ISSN 2501-0099

ROMANIAN MATHEMATICAL SOCIETY

Mehedinți Branch

R. M. M. - 31

ROMANIAN MATHEMATICAL MAGAZINE

WINTER EDITION 2021

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Mehedinți Branch

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Nr.31-WINTER EDITION 2021

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CONTENT

ABOUT AN INEQUALITY FROM RMM

By Flaviu Cristian Verde-Romania

In R.M.M. was published the inequality:

$$\left(\frac{y^2+z^2}{x^2}+\frac{z^2+x^2}{y^2}+\frac{x^2+y^2}{z^2}\right)(a^4+b^4+c^4)\geq 96F^2, \qquad (B-G,N)$$

where $x, y, z > 0$ and a, b, c are the lengths of the sides of triangle ABC with s −semiperimeter and F −area.

D.M.Bătinețu-Giurgiu, Dan Nănuți

The solution of this problem was published in [1], now we will developed this problem.

Solution 1. Let's denote: $x^2 = u, y^2 = v, z^2 = w$ and then we must show that:

$$\left(\frac{v+w}{u}+\frac{w+u}{v}+\frac{u+v}{w}\right)(a^4+b^4+c^4)\geq 96F^2; \ \ (1)$$

We have the inequality:

$$\sum_{cyc}\frac{v+w}{u}\geq 6, \forall u,v,w\in\mathbb{R}_+^* = (0,\infty); \ \ (*)$$

and G. Goldner Inequality: $a^4+b^4+c^4\geq 16F^2$, (G)

From (*),(G) we get:

$$\left(\sum_{cyc}\frac{v+w}{u}\right)(a^4+b^4+c^4)\geq 6\cdot 16F^2 = 96F^2$$

which is (B-G,N).

Solution 2. We have:

$$\left(\sum_{cyc}\frac{y^2+z^2}{x^2}\right)\left(\sum_{cyc}a^4\right)\geq\frac{1}{2}\left(\sum_{cyc}\left(\frac{y+z}{x}\right)^2\right)\left(\sum_{cyc}(a^2)^2\right)\overset{BCS}{\geq}$$

$$\geq\frac{1}{2}\left(\sum_{cyc}\frac{y+z}{x}\cdot a^2\right)^2\overset{Bătinețu-Giurgiu}{\geq}\frac{1}{2}\left(8\sqrt{3}F\right)^2 = \frac{1}{2}\cdot 64\cdot 3F^2 = 96F^2$$

Solution 3. We have:

$$\sum_{cyc}\frac{y^2+z^2}{x^2}\overset{AM-GM}{\geq}2\cdot\sum_{cyc}\frac{yz}{x^2}\overset{AM-GM}{\geq}2\cdot 3\cdot\sqrt[3]{\prod_{cyc}\frac{yz}{x^2}} = 2\cdot 3\cdot 1 = 6, \ \ (2)$$

and

$$a^4+b^4+c^4\overset{AM-GM}{\geq}3\cdot\sqrt[3]{(abc)^4} = 3\sqrt[3]{(4RF)^4} = 12F\sqrt[3]{4R^4F}\overset{Euler}{\geq}12F\sqrt[3]{4(2r)^2R^2F} =$$

$$= 12F\sqrt[3]{64r^2F\left(\frac{R}{2}\right)^2} = 48F\sqrt[3]{r^2F\left(\frac{R}{2}\right)^2}\overset{Mitrinovic}{\geq}48F\sqrt[3]{r^2F\left(\frac{s}{3\sqrt{3}}\right)^2} = 48F\sqrt[3]{\frac{r^2Fs^2}{27}} =$$

$$= 16F\sqrt[3]{F(sr)^2} = 16F\sqrt[3]{F^3} = 16F^2; \quad (3)$$

From (2),(3) we deduce that:

$$\left(\sum_{cyc}\frac{y^2+z^2}{x^2}\right)\left(\sum_{cyc}a^4\right) \geq 6\cdot 16F^2 = 96F^2$$

Generalization:

If $m \geq 0, x, y, z > 0$ and triangle ABC with s –semiperimeter, F –area, then:

$$\left(\sum_{cyc}\frac{y^{2m+2}+z^{2m+2}}{x^{2m+2}}\right)\left(\sum_{cyc}a^{4m+4}\right) \geq 2^{4m+5}\cdot 3^{1-m}\cdot F^{2m+2}, (**)$$

Proof. We have

$$\left(\sum_{cyc}\frac{y^{2m+2}+z^{2m+2}}{x^{2m+2}}\right)\left(\sum_{cyc}a^{4m+4}\right) \geq \frac{1}{2^{2m+1}}\left(\sum_{cyc}\left(\frac{x+y}{z}\right)^{2m+2}\right)\left(\sum_{cyc}a^{4m+4}\right) =$$

$$= \frac{1}{2^{2m+1}}\left(\sum_{cyc}\left(\frac{x+y}{z}\right)^{2(m+1)}\right)\left(\sum_{cyc}(a^2)^{2(m+1)}\right) \overset{Radon}{\geq}$$

$$\geq \frac{1}{2^{2m+1}}\cdot\frac{1}{3^m}\left(\sum_{cyc}\left(\frac{x+y}{z}\right)^2\right)^{m+1}\cdot\frac{1}{3^m}\left(\sum_{cyc}(a^2)^2\right)^{m+1} =$$

$$= \frac{1}{2^{2m+1}}\cdot\frac{1}{3^{2m}}\left(\sum_{cyc}\left(\frac{x+y}{z}\right)^2\right)^{m+1}\left(\sum_{cyc}(a^2)^2\right)^{m+1} \overset{BCS}{\geq}$$

$$\geq \frac{1}{2^{2m+1}\cdot 3^{2m}}\left(\sum_{cyc}\frac{x+y}{z}\cdot a^2\right)^{2m+2} \overset{Bătineţu-Giurgiu}{\geq}$$

$$\geq \frac{1}{2^{2m+1}\cdot 3^{2m}}\left(8\sqrt{3}F\right)^{2m+2} = \frac{2^{6m+6}}{2^{2m+1}\cdot 3^{2m}}\cdot 3^{m+1}\cdot F^{2m+2} = 2^{4m+5}\cdot 3^{1-m}\cdot F^{2m+2}$$

Note. If $m = 0$ then from relationship (**) we get (B-G,N) Inequality.

References:

[1] Chirciu Marin, About Bătineţu's Inequalities-R.M.M.No.20 Spring Edition 2021**, page 4-10.**

[2] Romanian Mathematical Magazine-www.ssmrmh.ro

ONE DROP FROM THE APPLIED MATHEMATICS

By Laviniu Bejenaru-Romania

Problem: Giving one symmetric quatratic matrix **A** with the property (Δ), decomposing it into a product $\mathbf{Q}\cdot\mathbf{Q}^T$, where **Q** is a lower-triangular matrix and $\mathbf{Q}^T$ is the transpose matrix.

(Δ) each k-dimmensional upper left-corner minor has his determinant strict positive

for $1\le k\le$dimention of matrix

Example: As one example, we can have the symmetric matrix

$A=\begin{pmatrix}4&2&1\\2&2&1\\1&1&1\end{pmatrix}$ with the minors $M_1=4>0$, $M_2=\begin{bmatrix}4&2\\2&2\end{bmatrix}=4>0$,

$M_3=\begin{vmatrix}4&2&1\\2&2&1\\1&1&1\end{vmatrix}=2>0$. So, the solution is:

$$Q=\begin{pmatrix}2&0&0\\1&1&0\\\frac{1}{2}&\frac{1}{2}&\frac{\sqrt{2}}{2}\end{pmatrix}$$

It can be verified that

$$\begin{pmatrix}4&2&1\\2&2&1\\1&1&1\end{pmatrix}=\begin{pmatrix}2&0&0\\1&1&0\\\frac{1}{2}&\frac{1}{2}&\frac{\sqrt{2}}{2}\end{pmatrix}\cdot\begin{pmatrix}2&1&\frac{1}{2}\\0&1&\frac{1}{2}\\0&0&\frac{\sqrt{2}}{2}\end{pmatrix}$$

Task:

Find the **Q** factor such that $\mathbf{A=Q\cdot Q^T}$ for $=\begin{pmatrix}9&3&1\\3&3&1\\1&1&1\end{pmatrix}$.

Mean:

The **Q** matrix from the factorization $A=Q\cdot Q^T$ is called Cholescky factor of A and this kind of square-look-like operation has many applications: efficiently compute of the determinants, gradients in stochastic models, samples form a Gaussian distribution.

References:

Deisenroth M.P., Faisal A.A., Ong C.S. - Mathematics for Machine Learning, Cambridge University Press, 2020, ISBN 9781108679930

ABOUT SOME INEQUALITIES IN TRIANGLE

By D.M.Bătineţu-Giurgiu, Daniel Sitaru-Romania

Let be $m \geq 0; x, y, z > 0$ and ΔABC with F −area, s −semiperimeter, then:

$$\frac{y+z}{x}\cdot a^{m+1}+\frac{z+x}{y}\cdot b^{m+1}+\frac{x+y}{z}\cdot c^{m+1}\geq 2^{m+2}\cdot\sqrt[4]{3^{3-m}}\cdot\sqrt{F^{m+1}};\ (*)$$

Proof. We have:

$$\sum_{cyc}\frac{y+z}{x}\cdot a^{m+1}\geq 2\cdot\sum_{cyc}\frac{\sqrt{yz}}{x}\cdot a^{m+1}\geq 2\cdot 3\cdot\sqrt[3]{\prod_{cyc}\frac{\sqrt{yz}}{x}\cdot a^{m+1}}=6\sqrt[3]{(abc)^{m+1}}=$$

$$=6\sqrt[3]{(4RF)^{m+1}}=6\sqrt[3]{2^{2m+2}\cdot R^{m+1}\cdot F^{m+1}}=6\sqrt[3]{2^{3(m+1)}\cdot\left(\frac{R}{2}\right)^{m+1}\cdot F^{m+1}}=$$

$$=6\cdot 2^{m+1}\sqrt[3]{\left(\sqrt{\frac{R}{2}}\cdot\sqrt{\frac{R}{2}}\cdot F\right)^{m+1}}\overset{Euler}{\geq}2^{m+2}\cdot 3\cdot\sqrt[3]{\left(\sqrt{\frac{R}{2}}\cdot\sqrt{r}\cdot F\right)^{m+1}}\overset{Mitrinovic}{\geq}$$

$$\geq 2^{m+2}\cdot 3\cdot\sqrt[3]{\left(\sqrt{\frac{s}{3\sqrt{3}}\cdot r\cdot F}\right)^{m+1}}=2^{m+2}\cdot 3\cdot\sqrt[3]{\left(\sqrt{\frac{F}{3\sqrt{3}}}\cdot F\right)^{m+1}}=$$

$$=2^{m+2}\cdot 3\cdot\sqrt[3]{\left(\sqrt{\frac{F}{\sqrt{3}}}\right)^{3(m+1)}}=2^{m+2}\cdot 3\cdot\frac{\left(\sqrt{F}\right)^{m+1}}{\left(\sqrt[4]{3}\right)^{m+1}}=2^{m+2}\cdot 3^{1-\frac{m+1}{4}}\cdot F^{\frac{m+1}{2}}=$$

$$=2^{m+2}\cdot 3^{\frac{3-m}{4}}\cdot F^{-\frac{m+1}{2}}$$

If $m=0$ we get:

$$\sum_{cyc}\frac{\boldsymbol{y+z}}{\boldsymbol{x}}\cdot\boldsymbol{a}\geq\mathbf{4}\cdot\sqrt[4]{\mathbf{27}}\cdot\sqrt{\boldsymbol{F}};\ \ (\mathbf{1})$$

If $m=1$ we get:

$$\sum_{cyc}\frac{\boldsymbol{y+z}}{\boldsymbol{x}}\cdot\boldsymbol{a}^{2}\geq\mathbf{8}\sqrt{\mathbf{3}}\cdot\boldsymbol{F};\quad(\boldsymbol{B}-\boldsymbol{G},\mathbf{1})$$

hence the first Bătineţu-Giurgiu Inequality.
If $m=3$ then we have

$$\sum_{cyc}\frac{\boldsymbol{y+z}}{\boldsymbol{x}}\cdot\boldsymbol{a}^{4}\geq\mathbf{32}\cdot\boldsymbol{F}^{2};\quad(\boldsymbol{B}-\boldsymbol{G},\mathbf{2})$$

hence the second Bătineţu-Giurgiu Inequality.
If $m=2$ then we have:

$$\sum_{cyc}\frac{\boldsymbol{y+z}}{\boldsymbol{x}}\cdot\mathbf{3}\geq\mathbf{16}\cdot\sqrt[3]{\mathbf{3}}\cdot\boldsymbol{F}\sqrt{\boldsymbol{F}}$$

If we denote $u=x^{\frac{1}{m+1}}, v=y^{\frac{1}{m+1}}, w=z^{\frac{1}{m+1}}$ then the inequality (*) becomes:

$$\sum_{cyc}\frac{y+z}{x}\cdot a^{m+1}=\sum_{cyc}\frac{v^{m+1}+w^{m+1}}{u^{m+1}}\cdot a^{m+1}\geq\frac{1}{2^m}\sum_{cyc}\left(\frac{v+w}{u}\right)^{m+1}\cdot a^{m+1}=$$

$$=\frac{1}{2^m}\sum_{cyc}\left(\frac{v+w}{u}\cdot a\right)^{m+1}\overset{Radon}{\geq}\frac{1}{2^m}\cdot\frac{1}{3^m}\left(\sum_{cyc}\frac{v+w}{u}\cdot a\right)^{m+1}\geq$$

$$\overset{(2)}{\geq}\frac{1}{6^m}\left(4\cdot\sqrt[4]{27}\cdot\sqrt{F}\right)^{m+1}=\frac{2^{2m+2}\cdot 3^{\frac{3}{4}(m+1)}\cdot F^{\frac{1}{2}(m+1)}}{6^m}=$$

$$=2^{2m+2-m}\cdot 3^{\frac{3}{4}(m+1)-m}\cdot F^{\frac{1}{2}(m+1)}=2^{m+2}\cdot 3^{\frac{1}{4}(3-m)}\cdot F^{\frac{1}{2}(m+1)};\quad(*)$$

hence we have the inequality (*).

If in (*) we take $m=5$, then:

$$\sum_{cyc}\frac{\boldsymbol{y+z}}{\boldsymbol{x}}\cdot\boldsymbol{a^6}\geq\boldsymbol{2^7\cdot 3^{\frac{1}{4}(3-5)}\cdot F^3=\frac{128}{\sqrt{3}}\cdot F^3};\quad\boldsymbol{(3)}$$

If in (*) we take $m=7$, then:

$$\sum_{cyc}\frac{\boldsymbol{y+z}}{\boldsymbol{x}}\cdot\boldsymbol{a^8}\geq\boldsymbol{2^9\cdot\sqrt[4]{3^{-4}}\cdot F^4=\frac{256}{3}\cdot F^4};\quad\boldsymbol{(4)}$$

Refference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

NEW INEQUALITIES IN TRIANGLE

By D.M.Bătineţu-Giurgiu, Daniel Sitaru, Neculai Stanciu-Romania

Let be ΔABC with F –area, s –semiperimeter.

Proposition. If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then

$$\frac{\boldsymbol{y+z}}{\boldsymbol{x}}\cdot(\boldsymbol{b+c})+\frac{\boldsymbol{z+x}}{\boldsymbol{y}}\cdot(\boldsymbol{c+a})+\frac{\boldsymbol{x+y}}{\boldsymbol{z}}\cdot(\boldsymbol{a+b})\geq\boldsymbol{8\sqrt[4]{27}\cdot\sqrt{F}}$$

Proof. We have:

$$\sum_{cyc}\frac{y+z}{x}\cdot(b+c)\geq 4\cdot\sum_{cyc}\frac{\sqrt{yz}}{x}\cdot\sqrt{bc}\geq 4\cdot 3\sqrt[3]{\prod_{cyc}\frac{\sqrt{yz}}{x}\cdot\sqrt{bc}}=$$

$$=12\sqrt[3]{abc}=12\sqrt[3]{4RF}=12\sqrt[3]{8\cdot\frac{R}{2}\cdot F}=24\sqrt[3]{\sqrt{\frac{R}{2}}\cdot\sqrt{\frac{R}{2}}\cdot F}\overset{Euler}{\geq}24\sqrt[3]{\sqrt{r}\cdot\sqrt{\frac{R}{2}}\cdot F}\overset{Mitrinovic}{\geq}$$

$$\geq 24\sqrt[3]{\sqrt{r}\cdot\sqrt{\frac{s}{3\sqrt{3}}}\cdot F} = 24\sqrt[3]{\frac{\sqrt{rs}}{\sqrt{3\sqrt{3}}}\cdot F} = 24\cdot\frac{\left(\sqrt[4]{3}\right)^4}{\sqrt[4]{3}}\cdot\sqrt{F} = 8\sqrt[4]{27}\cdot\sqrt{F}$$

From the inequality (1) we can prof some inequalities from [1] as follows:

Theorem. If $m\in[1,\infty); x,y,z>0$ then in any triangle ABC the following relationship holds:

$$\frac{y+z}{x}\cdot(b+c)^m+\frac{z+x}{y}\cdot(c+a)^m+\frac{x+y}{z}\cdot(a+b)^m\geq 2^{2m+1}\cdot 3^{\frac{1}{4}(4-m)}\cdot F^{\frac{1}{2}\cdot m};\ (**)$$

Proof. Let's denote $u=x^{\frac{1}{m}}, v=y^{\frac{1}{m}}, w=z^{\frac{1}{m}}$ and then

$$\sum_{cyc}\frac{y+z}{x}\cdot(b+c)^m=\sum_{cyc}\frac{v^m+w^m}{u^m}\cdot(b+c)^m\geq\frac{1}{2^{m-1}}\sum_{cyc}\left(\frac{v+w}{u}\cdot(b+c)\right)^m\geq$$

$$\overset{Radon}{\geq}\frac{1}{2^{m-1}\cdot 3^{m-1}}\left(\sum_{cyc}\frac{v+w}{u}\cdot(b+c)\right)^m\overset{(*)}{=}\frac{1}{6^{m-1}}\left(\sum_{cyc}\frac{v+w}{u}\cdot(b+c)\right)^m\overset{(*)}{\geq}$$

$$\geq\frac{1}{6^{m-1}}\left(8\sqrt[4]{27}\cdot\sqrt{F}\right)^m=\frac{2^{3m}\cdot 3^{\frac{3m}{4}}\cdot F^{\frac{1}{2}m}}{2^{m-1}\cdot 3^{1-m}}=2^{2m+1}\cdot 3^{\frac{3}{4}m-m+1}\cdot F^{\frac{1}{2}m}$$

$$=2^{2m+1}\cdot 3^{\frac{1}{4}(4-m)}\cdot F^{\frac{1}{2}m}$$

q.e.d.

If $m=2$ then the inequality (**) becomes as:

$$\sum_{cyc}\frac{y+z}{x}\cdot(b+c)^2\geq 2^5\cdot 3^{\frac{1}{2}}\cdot F=32\sqrt{3}\cdot F;\ \ (1)$$

If $m=4$ then the inequality (**) becomes as:

$$\sum_{cyc}\frac{y+z}{x}\cdot(b+c)^4\geq 2^9\cdot F^2=512\cdot F^2;\ \ (2)$$

If $m=8$ then the inequality (**) becomes as:

$$\sum_{cyc}\frac{y+z}{x}\cdot(b+c)^8\geq 2^{17}\cdot 3^{-1}\cdot F^4=2^{17}\cdot\frac{1}{3}\cdot F^4=\frac{2^{17}\cdot F^4}{3};\ \ (4)$$

Refference:

[1] Chirciu M.,About Bătineţu's inequalities, Romanian Mathematical Magazine-no.28-2021, page.4-10.

TRIGONOMETRIC SUBSTITUTIONS IN PROBLEM SOLVING

By Ioan Şerdean, Daniel Sitaru-Romania

Abstract: In this paper are indicated a few useful trigonometric substitutions for solving problems. Solved problems are also a part of this article.

Case 1: If $x,y,z>0;p,q,r\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$1+x^2=1+tan^2p;1+y^2=1+tan^2q;1+z^2=1+tan^2r$$

$$F(1+x^2,1+y^2,1+z^2)=G\left(\frac{1}{cos^2p},\frac{1}{cos^2q},\frac{1}{cos^2r}\right)$$

Case 2: If $x,y,z>0;p,q,r\in\left(0,\frac{\pi}{2}\right)$

$$\sqrt{1+x^2}=\frac{1}{cos\,p};\sqrt{1+y^2}=\frac{1}{cos\,q};\sqrt{1+z^2}=\frac{1}{cos\,r}$$

$$F\left(\sqrt{1+x^2},\sqrt{1+y^2},\sqrt{1+z^2}\right)=G\left(\frac{1}{cos\,p},\frac{1}{cos\,q},\frac{1}{cos\,r}\right)$$

Case 3: If $x,y,z>0;m\geq 0;p,q,r\in\left(0,\frac{\pi}{2}\right)$

$$\sqrt{x^2+m^2}=m\,tan\,p;\sqrt{y^2+m^2}=m\,tan\,q;\sqrt{z^2+m^2}=m\,tan\,r$$

$$F\left(\sqrt{x^2+m^2},\sqrt{y^2+m^2},\sqrt{z^2+m^2}\right)=G\left(\frac{m}{cos\,p},\frac{m}{cos\,q},\frac{m}{cos\,r}\right)$$

Case 4: If $x,y,z>0;p,q,r\in[0,2\pi]$

$$4x^3-3x=cos\,p;4y^3-3y=cos\,q;4z^3-3z=cos\,r$$

$$F(4x^3-3x,4y^3-3y,4z^3-3z)=F(cos\,p,cos\,q,cos\,r)$$

Case 5: If $x,y,z>0;p,q,r\in[0,2\pi]$

$$3x-4x^3=sin\,p,3y-4y^3=sin\,q,3z-4z^3=sin\,r$$

$$F(3x-4x^3,3y-4y^3,3z-4z^3)=F(sin\,p,sin\,q,sin\,r)$$

Case 6: If $x,y,z>0;p,q,r\in[0,2\pi]$

$$2x^2-1=cos\,p,2y^2-1=cos\,q,2z^2-1=cos\,r$$

$$F(2x^2-1,2y^2-1,2z^2-1)=F(cos\,p,cos\,q,cos\,r)$$

Case 7: If $x,y,z>0;p,q,r\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\frac{2x}{1-x^2}=tan\,p,\frac{2y}{1-y^2}=tan\,q,\frac{2z}{1-z^2}=tan\,r$$

$$F\left(\frac{2x}{1-x^2},\frac{2y}{1-y^2},\frac{2z}{1-z^2}\right)=G(tan\,p,tan\,q,tan\,r)$$

Case 8: If $x,y,z>0;p,q,r\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\frac{2x}{1+x^2}=tan\,p,\frac{2y}{1+y^2}=tan\,q,\frac{2z}{1+z^2}=tan\,r$$

$$F\left(\frac{2x}{1+x^2},\frac{2y}{1+y^2},\frac{2z}{1+z^2}\right)=G(tan\,p,tan\,q,tan\,r)$$

Case 9: If $x,y,z>0;p,q,r\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$

$$x=\frac{1}{cos\,p};y=\frac{1}{cos\,q};z=\frac{1}{cos\,r}$$

$$F(x^2-1,y^2-1,z^2-1)=G(tan^2p,tan^2q,tan^2r)$$

Case 10: If $x, y, z > 0; |x|, |y|, |z| \geq 1, p, q, r \in \left[0, \frac{\pi}{2}\right)$

$$F\left(\sqrt{x^2-1}, \sqrt{y^2-1}, \sqrt{z^2-1}\right) = G(\tan p, \tan q, \tan r)$$

$$x = \frac{1}{\cos p}, y = \frac{1}{\cos q}, z = \frac{1}{\cos r}$$

Case 11: If $x, y, z > 0; |x|, |y|, |z| \geq 1, p, q, r \in \left[0, \frac{\pi}{2}\right)$

$$x = \frac{m}{\cos p}, y = \frac{m}{\cos q}, z = \frac{m}{\cos r}$$

$$F\left(\sqrt{x^2-m^2}, \sqrt{y^2-m^2}, \sqrt{z^2-m^2}\right) = G(m\tan p, m\tan q, m\tan r)$$

Case 12: If $x, y, z > 0; xy \neq 1, yz \neq 1, zx \neq 1, p, q, r \in (0, \frac{\pi}{2})$

$$x = \tan q, y = \tan q, z = \tan r$$

$$F\left(\frac{x+y}{1-xy}; \frac{y+z}{1-yz}; \frac{z+x}{1-zx}\right) = G(\tan(p+q), \tan(q+r), \tan(r+p))$$

