – Romania

J.134 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{1}{2\sin^2 A + (\sin B + \sin C)^2} \le \frac{1}{24}\left(\frac{R}{r}\right)^4$$

Proposed by Marian Ursărescu – Romania

J.135 $a, b, c \in \mathbb{C}^*$ - different in pairs, $|a| = |b| = |c|, A(a), B(b), C(c)$. Prove that:

$$\sum_{cyc} \frac{|(a-b)(a-c)|}{|(a-b)|a-c| + (a-c)|a-b||^2} = 9\left(\sum_{cyc} |a-b|\right)^{-2} \Leftrightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

J.136 Let $\Delta A'B'C'$ be the circumcevian triangle of centroid in ΔABC. Prove that:

$$\frac{[A'B'C']}{[ABC]} \ge \left(\frac{2r}{R}\right)^2$$

Proposed by Marian Ursărescu – Romania

J.137 In ΔABC the following relationship holds:

$$\frac{R}{2r} + \frac{3ns^2}{(4R + r)^2} \ge n + 1, n \le \frac{3}{2}$$

Proposed by Marin Chirciu – Romania

J.138 In ΔABC the following relationship holds:

$$\left(\frac{r_a}{r_b}\right)^4 + \left(\frac{r_b}{r_c}\right)^4 + \left(\frac{r_c}{r_a}\right)^4 + \frac{2nr}{R} \ge n + 3, n \le 8$$

Proposed by Marin Chirciu – Romania

J.139 If $a, b, c > 1, n \in \mathbb{N}, n \ge 2$ then:

$$a \cdot b^{\sqrt[n]{\log_b c}} + b \cdot c^{\sqrt[n]{\log_c a}} + c \cdot a^{\sqrt[n]{\log_a b}} \le a^2 + b^2 + c^2$$

Proposed by Marin Chirciu – Romania

J.140 In ΔABC the following relationship holds:

$$n \cdot \frac{R}{2r} + \frac{w_a w_b w_c}{r_a r_b r_c} \ge n + 1, n \ge \frac{3}{8}$$

Proposed by Marin Chirciu – Romania

J.141 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$2\left(\frac{R}{r}-1\right)\sum_{cyc}\frac{h_a}{n_a}=\sum_{cyc}\left(\frac{n_a}{r_a}+\frac{r_a}{n_a}\right)$$

Proposed by Bogdan Fuștei-Romania

J.142 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\frac{h_a}{n_a}+\frac{h_b}{n_b}+\frac{h_c}{n_c}\ge\frac{3r}{R-r}$$

Proposed by Bogdan Fuștei-Romania

J.143 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne cevian, the following relationship holds:

$$\frac{2(R-r)}{r}\ge\frac{\sum r_a}{\sum h_a}+\frac{(\sum n_a)^2}{(\sum h_a)(\sum r_a)}$$

Proposed by Bogdan Fuștei-Romania

J.144 In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne cevian, the following relationship holds:

$$\frac{2\sqrt{2}}{3}\sum_{cyc}m_a\le\sum_{cyc}\frac{n_a^2+g_a^2+2rr_a}{\sqrt{4m_bm_c+3bc}}$$

Proposed by Bogdan Fuștei-Romania

J.145 In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\frac{2r_ah_a}{s-n_a}\ge\sum_{cyc}n_a+3r\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\sqrt{4-\frac{2r}{R}}$$

Proposed by Bogdan Fuștei-Romania

J.146 In ΔABC, the following relationship holds:

$$\sum_{cyc}\frac{m_a}{h_a}\ge\sum_{cyc}\sqrt{\frac{m_am_b}{r_ar_b}}$$

Proposed by Bogdan Fuștei-Romania

J.147 In ΔABC, the following relationship holds:

$$\sum_{cyc}\frac{m_a}{h_a}\ge\frac{1}{2}\sum_{cyc}\sqrt{\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)}$$

Proposed by Bogdan Fuștei-Romania

J.148 Let ΔDEF be the pedal triangle of $P\in Int\ (\Delta ABC), D\in(BC), E\in(CA)$, $F\in(AB)$. If $[PEF]=F_1,[PFD]=F_2,[PDE]=F_3$ then:

$$\frac{F_1}{R_1^2}+\frac{F_2}{R_2^2}+\frac{F_3}{R_3^2}\le\frac{[I_aI_bI_c]-2[ABC]}{8R^2}$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.149 Let ΔDEF be the pedal triangle of $P \in Int\ (\Delta ABC)$ in ΔABC. Prove that:

$$AP^2 + BP^2 + CP^2 \geq \sqrt{3} \cdot F + \frac{4F^2}{9R^2}$$

When equality holds?

Proposed by Mehmet Șahin – Ankara – Turkey

J.150 In ΔABC, g_a – Gergonne's cevian, the following relationship holds:

$$g_a + g_b + g_c \leq 3r + \sqrt{15R^2 - 12Rr}$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.151 In ΔABC, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\frac{R_a^2}{r_a^2 - r^2} + \frac{R_b^2}{r_b^2 - r^2} + \frac{R_c^2}{r_c^2 - r^2} \leq \frac{3R}{4r}$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.152 If r_V – inradii of pedal triangle of Bevan's point in ΔABC then:

$$r_V \geq \frac{r^3}{\sqrt{R^4 - 6Rr^3}}$$

Proposed by Mehmet Șahin – Ankara – Turkey

J.153 In ΔABC, O – circumcenter, $F, P \in BC$, $E, K \in CA$, $D, Q \in AB$, $\overline{FOK}, \overline{POQ}, \overline{DOE}$ – antiparallels. Prove that DE, FK, PQ can be sides in acute triangle.

Proposed by Mehmet Șahin – Ankara – Turkey

J.154 In ΔABC the following relationship holds:

$$\frac{r_a + r_b + r_c}{3} \geq R + r \geq \sqrt[3]{r_a r_b r_c}$$

Proposed by Alex Szoros – Romania

J.155 In ΔABC the following relationship holds:

$$\frac{2abc}{R} \leq \frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c}$$

Proposed by Alex Szoros – Romania

J.156 Solve in $\mathbb{R}$

$$\sqrt[3]{7 + \frac{3^x + 4^x + 12^x + 91^x}{13^x + 21^x + 28^x + 84^x}} + \sqrt[3]{9 - \frac{13^x + 21^x + 28^x + 84^x}{3^x + 4^x + 12^x + 91^x}} = 4$$

Proposed by Alex Szoros – Romania

J.157 In acute triangle ABC the following relationship holds:

$$\frac{(1 + \cos A)^3}{(a\cos A + b\cos C)(a\cos A + c\cos B)} \geq \frac{27}{8a^2}$$

Proposed by Alex Szoros – Romania

J.158 In ΔABC the following relationship holds:

$$\frac{(h_a+h_b+h_c)^3}{h_ah_bh_c}+5\geq\frac{16R}{R-r}$$

Proposed by Marin Chirciu – Romania

J.159 In ΔABC the following relationship holds:

$$\frac{a^2+b^2}{m_a^2+m_b^2}+\frac{b^2+c^2}{m_b^2+m_c^2}+\frac{c^2+a^2}{m_c^2+m_a^2}\leq 1+\frac{3R^2}{4r^2}$$

Proposed by Marin Chirciu – Romania

J.160 In ΔABC then following relationship holds:

$$\frac{h_a}{h_b}+\frac{h_b}{h_c}+\frac{h_c}{h_a}\leq\frac{1}{27}\left(1+\frac{4R}{r}\right)^2$$

Proposed by Marin Chirciu – Romania

J.161 In ΔABC the following relationship holds:

$$\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}\frac{1}{r_a}\right)+\frac{2\mu r}{R}\geq\mu+9,\mu\leq 8$$

Proposed by Marin Chirciu – Romania

J.162 In ΔABC, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ – incenter.

$$\sum\frac{m_a}{m_b+m_c}R_a^2\geq 2(r^2+3Rr-R^2)$$

Proposed by Marin Chirciu – Romania

J.163 In ΔABC

$$\sum\frac{s_a}{m_a+s_a}+\prod\frac{r_a}{w_a}\leq\frac{3}{2}+\frac{R}{2r}$$

Proposed by Marin Chirciu – Romania

J.164 In ΔABC

$$\frac{27}{4}R^2p\leq\sum\frac{r_a(r_b^3+r_c^3)}{a}\leq p(32R^2-101r^2)$$

Proposed by Marin Chirciu – Romania

J.165 In ΔABC, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ – incenter.

$$\sum\frac{h_a}{h_b+h_c}R_a^2\geq 2(r^2+3Rr-R^2)$$

Proposed by Marin Chirciu – Romania

J.166 In ΔABC

$$\frac{p}{3}\left(\frac{2r}{R}\right)^3(4R+r)^2\leq\sum\frac{h_a(h_b^3+h_c^3)}{a}\leq\frac{p}{3}\left(\frac{2r}{R}\right)(4R+r)^2$$

Proposed by Marin Chirciu – Romania

J.167 In ΔABC

$$\sum \frac{h_a(h_b^3 + h_c^3)}{a} \le \sum \frac{r_a(r_b^3 + r_c^3)}{a}$$

Proposed by Marin Chirciu – Romania

J.168 In $\Delta ABC, R_a, R_b, R_c$ – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ – incenter.

$$\sum \frac{r_a}{r_b + r_c} R_a^2 \ge 2(r^2 + 3Rr - R^2)$$

Proposed by Marin Chirciu – Romania

J.169 In $\Delta ABC, R_a, R_b, R_c$ – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB, I$ – incenter. If $x, y, z > 0$ then:

$$\sum \frac{x}{y + z} R_a^2 \ge 2(r^2 + 3Rr - R^2)$$

Proposed by Marin Chirciu – Romania

J.170 In ΔABC the following relationship holds:

$$\left(\frac{r_a r_b}{r_a + r_b}\right)^2 + \left(\frac{r_b r_c}{r_b + r_c}\right)^2 + \left(\frac{r_c r_a}{r_c + r_a}\right)^2 \ge \frac{27r^2}{4}$$

Proposed by Marian Ursărescu – Romania

J.171 In ΔABC the following relationship holds:

$$\frac{1}{\sin^3\frac{A}{2}\left(\sin\frac{B}{2} + \sin\frac{C}{2}\right)} + \frac{1}{\sin^3\frac{B}{2}\left(\sin\frac{C}{2} + \sin\frac{A}{2}\right)} + \frac{1}{\sin^3\frac{C}{2}\left(\sin\frac{A}{2} + \sin\frac{B}{2}\right)} \ge \frac{12R}{r}$$

Proposed by Marian Ursărescu – Romania

J.172 In ΔABC the following relationship holds:

$$\sum_{cyc} \sqrt{(r_a - r)(r_b - r)} \ge 6r$$

Proposed by Marian Ursărescu – Romania

J.173 In $\Delta ABC, I_a, I_b, I_c$ – excenters. Prove that:

$$\sqrt{BI_aC} + \sqrt{CI_bA} + \sqrt{AI_cB} \le \frac{3\sqrt{3\sqrt{3}}}{2} \cdot R$$

Proposed by Marian Ursărescu – Romania

J.174 $A(z_1), B(z_2), C(z_3), z_1, z_2, z_3 \in \mathbb{C} - \{0\}$ – different in pairs, $|z_1| = |z_2| = |z_3|$.

Prove that:

$$\sum_{cyc} \frac{|z_1 - z_2| + |z_1 - z_3|}{z_1} = 0 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

J.175 In acute $\Delta ABC, AA_1, BB_1, CC_1$ – internal bisectors, $\Delta A_2B_2C_2$ – circumcevian triangle of incenter, ΔDEF – medial triangle, $DA_3 \perp AA_2, EB_3 \perp BB_2, FC_3 \perp CC_2$. Prove that:

$$(R + r)^2 \le A_1A_2 \cdot AA_3 + B_1B_2 \cdot BB_3 + C_1C_2 \cdot CC_3 \le r^2 + 2R^2$$

Proposed by Marian Ursărescu – Romania

J.176 In $\Delta ABC, o_a$ – circumcevian, the following relationship holds:

$$\frac{h_ah_bh_c}{o_ao_bo_c} \le \left(\frac{s^2 + r^2 + 2Rr}{8R^2}\right)^4$$

Proposed by Daniel Sitaru,Luiza Cremeneanu – Romania

J.177 If in $\Delta ABC, \Omega$ – first Brocard point then:

$$\Omega A^2 + \Omega B^2 + \Omega C^2 \ge \frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2}$$

Proposed by Daniel Sitaru,Nineta Oprescu – Romania

J.178 If $a, b > 0$ then:

$$\frac{a^9 + a^4b^4\sqrt{ab} + b^9}{\left(a^5 + a^2b^2\sqrt{ab} + b^5\right)^2} \ge \frac{a^4 + a^2b^2 + b^4}{\left(a^3 + ab\sqrt{ab} + b^3\right)(a^2 + ab + b^2)}$$

Proposed by Daniel Sitaru,Roxana Popescu – Romania

J.179 In $\Delta ABC, o_a$ – circumcevian, the following relationship holds:

$$\frac{8r - R}{R} \le \frac{h_a}{o_a} + \frac{h_b}{o_b} + \frac{h_c}{o_c} \le 1 + \frac{3r}{R} + \frac{2r^2}{R^2}$$