Problem 1. If $x \in \mathbb{R}; |x| \leq 1; n \in \mathbb{N}$ then:

$$(1-x)^n + (1+x)^n \leq 2^n$$

Proof. $|x| \leq 1 \Rightarrow (\exists) t \in \left[0, \frac{\pi}{2}\right], x = \cos 2t$

Remains to prove: $(1-\cos 2t)^n + (1+\cos 2t)^n \leq 2^n \Leftrightarrow$

$$(2\sin^2 t)^n + (2\cos^2 t)^n \leq 2^n \Leftrightarrow \sin^{2n} t + \cos^{2n} t \leq 1$$

which is obviously because:

$$\begin{cases} \sin^{2n} t \leq \sin^2 t \\ \cos^{2n} t \leq \cos^2 t \end{cases} \Rightarrow \sin^{2n} t + \cos^{2n} t \leq \sin^2 t + \cos^2 t = 1$$

Problem 2. If $a \in (-\infty, -1) \cup (1, \infty)$ then:

$$\sqrt{a^2-1} + \sqrt{3} \leq 2|a|$$

Proof. $|a| > 1 \Rightarrow (\exists)|a| = \frac{1}{\cos \alpha}; \alpha \in \left[0, \frac{\pi}{2}\right)$

Remains to prove:

$$\sqrt{\frac{1}{\cos^2 \alpha} - 1} + \sqrt{3} \leq \frac{2}{\cos \alpha} \Leftrightarrow \tan \alpha + \sqrt{3} \leq \frac{2}{\cos \alpha}$$

$$\frac{1}{2}\sin \alpha + \sqrt{3}\cos \alpha \leq 1 \Leftrightarrow \sin\left(\alpha + \frac{\pi}{3}\right) \leq 1$$

Problem 3. If $a \in (0,1)$ then:

$$\left|4(a^3 - \sqrt{(1-a^2)^3} - 3\left(a - \sqrt{1-a^2}\right)\right| \leq \sqrt{2}$$

Proof. $|a| < 1 \Rightarrow (\exists) x \in \left[0, \frac{\pi}{2}\right); a = \cos x$

The inequality can be written:

$$\left|4(\cos^3 x - \sqrt{(1-\cos^2 x)^3} - 3\left(\cos x - \sqrt{1-\cos^2 x}\right)\right| \leq \sqrt{2}$$

$$|4(\cos^3 x - \sin^3 x) - 3(\cos x - \sin x)| \leq \sqrt{2} \Leftrightarrow$$

$$|(4\cos^3 x - 3\cos x) + (3\sin x - 4\sin^3 x)| \leq \sqrt{2} \Leftrightarrow |\sin 3x + \cos 3x| \leq \sqrt{2} \Leftrightarrow$$

$$\left|\cos 3x \cdot \frac{\sqrt{2}}{2} + \sin 3x \cdot \frac{\sqrt{2}}{2}\right| \leq 1 \Leftrightarrow \left|\sin\left(3x + \frac{\pi}{4}\right)\right| \leq 1$$

Problem 4. If $x, y \in \mathbb{R}$ then:

$$\left|\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}\right| \leq \frac{1}{2}$$

Proof. $x, y \in \mathbb{R} \Rightarrow (\exists) p, q \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); x = \tan p; y = \tan q$

$$\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} = \frac{(\tan p + \tan q)(1 - \tan p \tan q)}{(1+\tan^2 p)(1+\tan^2 q)} =$$

$$= \frac{\sin(p+q)\cos(p+q)\cos^2 p \cos^2 q}{\cos p \cos q \sin p \sin q} = \sin(p+q)\cos(p+q) = \frac{1}{2}\sin 2(p+q)$$

Remains to prove:

$$\left|\frac{1}{2}\sin 2(p+q)\right| \le \frac{1}{2} \Leftrightarrow |\sin 2(p+q)| \le 1$$

Problem 5. If $a, b \in \mathbb{R}$ then:

$$a^2(1+b^4) + b^2(1+a^4) \le (1+a^4)(1+b^4)$$

Proof. $a^2 \ge 0, b^2 \ge 0 \Rightarrow (\exists) p, q \in \left[0, \frac{\pi}{2}\right), a^2 = \tan p, b^2 = \tan q$

$$\tan p\,(1+\tan^2 q) + \tan q\,(1+\tan^2 p) \le (1+\tan^2 p)(1+\tan^2 q)$$

$$\tan p \cdot \frac{1}{\cos^2 q} + \tan q \cdot \frac{1}{\cos^2 p} \le \frac{1}{\cos^2 p} \cdot \frac{1}{\cos^2 q} \Leftrightarrow \tan p \cdot \cos^2 p + \tan q \cdot \cos^2 q \le 1$$

$$\sin p \cos p + \sin q \cos q \le 1 \Leftrightarrow 2\sin p \cos p + 2\sin q \cos q \le 2$$

$$\sin 2p + \sin 2q \le 2$$

which its obvious because $\sin 2p \le 1, \sin 2q \le 1$.

Problem 6. If $x, y \ge 0; x + y = 1$ then:

$$\left(x^2 + \frac{1}{x^2}\right) + \left(y^2 + \frac{1}{y^2}\right) \ge \frac{17}{2}$$

Proof. $x, y \ge 0 \Rightarrow (\exists) p \in [0, 2\pi); x = \sin^2 p; y = \cos^2 p$

$$\sin^4 p + \frac{1}{\cos^4 p} + \cos^4 p + \frac{1}{\cos^4 p} = (\sin^4 p + \cos^4 p)\left(1 + \frac{1}{\sin^4 p \cos^4 p}\right) =$$

$$= (1 - 2\sin^2 p \cos^2 p)\left(1 + \frac{16}{\sin^4 2p}\right) = \left(1 - \frac{\sin^2 2p}{2}\right)\left(1 + \frac{16}{\sin^4 2p}\right) \ge \left(1 - \frac{1}{2}\right)\left(1 + \frac{16}{1}\right)$$

$$= \frac{1}{2} \cdot 17 = \frac{17}{2}$$

Problem 7. If $a, b \in \mathbb{R}$ then:

$$a^2 + (a-b)^2 \ge \frac{3-\sqrt{5}}{2}(a^2+b^2)$$

Proof. If $b = 0$ inequality can be written:

$$2a^2 \ge \frac{3-\sqrt{5}}{2} \cdot a^2 \Leftrightarrow a^2(1+\sqrt{5}) \ge 0$$

If $b \ne 0$, dividing by b^2: $\frac{a^2}{b^2} + \left(\frac{a}{b} - 1\right)^2 \ge \frac{3-\sqrt{5}}{2}\left(\frac{a^2}{b^2} - 1\right)$

But: $\frac{a}{b} \in \mathbb{R} \Rightarrow (\exists) p \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); \frac{a}{b} = \tan p$

$$\tan^2 p + (\tan p - 1)^2 \ge \frac{3-\sqrt{5}}{2}(\tan^2 p + 1) \Leftrightarrow \sin^2 p(\sin p - \cos p)^2 \ge \frac{3-\sqrt{5}}{2} \Leftrightarrow$$

$$1 + \sin^2 p - 2\sin p \cos p \ge \frac{3-\sqrt{5}}{2} \Leftrightarrow 2 + 2\sin^2 p - 2\sin 2p \ge 3 - \sqrt{5} \Leftrightarrow$$

$$1 - 2\sin^2 p + 2\sin 2p \le \sqrt{5} \Leftrightarrow \cos 2p + 2\sin 2p \le \sqrt{5} \Leftrightarrow \frac{1}{\sqrt{5}}\cos 2p + \frac{2}{\sqrt{5}}\sin 2p \le 1 \Leftrightarrow$$

$$\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2 = 1 \Rightarrow (\exists) q \in \left(0, \frac{\pi}{2}\right); \frac{1}{\sqrt{5}} = \sin q; \frac{2}{\sqrt{5}} = \cos q$$

$$sin\,q\,cos\,2p + cos\,q\,sin\,2p \le 1 \Leftrightarrow sin(2p+q) \le 1$$

Problem 8. If $a,b \in [0,1]$ then:

$$\left|a\sqrt{1-b^2} + b\sqrt{1-a^2} + \sqrt{3}(ab - \sqrt{(1-a^2)(1-b^2)}\right| \le 2$$

Proof. $a,b \in [0,1] \Rightarrow (\exists)p,q \in \left[0,\frac{\pi}{2}\right]; a = \sin p, b = \sin q$

$$\left|\sin p\sqrt{1-sin^2q} + \sin q\sqrt{1-sin^2p} + \sqrt{3}(\sin p \sin q - \sqrt{(1-sin^2p)(1-sin^2q)}\right| \le 2$$

$$\Leftrightarrow \left|\sin p \cos q + \sin q \cos p + \sqrt{3}(\sin p \sin q - \sqrt{(1-sin^2p)(1-sin^2q)}\right| \le 2$$

$$\Leftrightarrow \left|\sin(p+q) - \sqrt{3}\cos(p+q)\right| \le 2 \Leftrightarrow \left|\frac{1}{2}\sin(p+q) - \frac{\sqrt{3}}{2}\cos(p+q)\right| \le 1 \Leftrightarrow$$

$$\left|\sin\left(p+q-\frac{\pi}{3}\right)\right| \le 1$$

Problem 9. Solve the following equation:

$$x^3 - 3x + a(1-3x^2) = 0; a \in \mathbb{R}$$

Proof. $x \in \mathbb{R} \Rightarrow (\exists)b \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right); x = \tan b$

$$a(1-3tan^2b) = 3\tan b - tan^3b \Rightarrow a = \frac{3\tan b - tan^3b}{1-3tan^2b}$$

$$a = 3\tan 3b \Rightarrow 3b = tan^{-1}a + k\pi; k \in \mathbb{Z} \Rightarrow b = \frac{1}{3}tan^{-1} + \frac{k\pi}{3}; k \in \mathbb{Z}$$

$$x = \tan b = \tan\left(\frac{1}{3}tan^{-1}a + \frac{k\pi}{3}\right); k \in \{-2,0,2\}$$

If $a = 1$ the equation: $x^3 - 3x^2 - 3x + 1 = 0$ has the solutions:

$$x = \tan\left(\frac{\pi}{12} + \frac{k\pi}{3}\right); k \in \{-2,0,2\}, x_1 = \tan\frac{\pi}{12}; x_2 = \tan\frac{\pi}{4}; x_3 = \tan\left(-\frac{7\pi}{4}\right)$$

If $a = 2$ the equation: $x^3 - 6x^2 - 3x + 2 = 0$ has the solutions:

$$x = \tan\left(\frac{1}{3}tan^{-1}2 + \frac{k\pi}{3}\right); k \in \{-2,0,2\}, x_1 = \tan\left(\frac{1}{3}tan^{-1}2\right), x_2 = \left(\frac{1}{3}tan^{-1}2 + \frac{2\pi}{3}\right),$$

$$x_3 = \tan\left(\frac{1}{3}tan^{-1}2 - \frac{2\pi}{3}\right)$$

Problem 10. Solve the equation:

$$4x^3 - 4x + a(x^4 - 6x^2 + 1) = 0; a \in \mathbb{R}$$

Proof. $x \in \mathbb{R} \Rightarrow (\exists)b \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right); x = \tan b; a(tan^4b - 6tan^2b + 1) = 4\tan b - 4tan^3b$

$$a = \frac{4\tan b - 4tan^3b}{tan^4b - 6tan^2b + 1} \Rightarrow a = \tan 4b \Rightarrow b = \frac{1}{4}\tan^{-1}a;$$

$$x = tan\left(\frac{1}{4}tan^{-1}a + \frac{k\pi}{4}\right); k \in \mathbb{Z}$$

If $a = 1$ the equation: $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ has the solutions:

$$x_1 = \tan\frac{\pi}{16}; x_2 = \tan\frac{5\pi}{16}; x_3 = \tan\left(-\frac{3\pi}{16}\right); x_4 = \tan\frac{9\pi}{16}$$

Problem 11. Find the formula of the general term of the sequence given by the relationships:

$$x_1 = a; a \in [-1,1], x_{n+1} = 2x_n^2 - 1; n \ge 1$$

Proof. $x_1 = a \in [-1,1] \Rightarrow (\exists)b \in [0,2\pi]; a = \cos b; b = cos^{-1}a$

$$x_2 = 2x_1^2 - 1 = 2cos^2b - 1 = \cos 2b;\ x_3 = 2x_2^2 - 1 = \cos(2^2b)$$

$$x_4 = 2x_3^2 - 1 = \cos(2^3b)$$

Through induction we can prove that: $x_n = \cos(2^{n-1}b) = \cos(2^{n-1}cos^{-1}a)$

Problem 12. Find the formula of the general term of the sequence given by the relationships:

$$x_1 = a; a \in [-1,1], x_{n+1} = 1 - 2x_n^2; n \geq 1$$

Proof. $x_1 = a \in [-1,1] \Rightarrow (\exists) b \in [0,2\pi]; a = \sin b\,; b = sin^{-1}a$

$$x_2 = 1 - 2x_1^2 = 1 - 2sin^2 b = \cos(2b)\,;\ x_3 = 1 - 2x_2^2 = \cos(2^2 b)$$

Through induction we can prove that: $x_n = \cos(2^{n-1}b) = \cos(2^{n-1}sin^{-1}a)$

Problem 13. Find the formula of the general term of the sequence given by the relationships:

$$x_1 = a; a \in [-1,1], x_{n+1} = x_n(3 - 4x_n^2); n \geq 1$$

Proof. $x_1 = a \in [-1,1] \Rightarrow (\exists) b \in [0,2\pi]; a = \sin b\,; b = sin^{-1}a$

$$x_2 = x_1(3 - 4x_1^2) = \sin b\,(3 - 4sin^2 3b) = \sin(3b)$$
$$x_3 = x_2(3 - 4x_2^2) = \sin 3b\,(3 - 4sin^2 3b) = \sin(3^2 b)$$
$$x_4 = x_3(3 - 4x_3^2) = \sin(3^2 b)\,(3 - 4sin^2(3^2 b)) = \sin(3^3 b)$$

Through induction we can prove that: $x_n = \cos(3^{n-1}b) = \cos(3^{n-1}sin^{-1}a)$

Problem 14. Find the formula of the general term of the sequence given by the relationships:

$$x_1 = a; a \in [-1,1], x_{n+1} = x_n(4x_n^2 - 3); n \geq 1$$

Proof. $x_1 = a \in [-1,1] \Rightarrow (\exists) b \in [0,2\pi]; a = \cos b\,; b = cos^{-1}a$

$$x_2 = x_1(4x_1^2 - 3) = \cos b\,(4cos^2 3b - 3) = \cos(3b)$$
$$x_3 = x_2(4x_2^2 - 3) = \cos 3b\,(4cos^2 3b - 3) = \cos(3^2 b)$$
$$x_4 = x_3(4x_3^2 - 3) = \cos(3^2 b)\,(4sin^2(3^2 b) - 3) = \cos(3^3 b)$$

Through induction we can prove that: $x_n = \cos(3^{n-1}b) = \cos(3^{n-1}cos^{-1}a)$

Problem 15. Find the formula of the general term of the sequence given by the relationships:

$$x_1 = a; a \in [-1,1], x_{n+1} = \frac{2x_n}{1 - x_n^2}; n \geq 1$$

Proof. $x_1 = a \in [-1,1] \Rightarrow (\exists) b \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); a = \tan b\,; b = tan^{-1}a$

$$x_2 = \frac{2x_1}{1 - x_1^2} = \frac{2\tan b}{1 - tan^2 b} = \tan(2b)\,;\ x_3 = \frac{2x_2}{1 - x_2^2} = \frac{2\tan(2b)}{1 - tan^2(2b)} = \tan(2^2 b)$$

Through induction we can prove that: $x_n = \tan(2^{n-1}b) = \tan(2^{n-1}tan^{-1}a)$

Problem 16. Find the formula of the general term of the sequence given by the relationships:

$$x_1 = a; a \in [-1,1], x_{n+1} = \frac{3x_n - x_n^3}{1 - 3x_n^2}; n \geq 1$$

Proof. $x_1 = a \in [-1,1] \Rightarrow (\exists) b \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); a = \tan b\,; b = tan^{-1}a$

$$x_2 = \frac{3x_1 - x_1^3}{1 - 3x_1^2} = \frac{3\tan b - tan^3 b}{1 - 3tan^2 b} = \tan(3b)\,;$$
$$x_3 = \frac{3x_2 - x_2^3}{1 - 3x_2^2} = \frac{3\tan(3b) - tan^3(3b)}{1 - 3tan^2(3b)} = \tan(3^2 b)$$

Through induction we can prove that: $x_n = \tan(3^{n-1}b) = \tan(3^{n-1}tan^{-1}a)$

Proposed Problems:

17. If $x \geq 0$ then: $\sqrt{x-1} + \sqrt{x\sqrt{x-1}} < x$

$$\left(Use: x = \frac{1}{cos^2 p}; p \in \left[0, \frac{\pi}{2}\right)\right)$$

18. If $|a| \le 1$ then: $\sqrt{1+\sqrt{1-a^2}}\left[\sqrt{(1+a)^3}-\sqrt{(1-a)^3}\right] \le 2\sqrt{2}+\sqrt{2-2a^2}$

$$(Use: a - \cos p\,; p \in [0,\pi])$$

19. If $|x| \le 1$ then: $\frac{\sqrt{3}-2}{2} \le \sqrt{3}x^2 + x\sqrt{1-x^2} \le \frac{\sqrt{3}+2}{2}$

$$(Use: x = \cos p\,; p \in [0,\pi])$$

20. If $a \in [0,2]$ then: $\left|\sqrt{2a-a^2}-\sqrt{3}a+\sqrt{3}\right| \le 2$

$$(Use: a - 1 = \cos p\,, p \in [0,\pi])$$

21. If $|a| \ge 1$ then: $\left|\frac{\sqrt{a^2-1}+\sqrt{3}}{a}\right| \le 2$

$$\left(Use: a = \frac{1}{\cos p}; p \in \left[0,\frac{\pi}{2}\right) \cup \left[\pi,\frac{3\pi}{2}\right)\right)$$

22. If $|a| \ge 1$ then: $-4 \le \frac{5-12\sqrt{a^2-1}}{a^2} \le 9$

$$\left(Use: a = \frac{1}{\cos p}; p \in \left[0,\frac{\pi}{2}\right) \cup \left[\pi,\frac{3\pi}{2}\right)\right)$$

Problem 23. If $x,y,z > 0, xy+yz+zx = 1$ then:

$$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{3z}{1+z^2} \le \sqrt{10}$$

Proof. $x,y,z > 0 \Rightarrow (\nexists)A,B,C \in \left(0,\frac{\pi}{2}\right); x = tan\frac{A}{2}, y = tan\frac{B}{2}, z = tan\frac{C}{2}$

$$\frac{x}{1+x^2} = \frac{tan\frac{A}{2}}{1+tan^2\frac{A}{2}} = \frac{sin\frac{A}{2}}{cos\frac{A}{2}}\cdot cos^2\frac{A}{2} = sin\frac{A}{2}cos\frac{A}{2}$$

$$\frac{y}{1+y^2} = sin\frac{B}{2}cos\frac{B}{2}; \frac{z}{1+z^2} = sin\frac{C}{2}cos\frac{C}{2}$$

$$sin\frac{A}{2}cos\frac{A}{2}+sin\frac{B}{2}sin\frac{B}{2}+3\,sin\frac{C}{2}cos\frac{C}{2} \le \sqrt{10} \Leftrightarrow sin\,A + sin\,B + 3\,sin\,C \le 2\sqrt{10}$$

$$\Leftrightarrow sin\,A + sin\,B = 2\,sin\frac{A+B}{2}cos\frac{A-B}{2} = 2\,cos\frac{C}{2}cos\frac{B-C}{2} \le 2\,cos\frac{C}{2}$$

$$2\,cos\frac{C}{2}+3\,sin\,C = 2\,cos\frac{C}{2}+6\,sin\frac{C}{2}cos\frac{C}{2} \le 2\,cos\frac{C}{2}+6\,sin\frac{C}{2} = 2\left(cos\frac{C}{2}+3\,sin\frac{C}{2}\right) \overset{CBS}{\le}$$

$$\le 2\cdot\sqrt{1^2+3^2}\cdot\sqrt{sin^2\frac{C}{2}+cos^2\frac{C}{2}} = 2\cdot\sqrt{10}\cdot 1 = 2\sqrt{10}$$

Problem 24. If $x,y,z > 0; xy+yz+zx = 1$ then:

$$\frac{x}{\sqrt{1+x^2}}+\frac{y}{\sqrt{1+y^2}}+\frac{z}{\sqrt{1+z^2}} \le \frac{3}{2}$$

Proof. $x,y,z > 0 \Rightarrow (\exists)A,B,C \in \left(0,\frac{\pi}{2}\right); x = cot\,A, y = cot\,B, z = cot\,C$

$$\frac{x}{\sqrt{1+x^2}} = cos\,A\,; \frac{y}{\sqrt{1+y^2}} = cos\,B\,; \frac{z}{\sqrt{1+z^2}} = cos\,C$$

Inequality to prove becomes a known one:

$$cos\,A + cos\,B + cos\,C \le \frac{3}{2}$$

Problem 25. If $x,y,z > 0; x+y+z = xyz$ then:

$$(x^2-1)(y^2-1)(z^2-1)\le\sqrt{(x^2+1)(y^2+1)(z^2+1)}$$

Proof. If $x,y,z>0\Rightarrow(\exists)A,B,C\in\left(0,\frac{\pi}{2}\right);x=cot\frac{A}{2},y=cot\frac{B}{2},z=cot\frac{C}{2}$

Inequality to prove becomes:

$$\frac{x^2-1}{x^2+1}\cdot\frac{y^2-1}{y^2+1}\cdot\frac{z^2-1}{z^2+1}\le\frac{1}{\sqrt{(x^2+1)(y^2+1)(z^2+1)}}$$

$$\frac{x^2-1}{x^2+1}=cos\,A;\ \frac{y^2-1}{y^2+1}=cos\,B;\frac{z^2-1}{z^2+1}=cos\,C$$

$$\frac{1}{\sqrt{x^2+1}}=sin\frac{A}{2};\frac{1}{\sqrt{y^2+1}}=sin\frac{B}{2};\frac{1}{\sqrt{z^2+1}}=sin\frac{C}{2}$$

Inequality to prove can be written:

$$cos\,A\,cos\,B\,cos\,C\le sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$

$$\cos A\cos B=\frac{1}{2}[\cos(A-B)+\cos(A+B)]=\frac{1}{2}[\cos(A-B)-\cos C]\le\frac{1}{2}(1-\cos C)$$
$$=\sin^2\frac{C}{2}$$

Analogous: $cos\,B\,cos\,C\le sin^2\frac{A}{2};cos\,C\,cos\,A\le sin^2\frac{B}{2}$ and multiplying its obtained the asked inequality.

Problem 26. If $x,y,z>0;xy+yz+zx=1$ then:

$$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}\ge\frac{3\sqrt{3}}{4}$$

Proof. $x,y,z>0\Rightarrow(\exists)A,B,C\in\left(0,\frac{\pi}{2}\right);x=tan\,A,y=tan\,B,z=tan\,C$

$$\frac{x}{1+x^2}=\frac{tan\,A}{1+tan^2A}=\frac{sin\,A}{cos\,A}\cdot\frac{cos^2A}{1}=sin\,A\,cos\,A$$

$$\frac{y}{1+y^2}=\frac{tan\,B}{1+tan^2B}=sin\,B\,cos\,B;\ \frac{z}{1+z^2}=sin\,C\,cos\,C$$

Inequality to prove can be written:

$$sin\,A\,cos\,A+sin\,B\,cos\,B+sin\,C\,cos\,C\ge\frac{3\sqrt{3}}{4}\Leftrightarrow sin\,A+sin\,B+sin\,C\ge\frac{3\sqrt{3}}{2}$$

$f:(0,\pi)\to\mathbb{R};f(x)=sin\,x;f'(x)=cos\,x;f''(x)=-sinx<0\Rightarrow f$ –concave.

By Jensen's inequality: $f\left(\frac{A+B+C}{3}\right)\ge\frac{1}{3}\big(f(A)+f(B)+f(C)\big)\Leftrightarrow$

$$3\,sin\frac{\pi}{3}\ge sin\,2A+sin\,2B+sin\,2C\Leftrightarrow sin\,2A+sin\,2B+sin\,2C\le\frac{3\sqrt{3}}{2}$$

Theorem 1. If $A,B,C\in(0,\pi);A+B+C=\pi$ then:

$$tan\frac{A}{2}tan\frac{B}{2}+tan\frac{B}{2}tan\frac{C}{2}+tan\frac{C}{2}tan\frac{A}{2}=1$$

Theorem 2. If $A,B,C\in(0,\pi);A+B+C=\pi$ then:

$$sin^2\frac{A}{2}+sin^2\frac{B}{2}+sin^2\frac{C}{2}+2\,sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}=1$$

Problem 27. If $x,y,z\in(0,\infty);x^2+y^2+z^2+2xyz=1$ then:

$$xy+yz+zx\le\frac{3}{4}$$

Proof. $x,y,z\in(0,\infty)\Rightarrow(\exists)A,B,C\in(0,\pi);x=sin\frac{A}{2},y=sin\frac{B}{2},z=sin\frac{C}{2}$

Inequality to prove becomes:

$$sin\frac{A}{2}sin\frac{B}{2}+sin\frac{B}{2}sin\frac{C}{2}+sin\frac{C}{2}sin\frac{A}{2}\le\frac{3}{4}$$

By Jensen's inequality: $\frac{1}{2}=sin\frac{\pi}{6}=sin\frac{A+B+C}{6}\ge\frac{1}{3}\left(sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\right)\Rightarrow$

$$sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\le\frac{3}{2}$$

On the other hand:

$$\sin\frac{A}{2}\sin\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}\le\frac{1}{3}\left(sin\frac{A}{2}+sin\frac{B}{2}+sin\frac{C}{2}\right)^2\le\frac{1}{3}\cdot\left(\frac{3}{2}\right)^2=\frac{9}{12}=\frac{3}{4}$$

Problem 28. If $x,y,z>0; x^2+y^2+z^2+2xyz=1$ then:

$$x+y+z\ge 4xyz+1$$

Proof. $x,y,z\in(0,\infty)\Rightarrow(\exists)A,B,C\in\left(0,\frac{\pi}{2}\right); x=cos\,A, y=cos\,B, z=cos\,C$

Inequality to prove can be written:

$$cos\,A+cos\,B+cos\,C\ge 4\,cos\,A\,cos\,B\,cos\,C+1\Leftrightarrow$$

$$2\,cos\frac{A+B}{2}cos\frac{A-B}{2}+1-2sin^2\frac{C}{2}\ge cos\,A\,cos\,B\,cos\,C+1\Leftrightarrow$$

$$2\,cos\frac{C}{2}cos\frac{A-B}{2}-2sin^2\frac{C}{2}\ge 4\,cos\,A\,cos\,B\,cos\,C\Leftrightarrow$$

$$2\,cos\frac{\pi-C}{2}cos\frac{A-B}{2}-2sin^2\frac{C}{2}\ge 4\,cos\,A\,cos\,B\,cos\,C\Leftrightarrow$$

$$2\,sin\frac{C}{2}\left(cos\frac{A-B}{2}-sin\frac{C}{2}\right)\ge 4\,cos\,A\,cos\,B\,cos\,C\Leftrightarrow$$

$$2\,sin\frac{C}{2}\cdot 2\cdot sin\frac{\frac{A-B}{2}+\frac{\pi-C}{2}}{2}sin\frac{\frac{\pi-C}{2}-\frac{A-B}{2}}{2}\ge 4\,cos\,A\,cos\,B\,cos\,C\Leftrightarrow$$

$$4\,sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\ge 4\,cos\,A\,cos\,B\,cos\,C$$

Remains to prove:

$$sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\ge cos\,A\,cos\,B\,cos\,C$$

$$cos\,Acos\,B\le\frac{(\cos A+\cos B)^2}{4}=sin^2\frac{C}{2}cos^2\frac{A-B}{2}\le sin^2\frac{C}{2}$$

Analogous:

$$\cos B\cos C\le sin^2\frac{A}{2};\cos C\cos A\le sin^2\frac{B}{2}$$

By multiplying:

$$cos^2Acos^2Bcos^2C\le sin^2\frac{A}{2}sin^2\frac{B}{2}sin^2\frac{C}{2}\Leftrightarrow sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}\ge cos\,A\,cos\,B\,cos\,C$$

Proposed Problems

29. If $a\in\mathbb{R}; |a|\ge 1$ then:

$$a+\frac{a}{\sqrt{a^2-1}}\ge 2\sqrt{2}$$

$$\left(Use: a=\frac{1}{\cos p}; p\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)\right)$$

30. If $a\in\mathbb{R}$ then:

$$\left|\frac{3a}{\sqrt{1+a^2}}-\frac{4a^2}{\sqrt{(1+a^2)^3}}\right|\le 1$$
$$\left(Use: a=\tan p\,;p\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right)$$

31. If $a\in\mathbb{R}$ then:
$$\frac{5}{2}\le\frac{12a^4+8a^2+3}{(1+2a^2)^2}\le 3$$
$$\left(Use: a=\frac{1}{\sqrt{2}}\tan p\,;p\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right)$$

32. If $a\in\mathbb{R}$ then:
$$-3\le\frac{6a+4|a^2-1|}{a^2+1}\le 5$$

33. If $a\in[1,3]$ then:
$$|4a^3-24a^2+45a-26|\le 1$$
$$(Use: a-2=\cos x)$$

34. If $x\in\mathbb{R}$ then:
$$\left|\frac{x(1-x^2)(x^4-6x^2+1)}{(1+x^2)^4}\right|\le\frac{1}{8}$$
$$(Use: x=\tan p)$$

35. If $a,b,c,d\in\mathbb{R}; a^2+b^2=c^2+d^2=1$ then:
$$-\sqrt{2}\le a(c+d)+b(c-d)\le 2$$
$$(Use: a=\sin x\,;b=\cos x\,;c=\sin y\,;d=\cos y)$$

36. If $a,b\in\mathbb{R}; a^2+b^2=1$ then:
$$\left(a^2+\frac{1}{a^2}\right)^2+\left(b^2+\frac{1}{b^2}\right)^2\ge\frac{25}{2}$$
$$(Use: a=\sin x\,;b=\cos x)$$

37. If $a,b\in\mathbb{R}; a^2+b^2-2a-4b+4=0$ then:
$$\left|a^2-b^2+2\sqrt{3}ab-2(1+2\sqrt{3})a+(4-2\sqrt{3})b+4\sqrt{3}-3\right|\le 2$$
$$(Use: a-1=\sin x\,;b-2=\cos x)$$

38. If $a\in[0,2]; b\in[0,3]$ then:
$$4\le a^2+b^2+ab+\sqrt{4-a^2}\cdot\sqrt{9-b^2}\le 19$$

39. If $m,n\in[1,\infty)$ then:
$$n\sqrt{m-1}+m\sqrt{n-1}\le mn$$
$$\left(Use: m=\frac{1}{cos^2x}\,;n=\frac{1}{cos^2y}\,;x,y\in\left[0,\frac{\pi}{2}\right]\right)$$

40. If $m,n\in(0,\infty); m>n$ then:
$$\sqrt{m^2-n^2}+\sqrt{2mn-n^2}\ge m$$
$$\left(Use:\frac{n}{m}=\sin x\,;x\in\left[0,\frac{\pi}{2}\right]\right)$$

41. If $x,y\in\mathbb{R}; |x|\le 1,|y|\le 1$ then:
$$\sqrt{1-x^2}+\sqrt{1-y^2}\le\sqrt{4-(x+y)^2}$$
$$(Use: x=\sin p\,;y=\sin q)$$

42. If $x_1,x_2,\dots,x_n\in[-1,1]; x_1^3+x_2^3+\dots+x_n^3=0$ then:

$$x_1+x_2+\dots+x_n\le\frac{n}{3}$$
$$(Use: x_i=\cos p_i\,; p_i\in[0,\pi]; i\in\overline{1,n})$$
$$a=y+z, b=z+x, c=x+y; s=x+y+z; S=\sqrt{xyz(x+y+z)}$$
$$R=\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}; r=\sqrt{\frac{xyz}{x+y+z}}$$
$$r_a=\sqrt{\frac{(x+y+z)yz}{x}}; r_b=\sqrt{\frac{(x+y+z)zx}{y}}; r_c=\sqrt{\frac{(x+y+z)xy}{z}}$$
$$\sin\frac{A}{2}=\sqrt{\frac{yz}{(x+y)(x+z)}}; \sin\frac{B}{2}=\sqrt{\frac{zx}{(y+z)(y+x)}}; \sin\frac{C}{2}=\sqrt{\frac{xy}{(z+x)(z+y)}}$$
$$\cos\frac{A}{2}=\sqrt{\frac{(x+y+z)x}{(x+y)(x+z)}}; \cos\frac{B}{2}=\sqrt{\frac{(x+y+z)y}{(y+x)(y+z)}}; \cos\frac{C}{2}=\sqrt{\frac{(x+y+z)z}{(z+x)(z+y)}}$$
$$\tan\frac{A}{2}=\sqrt{\frac{yz}{x}}; \tan\frac{B}{2}=\sqrt{\frac{zx}{y}}; \tan\frac{C}{2}=\sqrt{\frac{xy}{z}}$$
$$\cot\frac{A}{2}=\sqrt{\frac{x}{yz}}; \cot\frac{B}{2}=\sqrt{\frac{y}{zx}}; \cot\frac{C}{2}=\sqrt{\frac{z}{xy}}$$
$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\sqrt{xyz}; \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\frac{1}{\sqrt{xyz}}$$
$$\cos A=\frac{x(x+y+z)-yz}{(x+y)(x+z)}; \cos B=\frac{y(x+y+z)-xz}{(y+z)(y+x)}; \cos C=\frac{z(x+y+z)-xy}{(z+x)(z+y)}$$
$$\sin A=\frac{2\sqrt{xyz(x+y+z)}}{(x+y)(x+z)}; \sin B=\frac{2\sqrt{xyz(x+y+z)}}{(y+z)(y+x)}; \sin C=\frac{2\sqrt{xyz(x+y+z)}}{(z+y)(z+x)}$$

Problem 43. If $x,y,z>0$ then:
$$\frac{2xy}{(z+x)(z+y)}+\frac{2yz}{(x+y)(x+z)}+\frac{3zx}{(y+z)(y+x)}\ge\frac{5}{3}$$

Proof. $\cos A=1-2\sin^2\frac{A}{2}=1-\frac{2yz}{(x+y)(x+z)}; \frac{2yz}{(x+y)(x+z)}=\cos A$
$$\frac{2yz}{(x+y)(x+z)}=\frac{1}{2}-\frac{1}{2}\cdot\cos A; \frac{2xz}{(y+x)(y+z)}=\frac{1}{2}-\frac{1}{2}\cdot\cos B;$$
$$\frac{2xy}{(z+x)(z+y)}=\frac{1}{2}-\frac{1}{2}\cdot\cos C$$

Inequality to prove can be written:
$$2\left(\frac{1}{2}-\frac{1}{2}\cdot\cos C\right)+2\left(\frac{1}{2}-\frac{1}{2}\cdot\cos A\right)+3\left(\frac{1}{2}-\frac{1}{2}\cdot\cos B\right)\ge\frac{5}{3}\Leftrightarrow$$
$$1-\cos C+1-\cos A+\frac{3}{2}-\frac{3}{2}\cdot\cos B\ge\frac{5}{3}; \quad (1)$$
$$\cos A+\cos C+\frac{3}{2}\cos B=2\cos\frac{A+C}{2}\cos\frac{A-C}{2}+\frac{3}{2}\cos B$$
$$=2\sin\frac{B}{2}\cos\frac{A-C}{2}+\frac{3}{2}\left(1-2\sin^2\frac{B}{2}\right)$$

By (1), we need to prove:

$$\frac{7}{2}-2sin\frac{B}{2}cos\frac{A-C}{2}-\frac{3}{2}+3sin^2\frac{B}{2}\geq\frac{5}{3}\Leftrightarrow 3sin^2\frac{B}{2}-2\,sin\frac{B}{2}cos\frac{A-C}{2}+\frac{1}{3}\geq 0$$

$$3\left(sin\frac{B}{2}-\frac{1}{3}cos\frac{A-C}{2}\right)^2-\frac{1}{9}cos^2\frac{A-C}{2}+\frac{1}{3}\geq 0\Leftrightarrow$$

$$3\left(sin\frac{B}{2}-\frac{1}{3}cos\frac{A-C}{2}\right)^2+\frac{1}{9}-\frac{1}{9}cos^2\frac{A-C}{2}+\frac{2}{9}\geq 0$$

$$3\left(sin\frac{B}{2}-\frac{1}{3}cos\frac{A-C}{2}\right)^2+\frac{1}{9}sin^2\frac{A-C}{2}+\frac{2}{9}\geq 0$$

Problem 44. If $a,b,c>0; a+b+c=1$ then:

$$\sqrt{\frac{ab}{c+ab}}+\sqrt{\frac{bc}{a+bc}}+\sqrt{\frac{ca}{b+ca}}\leq\frac{3}{2}$$

Proof.

$$\sum_{cyc}\sqrt{\frac{ab}{c+ab}}\leq\frac{3}{2}\Leftrightarrow\sum_{cyc}\sqrt{\frac{ab}{(c+ab)\cdot 1}}\leq\frac{3}{2}\Leftrightarrow\sum_{cyc}\sqrt{\frac{ab}{(c+ab)(a+b+c)}}\leq\frac{3}{2}\Leftrightarrow$$

$$\sum_{cyc}\sqrt{\frac{ab}{ca+cb+c^2+ab(a+b+c)}}\leq\frac{3}{2}\Leftrightarrow\sum_{cyc}\sqrt{\frac{ab}{ca+cb+c^2+ab}}\leq\frac{3}{2}\Leftrightarrow$$

$$\sum_{cyc}\sqrt{\frac{ab}{(c+a)(c+b)}}\leq\frac{3}{2};\ \ (2)$$

Denote: $x=a+b; y=b+c; z=c+a; s=a+b+c$ and from (2) we must prove that:

$$\sum_{cyc}\sqrt{\frac{(s-y)(s-z)}{yz}}\leq\frac{3}{2}$$

$$f:(0,\pi)\to\mathbb{R}; f(x)=sin\frac{x}{2}; f'(x)=\frac{1}{2}cos\frac{x}{2}; f''(x)=-\frac{1}{4}sin\frac{x}{2}\leq 0$$

By Jensen's Inequality, we have:

$$f(A)+f(B)+f(C)\leq 3f\left(\frac{A+B+C}{3}\right)\Leftrightarrow\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq 3\sin\frac{\pi}{3}=\frac{3}{2}$$

$$\sin\frac{A}{2}=\sqrt{\frac{(s-y)(s-z)}{yz}};\sin\frac{B}{2}=\sqrt{\frac{(s-x)(s-z)}{zx}};\sin\frac{C}{2}=\sqrt{\frac{(s-x)(s-y)}{xy}}$$

Problem 45. If $x,y,z>0$ then:

$$\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)}\geq 2\sqrt{\frac{(x+y)(y+x)(z+x)}{x+y+z}}$$

(D.Grinberg)

Proof.

$$\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}+\sqrt{\frac{y(x+y+z)}{(y+z)(y+x)}}+\sqrt{\frac{z(x+y+z)}{(z+x)(z+y)}}\geq 2$$

$$a=x+y; b=y+z; c=z+x; s=x+y+z$$

$$\sqrt{\frac{s(s-b)}{ac}}+\sqrt{\frac{s(s-c)}{ab}}+\sqrt{\frac{s(s-a)}{bc}}\ge 2\Leftrightarrow \cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\ge 2$$

$$A=\pi-2A';B=\pi-2B';C=\pi-2C'$$

$$\cos\frac{\pi-2A'}{2}+\cos\frac{\pi-2B'}{2}+\cos\frac{\pi-2C'}{2}\ge 2\Leftrightarrow \sin A'+\sin B'+\sin C'\ge 2$$

By Jordan's Inequality:

$$\sin A'\ge\frac{2A'}{\pi};\sin B'\ge\frac{2B'}{\pi};\sin C'\ge\frac{2C'}{\pi}$$

By adding:

$$\sin A'+\sin B'+\sin C'\ge\frac{2(A'+B'+C')}{\pi}=\frac{2\pi}{\pi}=2$$

Problem 46. If $a,b,c\ge;a+b+c=abc$ then:

$$\sqrt{1+\frac{1}{a^2}}+\sqrt{1+\frac{1}{b^2}}+\sqrt{1+\frac{1}{c^2}}\ge 2\sqrt{3}$$

(A.Nicolaescu;C.Pătraşcu)

Proof. $a=\tan A;b=\tan B;c=\tan C;A,B,C\in\left(0,\frac{\pi}{2}\right)$

Inequality can be written:

$$\sqrt{1+\frac{1}{tan^2A}}+\sqrt{1+\frac{1}{tan^2B}}+\sqrt{1+\frac{1}{tan^2C}}\ge 2\sqrt{3}\Leftrightarrow\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\ge 2\sqrt{3}$$

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\overset{BCS}{\ge}\frac{(1+1+1)^2}{\sin A+\sin B+\sin C}\ge\frac{9}{\frac{3\sqrt{3}}{2}}=2\sqrt{3}$$

Problem 47. If $x,y,z>0;x+y+z=xyz$ then:

$$\sqrt{\frac{x^4}{3}}+\sqrt{\frac{y^4}{3}+1}+\sqrt{\frac{z^4}{3}+1}\ge 6$$

(George Apostolopoulos)

Proof. Denote: $a^2=\sqrt{3}\,tan\,A;\ b^2=\sqrt{3}\,tan\,B;c^2=\sqrt{3}\,tan\,C$

$$\sqrt{\frac{3tan^2A}{3}}+\sqrt{\frac{3tan^2B}{3}+1}+\sqrt{\frac{3tan^2C}{3}+1}\ge 6$$

$$\sqrt{1+tan^2A}+\sqrt{1+tan^2B}+\sqrt{1+tan^2C}\ge 6\Leftrightarrow\frac{1}{cos\,A}+\frac{1}{cos\,B}+\frac{1}{cos\,C}\ge 6$$

$$\frac{1}{cos\,A}+\frac{1}{cos\,B}+\frac{1}{cos\,C}\overset{BCS}{\ge}\frac{(1+1+1)^2}{cos\,A+cos\,B+cos\,C}=\frac{9}{\frac{3}{2}}=6$$

Problem 48. If $x,y,z>0;x+y+z=xyz$ then:

$$xy+yz+zx\ge 3+\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}$$

Proof. Denote: $x=tan\,A;y=tan\,B;z=tan\,C$

Inequality can be written:

$$tan\,A\,tan\,B+tan\,B\,tan\,C+tan\,C\,tan\,A\ge 3+\frac{1}{cos\,A}+\frac{1}{cos\,B}+\frac{1}{cos\,C}\Leftrightarrow$$

$$(tan\,A\,tan\,B-1)+(tan\,B\,tan\,C-1)+(tan\,C\,tan\,A-1)\ge\frac{1}{cos\,A}+\frac{1}{cos\,B}+\frac{1}{cos\,C}$$

$$\frac{\sin A\sin B-\cos A\cos B}{\cos A\cos B}+\frac{\sin B\sin C-\cos B\cos C}{\cos B\cos C}+\frac{\sin C\sin A-\cos C\cos A}{\cos C\cos A}\ge$$

$$\ge\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}\Leftrightarrow$$

$$\frac{\cos(A+B)}{\cos A\cos B}+\frac{\cos(B+C)}{\cos B\cos C}+\frac{\cos(C+A)}{\cos C\cos A}\ge\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}\Leftrightarrow$$

$$\frac{\cos C}{\cos A\cos B}+\frac{\cos A}{\cos B\cos C}+\frac{\cos B}{\cos C\cos A}\ge\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}\Leftrightarrow$$

$$\cos^2A+\cos^2B+\cos^2C\ge\cos A\cos B+\cos B\cos C+\cos C\cos A\Leftrightarrow$$

$$(\cos A-\cos B)^2+(\cos B-\cos C)^2+(\cos C-\cos A)^2\ge 0$$

Problem 49. If $x,y,z>0;xy+yz+zx=1$ then:

$$\frac{1-x^2}{1+x^2}+\frac{1-y^2}{1+y^2}+\frac{1-z^2}{1+z^2}\le\frac{3}{2}$$

(C.Popescu)

Proof. Denote $x=\tan\frac{A}{2};y=\tan\frac{B}{2};z=\tan\frac{C}{2}$

Inequality can be written:

$$\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}+\frac{1-\tan^2\frac{B}{2}}{1+\tan^2\frac{B}{2}}+\frac{1-\tan^2\frac{C}{2}}{1+\tan^2\frac{C}{2}}\le\frac{3}{2}\Leftrightarrow\cos A+\cos B+\cos C\le\frac{3}{2}$$

Problem 50. If $a,b,c>0;a+b+c=1$ then:

$$a^2+b^2+c^2+2\sqrt{2abc}\le 1$$

Proof. Denote $a=xy;b=yz;c=zx$

$$a+b+c=1\Leftrightarrow xy+yz+zx=1$$

For $x=\tan\frac{A}{2};y=\tan\frac{B}{2};z=\tan\frac{C}{2};A,B,C\in\left(0,\frac{\pi}{2}\right)$

Inequality can be written:

$$x^2y^2+y^2z^2+z^2x^2+2\sqrt{3}xyz\le 1\Leftrightarrow$$

$$(xy+yz+zx)^2-2xyz(x+y+z)+2\sqrt{3}+xyz\le 1$$

$$1-2xyz(x+y+z)+2\sqrt{3}xyz\le 1\Leftrightarrow x+y+z\ge\sqrt{3}$$

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\ge\sqrt{3}$$

Problem 51. If $x,y,z>1;\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$ then:

$$\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\le\sqrt{x+y+z}$$

Proof. Denote $x=a+1;y=b+1;z=c+1;a,b,c>0$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2\Leftrightarrow ab+bc+ca+2abc=1$$

Inequality can be written:

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\le\sqrt{a+b+c}$$

For $ab=\sin^2\frac{A}{2};bc=\sin^2\frac{B}{2};ca=\sin^2\frac{C}{2};A,B,C\in\left(0,\frac{\pi}{2}\right)$

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=1$$

By squaring inequality to prove becomes:

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\le\frac{3}{2}\Leftrightarrow\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\frac{3}{2}$$

Proposed Problems

52. If $a,b\in\mathbb{R}$; $15a+12b+7=13$ then:

$$a^2+b^2+2(b-a)\ge -1$$

$$(Use: a-1=R\sin p\,; b+1=R\cos p)$$

53. If $a,b\in\mathbb{R}$; $|a|\ge 1$; $|b|\ge 1$ then:

$$\left|\frac{\sqrt{a^2-1}+\sqrt{b^2-1}}{ab}\right|\le 1$$

$$\left(Use: a=\frac{1}{\cos p}; b=\frac{1}{\cos q}; p,q\in\left[0,\frac{\pi}{2}\right]\cup\left[\pi,\frac{3\pi}{2}\right]\right)$$

54. If $|x|\ge 1$; $|y|\ge 1$ then:

$$y\sqrt{x^2-1}+4\sqrt{y^2-1}+3\le xy\sqrt{26}$$

$$\left(Use: x=\frac{1}{\cos p}; y=\frac{1}{\cos q}; p,q\in\left(0,\frac{\pi}{2}\right)\right)$$

55. If $x,y,u,v\in\mathbb{R}$; $x^2+y^2=u^2+v^2=1$ then:

a. $|xu+yv|\le 1$

b. $|xv+yu|\le 1$

c. $-2\le(x-y)(u+v)+(x+y)(u-v)\le 2$

d. $-2\le(x+y)(u+v)-(x-y)(u-v)\le 2$

$$(Use: x=\cos a\,; y=\sin a\,; u=\cos b\,; v=\sin b\,; a,b\in(0,2\pi))$$

56. If $a,b\in\mathbb{R}$ then:

a. $(a+b)^4\le 8(a^4+b^4)$

b. $(a+b)^6\le 32(a^6+b^6)$

$$\left(Use: \tan x=\frac{b}{a}; x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right)$$

57. If $x,y\in\mathbb{R}$; $xy\ne 0$ then:

$$-2\sqrt{2}-2\le\frac{x^2-(x-4y)^2}{x^2+4y^2}\le 2\sqrt{2}-2$$

$$\left(Use: x=2y\tan p\,; p\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right)$$

58. If $x,y\in\mathbb{R}$; $36x^2+16y^2=9$ then:

$$\frac{15}{4}\le y-2x+5\le\frac{25}{4}$$

$$\left(Use: x=\frac{1}{2}\cos p\,; y=\frac{3}{4}\sin p\,; p\in[0,2\pi]\right)$$

59. If $x,y\in\mathbb{R}$; $3x+4y=5$ then $x^2+y^2\ge 1$

$$\left(Use: \sin p=\frac{3}{5}; \cos p=\frac{4}{5}\right)$$

60. If $a,b\in\mathbb{R}$; $4a^2+9b^2=25$ then:

$$|6a+12b|\le 25$$

$$\left(Use: \frac{2}{5}a=\sin p\,; \frac{3}{5}b=\cos p\,; p\in[0,2\pi]\right)$$

61. If $x,y,a,b,c>0$; $ax+by=0$; $a^2+b^2=c^2$ then: $x^2+y^2\ge 1$

$$\left(Use: \frac{a}{c}=\cos p\,; \frac{b}{c}=\sin p\,; p\in[0,2\pi)\right)$$

62. If $a > b > c > 0$ then:

$$\sqrt{c(a-c)}+\sqrt{c(b-c)}\le\sqrt{ab}$$

$$\left(Use: \sqrt{\frac{c}{a}}=\sin p;\ \sqrt{\frac{a-c}{a}}=\cos p;\ \sqrt{\frac{c}{b}}=\sin v;\ \sqrt{\frac{b-c}{b}}=\cos v;\ u,v\in\left[0,\frac{\pi}{2}\right]\right)$$

NEW GENERALIZATIONS OF INEQUALITIES IN TRIANGLE

By D.M.Bătineţu-Giurgiu, Claudia Nănuţi, Daniel Sitaru-Romania

Let be $\boldsymbol{m\ge 1; n,p\ge 0; n+p,x,y,z>0}$ and the triangle $\boldsymbol{ABC}$ with $\boldsymbol{F}$ –area, then:

$$\frac{y+z}{x}(nb+pc)^m+\frac{z+x}{y}(nc+pa)^m+\frac{x+y}{z}(na+pb)^m$$

$$\ge 2^{m+1}\cdot(n+p)^m\cdot\left(\sqrt[4]{3}\right)^{4-m}\cdot F^{\frac{m}{2}};\quad (*)$$

Proof. We have:

$$\sum_{cyc}\frac{y+z}{x}(nb+pc)^m=\sum_{cyc}\frac{v^m+w^m}{u^m}(nb+pc)^m\ge\frac{1}{2^{m-1}}\cdot\sum_{cyc}\left(\frac{v+w}{u}(nb+pc)\right)^m\ge$$

$$\overset{Radon}{\ge}\frac{1}{2^{m1}}\cdot\frac{1}{3^{m-1}}\left(\sum_{cyc}\frac{v+w}{u}(nb+pc)\right)^m=\frac{1}{6^m}\left(\sum_{cyc}\frac{v+w}{u}(nb+pc)\right)^m;(1)$$

where $x=u^m, y=v^m, z=w^m$.