Proposed by Daniel Sitaru,Aurelia Petrică – Romania

J.180 If F_n – Fibonacci numbers then:

$$\frac{F_{n+2}^5 - F_{n+1}^5 - F_n^5}{F_{n+2}^3 - F_{n+1}^3 - F_n^3} > 5\sqrt[3]{F_nF_{n+1}F_{2n+1}}, \qquad n \in \mathbb{N} - \{0,1\}$$

Proposed by Daniel Sitaru,Monica Stanca – Romania

J.181 If $a, b, c \in \mathbb{C}; |a| = |b| = |c| = 5$ then:

$$\sum_{cyc} |a + 5| + 5\sum_{cyc} |a^{10} + 1| + \sum_{cyc} |a^{11} + 5| \ge 30$$

Proposed by Daniel Sitaru,Nicola Cătălin – Romania

J.182 Find $x \in (0, \pi)$ such that:

$$\tan^{-1}(x + 1) + \tan^{-1} 2 + \tan^{-1}\left(\frac{x^2 \sin x - 2x\cos x - 2\sin x}{x^2\cos x - 2x\sin x - 2\cos x}\right) = x$$

Proposed by Daniel Sitaru,Aurel Chiriță – Romania

J.183 In acute ΔABC the following relationship holds:

$$\frac{1}{2}\sum_{cyc}\frac{r_a - r}{s_a}\sqrt{\frac{h_a}{r_a}} \geq \frac{\sum(r_a + AI)}{\sum(m_a + h_a - r)}$$

Proposed by Bogdan Fuștei – Romania

J.184 In ΔABC the following relationship holds:

$$2\sum_{cyc}\left(h_a \cdot \sqrt{\frac{m_c}{w_a h_a h_b}}\right) \geq 3 + \frac{a}{c} + \frac{b}{a} + \frac{c}{b}$$

Proposed by Bogdan Fuștei – Romania

J.185 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{n_a + h_a}{\sqrt{2r_a(n_a + h_a)}} \geq \cos\left(\frac{B - C}{2}\right)$$

Proposed by Bogdan Fuștei – Romania

J.186 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\frac{n_a^2}{2r_a + s_a} \leq 4R - 5r$$

Proposed by Bogdan Fuștei – Romania

J.187 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\sum_{cyc}\frac{n_a}{h_a} \geq \sum_{cyc}\sqrt{\frac{n_b n_c}{r_b r_c}}$$

Proposed by Bogdan Fuștei – Romania

J.188 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_b + h_c}{h_a} \leq \sqrt{\frac{2R}{r}} \cdot \sum_{cyc}\frac{s_a}{w_a}\sqrt{\frac{h_a}{r_a}}$$

Proposed by Bogdan Fuștei – Romania

J.189 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\left(\frac{h_b}{h_b - 2r} - 1\right)\left(\frac{h_c}{h_c - 2r} - 1\right)} \geq \frac{s}{r} + 3(2 - \sqrt{3})$$

Proposed by Bogdan Fuștei – Romania

J.190 In ΔABC the following relationship holds:

$$9 \cdot \max(a, b, c) \geq 2\sum_{cyc} n_a + 4r\sum_{cyc}\frac{2r_a + h_a}{s + n_a}$$

Proposed by Bogdan Fuștei – Romania

J.191 In ΔABC the following relationship holds:

$$\prod_{cyc}(n_a + r_a - AI) \geq 16rR^2$$

Proposed by Bogdan Fuștei – Romania

J.192 If $a, b, c > 0, abc = e$ then:

$$log\left(a^{\frac{a}{a+b}+\frac{a}{a+c}} \cdot b^{\frac{b}{b+c}+\frac{b}{b+a}} \cdot c^{\frac{c}{c+a}+\frac{c}{c+b}}\right) \geq 1$$

Proposed by Ionuț Florin Voinea-Romania

J.193 If $a, b, c, d > 1$ then:

$$\sum_{cyc} log_a\left(\frac{b^3 + c^3 + d^3}{b^2 + c^2 + d^2}\right) \geq 4$$

Proposed by Ionuț Florin Voinea-Romania

J.194 If $x, y, z \in (0,1), 2(x + y + z) > 3$ then:

$$2^x + 3^y + 4^z < 3x + 4y + 5z$$

Proposed by Ionuț Florin Voinea-Romania

J.195 If $0 < x, y < 1, x + y > 1$ then:

$$2^x + 3^y < 5\sqrt{x^2 + y^2}$$

Proposed by Ionuț Florin Voinea-Romania

J.196 In ΔABC the following relationship holds:

$$\sum_{cyc} sinAsin\frac{B}{2}sin\frac{C}{2} \leq \frac{\sqrt{3}(a^3 + b^3 + c^3)}{8abc}$$

Proposed by Ionuț Florin Voinea-Romania

J.197 If $x, z, y > 0$ then prove:

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 3\left(1 + \sqrt[3]{\frac{3(x + y + z)(x + y)(y + z)(z + x)}{(xy + yz + zx)^2}}\right)$$

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + x^3 + y^3 \geq \frac{32(x^2 + y^2)}{(x + y)^4} + x^2y + xy^2$$

Proposed by Nguyen Van Canh-Vietnam

J.198 In ΔABC the following relationship holds:

$$\frac{3\sqrt{6}}{2\sqrt{R}} \leq \frac{\sqrt{m_a}}{a} + \frac{\sqrt{m_b}}{b} + \frac{\sqrt{m_c}}{c} \leq \frac{3\sqrt{6}}{4} \cdot \frac{\sqrt{R}}{r}$$

Proposed by Nguyen Van Canh-Vietnam

J.199 In ΔABC the following relationship holds:

$$n_a^2 + n_b^2 + n_c^2 \geq m - a^2 + m_b^2 + m_c^2 + \frac{r}{4R}(R^2 - 4r^2)$$

$$3(w_a^3 + w_b^3 + w_c^3) \geq h_a^3 + h_b^3 + h_c^3 + \frac{324r^4}{R}$$

Proposed by Nguyen Van Canh-Vietnam

J.200 In ΔABC the following relationship holds:

$$\frac{3F}{R} \le \frac{w_a m_a}{a} + \frac{w_b m_b}{b} + \frac{w_c m_c}{c} < \frac{9R^2}{4r}$$

Proposed by Nguyen Van Canh-Vietnam

J.201 In ΔABC the following relationship holds:

$$\frac{h_a h_b h_c}{m_a m_b m_c} \le min\left\{\left(\frac{h_a}{w_a}\right)^3, \left(\frac{h_b}{w_b}\right)^3, \left(\frac{h_c}{m_c}\right)^3\right\}$$

Proposed by Nguyen Van Canh-Vietnam

J.202 In ΔABC the following relationship holds:

$$\sqrt{\frac{m_a + h_b}{r_c}} + \sqrt{\frac{m_b + h_c}{r_a}} + \sqrt{\frac{m_c + h_a}{r_b}} \le (4s^2 - 16Rr + 5r^2)\sqrt{\frac{2}{r}}$$

Proposed by Nguyen Van Canh-Vietnam

J.203 In ΔABC the following relationship holds:

$$\sqrt[4]{m_a h_a w_b r_c} + \sqrt[4]{m_b h_b w_a r_a} + \sqrt[4]{m_c h_c w_a r_b} \le 4R + r$$

Proposed by Nguyen Van Canh-Vietnam

J.204 In ΔABC the following relationship holds:

$$\sqrt{\frac{m_a w_a}{h_a}} + \sqrt{\frac{m_b w_b}{h_b}} + \sqrt{\frac{m_c w_c}{h_c}} \le \frac{4R + r}{\sqrt{3r}}$$

Proposed by Nguyen Van Canh-Vietnam

J.205 In ΔABC the following relationship holds:

$$\frac{\sqrt{h_a s_a} + \sqrt{h_b s_b} + \sqrt{h_c s_c}}{m_a m_b m_c} \le \frac{4R}{27r^3} + \frac{1}{s^2}$$

Proposed by Nguyen Van Canh-Vietnam

J.206 In ΔABC the following relationship holds:

$$\frac{4r}{R} + 2020\frac{m_a m_b m_c}{h_a h_b h_c} \ge 2022$$

Proposed by Nguyen Van Canh-Vietnam

J.207 If $u, v \in \mathbb{C}$ prove inequality

$$|u + v|^2 + |u - v|^2 \ge 2 \cdot max(|u|^2, |v|^2)$$

Proposed by Dorin Mărghidanu-Romania

J.208 For $a, b, c \ge 0$ then prove inequality:

$$\sqrt[4]{a(b + c)^3} + \sqrt[4]{b(c + a)^3} + \sqrt[4]{c(a + b)^3} \le \sqrt[4]{8}(a + b + c)$$

Proposed by Dorin Mărghidanu-Romania

J.209 Find the real value of x, such that

$$\sqrt{\sqrt{\frac{x+1}{x-1}}}+\sqrt{\sqrt{\frac{x-1}{x+1}}}=x+\frac{1}{x}$$

Proposed by Srinivasa Raghava –AIRMC- India

J.210 If $0<a_1,a_2,\dots,a_n<1, n\in\mathbb{N}, n\ge 2, 2\le k\le n, 2\le l\le n$ then:

$$\sqrt[k]{a_1a_2\cdot\ldots\cdot a_k}+\sqrt[l]{(1-a_1)(1-a_2)\cdot\ldots\cdot(1-a_l)}\le 1$$

Proposed by Lucian Tuțescu – Romania

J.211 $ABCD$ – cyclic quadrilateral, $AC\cap BD=\{0\}, r_1,r_2,r_3,r_4$ – inradii of $\Delta AOB,\Delta BOC,\Delta COD,\Delta DOA$. Prove that:

$$r_1=r_2=r_3=r_4\Leftrightarrow ABCD\text{ – rectangle}$$

Proposed by Cătălin Cristea – Romania

J.212 If $n\in\mathbb{N}, n\ge 5$ then:

$$2<\sqrt{2!+\sqrt[3]{3!+\sqrt[4]{4!}+\cdots+\sqrt[n]{n!}}}<\sqrt{5}$$

Proposed by Claudiu Coandă – Romania

J.213 In ΔABC the following relationship holds:

$$\frac{a^4}{r_ar_b}+\frac{b^4}{r_br_c}+\frac{c^4}{r_cr_a}\ge\frac{16F}{\sqrt{3}}$$

Proposed by D.M. Bătinețu-Giurgiu, Flaviu Cristian Verde – Romania

J.214 Solve for natural numbers:

$$\begin{cases}(x+2y+z)(3x+y+2z)=9\\(y-1)(y^2+y+1)=x(x^2+x+1)\end{cases}$$

Proposed by Daniel Sitaru,Nicolae Oprea – Romania

J.215 If $a,b>0$ then:

$$\sqrt[3]{\frac{a^3+b^3}{2}}\cdot\sqrt[4]{\frac{a^4+b^4}{2}}\cdot\sqrt[5]{\frac{a^5+b^5}{2}}\le\frac{a^5+b^5}{a^2+b^2}$$

Proposed by Daniel Sitaru,Iulia Selea – Romania

J.216 Solve for real numbers:

$$\begin{cases}\tan^2 x\,(1-\sin^8 x)+\cot^2 x\,(1-\cos^8 x)=\frac{15}{8}\\x+y=\pi\\\tan^2 y\,(1-\sin^{10} y)+\cot^2 y\,(1-\cos^{10} y)=\frac{31}{16}\end{cases}$$

Proposed by Daniel Sitaru,Ionuț Ivănescu – Romania

J.217 If $0 \le x, y, z < 1$ then:

$$\frac{(1-\sqrt[3]{x^2})(1-\sqrt[4]{y^3})(1-\sqrt[5]{z^4})}{(1-x)(1-y)(1-z)} \le \frac{\sqrt[3]{(1+x)^2}\cdot\sqrt[4]{(1+y)^3}\cdot\sqrt[5]{(1+z)^4}}{(1+x)(1+y)(1+z)}$$

Proposed by Daniel Sitaru,Amelia Curcă Năstăselu – Romania

J.218 Solve for $x, y, z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then:

$$\frac{\cos(5x)}{\cos x} + \frac{\cos(5y)}{\cos y} + \frac{\cos(5z)}{\cos z} = \frac{15}{4}$$

Proposed by Daniel Sitaru,Mirea Mihaela Mioara – Romania

J.219 If $a, b, c > 0, a^4 + b^4 + c^4 = 3(a^2 + b^2 + c^2)$ then:

$$\left(\sum_{cyc} a^{10}\right)\left(\sum_{cyc} a^3\right) \ge 3\left(\sum_{cyc} a^6\right)\left(\sum_{cyc} a^5\right)$$

Proposed by Daniel Sitaru,Claudiu Ciulcu – Romania

J.220 If $x, y, z > 0, \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} = 3$ then:

$$4(x^3 + y^3 + z^3) + 7\left(x^2 + y^2 + z^2 + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) \ge 54$$

Proposed by Daniel Sitaru,Lavinia Trincu – Romania

J.221 If $a, b, c, d, e, f \in \mathbb{C}, |a|^2 + |b|^2 + |c|^2 \le 3, |d|^2 + |e|^2 + |f|^2 \le \frac{1}{3}$ then:

$$|ad + be + cf|^2 \le 1$$

Proposed by Daniel Sitaru,Mihaela Nascu – Romania

J.222 O – circumcenter, I – incenter, R – circumradii, r – radii, a, b, c, d – sides in a bicentric quadrilateral. Prove that:

$$2OI^2 + r\sum_{cyc}\sqrt{4R^2 - a^2} = 2(R^2 + 2r^2)$$

Proposed by Daniel Sitaru,Corina Ionescu – Romania

J.223 If $a, b, c > 0$ then:

$$\frac{a+b+c}{4} + \frac{3abc}{4(ab+bc+ca)} \ge \sqrt[4]{\frac{abc(a+b+c)}{3}}$$

Proposed by Daniel Sitaru,Laura Zaharia – Romania

J.224 If $x, y \in \mathbb{N}$ then:

$$\frac{2^x(1+2^{x+2y})}{1+2^x} + \frac{2^y(1+2^{2x+y})}{1+2^y} + \frac{2^{x+y}(2^x+2^y)}{1+2^{x+y}} \ge 3\cdot 2^{x+y}$$

Proposed by Daniel Sitaru,Delia Schneider – Romania

J.225 If in $\Delta ABC, m(\sphericalangle A) = 27^\circ, m(\sphericalangle B) = 54^\circ$ then:

$$\frac{a^3 + ba^2 + c^3}{3abc} + \frac{8abc}{(a+b)(b+c)(c+a)} > \frac{5}{3}$$

Proposed by Daniel Sitaru,Elena Grigore – Romania

J.226 In $\Delta ABC, BB', CC'$ - symedians, $B' \in (AC), C' \in (AB), BB' \cap CC' = \{K\}$

$$AC'KB' \text{ - cyclic quadrilateral} \Leftrightarrow \frac{b^2}{a^2+c^2} + \frac{c^2}{a^2+b^2} = 1.$$

Proposed by Marian Ursărescu – Romania

J.227 In $\Delta ABC, N_a$ – Nagel's point, BB', CC' - Nagel's cevians, $B' \in (AC), C' \in (AB)$

$BB' \cap CC' = \{N_a\}$. Prove that: $AC'N_aB'$ - cyclic quadrilateral $\Leftrightarrow b + c = 2a$

Proposed by Marian Ursărescu – Romania

J.228 In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right)^3 \cdot \left(\sum_{cyc}\cot\frac{A}{2}\right)^{-1} \le 8\left(4\left(\frac{R}{r}\right)^2 - \frac{8R}{r} + 3\right)$$

Proposed by Marian Ursărescu – Romania

J.229 $z_1, z_2, z_3 \in \mathbb{C}^*$ - different in pairs, $|z_1| = |z_2| = |z_3|, A(z_1), B(z_2), C(z_3)$

$$\sum_{cyc}\left|\frac{(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2|}{2z_1 - z_2 - z_3}\right| = \sum_{cyc}|z_1 - z_2| \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

J.230 $z_1, z_2, z_3 \in \mathbb{C}^*$ - different in pairs, $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$

$$\sum_{cyc}\frac{1}{3|2z_1 - z_2 - z_3| + |2z_2 - z_1 - z_3|} = \frac{1}{2} \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu – Romania

J.231 In ΔABC the following relationship holds:

$$\sum_{cyc}(\sin A + \sin B)^3 \cdot \left(\sum_{cyc}\sin A\right)^{-1} \le 4\left(2 - \frac{r}{R}\right)$$

Proposed by Marian Ursărescu – Romania

J.232 In acute ΔABC the following relationship holds:

$$R^2 \cdot \prod_{cyc}(h_a^2 + h_a h_b + h_b^2) \ge 12 \cdot (3r)^8$$

Proposed by Marian Ursărescu – Romania

J.233 In ΔABC the following relationship holds:

$$\prod_{cyc}(r_a^2+r_b^2+m_c^2)\geq s^6$$

Proposed by Marian Ursărescu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.01 If $a,b,c>0$ then:

$$\left(1+a^{1+a^{1+a}}\right)\left(1+b^{1+b^{1+b}}\right)\left(1+c^{1+c^{1+c}}\right)\geq 8a^b b^c c^a$$

Proposed by Florică Anastase – Romania

S.02 $(a_n)_{n\geq 1},(b_n)_{n\geq 1},(c_n)_{n\geq 1}$ are sequences of real numbers such that:

$$a_n=\sum_{k=1}^{n}\binom{n}{k}\cdot k^{\frac{1}{k}};b_n=\sum_{k=1}^{n}\binom{n}{k}\cdot\left(\frac{1}{k}\right)^{\frac{1}{k}};c_n=\sum_{k=1}^{n}\frac{1}{2^k}\left(2\cos\frac{\pi}{2(k+1)}-\sin\frac{\pi(k+1)}{2(k+2)}\right)$$

Find:

$$\Omega=\lim_{n\to\infty}\left(a_n\cdot b_n\cdot c_n^{6n}\right)$$

Proposed by Florică Anastase – Romania

S.03 If $x\in\left(\pi,\frac{3\pi}{2}\right)$ then:

$$\frac{\left(\frac{1+\sin(\sin x)}{\sin x}\right)^{\frac{1}{1+\cot x}}\cdot\left(\frac{1+\sin(\cos x)}{\cos x}\right)^{\frac{1}{1+\tan x}}}{(1+\sin(\sin x))\sin x+(1+\sin(\cos x))\cos x}\leq 1$$

Proposed by Florică Anastase – Romania

S.04 Let $(a_n)_{n\geq 1}$ be sequence of real numbers such that: $a_1 = a > 0$,

$$na_{n+1} = (n+1)(a\cdot a_n + n\cdot a^{n+1}), \forall n\geq 1.$$ Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n^2]{\prod_{k=1}^{n} a_k}\right)^{\frac{n^2}{a_n}+\sum_{k=1}^{n}\left(\frac{k^2}{a_k}\right)}$$

Proposed by Florică Anastase – Romania

S.05 Find:

$$\Omega = \lim_{n\to\infty}\sum_{1\leq i\leq i\leq n}\frac{(n^2+ni+i^2)(n^2+nj+j^2)}{(n^2+i^2)(n^2+j^2)\sqrt{n^4+i^2+j^2}}\cdot e^{\tan^{-1}\left(\frac{n(i+j)}{n^2-ij}\right)}$$

Proposed by Florică Anastase – Romania

S.06 If $x_1, x_2, \dots, x_n > 0, n\geq 2, \lambda > 0$ then:

$$\frac{nx_1+4(1+\lambda)}{nx_2^2+n^2x_3^3+\dots+n^{n-1}x_n^n}+\frac{n^2x_2^2+4(1+\lambda)}{x_1+n^2x_3^3+\dots+n^{n-1}x_n^n}+\dots+$$

$$+\frac{n^nx_n^n+4(1+\lambda)}{x_1+nx_2^2+\dots+n^{n-2}x_{n-1}^{n-1}}\geq\frac{n^3x_1+n^4x_2^2+\dots+n^{n+2}x_n^n+4n^2(1+\lambda)}{(n-1)(x_1+nx_2^2+\dots+n^{n-1}x_n^n)}$$

Proposed by Florică Anastase – Romania

S.07 Find:

$$\Omega = \lim_{n\to\infty}\frac{\sum_{k=1}^{n-1}\left(\log(k^2+n^2)^n+k\tan^{-1}\frac{k}{n}+\frac{\pi}{4}\right)-2n(n-1)\log n}{n^2}$$

Proposed by Florică Anastase – Romania

S.08 Find:

$$\Omega = \lim_{n\to\infty}\frac{\sum_{k=1}^{n-1}\left(\log(k^2+n^2)^n+k\tan^{-1}\frac{k}{n}+\frac{\pi}{4}\right)-2n(n-1)\log n}{n^2}$$

Proposed by Florică Anastase – Romania

S.09 For $x\in[-1,1]$ define:

$$P(x) = \frac{\cos(n\cdot\cos^{-1}x)}{n}\cdot\sum_{k=1}^{n}\cos^{n-1}\frac{k\pi}{n}\cos\frac{(n-1)k\pi}{n}$$

Show that $P(x)$ is a polynomial of degree n with rational coefficients and coefficient to x^n is 1.

Proposed by Florică Anastase – Romania

S.10 If $(a_n)_{n\geq 1}, a_1 = e, n\in\mathbb{N}, n\geq 2, a_n = e^{\sqrt[n]{e}-1}\cdot a_{n-1}$ then find:

$$\Omega = \lim_{n\to\infty}\log_{n+1}a_n$$

Proposed by Florică Anastase – Romania

S.11 Let $(x_n)_{n\geq1},(y_n)_{n\geq1}$ be sequences of positive real numbers such that:

$$x_1>1, nx_{n+1}=(n-1)x_n+x_n^{1-n}; y_1>0, y_{n+1}=\frac{(n+x_n)n^n y_n}{y_n^n+n^n(n-x_n)}$$

Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{y^n}\sum_{k=1}^{n}\left(\cos^2\frac{4\pi x_k y_k}{2y_n+x_n}\right)$$

Proposed by Florică Anastase – Romania

S.12 Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\cdot\sqrt[n^2]{\prod_{k=1}^{n}k^k\cdot e^{\sum_{k=1}^{n}ke^{\frac{k}{n^2}}}}\right)$$

Proposed by Florică Anastase – Romania

S.13 If $t\geq0$, then in any ΔABC with the area F the following inequality holds:

$$m_a^{t+1}+m_b^{t+1}+m_c^{t+1}\geq2^{t+1}\left(\sqrt{3}\right)^{1-t}\left(\frac{F}{R}\right)^{t+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.14 $P_0(x)=1, P_1(x)=x, P_n(x)=2xP_{n-1}(x)-P_{n-2}(x), n\in\mathbb{N}, n\geq2$

Find the reminder of $P_n, n\geq3$ at x^3-x^2-x+1

Proposed by Marian Ursărescu – Romania

S.15 $L_0=1, L_1=1, L_{n+2}=L_{n+1}+L_n$. Find:

$$\Omega=\lim_{n\to\infty}\frac{\binom{n}{0}L_0^2-\binom{n}{1}L_1^2+\binom{n}{2}L_2^2+\cdots+(-1)^n\binom{n}{n}L_n^2}{n\cdot2^{n+1}+(-1)^n\cdot n\,L_n}$$

Proposed by Marian Ursărescu – Romania

S.16 $A\in M_4(\mathbb{R}), \det(A^2+3I_4)=\det(A^2+2A+2I_4)=0$. Find: $\Omega=\det A$

Proposed by Marian Ursărescu – Romania

S.17 $x_1=1, x_{n+1}=1+n\sqrt{x_n}, n\geq1$

Find:

$$\Omega_1=\lim_{n\to\infty}\frac{\sqrt{x_n}}{n}, \Omega_2=\lim_{n\to\infty}\left(\frac{\sqrt{x_n}}{n}\right)^{\sqrt{n}}$$

Proposed by Marian Ursărescu – Romania

S.18 If $A\in M_n(\mathbb{C}), n\ge 2, \det A\neq 0, a,b,c\in\mathbb{C}^*, |a|=|b|=|c|, aA+bA^{-1}=cI_n$ then:

$$\left(\frac{\sqrt{5}-1}{2}\right)^n\le|\det A|\le\left(\frac{\sqrt{5}+1}{2}\right)^n$$

Proposed by Marian Ursărescu – Romania

S.19 Find:

$$\Omega_n=\lim_{x\to 0}\frac{\ln(1+\sinh^n x)-\ln^n(1+\sinh x)}{x^{n+1}}, n\in\mathbb{N}, n\ge 2$$

Proposed by Marian Ursărescu – Romania

S.20 If $a,b,c\in\mathbb{R}, a\neq 0, \det(aA^2+bA+cI_{2018})\ge 0, \forall A\in M_{2018}(\mathbb{R})$

then: $b^2\le 4ac$

Proposed by Marian Ursărescu – Romania

S.21 If $A,B\in M_3(\mathbb{R}), Tr^2(AB)=Tr(AB)^2, (AB)^3=O_3$ then:

$$(BA)^3=O_3$$

Proposed by Marian Ursărescu – Romania

S.22 Solve for $M_3(\mathbb{R})$:

$$X+X^3+X^5+\dots+X^{2p-1}=\begin{pmatrix}p & p & 0\\ 0 & p & p\\ 0 & 0 & p\end{pmatrix}, p\in\mathbb{N}, p\ge 3$$

Proposed by Marian Ursărescu – Romania

S.23 Let $x_n, n\ge 2$ be the solution of equation:

$$x+\arctan(x-1)=\sqrt[n]{2018}$$

Find:

$$\Omega_1=\lim_{n\to\infty}x_n, \Omega_2=\lim_{n\to\infty}n(x_n-1)$$

Proposed by Marian Ursărescu – Romania

S.24 $f:[a,b]\to\mathbb{R}, f$ – continuous, f – nonconstant. Prove that:

$\exists c_1,c_2\in[a,b], c_1\neq c_2$ such that $c_1+c_2=a+b, c_1f(c_1)=c_2f(c_2)$

Proposed by Marian Ursărescu – Romania

S.25 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{\cos^{3n}\frac{A}{2}\left(\cos\frac{B}{2}+\cos\frac{C}{2}\right)}\ge\sqrt{3}\left(\frac{2}{\sqrt{3}}\right)^{3n}, n\in\mathbb{N}$$

Proposed by Marin Chirciu – Romania

S.26 If $a,b,c>0, a+b+c=3$ then:

$$\frac{1}{9}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+\sum_{cyc}\frac{a}{7bc+\sqrt{2(b^4+c^4)}}\ge\frac{2}{3}$$