But,

$$\sum_{cyc}\frac{v+w}{u}(nb+pc)=n\sum_{cyc}\frac{v+w}{u}\cdot b+p\sum_{cyc}\frac{v+w}{u}\cdot c=(n+p)\sum_{cyc}\frac{v+w}{u}\cdot a\ge$$

$$\ge(n+p)\sum_{cyc}\frac{2\sqrt{vw}}{u}\cdot a=2(n+p)\sum_{cyc}\frac{\sqrt{vw}}{u}\cdot a\ge 2(n+p)\cdot 3\cdot\sqrt[3]{\prod_{cyc}\frac{\sqrt{vw}}{u}\cdot a}=$$

$$=6(n+p)\sqrt[3]{abc}=6(n+p)\cdot\sqrt[3]{4RF}=6(n+p)\cdot\sqrt[3]{8\cdot\frac{R}{2}\cdot F}=12(n+p)\cdot\sqrt[3]{\sqrt{\frac{R}{2}}\cdot\sqrt{\frac{R}{2}}\cdot F}$$

$$\overset{Euler}{\ge}12(n+p)\cdot\sqrt[3]{\sqrt{r}\cdot\sqrt{\frac{R}{2}}\cdot F}\overset{Mitrinovic}{\ge}12(n+p)\cdot\sqrt[3]{\sqrt{r}\cdot\sqrt{\frac{s}{3\sqrt{3}}}\cdot F}=$$

$$=12(n+p)\cdot\sqrt[3]{\left(\frac{1}{\sqrt[4]{3}}\right)^3\cdot\sqrt{rs}\cdot F}=12(n+p)\cdot\frac{1}{\sqrt[4]{3}}\cdot\sqrt[3]{F\sqrt{F}}=\frac{12(n+p)\sqrt{F}}{\sqrt[4]{3}}=$$

$$= 4\cdot\frac{(\sqrt[4]{3})^4(n+p)\sqrt{F}}{\sqrt[4]{3}} = 4(n+p)(\sqrt[4]{3})^3\sqrt{F} = 4(n+p)\sqrt{3}\cdot\sqrt[4]{3}\cdot\sqrt{F};\ \ (2)$$

From (1),(2) we get:

$$\sum_{cyc}\frac{y+z}{x}(nb+pc)^m \ge \frac{1}{6^{m-1}}\cdot 2^{2m}\cdot(n+p)^m\cdot(\sqrt{3})^m\cdot(\sqrt[4]{3})^m\cdot F^{\frac{m}{2}} =$$

$$= \frac{2^{2m}}{2^{m-1}}\cdot(n+p)^m\cdot(\sqrt[4]{3})^{3m-4m+4}\cdot F^{\frac{m}{2}} = 2^{m+1}\cdot(n+p)^m\cdot(\sqrt[4]{3})^{4-m}\cdot F^{\frac{m}{2}}$$

q.e.d.

If $m = 1$ we get

$$\sum_{cyc}\frac{y+z}{x}(nb+pc) \ge 4(n+p)\cdot 3^{\frac{1}{4}(4-1)}\cdot\sqrt{F} = 4(n+p)(\sqrt[4]{3})^3\cdot\sqrt{F}$$

$$= 4(n+p)\cdot\sqrt{3}\cdot\sqrt[4]{3}\cdot\sqrt{F};\ \ (3)$$

which for $n = p = 1$ becomes

$$\boldsymbol{\sum_{cyc}\frac{y+z}{x}\cdot(b+c) \ge 8\sqrt{3}\cdot\sqrt[4]{3}\cdot\sqrt{F};\ \ (4)}$$

If $n = 1, p = 0$ we get

$$\boldsymbol{\sum_{cyc}\frac{y+z}{x}\cdot a \ge 4\sqrt{3}\cdot\sqrt[4]{3}\cdot\sqrt{F};\ \ (5)}$$

If $m = 2$ then the inequality $(*)$ becomes as

$$\boldsymbol{\sum_{cyc}\frac{y+z}{x}(nb+pc)^2 \ge 8\cdot(n+p)^2\cdot 3^{\frac{1}{4}(4-m)}\cdot F^{\frac{m}{2}} = 8\cdot(n+p)^2\cdot\sqrt{3}\cdot F^{\frac{m}{2}};\ (6)}$$

which for $n = p = 1$ becomes

$$\boldsymbol{\sum_{cyc}\frac{y+z}{x}(b+c)^2 \ge 32\sqrt{3}\cdot F;\ \ (7)}$$

and for $n = 1, p = 0$ we get first Bătineţu-Giurgiu's inequality

$$\boldsymbol{\sum_{cyc}\frac{y+z}{x}\cdot b^2 \ge 8\sqrt{3}\cdot F;\ \ (B-G,1)}$$

If $m = 3$ the inequality $(*)$ becomes

$$\boldsymbol{\sum_{cyc}\frac{y+z}{x}(nb+pc)^3 \ge 2^4\cdot(n+p)^3\cdot\sqrt[4]{3}\cdot F^{\frac{3}{2}};\ (8)}$$

which for $n = p = 1$ becomes as

$$\sum_{cyc}\frac{y+z}{x}(b+c)^3 \geq 2^7\cdot\sqrt[4]{3}\cdot F\sqrt{F} = 128\cdot\sqrt[4]{3}\cdot F\sqrt{F};\ \ (9)$$

and for $n = 1, p = 0$ we get

$$\sum_{cyc}\frac{y+z}{x}\cdot a^3 \geq 16\cdot\sqrt[4]{3}\cdot F\sqrt{F};\ \ (10)$$

If $m = 4$ the inequality $(*)$ becomes

$$\sum_{cyc}\frac{y+z}{x}(nb+pc)^4 \geq 2^5\cdot(n+p)^4\cdot F^2;\ (11)$$

which for $n = p = 1$ becomes

$$\sum_{cyc}\frac{y+z}{x}(b+c)^4 \geq 2^9\cdot F^2 = 512\cdot F^2;\ (11)$$

and for $n = 1, p = 0$ we get second Bătineţu-Giurgiu's inequality

$$\sum_{cyc}\frac{y+z}{x}\cdot b^4 \geq 2^5\cdot F^2 = 32\cdot F^2;\ \ (B-G,2)$$

Refference:

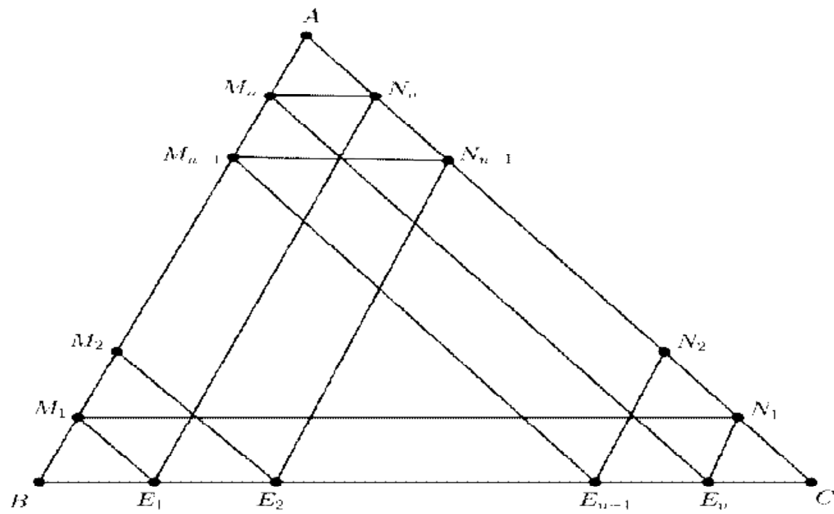
ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

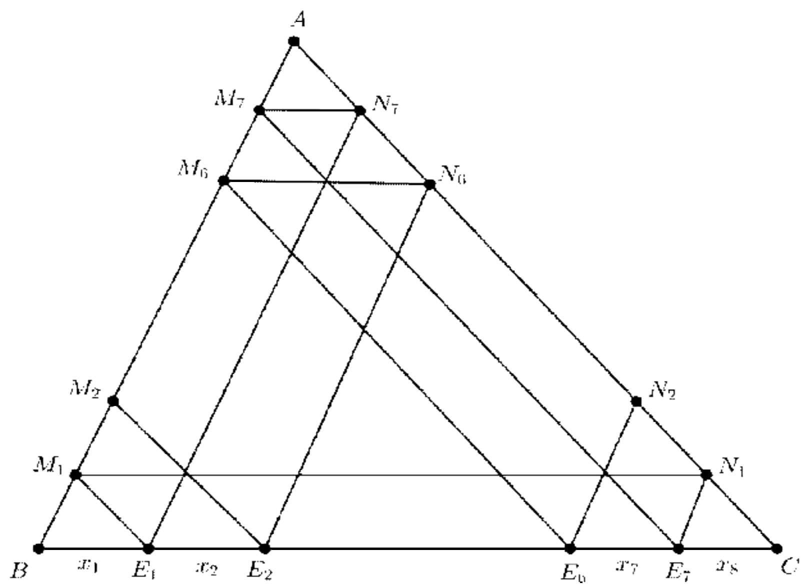
A COUNTING PROBLEM IN TRIANGLE

By Carmen-Victoriţa Chirfot –Romania

Let be a triangle ABC. Consider the points $M_1, M_2, \dots, M_n$ on the side (AB) and $N_1, N_2, \dots, N_3$ on the side (AC), $n \in \mathbb{N}^*$ such that $BM_1 = M_1M_2 = M_2M_3 = \cdots = M_{n-1}M_n = M_nA$ and $CN_1 = N_1N_2 = \cdots = N_{n-1}N_n = N_nA$. We also consider the points $E_1, E_2, \dots, E_n$ on the side (BC) such that $BE_1 = E_1E_2 = E_2E_3 = \cdots = E_{n-1}E_n = M_nC$.

Obviously, from the reciprocal of Thales' theorem, $M_kN_k \parallel BC, k = \overline{1,n}$, $E_kN_{n+1-k} \parallel AB, k = \overline{1,n}, E_kM_k \parallel AC, k = \overline{1,n}$. We join each point M_k with E_k, and each point M_k with N_k, and each point E_k with its corespondent N_{n+1-k}.





We intend, for a start, to determine the number of such triangles formed by the intersections of the given segments.

All quadrilaterals resulting at the intersection of parallel lines M_kN_k and $M_{k+1}N_{k+1}, k = \overline{1,n-1}$ with parallel lines E_kN_{n+1-k} and $E_{k+1}N_{n-k}, k = \overline{1,n-1}$, are parallelograms. Analogous to M_kN_k and $M_{k+1}N_{k+1}, k = \overline{1,n}$ with $E_kM_k \parallel E_{k+1}M_{k+1}$,

$k = \overline{1,n-1}$. Let be $(x_n)_{n\geq 1}$ the sequence with x_k representing the number of triangles inside the triangle BE_kM_k. Then $x_1 = 1, x_2 = x_1 + 2\cdot 1 + 1 + 1 = 5$. For the triangle BE_3M_3, the number of interior triangles is how many x_2 (as we have already found) to which we add two triangles similar to the vertex in M_3 and the bases on M_2N_2,

respectively on M_1N_1, 2 triangles similar to the vertex in E_3 and the bases on E_2N_{n-1}, respectively on E_1N_n and count once the large triangle BE_3M_3. So there are 3 innumerable triangles between the parallel lines E_2M_2 and E_3M_3. Thus, $x_3 = x_2 + 2\cdot 2 + 1 + 3 = 13$. For the triangle BE_4M_4, the number of interior triangles is how many x_3 we still count 3 triangles similar to the vertex in M_4 and the bases on M_3N_3, M_2N_2, respectively M_1N_1, 3 triangles similar to the vertex in E_4 and the bases on E_3N_{n-2},E_2N_{n-1} , respectively on E_1N_n and count once the large triangle BE_4M_4.

A triangle is also formed with the base on E_2M_2 and the tip on E_4M_4. We also have a triangle with the base on E_4M_4 and the tip on E_2M_2 (let's call it the opposite of the previous triangle). There are 5 countless triangles between the parallel lines E_3M_3 and E_4M_4. Thus, $x_4 = x_3 + 2\cdot 3 + 1 + (1) + (1) + 5 = 27$.

For the triangle BE_5M_5, the number of interior triangles is how much x_4 we count 4 triangles similar to the vertex in M_5 and the bases on M_4N_4, M_3N_3, M_2N_2 respectively M_1N_1, 4 triangles similar to the vertex in E_5 and the bases on $E_4N_{n-3}, E_3N_{n-2}, E_2N_{n-1}$ respectively on E_1N_n and we count the big triangle only once BE_5M_5. It also forms 2 triangles with base on $E_3M_3$and E_5M_5 tip on. We also have 2 triangles with base on E_5M_5 and the tip on E_3M_3 and a triangle with base on E_5M_5 and the E_2M_2 tip on. There are 7 countless triangles between the parallel lines E_4M_4 and E_5M_5. Thus, $x_5 = x_4 + 2\cdot 4 + 1 + (2 + 0) + (2 + 1) + 7 = 48$.

For the triangle BE_6M_6, the number of interior triangles is how much x_5 we count 5 triangles similar to the vertex in M_6 and the bases on M_5N_5, M_4N_4, M_3N_3 respectively M_1N_1, 5 triangles similar to the vertex in E_6 and the bases on $E_5N_{n-4}, E_4N_{n-3}, E_3N_{n-2}, E_2N_{n-1}$ respectively on E_1N_n and we count the big triangle only once BE_6M_6. It also forms 3 triangles with base on E_4M_4 and E_6M_6 tip on. We also have 2 triangles with base on E_6M_6 and the tip on E_3M_3 and a triangle with base on E_6M_6 and the E_2M_2 tip on. There are 9 countless triangles between the parallel lines E_5M_5 and E_6M_6. Thus, $x_6 = x_5 + 2\cdot 5 + 1 + (3 + 1) + (3 + 2 + 1) + 9 = 78$

For the triangle BE_7M_7, the number of interior triangles is how much x_6 we count 6 triangles similar to the vertex in M_7 and the bases on $M_6N_6, M_5N_5, M_4N_4, M_3N_3, M_2N_2$ respectively M_1N_1, 6 triangles similar to the vertex in E_7 and the bases on $E_6N_{n-5}, E_5N_{n-4}, E_4N_{n-3}, E_3N_{n-2}, E_2N_{n-1}$ respectively on E_1N_n and

we count the big triangle only once BE_7M_7. It also forms 2 triangles with base on E_4M_4 and E_7M_7 tip on. We also have 4 triangles with base on E_7M_7 and the tip on E_4M_4 and a triangle with base on E_7M_7 and the E_3M_3 tip on, and a triangle with base on E_7M_7 and the E_2M_2 tip on. There are 11 countless triangles between the parallel lines E_6M_6 and E_7M_7. Thus, $x_7 = x_6 + 2\cdot 6 + 1 + (4+2) + (4+3+2+1) + 11 = 118$.

Respecting the algorithm, we obtain that:

$$x_8 = x_7 + 2\cdot 7 + 1 + (5+3+1) + (5+4+3+2+1) + 13 = 170$$

We observe that, if $n = 2k, k\in\mathbb{N}^*, k\geq 2$, we get:

$$x_{2k} = x_{2k-1} + 2(2k-1) + 1 + \big((2k-3) + (2k-5) + \cdots + 1\big) + \big((2k-3) + (2k-4) + \cdots + 1\big) + 4k - 3 = x_{2k-1} + 3k^2 + k.$$

We observe that, if $n = 2k+1, k\in\mathbb{N}^*, k\geq 2$, we get:

$$x_{2k+1} = x_{2k} + 2(2k+1) + \big((2k-2) + (2k-4) + \cdots + 2\big) + \big((2k-2) + (2k-3) + (2k-4) + \cdots + 1\big) + 4k - 1 = x_{2k} + 3k^2 + 4k + 1.$$

Let's come back to the term x_n, we have: $x_1 = 1, x_2 = 5$,

$$x_n = \begin{cases} x_{n-1} + \dfrac{3n^2+2n}{4}, if\ n\in\mathbb{N}^* - even, n\geq 4 \\ x_{n-1} + \dfrac{3n^2+2n-1}{4}, if\ n\in\mathbb{N}^* - odd, n\geq 3 \end{cases}$$

So, $x_n = \begin{cases} 5 + \frac{3(3^2+4^2+\cdots+n^2)+2(3+4+5+\cdots+n)}{4} - \frac{n-2}{8}, if\ n - odd \\ 5 + \frac{3(3^2+4^2+\cdots+n^2)+2(3+4+5+\cdots+n)}{4} - \frac{n-1}{8}, if\ n - even \end{cases}$

Hence, $x_n = \begin{cases} \frac{2n^3+5n^2+2n}{8}, n\geq 4, n - odd \\ \frac{2n^3+5n^2+2n-1}{8}, n\geq 3, if\ n - even \end{cases}$

Consider $A = M_{n+1}$ and $C = E_{n+1}$, then x_{n+1} represent the number of interior triangles of triangle ABC formed according to the given rule.

This is the minimum number of triangles inside the triangle ABC formed by

- n lines parallel to each other $a_1, a_2, \dots, a_n$ and parallel to AB that intersect the sides (AC) and (BC),
- n lines parallel to each other $b_1, b_2, \dots, b_n$ and parallel to AB that intersect the sides (AB) and (AC),

- **n lines parallel to each other $c_1, c_2, \dots, c_n$ and parallel to AB that intersect the sides (AB) and (BC).**

The problem presented above is where any three straight lines a_m, b_n, c_p, and are concurrent, $m, n, p = \overline{1, n}$, i.e. any denote triangle $a_m b_n c_p$ is degenerate.

Refferences:

1. ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

2. https://math.stackexchange.com/questions/203873/how-many-triangles

ABOUT A RMM INEQUALITY-III

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\sum\left(\frac{1}{\tan\frac{A}{2}+\tan\frac{B}{2}}-\frac{1}{\cot\frac{A}{2}+\cot\frac{B}{2}}\right)\geq\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

Solution *We prove the following Lemmas:*

Lemma 1. 2) In ΔABC the following relationship holds:

$$\sum\frac{1}{\tan\frac{A}{2}+\tan\frac{B}{2}}=\frac{s^2+(4R+r)^2}{4Rs}$$

Proof. *Using* $\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{a(s-a)}{rs}$ *we obtain:*

$$\sum\frac{1}{\tan\frac{A}{2}+\tan\frac{B}{2}}=\sum\frac{rs}{a(s-a)}=rs\sum\frac{1}{a(s-a)}=\frac{s^2+(4R+r)^2}{4Rs}$$

which follows from $\sum\frac{1}{a(s-a)}=\frac{s^2+(4R+r)^2}{4Rrs^2}$

Lemma 2. 3) In ΔABC the following relationship holds:

$$\sum\frac{1}{\cot\frac{A}{2}+\cot\frac{B}{2}}=\frac{s^2+r^2+4Rr}{4Rs}$$

Proof. *Using* $\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{a}{r}$ *we obtain* $\sum\frac{1}{\cot\frac{A}{2}+\cot\frac{B}{2}}=\sum\frac{r}{a}=r\sum\frac{1}{a}=\frac{s^2+r^2+4Rr}{4Rs}$*, which follows from the following identity:* $\sum\frac{1}{a}=\frac{s^2+r^2+4Rr}{4Rrs}$*.Let's get back to the main problem.*

Using the above Lemmas the inequality can be written:

$$\frac{s^2+(4R+r)^2}{4Rrs}-\frac{s^2+r^2+4Rr}{4Rs}\geq\sqrt{3}\Leftrightarrow\frac{4R+r}{s}\geq\sqrt{3}\text{ (Doucet's inequality)}$$

Equality holds if and only if the triangle is equilateral.

Remark. *We propose in the same way:*

4) In ΔABC the following identity holds:

$$\sum\left(\frac{1}{1-\tan\frac{A}{2}\tan\frac{B}{2}}+\frac{1}{1-\cot\frac{A}{2}\cot\frac{B}{2}}\right)=3$$

Proposed by Marin Chirciu – Romania

Solution*We prove the following lemmas:*

Lemma 1.

5) In ΔABC the following relationship holds:

$$\sum\frac{1}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{s^2+r^2+4Rr}{4Rs}$$

Proof.*Using* $\tan\frac{B}{2}\tan\frac{C}{2}=\frac{s-a}{s}$ *we obtain* $\sum\frac{1}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\sum\frac{s}{a}=s\sum\frac{1}{a}=\frac{s^2+r^2+4Rr}{4Rr}$,

which follows from the following identity: $\sum\frac{1}{a}=\frac{s^2+r^2+4Rr}{4Rrs}$

Lemma 2.

6) In ΔABC the following relationship holds:

$$\sum\frac{1}{\cot\frac{A}{2}\cot\frac{B}{2}-1}=\frac{s^2+r^2-8Rr}{4Rr}$$

Proof *Using* $\cot\frac{B}{2}\cot\frac{C}{2}=\frac{s}{s-a}$ *we obtain*$\sum\frac{1}{\cot\frac{A}{2}\cot\frac{B}{2}-1}=\sum\frac{s-a}{a}=\frac{s^2+r^2-8Rr}{4Rrs}$ *which follows from the following identity*$\sum\frac{s-a}{a}=\frac{s^2+r^2-8Rr}{4Rs}$*. Let's get back to the main problem.*

Using the Lemmas we obtain:

$$\sum\left(\frac{1}{1-\tan\frac{A}{2}\tan\frac{B}{2}}+\frac{1}{1-\cot\frac{A}{2}\cot\frac{B}{2}}\right)=\frac{s^2+r^2+4Rr}{4Rr}-\frac{s^2+r^2-8Rr}{4Rr}=\frac{12Rr}{4Rr}=3$$

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

THE SERIES OF AN ODD NUMBER

By Mohammed Bouras-Moroco

Definition: **The series of an odd number x is the set of odd numbers which does not allow division by this number.**

I. **The construction of a series of an odd number a.**

-The number of series when can form: $N=\frac{a+1}{2}$

-Are (n,m) represents the couple in the series, with $a=\frac{n+m}{2}, n, m$ **–are pairs,** $n>m$**.**

The series becomes: $P_1=1\overset{+n}{\Rightarrow}P_2\overset{+m}{\Rightarrow}P_3\overset{+n}{\Rightarrow}P_4\overset{+m}{\Rightarrow}P_5\overset{+n}{\Rightarrow}P_6\overset{+m}{\Rightarrow}P_7$

The series is by form: $(n,m,n,m,\dots.)$ **on** a: $P_1=1, P_2=1+n, P_3=1+n+m$.

The series can be written by form: $P_k=P_{k-1}+P_{k-2}-P_{k-3}$ **then** $\frac{P_n}{a}\notin\mathbb{N}$.

We have: $\begin{cases}P_{2k}(m)=2ak-m+1\\ P_{2k+1}(m)=2ak+1\end{cases}$

Remark.The series says main if and only if $m=0, n=2a$.