Proposed by Marin Chirciu – Romania

S.27 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{H_n}\prod_{k=2}^{n}\left(\frac{k!!\cdot k^{k-1}\cdot(k-1)!!}{n!!\cdot(n-1)!!}\right)$$

Proposed by Daniel Sitaru,Dan Grigorie – Romania

S.28 If $a,b,c,d>0$, different in pairs then:

$$\sum_{cyc(a,b,c,d)}\frac{a^3}{(b+c+d)(a-b)(a-c)(a-d)}<\frac{(a+b+c+d)^3}{81abcd}$$

Proposed by Daniel Sitaru,Elena Alexie – Romania

S.29 If $n\in\mathbb{N}-\{0\}$ then:

$$\frac{\sin n}{\sin(n+1)}\sum_{k=1}^{n}\sin k\cdot\sin(k+1)\ge(n\cdot\sin 1)^2$$

Proposed by Daniel Sitaru– Romania

S.30 If $0<a\le b$ then:

$$\frac{\left(\frac{a+b}{2}+\sqrt{ab}-\frac{2ab}{a+b}\right)^{\frac{a+b}{2}+\sqrt{ab}-\frac{2ab}{a+b}}}{\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}}\ge\frac{\left(\sqrt{ab}\right)^{\sqrt{ab}}}{\left(\frac{2ab}{a+b}\right)^{\frac{2ab}{a+b}}}$$

Proposed by Daniel Sitaru,Elena Iacob Meda – Romania

S.31 Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\omega_k}{2\omega_{k+1}}-\log n\right),\ \omega_n=\lim_{x\to\frac{\pi}{2}}\prod_{k=1}^{n}\frac{1-\sin^k x}{\cos^2 x}$$

Proposed by Daniel Sitaru,Gilena Dobrică – Romania

S.32 Solve for real numbers:

$$2\cdot\sqrt[4]{e^{4^x}\cdot e^{2\cdot 6^x}\cdot e^{9^x}}=e^{4^x}+e^{9^x}$$

Proposed by Daniel Sitaru,Mihaela Dăianu – Romania

S.33 Solve for real numbers:

$$\begin{cases} x,y,z>0\\ \sqrt{1-x}+\dfrac{y}{\sqrt{1-y}}=2\sqrt{1+z}\\ \begin{vmatrix} yz & xy & zx\\ xy & zx & yz\\ zx & yz & xy\end{vmatrix}=0\end{cases}$$

Proposed by Daniel Sitaru,Mihaela Pupăză – Romania

S.34 Find:

$$\Omega=\lim_{n\to\infty}\left(\left(\sqrt[n]{\prod_{k=1}^{n}(k+n)}\right)^{-1}\cdot\sum_{k=1}^{n}\frac{1}{k}\cdot\sqrt[k]{\prod_{p=1}^{k}(k+p)}\right)$$

Proposed by Daniel Sitaru,Simona Radu – Romania

S.35

$$A\in M_n(\mathbb{C}),\det A\neq 0,3A+4A^{-1}+i(4A+3A^{-1}-5I_4)=O_4$$

Find: $\Omega_1=\min|\det A|,\Omega_2=\max|\det A|$

Proposed by Marian Ursărescu – Romania

S.36 If $A\in M_4(\mathbb{R}),\det A\neq 0,(TrA)^2=3Tr(A^2)$ then:

$$Tr\,(A^3)=3\cdot\det A\cdot Tr\,(A^{-1})$$

Proposed by Marian Ursărescu – Romania

S.37 $A\in M_4(\mathbb{R}),\det A=-1,\det(A^2+I_4)=0$. Find:

$$\Omega=Tr\,(A^*)$$

Proposed by Marian Ursărescu – Romania

S.38 If $A\in M_3(\mathbb{R}),\det(A^2-3A+3I_3)=0$ then:

$$2\det(A^2+3I_3)\geq 3(3+\det A)^2$$

Proposed by Marian Ursărescu – Romania

S.39 Let $x_n,y_n,z_n\subset\mathbb{R}$ such that:

$$\lim_{n\to\infty}\frac{x_n}{n^p}=a,;\lim_{n\to\infty}\frac{y_n}{n^q}=b,\lim_{n\to\infty}\frac{z_n}{n^r}=c,a,b,c\in\mathbb{R}^*,p,q,r\in\mathbb{N}^*$$

Find:

$$\lim_{n\to\infty}\frac{\sum_{1\leq i<j<k\leq n}x_ix_jx_k\cdot\sum_{1\leq i<j<k\leq n}y_iy_jy_k\cdot\sum_{1\leq i<j<k\leq n}z_iz_jz_k}{n^8\sum_{i=1}^{n}(x_iy_iz_i)^3}$$

Proposed by Marian Ursărescu – Romania

S.40 Find:

$$\lim_{n\to\infty}\sum_{1\leq i<j\leq n}\sin\left(\frac{\sqrt{ij}}{n^3}\right)$$

Proposed by Marian Ursărescu – Romania

S.41 If $a,b>0,ab=1$ then:

$$\left(\frac{a}{\sqrt{1+b^2}}\right)^n+\left(\frac{b}{\sqrt{1+a^2}}\right)^n\geq 2^{1-\frac{n}{2}},n\in\mathbb{N}$$

Proposed by Marin Chirciu – Romania

S.42

$$f(x) = \log[x^3 - (6n-11)x^2 + (12n^2 - 44n + 31)x - 8n^3 + 44n^2 - 62n + 21]$$

Find:

$$\Omega = \lim_{k\to\infty}\left(\frac{(-1)^{k+1}k \cdot 2^k}{k!} \cdot f^{(k)}(2n+1)\right)^{2^k}$$

Proposed by Costel Florea – Romania

S.43

$$f:\mathbb{R}\to\mathbb{R}, f(x) = x^n e^x, F'(x) = f(x), F(0) = (-1)^n \cdot n!, F(x) = e^x \cdot \sum_{k=0}^{n} u_k x^k$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{u_{n-100}}{n^2 \cdot u_{n-98}}\right)^{\frac{\phi n}{197}}, \phi - \text{golden ratio.}$$

Proposed by Costel Florea – Romania

S.44 If $x > \sqrt{\pi}$, then prove that

$$\tan^{-1}(\sqrt{x}) > \sin\left(\sec\left(\frac{1}{x}\right)\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

S.45 $x_0 \in \left(0, \frac{\pi}{2}\right), y_0 > 0, x_{n+1} = x_n \cdot \cos x_n, y_{n+1} = y_n + \frac{1}{y_n}$

Find:

$$\Omega = \lim_{n\to\infty}(x_n \cdot y_n)$$

Proposed by Marian Ursărescu – Romania

S.46 $x_n \geq 0, n \in \mathbb{N}, p + x_{n+1}^p < 1 + p x_n, p \in \mathbb{N}, p \geq 2$

Prove that x_n – convergent and find the limit.

Proposed by Marian Ursărescu – Romania

S.47 If $A \in M_3(\mathbb{R}), \det(A^2 - 3A + 3I_3) = 0$ then:

$$2\det(A^2 + 3I_3) \geq 3(3 + \det A)^2$$

Proposed by Marian Ursărescu – Romania

S.48 Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ \sqrt{1-x} + \dfrac{y}{\sqrt{1-y}} = 2\sqrt{1+z} \\ \begin{vmatrix} yz & xy & zx \\ xy & zx & yz \\ zx & yz & xy \end{vmatrix} = 0 \end{cases}$$

Proposed by Daniel Sitaru, Mădălina Giurgescu – Romania

S.49 Find:

$$\Omega = \lim_{n\to\infty} \frac{\left((2n)!\right)^2 \cdot (2n+1)}{(3^n \cdot n!)^4} \cdot \sum_{i=1}^{n} \frac{\binom{n}{i} \cdot \binom{n}{n-i}}{(2i+1)\left(\binom{2n-1}{2i-1} + \binom{2n-1}{2i}\right)}$$

Proposed by Daniel Sitaru– Romania

S.50 Find:

$$\Omega = \lim_{n\to\infty} \left((1 \cdot n + 3 \cdot (n-1) + 5 \cdot (n-2) + \dots + (2n-1) \cdot 1) \cdot \sin \frac{1}{n^3} \right)$$

Proposed by Daniel Sitaru,Dan Mitricoiu – Romania

S.51 Find:

$$\Omega = \lim_{n\to\infty} \left((n-1) \cdot \frac{1}{n} + (n-2) \dots \left(\frac{1}{n} + \frac{1}{n-1}\right) + (n-3) \cdot \left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2}\right) + \dots + 1 \cdot \left(\frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{2}\right) \right) \tan \frac{1}{n^2}$$

Proposed by Daniel Sitaru– Romania

S.52 Solve for real numbers:

$$x \cdot logx = x - x^{n+1}, n \in \mathbb{N}, n \text{ –fixed}$$

Proposed by Ionuț Florin Voinea-Romania

S.53 Find $x \in \left(0, \frac{\pi}{2}\right)$ such that:

$$e^{tanx+1} = log(tanx) + e^{tan^2x + tanx}$$

Proposed by Ionuț Florin Voinea-Romania

S.54 If $x \geq 1$ then:

$$e^x - e^{\frac{1}{x}} > e\left(x - \frac{1}{x}\right)$$

Proposed by Ionuț Florin Voinea-Romania

S.55 If $x, y, z > 0, tan^{-1}x + tan^{-1}y + tan^{-1}z = \pi$ then:

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq \sqrt{3xyz}$$

Proposed by Ionuț Florin Voinea-Romania

S.56 Solve for real numbers:

$$(3^x + 5^x + 7^x)(5^x + 7^x + 9^x) = (4^x + 5^x + 6^x)(6^x + 7^x + 8^x)$$

Proposed by Ionuț Florin Voinea-Romania

S.57 Let $f_m(x) = \frac{2x+m}{x-1}$, $g(x) = \frac{2x-m}{x+1}$. Find all real numbers m such that:

$$min f_m(x) + max g_m(x) = 2021, \forall x \in \left[0, \frac{1}{2}\right]$$

Proposed by Nguyen Van Canh-Vietnam

S.58 Let $f_m(x) = \frac{m}{sinx+2}$, $g(x) = \frac{m}{cosx+2}$. Find all real numbers m such that:

$$min f_m(x) + max g_m(x) = 1, \forall x \in \left[0, \frac{\pi}{2}\right]$$

Proposed by Nguyen Van Canh-Vietnam

S.59 Let $f_m(x) = \frac{x+m}{x-m} + \frac{5x+6}{5x-6m} + x^2 + 4mx + 5, m \in \mathbb{R}$. Find all positive real numbers m such that: $min f_m(x) + max g_m(x) = 2020, \forall x \in [-1,1]$

Proposed by Nguyen Van Canh-Vietnam

S.60 Let $a \geq b > 0$. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f^3(a^2x + b^2y) = 9f(ax^2 - by^2) - \frac{x^3y}{\sqrt{ab}}, \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

S.61 If $f: [0,1] \to \mathbb{R}$ is a continuous function such that $3\int_0^1 f(x)dx = 2$, then there is an $x_0 \in (0,1)$ such that $f(x_0) = \sqrt{x_0}$

Proposed by Dorin Mărghidanu-Romania

S.62 Let the function $f: (0, \infty) \to (0, \infty)$, having the properties:

$a)$ f – differentiable on $(0, \infty)$

$b)$ there is $\lim_{x\to\infty} x \cdot f'(x) = l, l \in \mathbb{R}$

Prove that for $p \in \mathbb{R}_+^*$, we have: $\lim_{x\to\infty} x \cdot [f(x+p) - f(x)] = p \cdot l$

Proposed by Dorin Mărghidanu-Romania

S.63 If $f: [0,1] \to [0, \infty)$ such that $\int_0^1 f(x)\,dx = 1$,

$$I_1 = \int_0^1 \frac{x^2 + x + 1}{x^2 + 1} \cdot e^{\tan^{-1} dx}, I_2 = \int_0^1 \left(x - \int_0^1 tf(t)\,dt\right)^2 f(x)dx$$

Then prove: $I_1 \geq e^{\pi I_2}$

Proposed by Florică Anastase – Romania

S.64 If $\lambda > 1, f: [0,1] \to [1, \lambda]$ continuous and convex function such that $f(0) = 0$ then prove:

$$\lambda^2 \int_0^{\frac{1}{\lambda}} f(x)\,dx \leq \int_0^1 f(x)\,dx \leq \lambda(\lambda + 1) \int_0^1 \frac{dx}{1 + f(x)}$$

Proposed by Florică Anastase – Romania

S.65 If $0 < a < b \le 2a, f:[a,b] \to [0,c), f$ – continuous, $f(a) = 0, f'(a) \ge 0$

f' - increasing, then:

$$\frac{(b-a)\big(c(b-a)+af(b)\big)}{bf(b)+c(b-a)-2\int_a^b f(x)\,dx} \ge c\big(c-f(b)\big)\int_a^b \frac{dx}{\big(c-f(x)\big)^2}$$

Proposed by Florică Anastase – Romania

S.66 If $n \in \mathbb{N}, n \ge 2$ then exists $f \in \mathbb{Z}_5[X], grad\ f = n$ such that f has not roots in $\mathbb{Z}_5$.