Example 1: The series of the number 3: **The principal series:** $1\overset{+6}{\Rightarrow}7\overset{+6}{\Rightarrow}13\overset{+6}{\Rightarrow}19\overset{+6}{\Rightarrow}25\overset{+6}{\Rightarrow}31\overset{+6}{\Rightarrow}37$

We pose: $P_1=1, P_2=7, P_3=13$**.The series it can be written as:** $P_n=P_{n-1}+P_{n-2}-P_{n-3}$ **then** $\frac{P_n}{3}\notin\mathbb{N}$**. The secondary series:** $1\overset{+4}{\Rightarrow}5\overset{+2}{\Rightarrow}7\overset{+4}{\Rightarrow}11\overset{+2}{\Rightarrow}13\overset{+4}{\Rightarrow}17\overset{+2}{\Rightarrow}19$

The series is by form: $(4,2,4,2,\dots)$ **The series it can be written as:** $P_n=P_{n-1}+P_{n-2}-P_{n-3}$ **then** $\frac{P_n}{3}\notin\mathbb{N}$.

Example 2: The series of the number 5: **The principal series:** $1 \overset{+10}{\Longrightarrow} 11 \overset{+10}{\Longrightarrow} 21 \overset{+10}{\Longrightarrow} 31 \overset{+10}{\Longrightarrow} 41 \overset{+10}{\Longrightarrow} 51 \overset{+10}{\Longrightarrow} 61$. **We pose:** $P_1 = 1, P_2 = 11, P_3 = 21$. **The series it can be written as:** $P_n = P_{n-1} + P_{n-2} - P_{n-3}$ **then** $5 \notin \mathbb{N}$.

1. **The secondary series:** $1 \overset{+6}{\Rightarrow} 7 \overset{+6}{\Rightarrow} 13 \overset{+6}{\Rightarrow} 19 \overset{+6}{\Rightarrow} 25 \overset{+6}{\Rightarrow} 31 \overset{+6}{\Rightarrow} 37$

The series is by form: $(6, 4, 6, 4, \dots)$. $P_1 = 1, P_2 = 7, P_3 = 13$.

2. **The series it can be written as:** $P_n = P_{n-1} + P_{n-2} - P_{n-3}$ **then** $\frac{P_n}{5} \notin \mathbb{N}$.
3. **The secondary series 2:** $1 \overset{+8}{\Rightarrow} 9 \overset{+2}{\Rightarrow} 11 \overset{+8}{\Rightarrow} 19 \overset{+2}{\Rightarrow} 21 \overset{+8}{\Rightarrow} 29 \overset{+2}{\Rightarrow} 21$

The series is by form: $(8, 2, 8, 2, \dots)$. **We pose:** $P_1 = 1, P_2 = 9, P_3 = 11$ **then** $\frac{P_n}{5} \notin \mathbb{N}$.

A. GENERALIZATION OF KOUTRAS' THEOREM

B. CHARACTERISTIC LINE (g) OF TRIANGLE

By Thanasis Gakopoulos-Farsala-Greece

A. GENERALIZATION of KOUTRAS' THEOREM

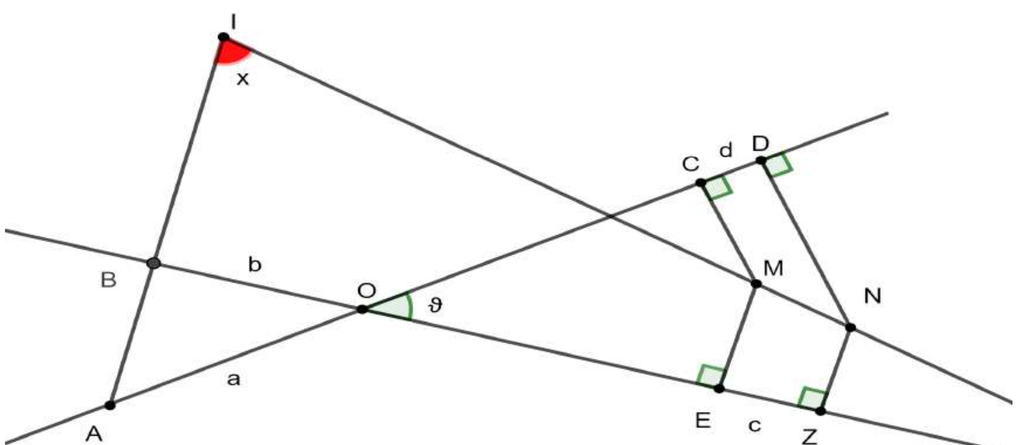


Figure-1

Let: $OA = a, OB = b, EZ = c. CD = d, \frac{OA}{OB} = \frac{a}{b} = m, \frac{\overline{EZ}}{\overline{CD}} = k \cdot \frac{\overline{OA}}{\overline{OB}} \Rightarrow \frac{c}{d} = k \cdot m, k \neq 0$

Then holds:

$$tanx = \frac{k \cdot m^2 - (k+1)m \cdot cos\vartheta + 1}{(k-1)m \cdot sin\vartheta}$$

- **If** $k = 1$, **then** $tanx = \infty \Rightarrow x = 90^\circ$

Stathis Koutras' theorem

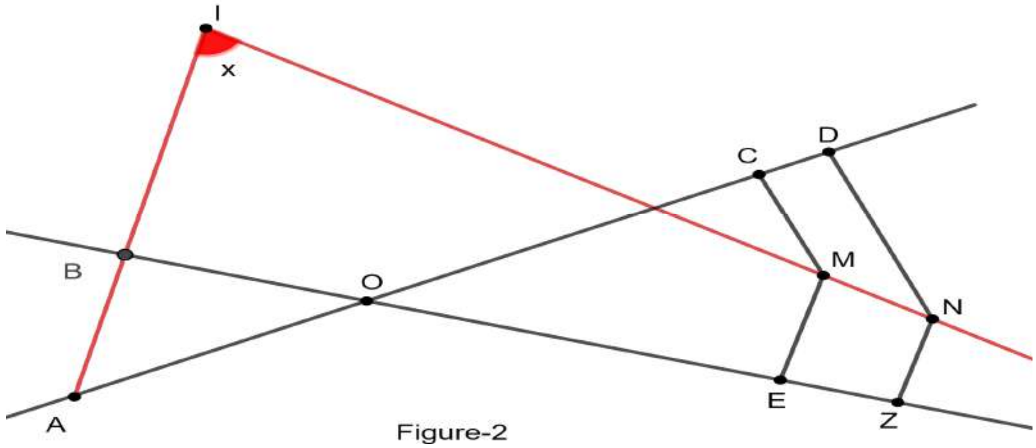


Figure-2

$$\frac{\boldsymbol{EZ}}{\boldsymbol{CD}} = \frac{\boldsymbol{OA}}{\boldsymbol{OB}} \Leftrightarrow \boldsymbol{MN} \perp \boldsymbol{AB}$$

Proof: (on figure 1). PLAGIOGONAL SYSTEM: $OE \equiv Ox, OC \equiv Oy$

Let: $OA = a, OB = b, OE = e, OZ = z, OC = c, OD = d$

$$AB: y = -\frac{a}{b}x - a, \lambda_{AB} = \lambda_1 = -\frac{a}{b};\ \ (E_1)$$

$$EM: y = -\frac{1}{cos\vartheta}x + \frac{e}{cos\vartheta};\ \ (1), CM: y = -cos\vartheta \cdot x + c;\ \ (2)$$

$$ZN: y = -\frac{1}{cos\vartheta} + \frac{z}{cos\vartheta};\ \ (3), DN: y = -cos\vartheta \cdot x + d;\qquad (4)$$

$$(1),(2): M\left(\frac{e - c \cdot cos\vartheta}{sin^2\vartheta}, \frac{c - e \cdot cos\vartheta}{sin^2\vartheta}\right), M(m_1, m_2)$$

$$(3),(4): N\left(\frac{z - d \cdot cos\vartheta}{sin^2\vartheta}, \frac{d - z \cdot cos\vartheta}{sin^2\vartheta}\right), N(n_1, n_2)$$

$$\lambda_{MN} = \lambda_2 = \frac{m_2 - n_2}{m_1 - n_1} \Rightarrow \lambda_2 = \frac{d - c \cdot cos\vartheta}{c - d \cdot cos\vartheta};\ \ (E_2)$$

$$tanx = \frac{(\lambda_2 - \lambda_1) \cdot sin\vartheta}{(\lambda_2 + \lambda_1) \cdot cos\vartheta + \lambda_2\lambda_1 + 1} \overset{E_1/E_2}{\Longleftrightarrow}$$

$$tanx = \frac{\left(1 + \frac{a}{b} \cdot \frac{c}{d}\right) - \left(\frac{c}{d} + \frac{a}{b}\right) \cdot cos\vartheta}{\left(\frac{c}{d} - \frac{a}{b}\right) \cdot sin\vartheta} = \frac{1 + k \cdot \frac{a^2}{b^2} - \frac{a}{b}(k+1) \cdot cos\vartheta}{\frac{a}{b}(k-1) \cdot sin\vartheta}$$

$$tanx = \frac{k \cdot m^2 - (k+1)m \cdot cos\vartheta + 1}{(k-1)m \cdot sin\vartheta}$$

So,

$$\frac{EZ}{CD} = k \cdot \frac{OA}{OB}, k \in \mathbb{R} - \{0,1\} \Leftrightarrow tanx = \frac{k \cdot m^2 - (k+1)m \cdot cos\vartheta + 1}{(k-1)m \cdot sin\vartheta}$$

B. CHARACTERISTIC LINE (g) of TRIANGLE

Given triangle ABC and circle (ω) with center K passes throw the vertices B, C and intersects the sides AB, AC at points D, E respectively. Let point G is the perpendicular projection of D on the side BC and line (g) is perpendicular bisector to the segment DG. Let circles (ω_1), (ω_2) with centers radom points K_1, K_2 belonging to the line (g) and radius GK_1, GK_2 respectively, intersect AB at points D_1, D_2 and ED at points C_1, C_2 respectively.

If $\sphericalangle ACB = \vartheta, \frac{AD}{DE} = m$, then the tio $\frac{C_1C_2}{D_1D_2} = \frac{c}{d}$ depends only on the parameters ϑ and m.

Holds that $\frac{c}{d} = k \cdot m$, where $k = \frac{m \cdot cos2\vartheta - cos\vartheta}{m(m \cdot cos\vartheta - 1)}$; $k \neq 0, cos\vartheta \neq \frac{1}{m}$

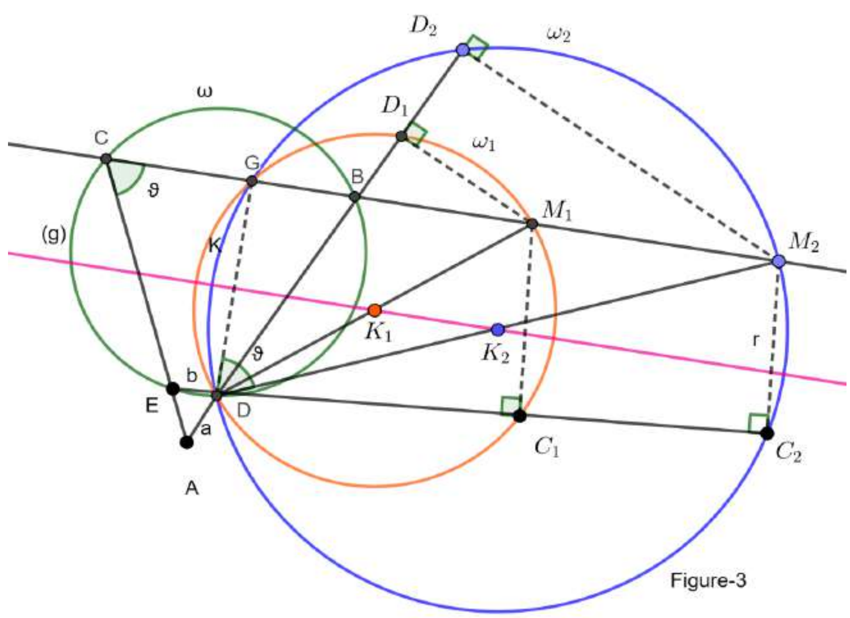


Figure-3

$$\text{Let } AD = a, ED = b, \frac{a}{b} = m, \sphericalangle ACB = x, \sphericalangle BDC_1 = \vartheta$$

$$DK_1 \cap (\omega_1) = M_1, DK_2 \cap (\omega_2) = M_2, \frac{c}{d} = k \cdot m, k \neq 1. \text{ Is: } BCED \text{ —cyclic} \Rightarrow x = \vartheta$$

$$DM_1 \text{ —diameter of } (\omega_1) \Rightarrow \sphericalangle DC_1M_1 = \sphericalangle DD_1M_1 = 90^\circ$$

$$DM_2 \text{ —diameter of } (\omega_2) \Rightarrow \sphericalangle DC_2M_2 = \sphericalangle DD_2M_2 = 90^\circ$$

$$\text{Is: } tanx = \frac{k \cdot m^2 - (k+1)m \cdot cos\vartheta + 1}{(k-1)m \cdot sin\vartheta} \Rightarrow \frac{sin\vartheta}{cos\vartheta} = \frac{k \cdot m^2 - (k+1)m \cdot cos\vartheta + 1}{(k-1)m \cdot sin\vartheta}$$

$$km \cdot sin^2\vartheta - m \cdot sin^2\vartheta = km^2 \cdot cos\vartheta - km \cdot cos^2\vartheta + cos\vartheta$$

$$km\cdot cos^2\vartheta + km\cdot sin^2\vartheta - km^2\cdot cos\vartheta = m\cdot sin^2\vartheta - m\cdot cos^2\vartheta + cos\vartheta$$

$$km - km^2\cdot cos\vartheta = m(sin^2\vartheta - cos^2\vartheta) + cos\vartheta$$

$$km(m\cdot cos\vartheta - 1) = m\cdot cos2\vartheta - cos\vartheta, \qquad k = \frac{m\cdot cos2\vartheta - cos\vartheta}{m(m\cdot cos\vartheta - 1)}$$

$$m\cdot cos\vartheta - 1 \neq 0 \Rightarrow cos\vartheta \neq \frac{1}{m} \Rightarrow cos\vartheta \neq \frac{a}{b}$$

If $\vartheta = 90^\circ \Leftrightarrow k = \frac{m\cdot(-1)-0}{m(m\cdot 0-1)} \Leftrightarrow k = 1$ and $\frac{c}{d} = \frac{a}{b}$

Koutras' theorem

Note: If more circles are written with centers $K_i, i = 1,2,3\ldots, K_i \in (g)$ and points C_i, D_i respectively, then holds that: $\frac{C_iC_{i+j}}{D_iD_{i+j}} = k\cdot\frac{a}{b}, j = 1,2,3,\ldots; i \neq j$

This is the characteristic property of line (g)

Application 1. In the figure 1 it is given that:

$\boldsymbol{\vartheta = 60^\circ, \frac{OA}{OB} = 2, \frac{EZ}{CD} = 3\cdot\frac{OA}{OB}}$. **Find angle $\boldsymbol{x}$.**

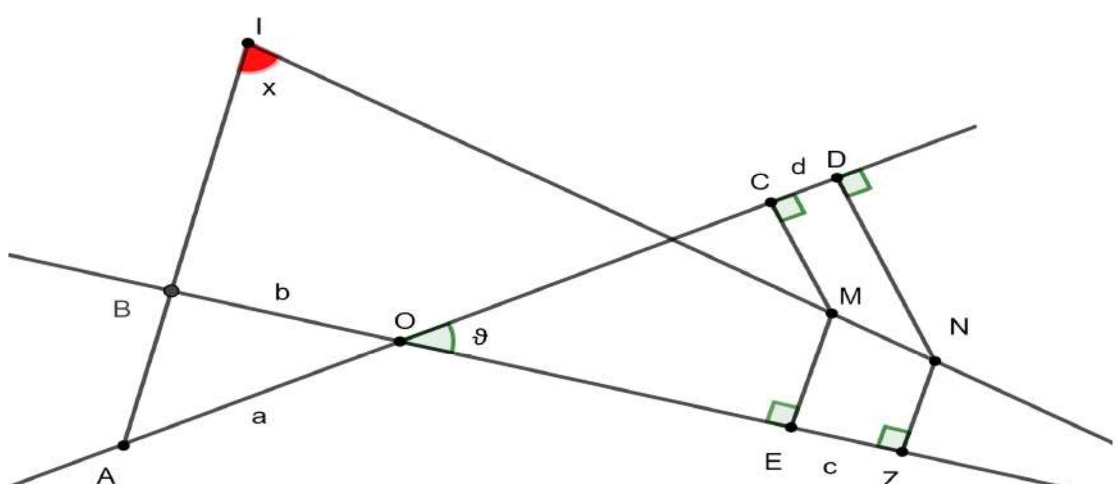


Figure-1

Solution. Let $OA = a, OB = b, EZ = c, CD = d$.Is: $\frac{a}{b} = m = 2, \frac{c}{d} = k\cdot\frac{a}{b} \Rightarrow k = 3$

$$tanx = \frac{k\cdot m^2 - (k+1)m\cdot cos\vartheta + 1}{(k-1)m\cdot sin\vartheta} = \frac{3\cdot 2^2 - (3+1)\cdot 2\cdot cos60^\circ + 1}{(3-1)\cdot 2\cdot sin60^\circ} = \frac{3\sqrt{3}}{2}$$

$$x = tan^{-1}\left(\frac{3\sqrt{3}}{2}\right) \approx 68{,}9\cdot 83^\circ$$

Application 2: In the figure 3 it is given that:

$$\vartheta = 30^\circ, \frac{OA}{OB} = 2, \frac{EZ}{CD} = k \cdot \frac{OA}{OB}, k = \frac{3}{4}\left(1+\sqrt{3}\right).$$ **Find angle x.**

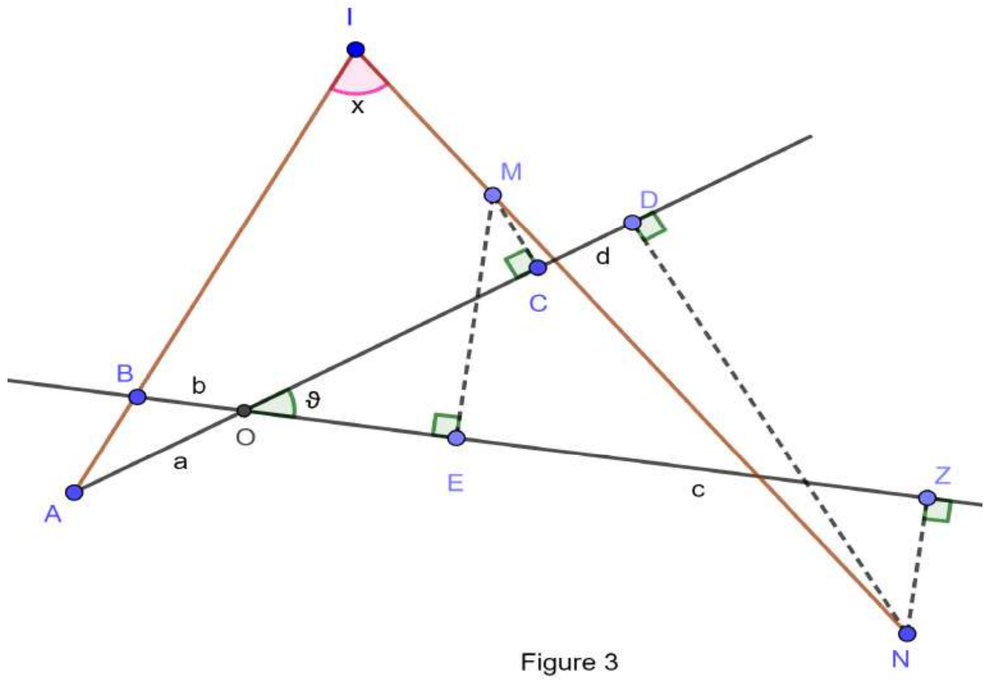


Figure 3

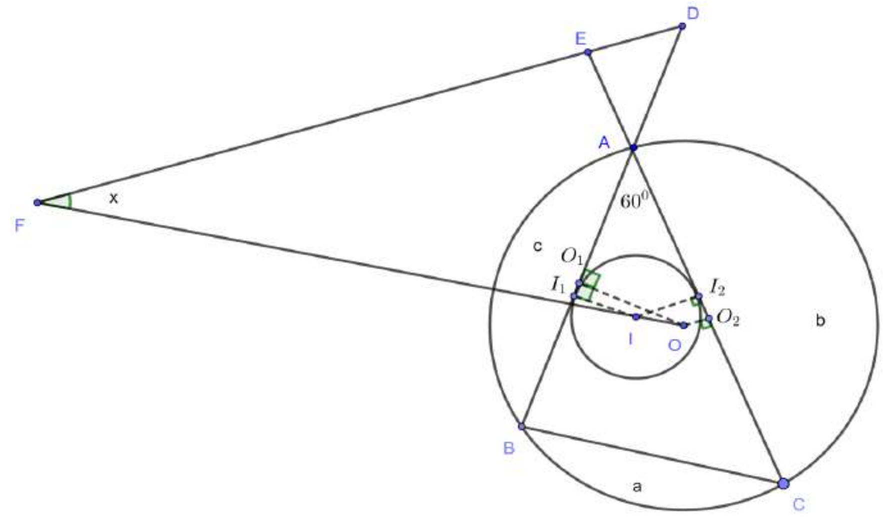
Solution. Let $OA = a, OB = b, EZ = c, CD = d$. Is $\frac{a}{b} = 2 = m, \frac{c}{d} = k \cdot \frac{a}{b} = k \cdot m$

$$tanx = \frac{k \cdot m^2 - (k+1)m \cdot cos\vartheta + 1}{(k-1)m \cdot sin\vartheta} =$$

$$= \frac{\frac{3}{4}(1+\sqrt{3}) \cdot 2^2 - \left[\frac{3}{4}(1+\sqrt{3}) + 1\right] \cdot 2cos30^\circ + 1}{\left[\frac{3}{4}(1+\sqrt{3}) - 1\right] \cdot 2sin30^\circ} = 2 + \sqrt{3} \Rightarrow tan\vartheta = 2 + \sqrt{3} \Rightarrow \vartheta = 75^\circ$$

Application 3. Given triangle ABC with lengths of sides a, b, c and $b > a > c, 2b + c = 3a$, $\sphericalangle BAC = 60^\circ$. Let points D, E on the extensions of the sides BA to point A and CA to point A respectively, such $2BD = 3a - c, CE = 3b - 2a$.

Denote I the incenter and O the circumcenter of ΔABC. DE and OI intersect at point F. Prove that: $\sphericalangle IFE = 30^\circ$



Solution. $2b+c=3a\Rightarrow\frac{b-a}{a-c}=\frac{1}{2},\ AI_1=\frac{-a+b+c}{2},OA_1=\frac{c}{2}\Rightarrow\overline{O_1I_1}=\frac{b-a}{2}$

$$AI_2=\frac{-a+b+c}{2},OA_2=\frac{b}{2}\Rightarrow\overline{O_2I_2}=-\frac{a-c}{2}\Rightarrow\frac{\overline{O_1I_1}}{\overline{O_2I_2}}=-\frac{b-a}{a-c}=-\frac{1}{2}$$

$$\overline{DA}=\overline{DB}-\overline{AB}=\frac{3a-c}{2}-c=\frac{3}{2}(a-c),\overline{EA}=\overline{EC}-\overline{AC}=3b-2a-b=2(b-a)$$

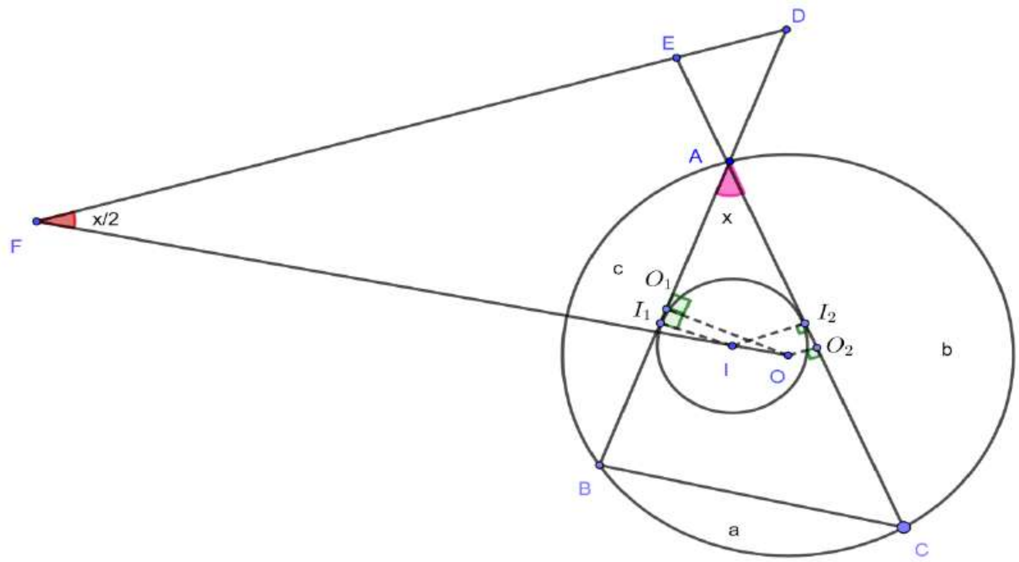
$$\Rightarrow\frac{\overline{AE}}{\overline{AD}}=\frac{4}{3}\cdot\frac{b-a}{a-c}=\frac{4}{3}\cdot\frac{1}{2}=\frac{2}{3}=m,\frac{\overline{O_1I_1}}{\overline{O_2I_2}}=k\cdot m\Rightarrow-\frac{1}{2}=k\cdot\frac{2}{3}\Rightarrow k=-\frac{3}{4}$$

$$tan(180^\circ-x)=-tanx=\frac{km^2-(k+1)m\cdot cos60^\circ}{(k-1)m\cdot sin60^\circ}$$

$$\Rightarrow-tanx=\frac{-\frac{3}{4}\left(\frac{2}{3}\right)^2-\left(1-\frac{3}{4}\right)\cdot\frac{2}{3}\cdot\frac{1}{2}+1}{\left(-\frac{3}{4}-1\right)\cdot\frac{2}{3}\cdot\frac{\sqrt{3}}{2}}=-\frac{\sqrt{3}}{3}\Rightarrow x=30^\circ$$

Application 4.Given triangle ABC with lengths sides a,b,c and $b>a>c,4b+3c=7a$. Let points D,E on the extensions of the sides BA to point A and CA to point A, such $\frac{AE}{AD}=\frac{16(b-a)}{21(a-c)}$.