Proposed by Florică Anastase – Romania

S.67 Find:

$$\Omega = \lim_{x\to 0}\left(\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k\binom{n}{k}}{n2^n+k}\cdot\prod_{k=0}^{n}\cos(2^{k-n}x)\right)$$

Proposed by Florică Anastase – Romania

S.68 If $0 < a \le b, f:[a,b] \to \mathbb{R}, f'(x) > 0, \forall x \in [a,b]$ then:

$$\int_a^b x^2 f(x)dx \ge ab\int_a^b f(x)\,dx$$

Proposed by Marian Ursărescu – Romania

S.69 Find:

$$\Omega(n) = \int_0^{\frac{\pi}{2}} \frac{2(2n+1)x+\cos\big((4n+2)x\big)}{\sin x+\cos x}dx, n \in \mathbb{N}, n \ge 1$$

Proposed by Marin Chirciu – Romania

S.70 Find:

$$\Omega(n) = \int_1^{2n-1} \frac{x+\sin(x-1)\,(x-2)\cdot\ldots\cdot(x-2n+1)}{x^2-2nx+n^2+1}dx, n \in \mathbb{N}, n \ge 2$$

Proposed by Marin Chirciu – Romania

S.71 Prove without softs:

$$\int_0^1 e^{x^2}dx\cdot\int_0^1 e^{-x^2}\,dx < \left(\frac{1+e}{2\sqrt{e}}\right)^2$$

Proposed by Daniel Sitaru, Anicuța Patricia Bețiu – Romania

S.72 If $f:[a,b] \to (0,\infty), 0 < a \le b, f$ - continuous then:

$$(b-a)^2\int_a^b f^3(x)\,dx + \left(\int_a^b f(x)\,dx\right)^3 \ge 2(b-a)\left(\int_a^b f(x)\,dx\right)\left(\int_a^b f^2(x)\,dx\right)$$

Proposed by Daniel Sitaru, Maria Lavinia Popa – Romania

S.73 Solve for real numbers:

$$\int_0^x \frac{t^2}{(t \cdot \sinh t - \cosh t)^2} dt = 0$$

Proposed by Daniel Sitaru,Sorin Pîrlea – Romania

S.74 If F_n – Fibonacci numbers then:

$$F_{n-1}^4 + F_n^4 + F_{n+1}^4 > F_{3n}\left(\frac{F_{n-1}F_n}{F_{n+1}} + F_{n+1}\right), n \geq 1$$

Proposed by Daniel Sitaru,Dorina Goiceanu – Romania

S.75 Find without any software:

$$\Omega = \int \frac{x^{2018}}{1 - x^{4038}} dx$$

Proposed by Daniel Sitaru,Iulia Sanda – Romania

S.76

$$x * y = \sqrt[5]{x^5y^5 - x^5 - y^5 + 2}, x, y \in \mathbb{R}$$

Solve for real numbers: $x * y * z - 1 = 0$

Proposed by Daniel Sitaru,Nicolae Radu – Romania

S.77 Find:

$$\Omega = \lim_{n\to\infty} \int_0^n \frac{4x^3 - 3n^2x}{n^3 + n^5 \cos^2 x} dx$$

Proposed by Daniel Sitaru,Mihaela Săncele – Romania

S.78 Solve for real numbers:

$$\int_0^x \frac{t^2}{(t \cdot \sinh t - \cosh t)^2} dt = \int_0^x \frac{t^2}{(t \cdot \cosh t - \sinh t)^2} dt$$

Proposed by Daniel Sitaru,Daniela Stoian – Romania

S.79 The equation: $x^8 + ax^7 + bx^6 + ax^5 + cx^4 + dx^3 + ex^2 + dx - 1 = 0$,
$a, b, c, d, e \in \mathbb{R}$ has all roots real numbers. Prove that:

$$|b + e - c| \geq 16$$

Proposed by Marian Ursărescu – Romania

S.80 Solve for $n \in \mathbb{N}^*$:

$$\int_0^{\log n} \frac{3e^{4x} - 2e^{3x} + e^{2x} - 1}{e^{4x} - e^{3x} + e^{2x} - e^x + 1} dx = 0$$

Proposed by Costel Florea – Romania

S.81

$$f_m(x)=(m+1)x^3-2(m+1)x^2-(m-2)x+2m-3, m\in\mathbb{R}-\{-1\}$$

Find the equation of the line which contains the three fixed points of f_m (the points do not depend of m)

Proposed by Costel Florea – Romania

S.82 If $\Omega(n)=\int_0^1 \tan^{-1}x\,dx+\int_0^1 x(\tan^{-1}x)^2\,dx+\cdots+\int_0^1 x^n(\tan^{-1}x)^{n+1}\,dx$

Find:

$$\lim_{n\to\infty}\frac{\Omega(n)}{n}$$

Proposed by Costel Florea – Romania

S.83 $A\in M_2(\mathbb{R}), Tr\,A+\det A=2$. Prove that:

$$\det(A^2+\det A\cdot A+Tr\,A\cdot I_2)\geq 4$$

Proposed by Marian Ursărescu – Romania

S.84 Find without any software:

$$\Omega=\int\frac{\sinh x-x\cosh x}{x^2-1+\cosh^2 x}dx$$

Proposed by Daniel Sitaru,Gigi Zaharia – Romania

S.85 Find:

$$\Omega(a)=\int_0^a\frac{\cos x\cdot\sin^7 x}{1+\sin^2 x+\sin^4 x}dx, a>0$$

Proposed by Daniel Sitaru,Alecu Orlando – Romania

S.86 If $0<a\leq b$ then find:

$$\Omega=\int_a^b\log\left(\frac{\left(1+\frac{x}{a}\right)^{x^{-1}\cdot e^{\frac{b}{x}}}}{\left(1+\frac{b}{x}\right)^{x^{-1}\cdot e^{\frac{x}{a}}}}\right)dx$$

Proposed by Daniel Sitaru,Ileana Stanciu – Romania

S.87 Find:

$$\Omega(n)=\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{x^2\cdot\log^{2n+1}x}{1+x^2}dx, n\in\mathbb{N}$$

Proposed by Daniel Sitaru – Romania

S.88

$$\omega(a)=\cosh a\cdot\int_{2a}^{3a}\frac{1}{\tanh x-\tanh a}dx,a>0$$

Find:

$$\Omega=\lim_{\substack{a\to 0\\ a>0}}e^{\omega(a)}$$

Proposed by Daniel Sitaru – Romania

S.89 Solve for real numbers:

$$\int_0^x\frac{t^2}{(t\cdot\sinh t-\cosh t)^2}dt=\int_0^x\frac{t^2}{(t\cdot\cosh t-\sinh t)^2}dt$$

Proposed by Daniel Sitaru – Romania

S.90

$$\Omega(a,n)=\int_{2a}^{3a}\frac{\cos^{n-1}\left(\frac{x+a}{2}\right)}{\sin^{n+1}\left(\frac{x-a}{2}\right)}dx,a>0,n\in\mathbb{N},n\geq 1$$

Find:

$$\Omega=\lim_{n\to\infty}\left(n\cdot\Omega\left(\frac{\pi}{3},n\right)\right)$$

Proposed by Daniel Sitaru – Romania

S.91 Find:

$$\Omega=\lim_{n\to\infty}\left(\int_{-1}^{1}\frac{(1-x^2)(1-2x+x^2)^n}{(1+x^2)^{n+1}}dx\right)^{\frac{1}{n}}$$

Proposed by Daniel Sitaru – Romania

S.92 Solve for real numbers:

$$\sum_{i=1}^{2020}\sum_{j=1}^{2021}(x+i^2)(x+j^2)=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\log\left(\frac{2\tan^3x+4\cot^5x}{2\cot^3x+4\tan^5x}\right)dx$$

Proposed by Daniel Sitaru – Romania

S.93 Solve for real numbers:

$$\int_0^x\frac{t^2}{(t\cdot\sinh t-\cosh t)^2}dt=0$$

Proposed by Daniel Sitaru – Romania

S.94 Find without any software:

$$\Omega = \int \frac{3x^2 - (1+2x)e^{2x}}{x^3 - xe^{2x}} dx$$

Proposed by Daniel Sitaru – Romania

S.95 Find without any software:

$$\Omega = \int \frac{1 + (1 - x^2 - e^x)e^x}{(1 + xe^x)\sqrt{(1 - x^2)(1 - e^{2x})}} dx$$

Proposed by Daniel Sitaru – Romania

S.96 Find:

$$\Omega(a) = \int_0^{\log a} x \cdot (1 + a^x \log a) \cdot a^{x+a^x} dx, a > 1$$

Proposed by Daniel Sitaru – Romania

S.97 If $0 < a \le b < \frac{\pi}{2}$ then:

$$\left(\int_a^b \frac{\sin x}{x} dx\right)^2 + \left(\int_a^b \frac{\cos x}{x} dx\right)^2 \le \log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

S.98 $f: \mathbb{R} \to \mathbb{R}, f -$ continuous, $a, b \in \mathbb{R}, a \le b$. Prove that:

$$\int_a^b \left(f^8(x) + f^2(x)\right) dx + b - a \ge \int_a^b \left(f^5(x) + f(x)\right) dx$$

Proposed by Daniel Sitaru – Romania

S.99 If $0 < a \le b < \frac{\pi}{2}$ then:

$$\frac{3}{2}\int_a^b \frac{\sin x}{x} dx \le b - a + \cos\left(\sqrt{ab}\right) \sin\left(\frac{b-a}{2}\right)$$

Proposed by Daniel Sitaru – Romania

S.100 For $x > 0$ and $m, n, p > 0$ – fixed, denote: $\Omega(m,n,p) = x_{\min}$, $x_{\min}$ – value of x such that $\Omega(m,n,p)(x) = \left(\sqrt{n + (p-x)^2} + \sqrt{m + x^2}\right)$ has minimum value.

Prove that: $\sum_{cyc}\left(\sqrt{m} + \sqrt{n}\right)\Omega(m,n,p) \ge 3\sqrt{mnp}$

Proposed by Daniel Sitaru – Romania

S.101 Solve for real numbers:

$$64x^5 - 112x^4 - 8x^3 + 105x^2 - 56x + 7 = 0$$

Proposed by Daniel Sitaru – Romania

S.102 Let Ω be area of pedal triangle of Lemoine's point in ΔABC. Prove that:

$$(a^2 + b^2 + c^2)^2 \cdot \Omega \geq 972\sqrt{3} \cdot r^6$$

Proposed by Daniel Sitaru – Romania

S.103 Find $X \in M_3(\mathbb{R})$ such that:

$$X^{2019} + X = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Proposed by Marian Ursărescu – Romania

S.104 $x_1 = 1, x_n = (1 + nx_n) \cdot x_{n+1}, n \in \mathbb{N}, n \geq 1$. Find:

$$\Omega = \lim_{n\to\infty} \left(\sqrt[n]{(n!)^2} \cdot x_n\right)$$

Proposed by Marian Ursărescu – Romania

S.105 Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{n}\tan^{-1} n - \frac{1}{n+1}\tan^{-1}(n+1)\right)\left(\sum_{k=2}^{n} \sqrt[k]{k!}\right)$$

Proposed by Marian Ursărescu – Romania

S.106 $A, B, C \in M_n(\mathbb{R}), n \in \mathbb{N}, n \geq 2, AB = BA, AC = CA, BC = CB$

If $A + B + C = I_n$ then $\det(AB + C) \cdot \det(BC + A) \cdot \det(CA + B) \geq 0$

Proposed by Marian Ursărescu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.01 If $(x_n)_{n\geq 1},(y_n)_{n\geq 1}; x_n = \sum_{k=1}^{n}\tan^{-1}\left(1+\frac{1}{k}\right)-\frac{n\pi}{4}; y_n = \sum_{k=1}^{n}\frac{1}{\cot^{-1}(2k+1)}; n$ – fixed

Find:

$$\Omega = \lim_{n\to\infty}\frac{\sum_{k=1}^{n}(H_1\cdot H_2\cdot\ldots\cdot H_k)^{\frac{1}{k}}}{\sum_{m=1}^{n}\left(\frac{x_m^2}{\pi}+\frac{\pi y_m^2}{4}\right)}$$

Proposed by Florică Anastase – Romania

U.02 In ΔABC the following relationship holds:

$$Si(A)+Si(B)+Si(C)<\frac{2\pi R+s}{3R}, Si(x)=\int_0^x\frac{\sin t}{t}dt$$

Proposed by Daniel Sitaru – Romania

U.03 If $a,b>1$ then:

$$\int_1^a\int_1^b\frac{dxdy}{\sqrt{x^2y+xy^2}}<\log\sqrt[4]{ab(a-1)(b-1)}+\left(\log\sqrt{a}\right)\left(\log\sqrt{b}\right)$$

Proposed by Daniel Sitaru – Romania

U.04 If $-2<a\leq b<2$ then:

$$\frac{1}{2}\left|\int_a^b\int_a^b\int_a^b\frac{xyz+4(x+y+z)}{xy+yz+zx+4}dxdydz\right|\leq(b-a)^3$$

Proposed by Daniel Sitaru – Romania

U.05 Find:

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_\varepsilon^1\frac{\log x}{x^3+x\sqrt{x}+1}dx$$

Proposed by Vasile Mircea Popa – Romania

U.06 Find:

$$\Omega(a)=\int_0^\infty\left(\frac{x^2}{(1-x^2+x^4)(1+ax)}\right)dx, a>0$$

Proposed by Vasile Mircea Popa – Romania

U.07 Find:

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_\varepsilon^1\frac{\log x\,dx}{1+x^2+x^4}$$

Proposed by Vasile Mircea Popa – Romania

U.08 Find a closed form:

$$\Omega(a)=\int_0^\infty \frac{x^2}{(1+ax)(1+x^2+x^4)}dx, a>0$$

Proposed by Vasile Mircea Popa – Romania

U.09 Find without softs:

$$\Omega=\int_0^\infty \frac{\arctan x}{x^5+1}dx$$

Proposed by Vasile Mircea Popa – Romania

U.10 Find:

$$\Omega=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_\varepsilon^1 \frac{\log x}{1-x^2+x^4}dx$$

Proposed by Vasile Mircea Popa – Romania

U.11 Find a closed form:

$$\Omega(a)=\int_0^\infty \frac{x^3}{(1-x^2+x^4)(1+a^2x^2)}dx, a\neq 0$$

Proposed by Vasile Mircea Popa – Romania

U.12 Find without softs:

$$\Omega=\lim_{n\to\infty}\int_0^\infty \frac{\tan^{-1}x}{1+x^n}dx$$

Proposed by Vasile Mircea Popa – Romania

U.13 Find:

$$\Omega=\lim_{\substack{t\to 1\\ t<1}}\left(\frac{1}{t-1}\int_{\sqrt{t}}^{t^2}\frac{1}{x}\sin^{-1}x\,dx\right)$$

Proposed by Vasile Mircea Popa – Romania

U.14 Find a closed form:

$$\Omega(a)=\int_0^\infty \frac{x^3}{(x^4+x^2+1)(1+ax)}dx, a>0$$

Proposed by Vasile Mircea Popa – Romania

U.15 Find without softs:

$$\Omega = \int_0^2 \frac{\arctan x}{2x^2 + 2x + 3} dx$$

Proposed by Vasile Mircea Popa – Romania

U.16 Find without softs:

$$\Omega = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_\varepsilon^1 \frac{\ln x}{x^4 + 1} dx$$

Proposed by Vasile Mircea Popa – Romania

U.17 Find without any software:

$$\Omega = \int_0^2 \frac{\arctan x}{x^2 + 4x + 3} dx$$

Proposed by Vasile Mircea Popa – Romania

U.18 Find a closed form:

$$\Omega(a) = \int_0^\infty \frac{x^2}{(1 + x^4)(1 + a^2x^2)}, a \in \mathbb{R}$$

Proposed by Vasile Mircea Popa – Romania

U.19 Find a closed form:

$$\Omega(a) = \int_0^\infty \frac{x^2}{(x^4 - x^2 + 1)(1 + a^2x^2)} dx, a > 0$$

Proposed by Vasile Mircea Popa – Romania

U.20 Let $J_n(x)$ – Bessel function and $\mathcal{L}_x[f](y)$ – Laplace Transform and if

$$S(n) = \int_0^1 \mathcal{L}_x[e^{-x}J_n(x)]\,(y)dy$$

then prove that

$$\int_0^\infty S(n)\,dn = \log\left(\frac{\log(2 + \sqrt{5})}{\log(1 + \sqrt{2})}\right)$$

$$\sum_{n=1}^\infty S(n) = \log\left(\frac{\sqrt{5} - 3}{\sqrt{2} - 2}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.21 For $n > 1$ we have

$$\int_0^\infty \frac{\sqrt[n]{\sinh(x)}}{\cosh(x)+1}dx = \frac{2n+1}{2n-2}\frac{\Gamma\left(\frac{2n-1}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)}{2^{\frac{1}{n}}\Gamma\left(\frac{4n+1}{2n}\right)}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.22 Show that:

$$\sum_{n=-\infty}^{\infty}\frac{\left(\frac{5n+1}{13}\right)}{(2n+1)^3} =$$

$$= \frac{\pi^3}{17576}\left\{-\left(\frac{\sin\left(\frac{3\pi}{13}\right)}{\sin^4\left(\frac{3\pi}{26}\right)} + \frac{\sin\left(\frac{\pi}{13}\right)}{\cos^3\left(\frac{\pi}{13}\right)} - \frac{2\sin\left(\frac{3\pi}{13}\right)}{\cos^3\left(\frac{3\pi}{13}\right)} + \frac{2\cos\left(\frac{5\pi}{26}\right)}{\sin^3\left(\frac{5\pi}{26}\right)}\right)\right\}$$

Here in the summation $\left(\frac{n}{m}\right)$ is Jacobi Symbol

Proposed by Srinivasa Raghava-AIRMC-India

U.23 If we have the relation

$$\alpha\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}F_{4n+2}F_{6n+3}}{\varphi^{8n}} + \sum_{n=0}^{\infty}\frac{(-1)^n L_{2n+1}L_{4n+2}L_{6n+3}}{\varphi^{8n}} = 0$$

then find the value of α.

Here φ – Golden Ratio

Proposed by Srinivasa Raghava-AIRMC-India

U.24 Show that

$$\int_{-\infty}^{\infty} e^{z-z^2}\left(\mathcal{L}_y\left[\mathcal{F}_x\left[e^{x-x^2}\right](y)\right](z)\right)dz = (1+i)\sqrt{\frac{e\pi}{2}}$$

$\mathcal{L}_x[f](y)$ is Laplace Transform and $\mathcal{F}_x[f](y)$ is Fourier Transform

Proposed by Srinivasa Raghava-AIRMC-India

U.25. Prove the summation

$$\sum_{n=1}^{\infty}\frac{(F_n+\phi^{n-1})(L_n+\phi^{n-1})}{\phi^{6n-1}} = \frac{4\phi}{15}$$

F_n – Fibonnacci number; L_n – Lucas number; ϕ – Golden Ratio

Proposed by Srinivasa Raghava-AIRMC-India

U.26 Evaluate the integral in a closed – form

$$\int_1^{\infty}\left(\sum_{n=0}^{\infty}\left(\left(2+i\sqrt{2}\right)^n+3^n+\left(2-i\sqrt{2}\right)^n\right)x^n\right)dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.27 Compute the integral

$$\int_0^1\int_0^1 \frac{\tan^{-1}(\min(x,y))}{\sqrt{x+y}}dydx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.28

$$\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}+L_{2n+1}}{(2n+1)\phi^{4n}}=\frac{\pi(\phi+1)}{4\sqrt{5}}+\phi^2\coth^{-1}\left(\sqrt{5}\right)$$

ϕ – Golden Ratio, F_n – Fibonacci numbers, L_n – Lucas numbers

Proposed by Srinivasa Raghava-AIRMC-India

U.29 If we have the relation

$$\sum_{n=0}^{\infty}\frac{\phi^{3n+1}+(-1)^n(3n+1)}{\phi^{4n+1}}=\frac{1+k}{k}\sum_{n=0}^{\infty}\frac{(-1)^n\left(n\phi+\sqrt{5}\right)+\phi^{2n+1}}{\phi^{3n+2}}$$

then find the value of k. ϕ – Golden Ratio

Proposed by Srinivasa Raghava-AIRMC-India

U.30 For $n>1$, prove the relation

$$\int_0^{\frac{\pi}{2}}\sqrt[n]{\tan(x)}\log(\sin(x))dx=\frac{H_{\frac{1}{2}\left(\frac{1}{n}-1\right)}}{2}\int_0^{\frac{\pi}{2}}\sqrt[n]{\tan(x)}\,dx$$

H_k – Harmonic Number

Proposed by Srinivasa Raghava-AIRMC-India

U.31 If we define the function $f(m,n)$ for $m,n>0$

$$f(m,n)=\int_{-\infty}^{\infty}\frac{\log(1+mx^2)}{1+nx^2}dx$$

then show that

$$\int_0^1\int_0^1 f(m,n)\,dndm=\frac{2}{3}\pi(1+8\log(2))$$

Proposed by Srinivasa Raghava-AIRMC-India

U.32 Prove that

$$e^{-\pi}=\int_{-\infty}^{\infty}\frac{\left(1+\frac{x}{\pi}\right)\sin(\pi x)}{x^2+4x+5}dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.33 Establish the inequality

$$0\le\frac{\partial\pi^x}{\partial x^\pi}\le\left(\frac{e}{\pi-1}\right)^{1-\pi}\frac{\pi}{\log^\pi(\pi)}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.34 Prove the integral

$$\int_{-\infty}^{\infty}\frac{(1+x+x^2+x^3+x^4)\cos(\pi x)}{e^{\pi(x^2+x)}}dx=\frac{\pi-3}{4\pi}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.35 For $n\ge 1$. Prove that

$$\int_{-\infty}^{\infty}\frac{e^{-2\pi\sin x}\sin\left(\frac{\pi x}{n}\right)}{\cosh(\pi\sqrt{n}x)}dx=\frac{i\left(\operatorname{sech}\left(\frac{\pi(2n^2+1)}{2n^{\frac{3}{2}}}\right)-\operatorname{sech}\left(\frac{\pi(2n^2-1)}{2n^{\frac{3}{2}}}\right)\right)}{2\sqrt{n}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.36 Prove that

$$\int_0^1\tan^{-1}\left(\frac{1-x}{1+x}\right)\log\left(\frac{1+x^2}{1-x^2}\right)dx=\frac{5\pi^2}{48}-G$$

G – Catalan's constant

Proposed by Srinivasa Raghava-AIRMC-India

U.37 Find a closed form:

$$\Omega=\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)(4n+3)}$$

Proposed by Daniel Sitaru – Romania

U.38 Find:

$$\Omega=\left(\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}}\sin^n x\cdot\sin nx\,dx\right)\left(\sum_{n=0}^{\infty}\int_0^{\pi}\cos^n x\cdot\cos nx\,dx\right)^{-1}$$

Proposed by Daniel Sitaru – Romania

U.39 Find $x, y, z > 0$ such that:

$$\begin{cases} \sum_{n=0}^{\infty} \frac{\cos^{2n} x}{n!} = \sqrt[4]{e} \\ x + y + z = \frac{8}{\log 2} \int_0^1 \frac{\tan^{-1} t}{t+1} dt \\ \sum_{n=0}^{\infty} \frac{\sin^{2n} z}{n!} = \sqrt[4]{e^3} \end{cases}$$

Proposed by Daniel Sitaru – Romania

U.40 Find a closed form:

$$\Omega = \int_0^{\frac{\pi}{2}} \log\left(1 + \frac{2\sin x}{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}\right)^{\log(\sin x)^{\cos x}} dx$$

Proposed by Daniel Sitaru – Romania

U.41 Find a closed form:

$$\Omega = \int_0^{\infty}\int_0^{\infty} \frac{\log\left(x^{\log y}\right) \cdot \log\left(y^{\sqrt{x}}\right) \cdot \log\left(x^{\sqrt[3]{y}}\right)}{(1+x^2)(1+y^2)} dxdy$$

Proposed by Daniel Sitaru – Romania

U.42 If $a, b \geq 0$ then:

$$(a+b)\sqrt{e^{-2a\int_0^{\infty}\frac{\log x}{e^x}dx} + e^{-2b\int_0^1 \log\left(\log\frac{1}{x}\right)dx}} \geq \sqrt{2}(2ab\gamma + a + b)$$

Proposed by Daniel Sitaru – Romania

U.43 Find without any software:

$$\Omega = \int_0^{\infty} \frac{(\cosh x - \sinh x)(4\sinh^2 x - \sinh 2x + 2)}{(\cosh x + \sinh x)\left((4\sinh^2 x + \sinh 2x + 2)\right)} dx$$

Proposed by Daniel Sitaru – Romania

U.44 Without the use of generating function of harmonic number prove that:

$$\int_0^1 \frac{Li_2(x)}{1+x} dx = \ln 2\zeta(2) + \sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^2} = \ln 2\zeta(2) - \frac{5}{8}\zeta(3)$$

where $Li_2(x)$ and H_n are dilogarithm function and nth harmonic number respectively

Proposed by Narendra Bhandari – Bajura – Nepal

U.45 For all $n \geq 0$, prove that:

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\int_{0}^{1}\log^2(1+x)\,x^k\,dx = \frac{2^{n+1}-1}{(n+1)^3}+\frac{2^n}{n+1}\log^2(2)-\frac{2^{n+1}}{(n+1)^2}\log(2)$$

Proposed by Narendra Bhandari – Bajura – Nepal

U.46 Prove the following:

$$\int_{0}^{1}\frac{\ln^3(1-x)\ln^2(x)}{1-x}dx-3\int_{0}^{1}\frac{\ln(1-x)\,Li_2^2(1-x)}{1-x}dx = 6\zeta^2(3)-8\zeta(6)$$

where $Li_2(x)$ is dilogarithm function and $\zeta(x)$ is Riemann zeta function

Proposed by Narendra Bhandari – Bajura – Nepal

U.47 Prove that:

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\zeta(2)-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\cdots-\frac{1}{n^2}\right)=\frac{9}{2}\zeta(5)-2\zeta(2)\zeta(3)$$

where $\zeta(z)$ is Riemann zeta function.