Denote I –the incenter and O –the circumcenter of ΔABC. DE and OI intersect at point F. If $\sphericalangle BAC=2\cdot\sphericalangle IFC=x$, find the value of x.



Solution. $4b + 3c = 7a \Rightarrow \frac{b-a}{a-c} = \frac{3}{4}$; (1)

$$AI_1 = \frac{-a+b+c}{2}, AO_1 = \frac{c}{2} \Rightarrow \overline{O_1I_1} = \frac{b-a}{2}, AI_2 = \frac{-a+b+c}{2}, OA_2 = \frac{b}{2} \Rightarrow \overline{O_2I_2} = -\frac{a-c}{2}$$

$$\Rightarrow \frac{\overline{O_1I_1}}{\overline{O_2I_2}} = -\frac{b-a}{a-c} = -\frac{3}{4};\ (2) \Rightarrow \frac{\overline{AE}}{\overline{AD}} = \frac{16}{21}\cdot\frac{b-a}{a-c} \overset{(2)}{=} \frac{16}{21}\cdot\frac{3}{4} = \frac{4}{7} = m;\ (3)$$

$$\frac{\overline{O_1I_1}}{\overline{O_2I_2}} = k\cdot m \xrightarrow{(2)/(3)} -\frac{3}{4} = k\cdot\frac{4}{7} \Rightarrow k = -\frac{21}{16};\ (4)$$

$$tan\left(180^\circ - \frac{x}{2}\right) = -tan\frac{x}{2} = \frac{km^2 - (k+1)m\cdot cosx}{(k-1)m\cdot sinx}$$

$$\Rightarrow -tanx = \frac{-\frac{21}{16}\left(\frac{4}{7}\right)^2 - \left(1-\frac{21}{16}\right)\cdot\frac{4}{7}\cdot cosx + 1}{\left(-\frac{21}{16}-1\right)\cdot\frac{4}{7}\cdot sinx} = -\frac{\sqrt{3}}{3} \Rightarrow x = 30^\circ$$

$$\Rightarrow \frac{3}{2}\left(cos^2\frac{x}{2} - sin^2\frac{x}{2}\right) = \frac{3}{4} \Rightarrow cosx = \frac{1}{2} \Rightarrow x = 60^\circ$$

Application 5. Cyclic quadrilateral $CBED$ is given. The extension of side CE to point E and the extension of side BD to point D intersect at point A is $\frac{DA}{DE} = \frac{3}{2}$.

Let G be the vertical projection of the point D on the CB and line (g) is the perpendicular bisector of the segment DG. Random distinct points K_1, K_2 belonging to the line (g) are the centers of circles $(\omega_1), (\omega_2)$ with radius GK_1, GK_2 respectively.

Circles $(\omega_1),(\omega_2)$ intersect the line BD at points D_1,D_2 and the line ED at ponts C_1,C_2 respectively. If $\frac{C_1C_2}{D_1D_2}=S$, find the angle BCE.

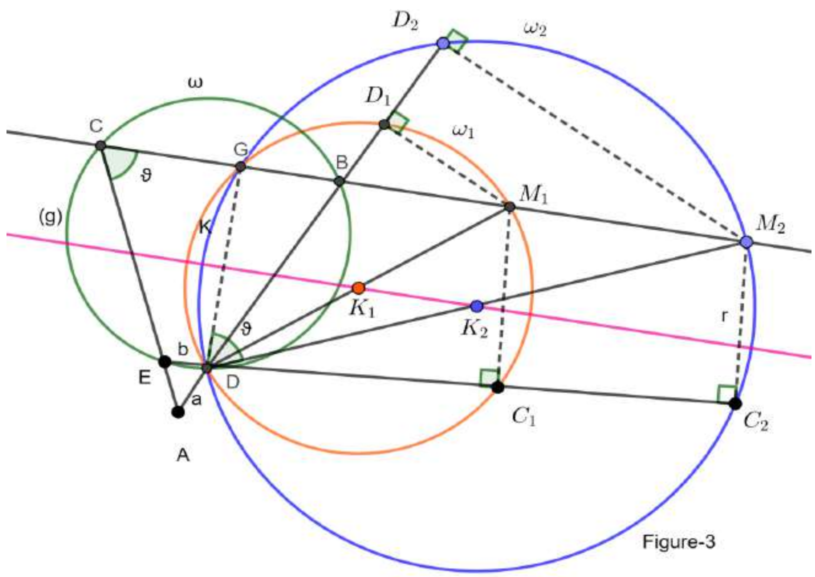


Figure-3

Solution. Let $\sphericalangle BCE=\sphericalangle BDC_1$, let $BC\cap(\omega_1)=M_1, \sphericalangle G=90^\circ \Rightarrow M_1\in OK_1$

Let $BC\cap(\omega_2)=M_2,, \sphericalangle G=90^\circ \Rightarrow M_2\in OK_2$. Is $M_1D_1\perp AD, M_2D_2\perp AD, M_1C_1\perp ED$, $M_2C_2\perp ED$.Let $\frac{DA}{DE}=m=\frac{3}{2}, \frac{C_1C_2}{D_1D_2}=S$ and let $k:\frac{c}{d}=k\cdot m\Rightarrow k=\frac{10}{3}$

Is $k=\frac{m\cdot cos2\vartheta-cos\vartheta}{m(m\cdot cos\vartheta-1)}\xRightarrow{k=\frac{10}{3};m=\frac{3}{2}}6\cdot cos^2\vartheta-17cos\vartheta+7=0\xRightarrow{cos\vartheta<1}cos\vartheta=\frac{1}{2}\Rightarrow\sphericalangle BCE=60^\circ$

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

PROPOSED PROBLEMS

5-CLASS-STANDARD

RMM Romanian Mathematical Magazine

V.1. If $a, b, c, d > 0; \frac{1}{a+1} + \frac{2}{b+1} + \frac{3}{c+1} + \frac{4}{d+1} = 1$ then find:

$$\frac{a}{a+1} + \frac{2b}{b+1} + \frac{3c}{c+1} + \frac{4d}{d+1}$$

Proposed by Daniel Sitaru,Paula Țuinea – Romania

V.2. Find $a, b \in \mathbb{Z}$ such that $ab + 3a - 2b = 7$.

Proposed by Daniel Sitaru,Alina Țigae – Romania

V.3. Find all $\Omega = \overline{abcdef}$ such that:

$$abcdef = abc + def$$

Proposed by Daniel Sitaru,Oana Preda – Romania

V.4. If $\overline{ab} \cdot \overline{cd} = 899$ find $\Omega = \overline{abcd} + \overline{cdab}$.

Proposed by Daniel Sitaru,Camelia Dană – Romania

V.5. Compare the numbers:

$$\Omega_1 = 2018^{2018} + 2019^{2018} \text{ and } \Omega_2 = 2018^{2019} + 2019^{2018}$$

Proposed by Daniel Sitaru,Nineta Oprescu – Romania

V.6. Find Ω_1, Ω_2 natural numbers such that $\Omega_1 + \Omega_2 = 876$ and great common divisor of Ω_1, Ω_2 is 169.

Proposed by Daniel Sitaru,Luiza Cremeneanu – Romania

V.7. If $\Omega_1 = 36 + 36^2 + \cdots + 36^{2018}$

$$\Omega_2 = 25 + 25^2 + \cdots + 25^{2018}$$

then $\Omega_1 - \Omega_2$ is divisible with 11.

Proposed by Daniel Sitaru,Roxana Vasile – Romania

V.8. Find all numbers $\Omega = \overline{2058abc}$ divisible with 343.

Proposed by Daniel Sitaru,Eugenia Turcu – Romania

V.9. Find last two digits of the number:

$$\Omega = 9 + 9^2 + 9^3 + \cdots + 9^{2020}$$

Proposed by Daniel Sitaru,Carina Viespescu – Romania

V.10. Solve for real numbers:

$$\frac{2x-1}{2017} + \frac{2x-2}{2016} + \frac{2x-3}{2015} + \cdots + \frac{2x-10}{2008} = \frac{10x}{1009}$$

Proposed by Daniel Sitaru,Mihai Ionescu – Romania

V.11. Prove that:

$$\Omega = \overline{abcd2} + \left(\overline{abcd2}\right)^2 + \left(\overline{abcd2}\right)^2 + \cdots + \left(\overline{abcd2}\right)^{2020}$$

is divisible with 10.

Proposed by Daniel Sitaru,Marian Voinea – Romania

V.12. If $\Omega = \overline{abbc} - \overline{cbbba}$; $a > c$ then Ω can't be a perfect square.

Proposed by Daniel Sitaru,Delia Popescu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

6-CLASS-STANDARD

VI.1. Find all four digit positive integers, $\overline{MANZU}$ with distinct digits M, A, N, Z, U and holds

$$\begin{cases} \overline{MANZU} - N = \overline{MANZX} = NY^2 \\ XZNA = ZA^2 \end{cases}$$

where $0 \le X, Y \in \mathbb{N} \le 10$ and $Z^2 - Y = M$

Proposed by Naren Bhandari-Nepal

VI.2. Let be $x, y, z, a, b, c > 0$, such that:

$$x + 2y = \frac{a}{x}, y + 2z = \frac{b}{y}, z + 2x = \frac{c}{z}.$$

Compute $x + y + z$ in function of a, b, c.

Proposed by Marin Chirciu – Romania

VI.3. Let be the set $M = \left\{\frac{n+301}{n+10} | n \in \mathbb{N}\right\}$. How many natural numbers does the set M contains?

Proposed by Marin Chirciu – Romania

VI.4. Let be $a = 3k, b = 3k + 1, c = 3k + 2$, where $k \in \mathbb{N}$.

Prove that the number $x = (n + a)(n + b)(n + c)$ is divisible with 3, for any $n \in \mathbb{N}$.

Proposed by Marin Chirciu – Romania

VI.5. Let be $n \in \mathbb{N}^*$. Prove that the number

$$7^{2n} - 7^{2n-1} - 7^{2n-2}$$

can be written as a sum of three nonzero distinct squares.

Proposed by Marin Chirciu – Romania

VI.6. Let be $n \in \mathbb{N}^*$. Prove that the number

$$6^{2n} - 6^{2n-1} - 6^{2n-2}$$

can be written as a sum of three nonzero distinct squares.

Proposed by Marin Chirciu – Romania

VI.7. Let be $n \in \mathbb{N}^*$. Prove that the number

$$5^{2n} - 5^{2n-1} - 5^{2n-2}$$

can be written as a sum of three nonzero distinct squares.

Proposed by Marin Chirciu – Romania

VI.8. If $n \in \mathbb{N}^*$ such that $2n+3$ and $3n+3$ are perfect squares, prove that $5n+9$ is a composed number.

Proposed by Marin Chirciu – Romania

VI.9. Let be $a, b, c \in \mathbb{N}^*$ and $n \in \mathbb{N}$. Prove that the fraction

$$\frac{b^n c^{n+1} + a}{b^{n+1} c^{n+1} + ab - 1}$$

is irreducible.

Proposed by Marin Chirciu – Romania

VI.10. Let be $n \in \mathbb{N}$. Prove that:

$$\frac{1}{6}(7^n + 3^{n+1} + 2) \in \mathbb{N}.$$

Proposed by Marin Chirciu – Romania

VI.11. Let be $n \in \mathbb{N}$. Prove that:

$$\frac{1}{4}(7^n + 47^n - 2) \in \mathbb{N}.$$

Proposed by Marin Chirciu – Romania

VI.12. Let be $n \in \mathbb{N}$. Prove that:

$$\frac{1}{8}(7^n + 75^n - 2) \in \mathbb{N}.$$

Proposed by Marin Chirciu – Romania

VI.13. Let be $n \in \mathbb{N}$. Prove that:

$$\frac{1}{8}(7^n + 83^n - 2) \in \mathbb{N}.$$

Proposed by Marin Chirciu – Romania

VI.14. Let be $n \in \mathbb{N}$. Prove that the number

$$1 + 3 + 3^2 + \cdots + 3^{4n+1}$$

can be divided with 4, but can't be divided with 8.

Proposed by Marin Chirciu – Romania

VI.15. Prove that the number:

$$39^{39} + 38^{34}$$

can be divided with 11.

Proposed by Marin Chirciu – Romania

VI.16. Let be $a, b, c, d \in \mathbb{N}$. Prove that the fraction

$$\frac{2^{3a+1} + 3^{6b+3} + 6}{4^{3c+2} + 5^{6d} + 4}$$

is reducible.

Proposed by Marin Chirciu – Romania

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7-CLASS-STANDARD

VII.1. If $n \in \mathbb{N}$ then Ω is divisible with 191919

$$\Omega = \frac{n^{37} - n}{10}$$

Proposed by Jalil Hajimir-Canada

VII.2. Let $\sigma(n)$ is the divisor function and observe that

$$\sigma(11) = 12, \sigma(6) = 12$$
$$\sigma(17) = 18, \sigma(10) = 18$$

then find all $n \in \mathbb{N}$ such that $\sigma(n) = 158$ where 11,17 are primes.

Proposed by Naren Bhandari-Nepal

VII.3. Find $(x, y, z) \in \mathbb{N}^3$ such that:

$$\begin{cases} \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = \frac{7}{2} \\ x + y + z = 2\gcd(x + y, z) \\ x \leq y \leq z \ \ and\ z\ prime\ number \end{cases}$$

Proposed by Mokhtar Khassani-Algerie

VII.4 Let x, y be positive rational numbers which simultaneous verify the conditions:

i). $2(x - y)^2 + 4y^2 = 4xy$

ii). $\sqrt{\frac{11x+3y}{7x+2y}} \in \mathbb{Q}$.

Compute the value of the rapport: $\frac{2x+3y}{4x+5y}$. ***Proposed by Marin Chirciu – Romania***

VII.5. Find $x \in \mathbb{Z}, x \neq 1$, for which

$$\sqrt{\frac{5x - 9}{x - 1}} \in \mathbb{Z}$$

Proposed by Marin Chirciu – Romania

VII.6. Let $a, b \in \mathbb{N}^*$ and $n \in \mathbb{N}$. Prove that the number

$$P(n) = (a + b)^{4n+1} - a(ab + b^2)^{2n} - b^{4n+1}$$

is divisible with $a(a + 2b)^2$.

Proposed by Marin Chirciu – Romania

VII.7. Prove that the number 13^n can be written as a sum of four nonzero perfect squares, for any $n \in \mathbb{N}^*$.

Proposed by Marin Chirciu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

8-CLASS-STANDARD

VIII.1. If $x \in \mathbb{R}_+^* = (0,\infty), m \in \mathbb{R}_+ = [0,\infty), [x]$ –great integer function, $\{x\} = x - [x]$, then:

$$2^{m+1}([x]\cdot\{x\})^{\frac{m+1}{2}} \le x^{m+1} \le 2^m([x]^{m+1} + \{x\}^{m+1})$$

Proposed by D.M.Bătinețu-Giurgiu – Romania

VIII.2. Find last 3 digits of:

$$\Omega = 2018^{\underbrace{201920192019\ldots201953}_{50\ times\ "2019"}} + 2019^{\underbrace{201820182018..201835}_{50\ times\ "2018"}}$$

Proposed by Naren Bhandari-Bajura-Nepal

VIII.3. If $x, y, z \ge 0$ then:

$$\sum_{cyc}\frac{(x+1)(y+1)}{(x+2)(y+2)} = \frac{3}{4} \Rightarrow \sum_{cyc}\sqrt{(x+1)(y+1)} \ge 3$$

Proposed by Daniel Sitaru,Aurel Chiriță – Romania

VIII.4. If $a, b, c > 0, \frac{1}{a^3+1} + \frac{1}{b^3+1} + \frac{1}{c^3+1} = \frac{8}{3}$ then:

$$(a+b)(b+c)(c+a) \le 1$$

Proposed by Rahim Shahbazov-Azerbaijan

VIII.5. Solve for real numbers:

$$\begin{cases} x^2 = 2 + y \\ y^2 = 2 + z \\ z^2 = 2 + x \end{cases}$$

Proposed by Rahim Shahbazov-Azerbaijan

VIII.6. If $a, b, c > 0, (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{49}{4}$ then:

$$\frac{a}{b},\frac{b}{c},\frac{c}{a}\in\left[\frac{1}{4},4\right]$$

Proposed by Rahim Shahbazov-Azerbaijan

VIII.7. If $a,b,c>0, a+b+c=1, 0\le n\le\frac{9}{7}$ then:

$$ab+bc+ca-nabc\le\frac{9-n}{27}$$

Proposed by Marin Chirciu – Romania

VIII.8. Solve for natural numbers:

$$\begin{cases} 7x=yz(y+z)+6\max(x,y,z) \\ 7y=xz(x+z)+\min(x,y,z) \\ 7z=xy(x+y)+\max(\min(x,y),\min(x,z),\min(y,z)) \end{cases}$$

Proposed by Mokhtar Khassani-Algerie

VIII.9. If $0<x,y,z$ then:

$$x(y^2[x]+z^2\{x\})\ge(y[x]+z\{x\})^2,$$

$\{x\}=x-[x],[*]$ - great integer function

Proposed by Daniel Sitaru,Nicolae Oprea – Romania

VIII.10. If $a,b,c>0, ab+bc+ca=3$ then:

$$a^2+b^2+c^2+abc(a+b+c)\ge 6$$

Proposed by George Apostolopoulos – Greece

VIII.11. Solve:

$$\frac{[x]}{[x]+1}+\frac{8[2x]}{[2x]+8}=\frac{[x][2x]+8[2x]+[x]+8}{[2x]+[x]+9}$$

Proposed by Jalil Hajimir-Canada

VIII.12. If $x,y,z\ge 2$ then:

$$\sum_{cyc}\frac{1}{x+1}=1\Rightarrow\sum_{cyc}\frac{3x^2+x+4}{(x+1)(x^4+2)}+2\le 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Proposed by Daniel Sitaru,Ramona Nălbaru – Romania

VIII.13. If $a,b,t,x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then:

$$\frac{1}{t(at+bx)}+\frac{1}{x(ax+by)}+\frac{1}{y(ay+bz)}+\frac{1}{z(az+bt)}\ge\frac{64}{(a+b)(t+x+y+z)^2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

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9-CLASS-STANDARD

IX.1. If $m \geq 0; x, y, z \geq 0$, then in triangle ABC with F –area the following relationship holds:

$$\sum_{cyc} \left(\frac{y+z}{x}\right)^{m+1} \cdot \frac{a^{2m}}{h_a^2} \geq 2^{3m+1} \cdot \left(\sqrt{3}\right)^{1-m} \cdot F^{m-1}$$

Proposed by D.M.Bătinețu-Giurgiu, Flaviu-Cristian Verde – Romania

IX.2. If in $\Delta ABC, M \in Int(ABC)$ and $x = MA, y = MB, z = MC$ then:

$$\sum_{cyc} \left(\frac{x}{a} + \frac{y}{b}\right) \sqrt{\left(\frac{z}{c} + \frac{x}{a}\right)\left(\frac{z}{c} + \frac{y}{b}\right)} \geq 4$$

Proposed by D.M.Bătinețu-Giurgiu, Flaviu-Cristian Verde – Romania

IX.3. If $x, y, z > 0; u \geq 0$ then in any ΔABC the following relationship holds:

$$\sum_{cyc} \frac{(y+z+m)}{(x+u)h_a} \cdot a^3 \geq 16F,$$

where F –area of triangle ABC.