Proposed by Narendra Bhandari – Bajura – Nepal

U.48 Prove or disprove:

$$\int_{0}^{1}\ln^3(1+x)\ln(1-x)dx = 6\left(\ln^2(2)-\frac{1}{8}\right)\zeta(3)+\frac{3}{2}\ln^4(2)-3\ln^3(2)+$$

$$+\ln^2(2)\left(18-3\zeta(2)\right)+(\zeta(2)-6)6\ln(2)+$$

$$+24-33\zeta(4)-6\zeta(2)-6\zeta(\overline{3},\overline{1})$$

where $\zeta(\overline{s}_1,\overline{s}_2,\ldots,\overline{s}_k)$ is AMZV, Alternating Multiple zeta Value.

Proposed by Narendra Bhandari – Bajura – Nepal

U.49 Prove or disprove

$$\int_{0}^{1}\ln^4(1+x)\ln(1-x)\,dx = 48\zeta(5)+\zeta(\overline{3},\overline{1})+24\ln^2(2)\zeta(3)-48\ln(2)\zeta(3)+$$

$$+6\zeta(3)+2\ln^5(2)-8\ln^4(2)-\frac{4\pi^2\ln^3(2)}{3}+32\ln^3(2)+2\pi^2\ln^2(2)-96\ln^2(2)-$$

$$-\frac{8\pi^4\ln(2)}{15}-24\zeta(2)\ln(2)+192\ln(2)+36\zeta(4)-48Li_5\left(\frac{1}{2}\right)+4\pi^2-120$$

where $\zeta(\overline{s}_1,\overline{s}_2,\ldots,\overline{s}_k)$ are Alternating Multi Zeta Values

Proposed by Narendra Bhandari – Bajura – Nepal

U.50 Prove or disprove

$$\sum_{p=1}^{\infty}\int_0^{\infty} x^2 \log^2\left(x^{\sqrt{p}}\right)\frac{dx}{\sec(px)} = \frac{17}{2}\pi\zeta(2) - 5\gamma\pi\zeta(2) + 5\pi\zeta(2)\ln\left(\frac{2e^{\gamma}\pi}{A^{12}}\right)$$

where $\zeta(.)$ is being Riemann zeta function and A is Glashier – kinkelin constant.

Proposed by Narendra Bhandari – Bajura – Nepal

U.51 Prove that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\sqrt[3]{k}} - \frac{3n+1}{2\sqrt[3]{n}}\right) = \zeta\left(\frac{1}{3}\right)$$

where $\zeta(z)$ is Riemann zeta function

Proposed by Narendra Bhandari – Bajura – Nepal

U.52 Evaluate the integral in a closed – form

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x) + \frac{\cos(x)}{\sin(x)+\frac{\cos(x)}{\sin(x)+\cot(x)}}}dx$$

Proposed by Srinivasa Raghava –AIRMC- India

U.53 If $\alpha = \frac{\phi}{1+}\frac{\phi}{\sqrt{5}+}\frac{\phi}{1+}\frac{\phi}{\sqrt{5}+}\frac{\phi}{1+}\frac{\phi}{\sqrt{5}+}\frac{\phi}{1+\cdots\ldots}$ then evaluate this continued fraction

$$1 + \frac{\alpha}{\phi+}\frac{\alpha}{1+}\frac{\alpha}{\phi+}\frac{\alpha}{1+}\frac{\alpha}{\phi+}\frac{\alpha}{1+}\frac{\alpha}{\phi+\cdots\ldots}.\quad \phi - \text{Golden Ratio}$$

Proposed by Srinivasa Raghava –AIRMC- India

U.54.

$$\int_0^{\infty}\sqrt[5]{x}\log^5\left(\coth\left(\sqrt{5}x\right)\right)dx$$

Proposed by Srinivasa Raghava –AIRMC- India

U.55 Let $f\left(\begin{pmatrix} x \\ y \\ z\end{pmatrix}\right) = \begin{pmatrix} xy + \sin(z) \\ zx + \sin(y) \\ yz + \sin(x)\end{pmatrix}$ and if, $J_f(x,y,z)$ is the Jacobian Matrix of $f\left(\begin{pmatrix} x \\ y \\ z\end{pmatrix}\right)$

then prove that $\frac{\partial^6}{\partial x^2\partial y^2\partial z^2}\left|J_f(x,y,z)\right| = \cos(x)\cos(y)\cos(z)$

where, $|*|$ is the Determinant of the matrix

Proposed by Srinivasa Raghava –AIRMC- India

U.56 Let, $H_{\{x,y\}}\left(e^{-\pi(x^2+y^2)}\right)$ is the Hessian matrix of the function $e^{-\pi(x^2+y^2)}$

then show that

$$\int_0^\infty\int_0^\infty \left|H_{\{x,y\}}\left(e^{-\pi(x^2+y^2)}\right)\right| e^{-\pi(x^2+y^2)}\,dydx = \frac{\pi^2}{9}$$

Here $|*|$ is the determinant of the matrix

Proposed by Srinivasa Raghava –AIRMC- India

U.57 Evaluate the integral

$$\int_1^\infty\int_1^\infty \frac{\sqrt[5]{x}+\sqrt[5]{y}}{x^5\sqrt[5]{y}+y^5\sqrt[5]{x}}\,dy\,dx$$

Proposed by Srinivasa Raghava –AIRMC- India

U.58

$$\int_{-\infty}^{\infty} \frac{x^2+\frac{x}{\phi}+\frac{1}{\phi^2}}{(1+x^2+x^4)\left(x^4+\frac{x^2}{\phi}+\frac{1}{\phi^2}\right)}\,dx = 2\pi\sqrt{\phi-\frac{2}{3}\left(\sqrt{5\phi+3}-1\right)}$$

ϕ – Golden Ratio

Proposed by Srinivasa Raghava –AIRMC- India

U.59 Let for $z \ge 0$

$$\psi(z) = \int_0^\infty\int_0^\infty \frac{\sin\left(z(x+y)\right)}{(x^2+1)(y^2+1)}\,dy\,dx$$

then prove that

$$\int_0^\infty \frac{\psi(z)}{\sqrt{z}}\,dz = \pi^{\frac{3}{2}}\left(1+\frac{\log\left(3-2\sqrt{2}\right)}{2\sqrt{2}}\right)$$

$$\int_0^\infty \frac{\psi(z^2)}{z^2}\,dz = -\frac{\pi^{\frac{3}{2}}\log\left(17-12\sqrt{2}\right)}{2\sqrt{2}}$$

Proposed by Srinivasa Raghava –AIRMC- India

U.60 Prove that

$$\int_0^{\frac{\pi}{5}} \frac{\sqrt{5}-\tan\left(\frac{x}{5}\right)}{\sqrt{5}+\tan\left(\frac{x}{5}\right)}\,dx = \frac{2\pi}{15}+\frac{5}{6}\sqrt{5}\log\left(\frac{1}{5}\left(3+\sqrt{5}\sin\left(\frac{2\pi}{25}\right)+2\cos\left(\frac{2\pi}{25}\right)\right)\right)$$

Proposed by Srinivasa Raghava –AIRMC- India

U.61 Prove that

$$\int_0^1 \frac{\log(x)\log\left(\frac{x}{1-x}\right)}{\sqrt[3]{\frac{x}{1-x}}}dx$$

$$=\frac{\pi^3}{9\sqrt{3}}-\frac{5\pi^2}{9}-\frac{\pi^2\log(3)}{3}+\frac{2\pi}{\sqrt{3}}+\pi\sqrt{3}\log(3)+\frac{2\pi\psi^{(1)}\left(\frac{2}{3}\right)}{3\sqrt{3}}$$

Proposed by Srinivasa Raghava –AIRMC- India

U.62 Find the Co-variance of this 4 × 4 matrix and compute its Trace

$$\begin{pmatrix} \frac{5}{16} & -\frac{1}{8}-\frac{3i}{16} & \frac{1}{16}+\frac{i}{2} & -\frac{1}{4}-\frac{5i}{16} \\ -\frac{1}{8}-\frac{3i}{16} & -\frac{1}{16}+\frac{i}{4} & \frac{1}{8}-\frac{7i}{16} & \frac{1}{16}+\frac{3i}{8} \\ \frac{1}{16}+\frac{i}{2} & \frac{1}{8}-\frac{7i}{16} & \frac{5}{16} & -\frac{1}{2}-\frac{i}{16} \\ -\frac{1}{4}-\frac{5i}{16} & \frac{1}{16}+\frac{3i}{8} & -\frac{1}{2}-\frac{i}{16} & \frac{11}{16} \end{pmatrix}$$

Proposed by Srinivasa Raghava –AIRMC- India

U.63

$$\sum_{n=0}^{\infty}\frac{(-1)^{1+2+3+4+5+\cdots+n}}{1\times 2\times 3\times 4\times 5\ldots n}=\cos(1)-\sin(1)$$

Proposed by Srinivasa Raghava –AIRMC- India

U.64 Prove the integral relation

$$\int_0^1\int_0^1\int_0^1 \frac{\log\left(\max(x,y,z)+\min\left(\sqrt{x},\sqrt{y},\sqrt{z}\right)\right)}{\max(x,y,z)+\min\left(\sqrt{x},\sqrt{y},\sqrt{z}\right)}dzdydx=\frac{43}{16}-\frac{\pi^2}{12}-\frac{1}{3}(8\log(2))$$

Proposed by Srinivasa Raghava –AIRMC- India

U.65 Let, for $0\le z\le 1$

$$\psi(z)=\int_z^1\int_z^1\frac{x+y}{(x+y)(xy+1)}dy\,dx$$

then prove that

$$\int_0^1\left(\psi(z)+\psi(\sqrt{z})\right)dz=\frac{7}{2}-\pi,\int_0^1\left(\psi(z)+\psi(\sqrt{z})\right)\log\left(\frac{1}{z}\right)dz=\frac{1}{2}(23-8G)-\frac{\pi^2}{12}-2\pi$$

where G is Catalan Constant

Proposed by Srinivasa Raghava –AIRMC- India

U.66 Let the function

$$R(x) = \int_{\log(\sin(x))}^{\infty} \frac{x}{e^x + 1} dx$$

then show that

$$\int_0^{\frac{\pi}{2}} R(x)^2 \sin(2x)\, dx = 3\zeta(3) - \frac{11\zeta(4)}{4}$$

Proposed by Srinivasa Raghava –AIRMC- India

U.67 Prove the relation

$$\frac{\mathcal{F}_x[\sin^2(e^{-x})](\alpha)}{\mathcal{F}_x[(e^{-x}\sin(e^{-x}))^2](\alpha)} = \frac{4}{\alpha^2 + i\alpha}$$

$\mathcal{F}_x[\dots](\alpha)$ is Fourier Transform

Proposed by Srinivasa Raghava –AIRMC- India

U.68 If we have the integrals

$$\int_0^{\infty} \frac{\tanh\left(\frac{x}{2}\right) + \tanh(2x)}{x}\left(e^{\frac{3x}{2}} - 1\right)^2 e^{-4x} dx = \log(A)$$

$$\int_0^{\infty} \frac{\tanh\left(\frac{x}{2}\right) + \tanh(2x)}{x}\left(e^{\frac{3x}{2}} + 1\right)^2 e^{-4x} dx = \log(B)$$

then prove that $AB = \left(\frac{\sqrt{\pi}\Gamma\left(\frac{1}{8}\right)}{3\Gamma\left(\frac{5}{8}\right)}\right)^4$

Proposed by Srinivasa Raghava –AIRMC- India

U.69 If $a, b, c \geq 0$ then:

$$\int_a^{2a}\int_b^{3b}\int_c^{4c}\left(\sqrt[6]{\frac{x+1}{y+1}} + \sqrt[8]{\frac{y+1}{z+1}} + \sqrt[10]{\frac{z+1}{x+1}}\right) \geq 15abc$$

Proposed by Daniel Sitaru – Romania

U.70 If $0 < a \leq b$ then:

$$\int_a^b\int_a^b\int_a^b \left|\frac{x^3 - y^3}{x + y} + \frac{y^3 - z^3}{y + z} + \frac{z^3 - x^3}{z + x}\right| dxdydz \leq \frac{(b-a)^5}{8}$$

Proposed by Daniel Sitaru – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-SPRING 2022

PROBLEMS FOR JUNIORS

JP.346 Find all values of k such that the following inequality:

$$\frac{a^2}{b}+\frac{b^2}{a}+\frac{kab}{a+b}\geq\left(1+\frac{k}{4}\right)(a+b)$$

holds for all positive real numbers a,b.

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

JP.347 Let a,b,c be non-negative real numbers, no two of which are zero. Prove that

$$\frac{2(a+b)(a+c)}{b+c}+\frac{2(b+c)(b+a)}{c+a}+\frac{2(c+a)(c+b)}{a+b}>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c}$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

JP. 348. If $a,b,c>0$ then:

$$\left(\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}\right)\left(\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\right)\geq\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2$$

Proposed by Daniel Sitaru – Romania

JP.349. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{a^6}{a^2+b}+\frac{b^6}{b^2+c}+\frac{c^6}{c^2+a}\geq\frac{3}{2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

JP.350. For $1\leq a,b,c\leq\frac{2\sqrt{3}}{3}$, prove that:

$$\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2}+(a+b+c)^2-3(a+b+c)\leq 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

JP.351 In ΔABC the following relationship holds:

$$\prod_{cyc} sin^2 A \geq 4\prod_{cyc} cosA - 5\prod_{cyc} cos^2 A$$

Proposed by Cristian Miu-Romania

JP.352. If $a, b, c \in \mathbb{C}; |a| = |b| = |c| = 1$ then:

$$3|a+b+c| + 2(|a-b| + |b-c| + |c-a|) \geq 9$$

Proposed by Daniel Sitaru – Romania

JP.353. In $\Delta ABC, P \in Int(\Delta ABC), \mu(\widehat{ABP}) = 20^o$,

$$\mu(\widehat{PBC}) = \mu(\widehat{PCB}) = 10^o, \mu(\widehat{PCA}) = 40^0.$$

Prove that: $|AP| + |BC| = \sqrt{3}|AB|$

Proposed by Mehmet Şahin-Ankara-Turkey

JP.354 In acute $\Delta AB, O$ –circumcenter, $F, K \in (AB), M, L(BC), E, N \in\in (CA)$

$\overline{FOE}, \overline{MON}, \overline{LOK}$ –are the antiparallels. Let ρ_a, ρ_b, ρ_c –inradii of $AFE, \Delta\Delta BLK, \Delta CMN$.

Prove that: $\rho_a + \rho_b + \rho_c = R$

Proposed by Mehmet Şahin-Ankara-Turkey

JP.355 In $\Delta ABC, A_1, B_1, C_1$ are contact points by inscribed circle. Prove that:

$$\left(\frac{AB}{A_1B_1}\right)^2 + \left(\frac{BC}{B_1C_1}\right)^2 + \left(\frac{CA}{C_1A_1}\right)^2 \geq \frac{6R}{r}$$

Proposed by Marian Ursărescu-Romania

JP.356 In $\Delta ABC, I$ –incenter and R_a, R_b, R_c –circumradius in $\Delta IBC, \Delta IAB, \Delta IAC$.

Prove that:

$$\left(\frac{R_a}{a}\right)^2 + \left(\frac{R_b}{b}\right)^2 + \left(\frac{R_c}{c}\right)^2 \geq 1$$

Proposed by Marian Ursărescu-Romania

JP.357 In $\Delta ABC, N_a$ –Nagel's point. Prove that inscribed circle of ΔABC passes through point N_a if and only if $s^2 + 4r^2 = 16Rr$

Proposed by Marian Ursărescu-Romania

JP.358. If $x, y, z > 0, xyz = 1$ then in ΔABC the following relationship holds:

$$a^4\left(y + 1 + \frac{1}{x}\right) + b^4\left(z + 1 + \frac{1}{y}\right) + c^4\left(x + 1 + \frac{1}{z}\right) \geq 1296r^4$$

Proposed by Daniel Sitaru-Romania

JP.359 Find all n natural numbers such that:

$$\sqrt[3]{\frac{n+27}{(n+8)(n+1)}} \in \mathbb{Q}$$

Proposed by George Florin Şerban-Romania

JP.360. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\sum_{cyc} \frac{\tan^2 x}{\tan^3 x + \cot x} + \sum_{cyc} \frac{\cot^2 x}{\cot^3 x + \cot x} \geq 2 \sum_{cyc} \frac{1}{\tan^2 x + \cot^2 x}$$

Proposed by Daniel Sitaru – Romania

PROBLEMS FOR SENIORS

SP.346 Determine all functions $f: (0, \infty) \to \mathbb{R}$ such that:

$$f(xy) \leq xf(x) + yf(y) \leq log(xy), \forall x, y > 0$$

Proposed by Marian Ursărescu-Romania

SP.347 If $A, B \in M_3(\mathbb{R})$ such that $tr((AB - BA)^2) = 0$. Prove that:

$$det((AB - BA)^2 + AB - BA + I_3) = \left(1 - det(AB - BA)\right)^2$$

Proposed by Marian Ursărescu-Romania

SP.348 In ΔABC prove that inscribed circle of ΔABC passes through to G if and only if

$$s^2 = 16Rr + 4r^2$$

Proposed by Marian Ursărescu-Romania

SP.349 If $a \in \left(0, \frac{\pi}{2}\right)$ then prove:

$$(sina)^{\sqrt{log_{sina}cosa}} + (cosa)^{\sqrt{log_{cosa}sina}} \leq \sqrt{2}$$

Proposed by Ionuț Florin Voinea-Romania

SP.350 If $x, y, z > 0, xy + yz + zx = 1$ and $\lambda \geq \frac{2}{3}$, then:

$$\frac{1}{x^2(x^2 + \lambda)} + \frac{1}{y^2(y^2 + \lambda)} + \frac{1}{z^2(z^2 + \lambda)} \geq \frac{27}{3\lambda + 1}$$

Proposed by Marin Chirciu-Romania

SP.351. If $x, y \in \left(0, \frac{\pi}{2}\right)$; $\sqrt[3]{1 + \tan x} + \sqrt[3]{1 + \tan y} = 2\sqrt[3]{2}$ then:

$$\sqrt[3]{1 - \tan x} + \sqrt[3]{1 - \tan y} \leq 4 - 2\sqrt[3]{2}$$

Proposed by Daniel Sitaru - Romania

SP.352. Let $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}$ be sequences of real numbers with $x_1 = 0, y_1 = 1,$

$$x_{n+1} = \frac{ax_n + by_n}{a+b}, y_{n+1} = \frac{cx_n + dy_n}{c+d}, \forall n \geq 1, a, b, c, d > 0, ad \neq bc.$$

Prove that if $(z_n)_{n\geq 1}, z_n = y_n - x_n$, then $(z_n)_{n\geq 1}$ –geometric progression, and if $q < 1, q$ –ratio of progression, then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

Proposed by Marin Chirciu-Romania

SP.353. Let $\lambda > 0$ fixed. Solve for real numbers:

$$\begin{cases} \lambda x = \sqrt{\lambda^2 y^2 - 1} + \sqrt{\lambda^2 z^2 - 1} \\ \lambda y = \sqrt{\lambda^2 z^2 - 1} + \sqrt{\lambda^2 x^2 - 1} \\ \lambda z = \sqrt{\lambda^2 x^2 - 1} + \sqrt{\lambda^2 y^2 - 1} \end{cases}$$

Proposed by Marin Chirciu-Romania

SP.354. If $x, y, z \geq 1$ then:

$$y \cdot x^x + z \cdot y^y + x \cdot z^z \geq x + y + z + \ln(x^{xy} \cdot y^{yz} \cdot z^{zx})$$

Proposed by Daniel Sitaru – Romania

SP. 355 Let I_a, I_b, I_c and r_a, r_b, r_c denote the excenters and exradii of the triangle ABC, respectively. Let ρ_a –be the radius of the circle that lies inside and touches internally the excircle opposite A and touches the sides I_aI_b, I_aI_c of triangle $I_aI_bI_c$ externally. Let ρ_b, ρ_c be defined similarly. Prove that:

$$\frac{\rho_a}{r_a} + \frac{\rho_b}{r_b} + \frac{\rho_c}{r_c} \leq 1$$

Proposed by Mehmet Şahin-Ankara-Turkey

SP.356 If $a, b, c > 0$ such that $abc = 1$ then:

$$\frac{(1+ab)^3}{(c+a)(a+b)} + \frac{(1+bc)^3}{(a+b)(b+c)} + \frac{(1+ca)^3}{(b+c)(c+a)} \geq \frac{54}{(a^2+b^2+c^2)^2}$$

Proposed by Pedro Pantoja-Natal-Brazil

SP.357 Let $S(n)$ be the sum of the digits of the positive integer n. Determine all pairs of positive integers $(a, b), a \geq b$ such that the equation $S(a^2) = (b - 2018)^4$ has only finite solutions in positive integers.

Proposed by Pedro Pantoja-Natal-Brazil

SP.358 If $x, y, z > 0$ then:

$$4\sum_{cyc} \frac{x^3}{(y+1)(z+1)} + 3 \geq 6\sqrt[3]{xyz}$$

Proposed by Daniel Sitaru-Romania

SP.359 If $A \in M_3(\mathbb{R}), p \in \mathbb{R}^*$ such that $det(A^2 - pA + p^2 I_3) = 0$.

Prove that:

$$2det(A^2 + p^2 I_3) \geq (detA + p^3)^2$$

Proposed by Marian Ursărescu-Romania

SP.360 $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs such that $|z_1| = |z_2| = |z_3|$. If

$$\sum_{cyc} \left| \frac{2z_1 - z_2 - z_3}{(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2|} \right| = \frac{1}{|z_1 - z_2|} + \frac{1}{|z_2 - z_3|} + \frac{1}{|z_3 - z_1|},$$

then z_1, z_2, z_3 are affixes on equilateral triangle.

Proposed by Marian Ursărescu-Romania

UNDERGRADUATE PROBLEMS

UP.346. Solve for real numbers:

$$2\int_0^x \frac{x^2 \cdot e^{\arctan x}}{\sqrt{1 + x^2}} dx = 1$$

Proposed by Daniel Sitaru – Romania

UP.347. Solve for complex numbers:

$$\begin{cases} \frac{|x|^2}{3} + \frac{|y|^2}{5} = \frac{|x + y|^2}{8} \\ 10x + y = 7 + 14i \end{cases}$$

Proposed by Daniel Sitaru – Romania

UP.348. If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and ABC is a triangle with the area F, then:

$$\frac{(x + 3y)(y + 4z + 3x)}{(y + 3z)(z + 3x)} \cdot a^4 + \frac{(y + 3z)(z + 4x + 3y)}{(z + 3x)(x + 3y)} \cdot b^4 + \frac{(z + 3x)(x + 4y + 3z)}{(x + 3y)(y + 3z)} c^4 \geq$$
$$\geq 32F^2$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

UP.349. If $m \geq 0; x, y, z > 0$, then in any ΔABC with the area F the following inequality holds:

$$\frac{y + z}{x \cdot h_a^{m+1}} + \frac{z + x}{y \cdot h_b^{m+1}} + \frac{x + y}{z \cdot h_c^{m+1}} \geq \frac{2}{\left(\sqrt{F}\right)^{m+1}} \left(\sqrt[4]{3}\right)^{3-m}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

UP.350. If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ΔABC with the area F the following inequality holds:

$$\sum_{cyc}\frac{y+z}{x+\sqrt{(x+2y)(x+2z)}}a^2\ge 2\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

UP. 351. If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$ and ABC is a triangle, then:

$$\frac{(x+2y)(2x+y+3z)\cdot a^2}{(y+z)(z+2x)\cdot h_a^2}+\frac{(y+2z)(2y+z+3x)\cdot b^2}{(z+2x)(x+2y)\cdot h_b^2}+$$
$$+\frac{(z+2x)(2z+x+3y)\cdot c^2}{(x+2y)(y+2z)\cdot h_c^2}\ge 8$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

UP. 352. If $x,y,z>0$ and ΔABC have the semiperimeter s the following inequality holds:

$$\frac{(y+z)a}{x\cdot h_a(s-a)}+\frac{(z+x)b}{y\cdot h_b(s-b)}+\frac{(x+y)c}{z\cdot h_c(s-c)}\ge\frac{12\sqrt{3}}{s}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

UP.353. Let $(x_n)_{n\ge1}$ be a sequence of real numbers with $x_1=\sqrt[3]{a^3-a}, a\ge 2$ and $x_{n+1}=\sqrt[3]{a^3-a+x_n}, n\ge 1$. Prove that $(x_n)_{n\ge1}$ –convergent and find $\Omega_1=\lim\limits_{n\to\infty}x_n, \Omega_2=\lim\limits_{n\to\infty}\{x_n\}$, where $\{*\}$ –fractional part.

Proposed by Marin Chirciu-Romania

UP.354. Find:

$$\Omega=\lim_{n\to\infty}\left(1+\left(\sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k\pi}{n}\right)^{-1}\right)^n$$

Proposed by Florică Anastase-Romania

UP.355. Find:

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n}}\left(\frac{\cot x}{2}\cdot\left(\sum_{k=1}^{n-1}\frac{k}{n}\sin\frac{k\pi}{n}\right)^{-1}\right)^{\frac{1}{\tan(2nx)}}\right)$$

Proposed by Florică Anastase-Romania

UP.356. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} cos\frac{(n-1)k\pi}{n} \cdot cos^{n-1}\left(\frac{k\pi}{n}\right)}$$

Proposed by Florică Anastase-Romania

UP.357.

$$\text{If } f:\left[0,\frac{\pi}{2}\right) \to \mathbb{R}, f(x) = -\int_0^x log(cosy)\,dy$$

Prove that:

$$\lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} f(x) = -\int_0^{\frac{\pi}{2}} log(siny)\,dy$$

Proposed by Florică Anastase-Romania

UP.358. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\left(\int_a^b \frac{sinx}{x}dx\right)^2 + \left(\int_a^b \frac{cosx}{x}dx\right)^2 \leq log^2\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru-Romania

UP.359 Find:

$$\Omega = \int_0^\infty \frac{tan^{-1}x}{x^3+1}dx$$

Proposed by Vasile Mircea Popa-Romania

UP.360 Find $x, y > 0$ such that:

$$\begin{cases} x + y = \frac{2}{3} \\ \frac{(x+1)^2}{3x^2-2x+1} + \frac{(y+1)^2}{3y^2-2y+1} = \frac{16}{3} \end{cases}$$

Proposed by Daniel Sitaru-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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NOTĂ: Pentru a publica probleme propuse, articole și note matematice în RMM puteți trimite materialele pe mailul: dansitaru63@yahoo.com