Proposed by D.M.Bătinețu-Giurgiu, Flaviu-Cristian Verde – Romania

IX.4. In $\Delta ABC, M \in (BC), N \in (CA), P \in (AB)$ the following relationship holds:

$$\left(\frac{AM^3}{h_b + h_c} + \frac{BN^3}{h_c + h_a} + \frac{CP^3}{h_a + h_b}\right)\left(\frac{1}{(h_a + h_b)^2} + \frac{1}{(h_b + h_c)^2} + \frac{1}{(h_c + h_a)^2}\right) \geq \frac{9}{8}$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

IX.5 If $a, b, c, x, y \in \mathbb{R}_+^* = (0, \infty); m \in \mathbb{N}$ and $abc = 1$, then:

$$3m+\frac{(ax+by)^{2m+2}}{(a+b+2c)^{m+1}}+\frac{(ay+bz)^{2m+2}}{(2a+b+c)^{m+1}}+\frac{(az+bx)^{2m+2}}{(a+2b+c)^{m+1}}\ge\frac{3}{4}(m+1)(x+y)^2$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

IX.6 If in $\Delta ABC, M\in Int(ABC), x_A=MA, x_B=MB, x_C=MC$ the following relationship holds:

$$3m+\left(\frac{x_A}{h_a}\right)^{m+1}+\left(\frac{x_B}{h_b}\right)^{m+1}+\left(\frac{x_C}{h_c}\right)^{m+1}\ge 2(m+1)$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

IX.7. Let be $m,n\in\mathbb{R}_+=[0,\infty); p\in\mathbb{N}$ in $\Delta ABC, M\in(ABC), x,y,z-$the distances by point M to the tips A,B,C and $u,v,w-$the distances by point M to the sides of triangle $[BC],[CA],[AB]$. Prove that:

$$3p+\frac{(mx+ny)^{2p+2}}{(uv)^{p+1}}+\frac{(my+nz)^{2p+2}}{(vw)^{p+1}}+\frac{(mz+nx)^{2p+2}}{(wu)^{p+1}}\ge 12(p+1)(m+n)^2$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

IX.8. Let be $x,y,z\in\mathbb{R}_+^*=(0,\infty), t\in\mathbb{R}_+=[0,\infty)$ and the triangle ABC with $F-$area, the following relationship holds:

$$\frac{4x+3y+z+2t}{y+3z+t}\cdot a^2+\frac{x+4y+3z+2t}{z+3x+t}\cdot b^2+\frac{3x+y+4z+2t}{x+3y+t}\cdot c^2\ge 8\sqrt{3}F$$

Proposed by D.M.Bătinețu-Giurgiu – Romania

IX.9. If in $\Delta ABC, F-$area, $M\in Int(ABC)$ and $x=MA, y=MB, z=MC$ the following relationship holds:

$$\frac{x^2\cdot h_a}{a}+\frac{y^2\cdot h_b}{b}+\frac{z^2\cdot h_c}{c}\ge 2F$$

Proposed by D.M.Bătinețu-Giurgiu – Romania

IX.10. In ΔABC the following relationship holds:

$$\frac{m_b}{h_c}+\frac{m_c}{h_b}\ge\frac{2m_a}{h_a}$$

Proposed by Bogdan Fuștei – Romania

IX.11. In $\Delta ABC, I-$inenter the following relationship holds:

$$\sum_{cyc}\frac{m_a}{s_a}\le\min\left(2\sum_{cyc}\frac{m_a}{w_a}-3;\frac{1}{2r}\sum_{cyc}AI\right)$$

Proposed by Bogdan Fuștei – Romania

IX.12. In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$\prod_{cyc}\frac{n_a^2}{r_a}\geq\prod_{cyc}(4m_a-2h_a-r_a)$$

Proposed by Bogdan Fuștei – Romania

IX.13. In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian the following relationship holds:

$$6R\sum_{cyc}\frac{h_bh_c}{h_c}\geq\sum_{cyc}(n_a^2+2w_a^2+g_a^2)$$

Proposed by Bogdan Fuștei – Romania

IX.14. In ΔABC the following relationship holds:

$$\frac{m_a}{a}+\frac{m_b}{b}+\frac{m_c}{c}\geq\frac{3\sqrt{3}}{2}\geq\frac{h_a+h_b}{a+b}+\frac{h_b+h_c}{b+c}+\frac{h_c+h_a}{c+a}$$

Proposed by Bogdan Fuștei – Romania

IX.15 In acute $\Delta ABC, n_a$ – Nagel's cevian the following relaitionship holds:

$$n_an_bn_c\geq s^2\sqrt{\frac{2(m_a-2r)(m_b-2r)(m_c-2r)}{R}}$$

Proposed by Bogdan Fuștei – Romania

IX.16. In $\Delta ABC, n_a$ – Nagel's cevian the follwoing relationship holds:

$$\frac{n_a+m_a+w_b+w_c+\sqrt{2r_ah_a}}{h_a+h_b+h_c}\leq\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)\sqrt{\frac{R}{r}}$$

Proposed by Bogdan Fuștei – Romania

IX.17. In ΔABC the following relationship holds:

$$2\sum_{cyc}\frac{r_ar_b}{w_a^2}\geq\sum_{cyc}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)$$

Proposed by Bogdan Fuștei – Romania

IX.18. In ΔABC the following relationship holds:

$$\frac{r_a}{m_a}+\frac{r_b}{m_b}+\frac{r_c}{m_c}\geq 4-\frac{2r}{R}$$

Proposed by Bogdan Fuștei – Romania

IX.19. If $x,y\in\mathbb{R}, x^2+y^2-6x-8y+24\leq 0$ then:

$$16 \leq x^2 + y^2 \leq 36$$

Proposed by Daniel Sitaru,Ionuț Ivănescu – Romania

IX.20. Solve for real numbers:

$$\begin{cases} 2x + 2y - 3z = -6 \\ 3x^2 + 3y^2 - 4z^2 = -10 \\ 4x^3 + 4y^3 - 5z^3 = -40 \end{cases}$$

Proposed by Radu Diaconu – Romania

IX.21. Solve for real numbers:

$$(2m - 1)\sin 2x + (m - 2)\cos 2x + 2 = 0, m \in \mathbb{R}$$

Proposed by Radu Diaconu – Romania

IX.22. Solve for real numbers:

$$4(\sin x + 2\cos y) + 3(\cos x + 2\sin y) = 15$$

Proposed by Daniel Sitaru,Amelia Curcă Năstăselu– Romania

IX.23. If $a, b > 0$ then:

$$\left(\frac{a+b}{2} + \sqrt{ab} + \frac{2ab}{a+b}\right)^4 \geq \frac{(a+b)^4}{16} + 15a^2b^2 + 65\left(\frac{2ab}{a+b}\right)^4$$

Proposed by Daniel Sitaru,Mirea Mihaela Mioara– Romania

IX.24. Let $\phi(m,n) = \frac{n^{2^m}-1}{2^{m+2}}, m \in \mathbb{N}^*, n$ odd number. Prove that: $\phi(m,n) \in \mathbb{N}$

Proposed by Mohammed Bouras-Morocco

IX.25. Let: $Q = \frac{1+\tan\left(\frac{3\pi}{8}\right)\cdot\tan\left(\frac{\pi}{10}\right)}{1-\tan\left(\frac{\pi}{8}\right)\cdot\tan\left(\frac{\pi}{10}\right)}$

Prove that: $\frac{Q-1}{Q+1} = \sqrt{7 - 3\sqrt{5} - \sqrt{85 - 38\sqrt{5}}}$

Proposed by Mohammed Bouras-Morocco

IX.26. In ΔABC the following relationship holds:

$$\frac{(h_a + h_b + h_c)^3}{h_a h_b h_c} + 5 \geq \frac{16R}{R - r}$$

Proposed by Marin Chirciu – Romania

IX.27. In $\Delta ABC, O$ – circumcentre, I – incentre the following relationship holds:

$$(w_a - w_b)^2 + (w_b - w_c)^2 + (w_c - w_a)^2 \leq n \cdot OI^2, n \geq \frac{35}{2}$$

Proposed by Marin Chirciu – Romania

IX.28. In ΔABC the following relationship holds:

$$\frac{s^2}{27r^2}+\frac{3ns^2}{(4R+r)^2}\geq n+1, n\leq\frac{16}{9}$$

Proposed by Marin Chirciu – Romania

IX.29. If $a,b,c>0, n\geq 1$ then:

$$\frac{a}{na+b+c}+\frac{b}{nb+c+a}+\frac{c}{nc+a+b}\geq\frac{27}{(n+2)(a+b+c)(ab+bc+ca)}$$

Proposed by Marin Chirciu – Romania

IX.30. In ΔABC the following relationship holds:

$$a+b+c\leq\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\leq(a+b+c)\frac{R}{2r}$$

Proposed by Marin Chirciu – Romania

IX.31. In acute ΔABC the following relationship holds:

$$\frac{1}{1-\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{1}{1-\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{1}{1-\tan\frac{A}{2}\tan\frac{B}{2}}\leq\frac{3R^2-8Rr-5r^2}{2R^2-4Rr-2r^2}$$

Proposed by Marin Chirciu – Romania

IX.32. If $a,b,c>0$ then:

$$\sum_{cyc}\frac{c+\sqrt{ab}}{\sqrt{ab}(a+b+2c)}\geq\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$$

Proposed by Daniel Sitaru,Claudiu Ciulcu – Romania

IX.33. If $u,v,w,x,y,z\in\mathbb{R}_+^*$ and ABC is a triangle having the area F, then:

$$\frac{ux+(y+z)(v+w)}{u(y+z)h_a^2}+\frac{uy+(z+x)(w+u)}{v(z+x)h_b^2}+\frac{uz+(x+y)(u+v)}{w(x+y)h_c^2}\geq\frac{5\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

IX.34. Let $x,y,z\in\mathbb{R}_+^*\setminus(0,\infty)$ and d_a,d_b,d_c the centroid's G distances of ABC triangle to it's sides and F the area of triangle.

$$\left(\frac{x}{d_a^2}+\frac{y}{d_b^2}+\frac{z}{d_c^2}\right)^2\geq\frac{81}{p^2}(xy+yz+zx)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

IX.35. If d_a,d_b,d_c are the centroid's G distances of ABC triangle, having the area F, then:

$$\frac{1}{d_a d_b}+\frac{1}{d_b d_c}+\frac{1}{d_c d_a}\geq\frac{9\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania, Martin Lukarevski – Macedonia

IX.36. If $n\in\mathbb{N}$ and m_a, m_b, m_c are the medians lengths of ABC triangle, then:

$$\frac{m_a^{4m+4}}{(m_b\cdot m_c)^{n+1}}+\frac{m_b^{4n+4}}{(m_c m_a)^{n+1}}+\frac{m_c^{4n+4}}{(m_a m_b)^{n+1}}\geq 27r^2-3n$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

IX.37. If $m\in\mathbb{R}_+=[0,\infty); x,y\in\mathbb{R}_+^*=(0,\infty)$ then in any ABC triangle the following inequality holds:

$$\frac{r_a}{(xr_b+yr_c)^{m+1}}+\frac{r_b}{(xr_c+yr_a)^{m+1}}+\frac{r_c}{(xr_a+yr_b)^{m+1}}\geq\frac{3^{m+1}}{(x+y)^{m+1}(yR+r)^m}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

IX.38 If $x,y,z\in\mathbb{R}_+^*=(0,\infty)$, then in any ABC triangle having the area F the following inequality holds:

$$\frac{y+z}{xh_b h_c}+\frac{z+x}{yh_c h_a}+\frac{x+y}{zh_a h_b}\geq\frac{2\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

IX.39. In ΔABC the following relationship holds:

$$\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{4r}{R}\geq 5$$

Proposed by Rahim Shahbazov-Azerbaijan

IX.40. If $x,y,z>0$ then:

$$9\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)^2+\frac{2(x^3+y^3+z^3)}{xyz}\geq 15$$

Proposed by Rahim Shahbazov-Azerbaijan

IX.41. In ΔABC the following relationship holds:

$$8\cos A\cos B\cos C\leq\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2$$

Proposed by Rahim Shahbazov-Azerbaijan

IX.42. If $a,b,c>0, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ then:

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\geq 3$$

Proposed by Rahim Shahbazov-Azerbaijan

IX.43. In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\frac{27r^2}{4s^2} \le \left(\frac{AI}{b+c}\right)^2 + \left(\frac{BI}{c+a}\right)^2 + \left(\frac{CI}{a+b}\right)^2 \le \frac{1}{4}$$

Proposed by Marin Chirciu – Romania

IX.44. In ΔABC the following relationship holds:

$$\frac{27R^2}{4s^2} + \frac{3ns^2}{(4R+r)^2} \ge n+1, n \le \frac{11}{16}$$

Proposed by Marin Chirciu – Romania

IX.45. If $a, b, c > 0, a^2 + b^2 + c^2 + 2abc = 1$ then:

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} \ge 6$$

Proposed by Marin Chirciu – Romania

IX.46. In ΔABC the following relationship holds:

$$\prod_{cyc}\left(\frac{1}{a+b} + \frac{1}{b+c} - \frac{1}{c+a}\right) \le \frac{1}{(a+b)(b+c)(c+a)}$$

Proposed by Daniel Sitaru,Lavinia Trincu– Romania

IX.47. If $x, y, z, t > 0$ then:

$$\frac{(xz-yt)^2 + (xz-yt)(xt+yz+yt) + (xt+yz+yt)^2}{xyzt} \ge 9$$

Proposed by Daniel Sitaru,Mihaela Nascu – Romania

IX.48. In ΔABC the following relationship holds:

$$2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \le 3\sqrt{\frac{3abc}{4Rr + r^2}}$$

Proposed by Daniel Sitaru,Nicolae Tomescu– Romania

IX.49. In ΔABC the following relationship holds:

$$3(a^3 + b^3 + 8m_c^3 + 6abm_c) \le 2(a + b + 2m_c)(3a^2 + 3b^2 - c^2)$$

When the equality does hold?

Proposed by Daniel Sitaru,Seinu Cristina – Romania

IX.50. In ΔABC the following relationship holds:

$$\frac{m_a}{h_c} + \frac{m_c}{h_b} \ge \frac{2m_a}{h_a}$$

Proposed by Bogdan Fuștei – Romania

IX.51. In $\Delta ABC, I$ – incenter the following relationship holds:

$$\sum_{cyc}\frac{m_a}{s_a}\le \min\left(2\sum_{cyc}\frac{m_a}{w_a}-3, \frac{1}{2r}\sum_{cyc}AI\right)$$

Proposed by Bogdan Fuștei – Romania

IX.52. In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$\prod_{cyc}\frac{n_a^2}{r_a}\ge \prod_{cyc}(4m_a-2h_a-r_a)$$

Proposed by Bogdan Fuștei – Romania

IX.53. In $\Delta ABC, n_a$- Nagel's cevian, g_a – Gergonne's cevian the following relationship holds:

$$6R\sum_{cyc}\frac{h_b h_c}{h_c}\ge \sum_{cyc}(n_a^2+2w_a^2+g_a^2)$$

Proposed by Bogdan Fuștei – Romania

IX.54. In ΔABC the following relationship holds:

$$\frac{m_a}{a}+\frac{m_b}{b}+\frac{m_c}{c}\ge \frac{3\sqrt{3}}{2}\ge \frac{h_a+h_b}{a+b}+\frac{h_b+h_c}{b+c}+\frac{h_c+h_a}{c+a}$$

Proposed by Bogdan Fuștei – Romania

IX.55. In acute $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$n_a n_b n_c\ge s^2\sqrt{\frac{2(m_a-2r)(m_b-2r)(m_c-2r)}{R}}$$

Proposed by Bogdan Fuștei – Romania

IX.56. In ΔABC the following relationship holds:

$$(r_a+r_b+r_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_b}\right)\ge \frac{9(a+b)(b+c)(c+a)}{8abc}$$

Proposed by Adil Abdullayev-Azerbaijan

IX.57. In ΔABC the following relationship holds:

$$\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}=\frac{(a-b)(b-c)(c-a)}{16R^2r}$$

Proposed by Adil Abdullayev-Azerbaijan

IX.58. Show that:

$$\frac{\sqrt{4-\sqrt{8+\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}}}}}{\sqrt{2^{\frac{3}{2}}-\sqrt{4+\sqrt{8}+\sqrt{3}+\sqrt{10-2\sqrt{5}}}}}>2\sin 1^\circ$$

Proposed by Naren Bhandari-Nepal

IX.59. We know that

$$1 \times 2 \times 3 \times \dots \times 7 = 7 \times \dots \times 10.$$

Now, for $n > 7$, can the following equality ever hold true:

$$1 \times 2 \times 3 \times \dots \times (n-1) \times n = n \times \dots \times (n+k)$$

for some positive integer k?

Proposed by Naren Bhandari-Nepal

IX.60. If $a, b, c, d > 0, a + b + c + d + 1 = 5abcd$ then:

$$\frac{a^3}{a^3 + b^4 + c^4 + d^4} + \frac{b^3}{b^3 + c^4 + d^4 + a^4} + \frac{c^3}{c^3 + d^4 + a^4 + b^4} + \frac{d^3}{d^3 + a^4 + b^4 + c^4} \le 1$$

Proposed by Rahim Shahbazov-Azerbaijan

IX.61. In ΔABC the following relationship holds:

$$4\cos\frac{A}{2}\cos\frac{B}{2} \le 1 + \sqrt{\left(1 + \frac{a}{c}\right)^2 + \left(1 + \frac{b}{c}\right)^2 - 2\left(1 + \frac{a}{c}\right)\left(1 + \frac{b}{c}\right)\cos C}$$

Proposed by Adil Abdullayev-Azerbaijan

IX.62. In ΔABC the following relationship holds:

$$\prod_{cyc} \frac{r_a + r_b}{2r_c} \ge \frac{9R^2}{a^2 + b^2 + c^2} \ge \frac{4(2R^2 + r^2)}{a^2 + b^2 + c^2}$$

Proposed by Adil Abdullayev-Azerbaijan

IX.63. In ΔABC the following relationship holds:

$$\frac{9(a+b)(b+c)(c+a)}{8abc} \le 1 + \frac{4R}{r}$$

Proposed by Adil Abdullayev-Azerbaijan

IX.64. Find last 3 digits of:

$$\Omega = 2019^{\underbrace{201920192019\dots201953}_{50\ times\ "2019"}}$$

Proposed by Naren Bhandari-Nepal

IX.65. If $a, b, c > 0$ then:

$$\sum_{cyc} \sqrt{\frac{(b+c)^3}{a^3 + abc}} \ge 6$$

Proposed by Rahim Shahbazov-Azerbaijan

IX.66. In ΔABC the following relationship holds:

$$\frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \le \frac{R}{2r}$$

Proposed by Rahim Shahbazov-Azerbaijan

IX.67. In ΔABC the following relationship holds:

$$7s\sum_{cyc} s_a^3 > (2\sqrt{2}+1)\left(\sum_{cyc} s_a^2\right)\left(\sum_{cyc} h_a^2\right)$$

Proposed by Daniel Sitaru,Delia Schneider– Romania

IX.68. In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\frac{n_a + r_a}{AI} + \frac{n_b + r_b}{BI} + \frac{n_c + r_c}{CI} \le \left(\sqrt{3} - \sqrt{\frac{r}{R}}\right)\left(1 + \frac{4R}{r}\right)$$

Proposed by Bogdan Fuștei – Romania

IX.69. In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{r_a}{4m_a - r_a}} \ge \sum_{cyc}\tan\frac{A}{2} \ge \sqrt{4 - \frac{2r}{R}} \ge \sqrt{3}$$

Proposed by Bogdan Fuștei – Romania

IX.70. In $\Delta ABC, g_a$ – Gergonne's cevian the following relationship holds:

$$\sum_{cyc}\frac{r_a + r}{r_a - r} > \sum_{cyc}\frac{g_a - h_a}{w_a - g_a}, \sum_{cyc}\frac{w_a - g_a}{r_a - r} > \sum_{cyc}\frac{g_a - h_a}{r_a + r}$$

Proposed by Bogdan Fuștei – Romania

IX.71. In ΔABC the following relationship holds:

$$\max\left(\sum_{cyc}\frac{m_a - w_a}{h_a}, \sum_{cyc}\frac{m_a - w_a}{r_a}\right) \le \frac{s\sqrt{3} - w_a - w_b - w_c}{r}$$

Proposed by Bogdan Fuștei – Romania

IX.72. In ΔABC the following relationship holds:

$$\frac{\sqrt{m_a r_a}}{w_a} + \frac{\sqrt{m_b r_b}}{w_b} + \frac{\sqrt{m_c r_c}}{w_c} \le 1 + \frac{R}{r}$$

Proposed by Bogdan Fuștei – Romania

IX.73. In ΔABC the following relationship holds:

$$\left(\sum_{cyc}\sqrt{\frac{r_a}{w_a}}\right)^2 \ge 4 + 5\sqrt[5]{\left(\frac{r_a + r_b + r_c}{m_a + m_b + m_c}\right)^6}$$

Proposed by Bogdan Fuștei – Romania

IX.74. In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian the following relationship holds:

$$\frac{2m_a + n_a + g_a}{h_a} + \sqrt{\frac{r_b + r_c}{h_a}} \le \frac{(1+\sqrt{3})R}{r}$$

Proposed by Bogdan Fuștei – Romania

IX.75. In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{n_a}{a} + \frac{n_b}{b} + \frac{n_c}{c} \le \left(\frac{R}{r} + 3\sqrt{3} - 4\right)\left(\frac{R}{r} - 1\right)$$

Proposed by Bogdan Fuștei – Romania

IX.76. Solve for real numbers:

$$4(\sin x + 2\cos y) + 3(\cos x + 2\sin y) = 15$$

Proposed by Daniel Sitaru,Alecu Orlando– Romania

IX.77. Solve for real numbers:

$$\begin{cases} xy(4xy-1)^2 + 16xy = 16z^2 \\ yz(4yz-1)^2 + 16yz = 16x^2 \\ zx(4zx-1)^2 + 16zx = 16y^2 \end{cases}$$

Proposed by Daniel Sitaru,Dan Grigorie – Romania

IX.78. Let be $m, n \in \mathbb{R}_+^* = (0, \infty)$ and M an interior point to ABC triangle. If x, y, z are the distances of point M to the apices A, B, C and u, v, w the distances of point M to the sides BC, CA, AB then:

$$\frac{m^2y^2 + n^2z^2}{v^2 + 2wu} + \frac{m^2z^2 + n^2x^2}{w^2 + 2uv} + \frac{m^2x^2 + n^2y^2}{u^2 + 2vw} \ge 2(m+n)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.79. If $m \in \mathbb{R}_+ = [0, \infty)$; $x, y, z \in \mathbb{R}_+ = (0, \infty)$ then in any ABC triangle the following inequality holds:

$$\left(\frac{x \cdot a^4}{y+z}\right)^{m+1} + \left(\frac{y \cdot b^4}{z+x}\right)^{m+1} + \left(\frac{z \cdot c^4}{x+y}\right)^{m+1} \ge \frac{8^{m+1}}{3^m} \cdot F^{2m+2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.80. If $p \in \mathbb{R}_+ = [0, \infty)$; $m, n, x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then in any ABC triangle the following inequality holds:

$$\left(\frac{my + nz}{x} \cdot a^4\right)^{p+1} + \left(\frac{mz + nx}{y} \cdot b^4\right)^{p+1} + \left(\frac{mx + ny}{z} \cdot c^4\right)^{p+1} \ge$$

$$\geq \frac{2^{6p+6}}{3^p} \cdot \frac{m^{p+1} \cdot n^{p+1}}{(m+n)^{p+1}} \cdot F^{2p+2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.81. If $m \in \mathbb{R}_+ = [0, \infty)$ and a, b, c are the sides lengths of ABC triangle having the area F, then:

$$\frac{a^{m+2}}{(2a+b+c)^m} + \frac{b^{m+2}}{(a+2b+c)^m} + \frac{c^{m+2}}{(a+b+2c)^m} \geq \frac{\sqrt{3}}{4^{m+1}} F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.82. If $m \in \mathbb{N}; n, p \in \mathbb{R}_+^* = (0, \infty), F$ is the area and s the semiperimeter of ABC triangle, then:

$$m + 2^m((ns^2)^{m+1} + (pr)^{m+1} \cdot (4R+r)^{m+1}) \geq (m+1)(3n+p)\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.84. Let $m, n \in \mathbb{N}, n \geq 2$, then in ABC triangle having the semiperimeter s and area F the following inequality holds:

$$3m + a^{m+1} r_b^{n(m+1)} + b^{m+1} \cdot r_c^{n(m+1)} + c^{m+1} r_a^{n(m+1)} \geq 2(m+1)F^2 \cdot s^{n-3} \left(\sqrt{3}\right)^{6-n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.85. If $m \in \mathbb{N}$ and ABC triangle is a triangle having the semiperimeter s, then:

$$\sqrt{\left(\frac{a}{s-a}\right)^{m+1}} + \sqrt{\left(\frac{b}{s-b}\right)^{m+1}} + \sqrt{\left(\frac{c}{s-c}\right)^{m+1}} + 3m \geq 3(m+1)\sqrt{2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.86. Let be $n \in \mathbb{N}, n \geq 2, x_k \in \mathbb{R}_+^* = (0, \infty), \forall k = \overline{1, n}$ and $\sigma \in S_n$, and $a, b \in \mathbb{R}_+^*$. Then:

$$\sum_{k=1}^{n} \left(a + \frac{b \cdot x_k}{x_{\sigma(k)}}\right)^2 \geq (a+b)^2 n$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.87. Let $m \in \mathbb{R}_+ = [0, \infty), n \in \mathbb{N}, n \geq 3, x_k \in \mathbb{R}_+^* = (0, \infty), \forall k = \overline{1, n}$ and $X_n = \sum_{k=1}^{n} x_k$ then:

$$\sum_{k=1}^{n} x_k^{m+1} + \frac{1}{(n-1)^m} \sum_{k=1}^{n} (X_n - x_k)^{m+1} \geq \frac{X_n^{m+1}}{n^{m-1}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.88. If $x, y \in \mathbb{R}_+^* = (0, \infty)$ and a, b, c are the sides lengths and h_a, h_b, h_c are the heights lengths of ABC triangle, then:

$$\frac{(2x-y)xa}{h_a} + \frac{(2y-x)yb}{h_b} + \frac{xyc}{h_c} \geq 2\sqrt{3}xy$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.89. In ABC triangle having the area F let w_a, w_b, w_c be the interior bisectors and the other notations being the usual ones, then:

$$\frac{a \cdot w_a}{h_a} + \frac{b \cdot w_b}{h_b} + \frac{c \cdot w_c}{h_c} \geq 2\sqrt{3\sqrt{3}F}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.90. In ABC triangle having the area F, with the usual notations the following inequality holds: $(a^2+b^2+c^2)^{\frac{3}{2}} \cdot \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq 18F$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.91. Let $m \in \mathbb{N}$ and M be an interior point in ABC triangle and x, y, z the sides of point M to A, B, C apices and u, v, w the distances of point M to the sides BC, CA, AB. Prove that:

$$3m + \frac{x^{2m+2}}{(vw)^{m+1}} + \frac{y^{2m+2}}{(wu)^{m+1}} + \frac{z^{2m+2}}{(uv)^{m+1}} \geq 12(m+1)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

IX.92. If $a, b, c, d, e \in \mathbb{R}_+^* = (0, \infty)$ and $a^2+b^2+c^2+d^2 = e^2$, then:

$$(a+c)(b+d) \leq e^2$$

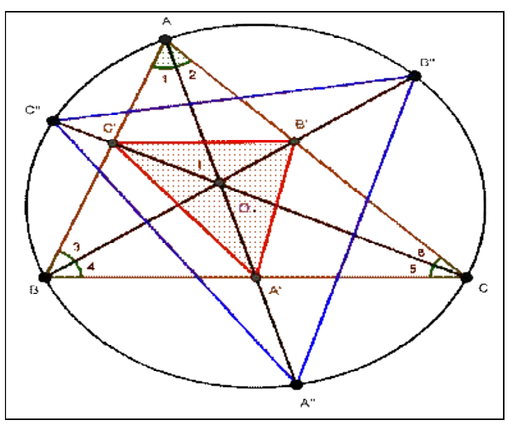
Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți – Romania

IX.93. Find all functions $f: (0, +\infty) \to \mathbb{R}$ such that:

$$f(xy) \leq xf(x) + yf(y) \leq log(xy), \forall x, y > 0$$

Proposed by Marian Ursărescu-Romania

IX.94. In $\Delta ABC, AA', BB', CC'$ —internal bisectors, $\Delta A''B''C''$ —circumcevian triangle of incenter. Prove that: $\frac{[A'B'C']}{[A''B''C'']} \leq \frac{r}{2R}$.



Proposed by Marian Ursărescu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

10-CLASS-STANDARD

X.1. If $x_k \in \mathbb{R}_+^* = (0,\infty), k = \overline{1,n}$, then:

$$\sum_{k=1}^{n}\left(\frac{[x_k]}{\{x_{k+2}\}} + \frac{[x_k]^2}{[x_k]\{x_{k+2}\}}\right) \ge \frac{\frac{1}{2}\left(\sum_{k=1}^{n} x_k\right)^2}{\sqrt{\left(\sum_{k=1}^{n}[x_k]^2\right)\left(\sum_{k=1}^{n}\{x_k\}^2\right)}}$$

where $[x]$ –GIF, $\{x\} = x - [x], x \in \mathbb{R}, x_{n+1} = x_n, x_{n+2} = x_1$.

Proposed by D.M.Bătinețu-Giurgiu– Romania

X.2. If $m,n,x,y,z \in \mathbb{R}_+ = [0,\infty), m + n = 2$, then in any ΔABC with F –area the following relationship holds:

$$yz \cdot \frac{a^m}{h_a^n} + zx \cdot \frac{b^m}{h_b^n} + xy \cdot \frac{c^m}{h_c^n} \le \frac{(x+y+z)^2 R^{m+n}}{(abc)^n}$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

X.3. If $m \in \mathbb{N}, t \in \mathbb{R}_+ = [0,\infty)$ and $x,y,z \in \mathbb{R}_+^* = (0,\infty)$ then in any triangle ABC the following relationship holds:

$$3m + \left(\frac{y+z+6t}{(x+3t)a}\right)^{m+1} + \left(\frac{z+x+6t}{(y+3t)b}\right)^{m+1} + \left(\frac{x+y+6t}{(z+3t)c}\right)^{m+1} \ge \frac{2\sqrt{3}(m+1)}{R}$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

X.4. In any $\Delta ABC, M \in Int(ABC), x = MA, y = MB, z = MC$ the following relationship holds:

$$\sum_{cyc}\left(\frac{x}{a}\right)^4 + \sum_{cyc}\frac{x^3 y}{a^3 b} \ge \frac{2}{3}$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

X.5. Let be $m,n \in \mathbb{R}_+ = [0,\infty); m + n = 4; x,y,z \in \mathbb{R}_+^* = (0,\infty)$ and ΔABC with F –area, then the following relationship holds:

$$\left(\frac{x^2a^m}{h_a^n}+\frac{y^2b^m}{h_b^n}+\frac{z^2c^m}{h_c^n}\right)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\ge 3\cdot 2^{m-2}\cdot F^{m-2}$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

X.6. If $m,n\in\mathbb{R}_+=[0,\infty), m+n=2$, ΔABC with F –area and $M\in(BC), N\in(CA), P\in(AB)$ the following relationship holds:

$$\frac{(a^2+b^2)\cdot CP^m}{c^n}+\frac{(b^2+c^2)\cdot AM^m}{a^n}+\frac{(c^2+a^2)\cdot BN^m}{b^n}\ge 2^{n+1}\cdot 3F^m$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

X.7. If $x,y\in\mathbb{R}_+=[0,\infty), x+y=2$ then in any triangle ABC with F –area the following relationship holds:

$$\frac{(a^2+b^2)\cdot h_c^x}{c^y}+\frac{(b^2+c^2)\cdot h_a^x}{a^y}+\frac{(c^2+a^2)\cdot h_b^x}{b^y}\ge 2^{x+1}\cdot 3F^x$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

X.8. If $m,n\in\mathbb{R}_+=[0,\infty), m+n=2$, then in any ΔABC with F –area the following relationship holds:

$$\frac{(a+b)^2\cdot h_c^m}{c^n}+\frac{(b+c)^2\cdot h_a^m}{a^n}+\frac{(c+a)^m\cdot h_b^m}{b^n}\ge 2^m\cdot 6F^m$$

Proposed by D.M.Bătinețu-Giurgiu– Romania

X.9. In ΔABC the follwoing relationship holds:

$$\sqrt[3]{\prod_{cyc} n_a^2}+2\sqrt[3]{\prod_{cyc} r_a h_a}\le s^2$$

Proposed by Bogdan Fuștei – Romania

X.10. If in $\Delta ABC, I$ – incenter, n_a – Nagel's cevian, g_a – Gergonne's cevian then:

$$\frac{AI}{h_a}+\frac{BI}{h_b}+\frac{CI}{h_c}\le\frac{R}{r},\ \sum_{cyc}\frac{n_a^2+g_a^2}{b^2+c^2}\ge 2+\frac{r}{2R}$$

Proposed by Bogdan Fuștei – Romania

X.11. In ΔABC the following relationship holds:

$$m_a\ge\frac{1}{2\sqrt{2}}\left((b+c)\cos\frac{A}{2}+|b-c|\sin\frac{A}{2}\right)$$

Proposed by Bogdan Fuștei – Romania

X.12. In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$m_a \geq \frac{1}{2}\left(\frac{h_b + h_c}{2} + |b - c| \sin^2 \frac{A}{2}\right) \sqrt{\frac{n_a + h_a}{r_a}}$$

Proposed by Bogdan Fuștei – Romania

X.13 Let $\alpha, \beta > 0$. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$\alpha f(x) f(y) = \beta f(x + y) + \alpha\beta xy, \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

X.14. Solve for complex numbers:

$$3x^6 - 9x^5 + 18x^4 - 21x^3 + 15x^2 - 6x + 1 = 0$$

Proposed by Daniel Sitaru,Lucian Lazăr– Romania

X.15. In ΔABC: $a + b + c = 1$. Prove that:

$$\sum_{cyc} \left(\frac{1}{9} \cdot \mu^2(A) + \frac{2}{3} \cdot \frac{\mu(A)}{\tan\frac{A}{3}} + \frac{ab}{c}\right) > 7$$

Proposed by Radu Diaconu – Romania

X.16. If in $\Delta ABC, r = 1$ then the following relationship holds:

$$\left(\sum_{cyc} \tan\frac{A}{6^n} - \sum_{cyc} \sin\frac{A}{6^n}\right)\left(\sum_{cyc} \frac{1}{w_a}\right) < \frac{1}{6^n}, n \geq 2$$

Proposed by Radu Diaconu – Romania

X.17. In $ABCD$ convexe quadrilateral the following relationship holds ($m > 0, n \geq 0$):

$$\left(\sum_{cyc} \cos^2 A\right)\left(\sum_{cyc} \frac{\mu^{m+1}(A)}{(a + nb)^m}\right) \geq \frac{8\pi^{m+1}}{s^m(n + 1)^m} \cos^2\frac{A + B}{2} \cos^2\frac{B + C}{2} \cos^2\frac{C + A}{2}$$

Proposed by Radu Diaconu – Romania

X.18. In ΔABC the following relationship holds:

$$(w_a + w_b + w_c)\left(\frac{A}{b + c} + \frac{B}{c + a} + \frac{C}{a + b}\right) \geq \frac{27\pi r}{4s}$$

Proposed by Radu Diaconu – Romania

X.19. In ΔABC the following relationship holds:

$$3(1 + 3r) < \sum_{cyc} \left(h_a + \frac{1}{4} \cdot \mu(A) \cdot \csc\frac{A}{4}\right) < \frac{3(\pi + 3R)}{2}$$

Proposed by Radu Diaconu – Romania

X.20. Prove that:

$$10^{2^n}\left(\prod_{k=1}^{n}\frac{1}{10^{2^{n-k}}+1}\right)\left(\sum_{k=1}^{2^{n-1}}10^{-2k}\right)=\frac{1}{11}$$

Proposed by Mohammed Bouras-Morocco

X.21. Solve for real numbers:

$$\begin{cases} 6x+3y+2z=18 \\ 108(x+y+z)^{x+y+z}=xy^2z^3\cdot 6^{x+y+z} \end{cases}$$

Proposed by Daniel Sitaru,Daniela Beldea– Romania

X.22. In ΔABC the following relationship holds:

$$\frac{(m_b+m_c)\sin A}{m_a\sin B\sin C}+\frac{(m_c+m_a)\sin B}{m_b\sin C\sin A}+\frac{(m_a+m_b)\sin C}{m_c\sin A\sin B}\geq 4\sqrt{3}$$

Proposed by Daniel Sitaru,Simona Miu– Romania

X.23.

$\Omega(a,b,c)=\frac{(a+b+c)^3}{(4a^3+1)(4b^3+1)(4c^3+1)}, a,b,c\in\mathbb{R}$

Find: $\Omega=\max(\Omega(a,b,c))$

Proposed by Marin Chirciu – Romania

X.24. If $x,y,z>0, x+y+z=1, n\geq 2$ then:

$$\frac{x}{\sqrt{nx+y}}+\frac{y}{\sqrt{ny+z}}+\frac{z}{\sqrt{nz+x}}\leq\sqrt{\frac{3}{n+1}}$$

Proposed by Marin Chirciu – Romania

X.25. Solve for real numbers:

$$x+\sqrt{a^2-x^2+1}+x\sqrt{a^2-x+1}=2a+1, a\in[0,7)$$

Proposed by Marin Chirciu – Romania

X.26. In ΔABC the following relationship holds:

$$\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}\frac{1}{r_a}\right)+\frac{2\mu r}{R}\geq\mu+9, \mu\leq 8$$

Proposed by Marin Chirciu – Romania

X.27. In ΔABC the following relationship holds:

$$a^3+b^3+c^3\geq 8\sqrt[4]{3S^6}$$

Proposed by Daniel Sitaru,Alina Georgiana Ghiță– Romania

X.28. In $\Delta ABC, K$ – Lemoine's point, the following relationship holds:

$$\frac{aAK + bBK + cCK}{m_a + m_b + m_c} \le \frac{2R\sqrt{3}}{3}$$

Proposed by Daniel Sitaru, Mihaela Dăianu – Romania

X.29. In ΔABC the following relationship holds:

$$\left(\frac{a^4 m_a^2}{m_b m_c}\right)^5 + \left(\frac{b^4 m_b^2}{m_c m_a}\right)^5 + \left(\frac{c^4 m_c^2}{m_a m_b}\right)^5 \ge \frac{(4S)^{10}}{81}$$

Proposed by Daniel Sitaru, Doina Cristina Călina – Romania

X.30. If $x, y, z, u, v, w > 0, uv + vw + wu = 3$ then:

$$\sum_{cyc} \frac{(x^2 + y^2 + z^2 + 2xy + 2zy)u^2}{xz} \ge 18 + u^2 + v^2 + w^2$$

Proposed by Daniel Sitaru, Simona Radu – Romania

X.31. Let be $m \in \mathbb{R}_+ = [0, \infty), n \in \mathbb{N}$ then in ABC triangle with the area F the following inequality holds:

$$3n + \frac{a^{(m+2)(n+1)}}{(b+c)^{m(n+1)}} + \frac{b^{(m+2)(n+1)}}{(c+a)^{m(n+1)}} + \frac{c^{(m+2)(n+1)}}{(a+b)^{m(n+1)}} \ge \frac{(n+1)\sqrt{3}F}{2^{m-2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți – Romania

X.32. In ABC triangle having the area F the following inequality holds:

$$3(a^2 + b^2 + c^2)^2 \ge \sum_{cyc} (a^2 + b^2 - c^2)^2 + 128F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

X.33. If $m \in \mathbb{N}^*, x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and ABC is a triangle having the area F, then:

$$3m + \left(\frac{y+z}{x} \cdot a^2\right)^{m+1} + \left(\frac{z+x}{y} \cdot b^2\right)^{m+1} + \left(\frac{x+y}{z} \cdot c^2\right)^{m+1} \ge 4(3m+2)\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

X.34. If $m, n \in \mathbb{N}^*, x, y, z, u, v \in \mathbb{R}_+^* = (0, \infty)$ and ABC is a triangle having the area F then:

$$(m + (u(y+z) \cdot a)^{m+1})\left(n + \left(\frac{v}{x} \cdot a\right)^{n+1}\right) + \left(m + \left(\frac{v}{y} \cdot b\right)^{m+1}\right) + (n + (u(z+x)b)^{n+1}) +$$

$$+(m+1)v\left(\left(\frac{x+y}{z}\right)c^2 u\right)^{n+1} \ge 8(m+1)(n+1)uv\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

X.35. If $m \in \mathbb{R}_+ = [0,\infty); n \in \mathbb{N}^*; x,y,z \in R_+^* = (0,\infty)$ then in ABC triangle having the area F the following inequality holds:

$$(3n)^{m+1} + \left(\frac{(y+z)a^2}{x}\right)^{(m+1)(n+1)} + \left(\frac{(z+x)b^2}{y}\right)^{(m+1)(n+1)} + \left(\frac{(x+y)c^2}{z}\right)^{(m+1)(n+1)} \ge \frac{(n+1)\sqrt{3}}{2^{2m-3}}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

X.36. If $x,y \in \mathbb{R}_+^* = (0,\infty)$ then in any ABC triangle the following inequality holds:

$$\frac{xa}{(y+z)h_a} + \frac{yb}{(z+x)h_b} + \frac{zc}{(x+y)h_c} \ge \sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

X.37. Let be $m,n \in \mathbb{R}_+^* = (0,\infty)$ and ABC triangle. If M,N,P are arbitrary points on BC, CA respectively AB, then:

$$\frac{AM \cdot BN}{mh_a + nh_b} + \frac{BN \cdot CP}{mh_b + nh_c} + \frac{CP \cdot AB}{mh_c + nh_a} \ge \frac{18r}{m+n}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

X.38 If $x,y,z \in \mathbb{R}_+^*$, then in any ABC triangle having the area F the following inequality holds:

$$\left(\frac{x}{h_a^2} + \frac{y}{h_b^2} + \frac{z}{h_c^2}\right)^2 \ge \frac{xy + y + zx}{F^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

X.39 If $x,y,z \in \mathbb{R}_+^* = (0,\infty)$ then in ABC triangle the following inequality holds:

$$\frac{(y+z)a}{xh_a} + \frac{(z+x)b}{yh_b} + \frac{(x+y)c}{zh_c} \ge 4\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania, Martin Lukarevski – Macedonia

X.40. If ABC is a triangle having the area F, then:

$$\frac{a^4b^2}{h_b^2} + \frac{b^4c^2}{h_c^2} + \frac{c^4a^2}{h_a^2} \ge \frac{64}{3} \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți – Romania

X.41. If $m \in [1,\infty)$ and ABC is a triangle having the area F then:

$$\frac{a^{2m}b^m}{h_b^m} + \frac{b^{2m}c^m}{h_c^m} + \frac{c^{2m}a^m}{h_a^m} \ge 2^{3m} \cdot 3^{1-m} \cdot F^m$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

X.42. If $m \in [1,\infty), x \in \mathbb{R}$ then in any ABC triangle having the area F the following inequality holds:

$$\frac{a^{m(2-x)}}{h_a^{mx}} + \frac{b^{m(2-x)}}{h_b^{mx}} + \frac{c^{m(2-x)}}{h_c^{nx}} \ge 2^{(2-x)m}\left(\sqrt{3}\right)^{2-m} \cdot F^{(1-x)m}$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți – Romania

X.43. In any ABC triangle having the area F the following inequality holds:

$$a^3 + b^3 + c^3 \ge 8\sqrt[4]{3}F\sqrt{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

X.44. P and q are distinct prime numbers. Show $\sqrt[p]{q} + \sqrt[q]{p}$ is an irrational number.

Proposed by Jalil Hajimir-Canada

X.45. Solve:

$$4^{[x]} + 3^x = 5^{[x]} + 2^x$$

$[x]$ is the greatest integer part of x.

Proposed by Jalil Hajimir-Canada

X.46. Solve:

$$(6x^2+1)^6 + 2(3x^2+1)^3 + 3(2x^2+1)^2 = 6(6x^2+1)(3x^2+1)(2x^2+1)$$

Proposed by Jalil Hajimir-Canada

X.47. If $a,b,c > 0, \frac{1}{2a+2019} + \frac{1}{2b+2019} + \frac{1}{2c+2019} = \frac{1}{2019}$ then:

$$\sqrt[3]{abc} \ge 2019$$

Proposed by Rahim Shahbazov-Azerbaijan

X.48. If in $\Delta ABC, m(\sphericalangle A) > 152°$ then:

$$h_a < \frac{7}{50}(b+c)$$

Proposed by Rovsen Pirguliyev-Azerbaijan

X.49. Solve for real numbers:

$$[\tan x \cdot \{\cos x\}] = [\cot x \cdot \{\sin x\}]$$

$\{x\} = x - [x], [x]$ – great integer function

Proposed by Rovsen Pirguliyev-Azerbaijan

X.50. If $a,b,c,d > 0$ then:

$$16(a+b+c+d) \ge \sqrt[4]{\frac{a^4+b^4+c^4+d^4}{4}} + 63\sqrt[4]{abcd}$$

Proposed by Rahim Shahbazov-Azerbaijan

X.51. In ΔABC the following relationship holds:

$$\frac{h_a h_b}{h_c}+\frac{h_b h_c}{h_a}+\frac{h_c h_a}{h_b}\le m_a+m_b+m_c$$

Proposed by Rahim Shahbazov-Azerbaijan

X.52. If $x,y,z>0$ then:

$$\frac{x}{y+z+\sqrt[4]{\frac{y^4+z^4}{2}}}+\frac{y}{z+x+\sqrt[4]{\frac{z^4+x^4}{2}}}+\frac{z}{x+y+\sqrt[4]{\frac{x^4+y^4}{2}}}\ge 1$$

Proposed by Rahim Shahbazov-Azerbaijan

X.53. If $a,b,c>0, abc=1$ then:

$$\frac{1}{a^{100}+b^{99}+c^{98}+3}+\frac{1}{b^{100}+c^{99}+a^{98}+3}+\frac{1}{c^{100}+a^{99}+b^{98}+3}\le\frac{1}{2}$$

Proposed by Rahim Shahbazov-Azerbaijan

X.54. If $x,y,z>0, xyz=1$ then:

$$\frac{1}{x^5+x^3+x}+\frac{1}{y^5+y^3+y}+\frac{1}{z^5+z^3+z}\ge 1$$

Proposed by Rahim Shahbazov-Azerbaijan

X.55. Solve for real numbers:

$$x^2(2-x)^2=1+(a^2-2)(1-x)^2, a\in\mathbb{R}, a-\text{fixed}$$

Proposed by Marin Chirciu – Romania

X.56. $ABCD$ – tangential quadrilateral with inradii $r=1, A'B'C'D'$ - contact cyclic quadrilateral of $ABCD$. Prove that:

$$[A'B'C'D']\cdot\sum_{cyc}\frac{\mu^2(A)}{\mu(A)\mu(B)\mu(C)\mu(D)+1}\ge\frac{32\pi^2}{16+\pi^4}\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}$$

Proposed by Radu Diaconu – Romania

X.57. In acute $\Delta ABC, H$ – orthocenter, the following relationship holds:

$$(A^2+B^2+C^2)\left(\frac{a^5}{AH}+\frac{b^5}{BH}+\frac{c^5}{CH}\right)\ge\frac{32\pi^2 s^5}{243R}$$

Proposed by Radu Diaconu – Romania

X.58. Solve for real numbers:

$$x+x^{\log_a b}=x^{\log_a(a+b)}, 1<a<b$$

Proposed by Marin Chirciu – Romania

X.59. In ΔABC the following relationship holds:

$$\frac{n(a^2+b^2+c^2)}{ab+bc+ca}+\sum_{cyc}\frac{w_a^2}{bc}\le n+\frac{9}{4}, n\le\frac{5}{4}$$

Proposed by Marin Chirciu – Romania

X.60. In ΔABC the following relationship holds:

$$\frac{16(2R-r)^2}{r}\le\sum_{cyc}\frac{r_a}{\sin^4\frac{A}{2}}\le\frac{4R^2(2R-r)^2}{r^3}$$

Proposed by Marin Chirciu – Romania

X.61. In ΔABC the following relationship holds:

$$\left(\frac{a}{w_bw_c}\right)^{2n}+\left(\frac{b}{w_cw_a}\right)^{2n}+\left(\frac{c}{w_aw_b}\right)^{2n}\ge\frac{1}{3^{n-1}}\left(\frac{4}{3R}\right)^{2n}, n\in\mathbb{N}^*$$

Proposed by Marin Chirciu – Romania

X.62. In ΔABC the following relationship holds:

$$\frac{b^2+c^2}{(s-a)^2}+\frac{c^2+a^2}{(s-b)^2}+\frac{a^2+b^2}{(s-c)^2}\le 6\left(\frac{R}{r}\right)^2$$

Proposed by Marin Chirciu – Romania

X.63. In ΔABC the following relationship holds:

$$\left(\frac{h_a}{r_a}\right)^2+\left(\frac{h_b}{r_b}\right)^2+\left(\frac{h_c}{r_c}\right)^2+\frac{2\mu r}{R}\ge\mu+1, \mu\le 5$$

Proposed by Marin Chirciu – Romania

X.64. In ΔABC the following relationship holds:

$$\sqrt{\frac{bc}{r_b+r_c}}+\sqrt{\frac{ca}{r_c+r_a}}+\sqrt{\frac{ab}{r_a+r_b}}\le\frac{3R}{2r}\sqrt{R}$$

Proposed by Marin Chirciu – Romania

X.65. Solve for real numbers:

$$\begin{cases} x+y+z=11 \\ \dfrac{yz+36x}{x(y-x)(z-x)}+\dfrac{zx+36y}{y(x-y)(z-y)}+\dfrac{xy+36z}{z(x-z)(y-z)}=1 \\ xyz=36 \end{cases}$$

Proposed by Daniel Sitaru,Virginia Grigorescu– Romania

X.66. Find $x,y,z\ge 0$ such that:

$$\begin{cases} x-y-z=\sin x-\sin y-\sin z \\ x^2-y^2-z^2=\sin^2 x-\sin^2 y-\sin^2 z \\ x^3-y^3-z^3=\sin^3 x-\sin^3 y-\sin^3 z \end{cases}$$

Proposed by Daniel Sitaru,Ileana Duma– Romania

X.67.

$$\Omega_1 = |z_1 + z_2 + z_3|, z_1, z_2, z_3 \in \mathbb{C}$$

$$\Omega_2 = |z_1 + z_2 - z_3 + 4i| + |z_1 - z_2 + z_3 + 2i| + |-z_1 + z_2 + z_3 - 6i|$$

Prove that: $\Omega_1 \leq \Omega_2$

Proposed by Daniel Sitaru, Alexandrina Năstase– Romania

X.68. In ΔABC the following relationship holds:

$$\sqrt[3]{\prod_{cyc} n_a^2} + 2\sqrt[3]{\prod_{cyc} r_a h_a} \leq s^2$$

Proposed by Bogdan Fuștei – Romania

X.69. If in $\Delta ABC, I$ – incenter, n_a – Nagel's cevian, g_a – Gergonne's cevian then:

$$\frac{AI}{h_a} + \frac{BI}{h_b} + \frac{CI}{h_c} \leq \frac{R}{r}, \sum_{cyc} \frac{n_a^2 + g_a^2}{b^2 + c^2} \geq 2 + \frac{r}{2R}$$

Proposed by Bogdan Fuștei – Romania

X.70. In ΔABC the following relationship holds:

$$m_a \geq \frac{1}{2\sqrt{2}}\left((b + c)\cos\frac{A}{2} + |b - c|\sin\frac{A}{2}\right)$$

Proposed by Bogdan Fuștei – Romania

X.71. In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$m_a \geq \frac{1}{2}\left(\frac{h_b + h_c}{2} + |b - c|\sin^2\frac{A}{2}\right)\sqrt{\frac{n_a + h_a}{r_a}}$$

Proposed by Bogdan Fuștei – Romania

X.72. In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$\frac{n_a + m_a + w_b + w_c + \sqrt{2r_a h_a}}{h_a + h_b + h_c} \leq \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right)\sqrt{\frac{R}{r}}$$

Proposed by Bogdan Fuștei – Romania

X.73. In ΔABC the following relationship holds:

$$2\sum_{cyc} \frac{r_a r_b}{w_a^2} \geq \sum_{cyc}\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right)$$

Proposed by Bogdan Fuștei – Romania

X.74. In ΔABC the following relationship holds:

$$\frac{r_a}{m_a}+\frac{r_b}{m_b}+\frac{r_c}{m_c}\ge 4-\frac{2r}{R}$$

Proposed by Bogdan Fuștei – Romania

X.75. In ΔABC the following relationship holds:

$$\left(9\tan^2\frac{A}{2}+1\right)\left(9\tan^2\frac{B}{2}+1\right)\left(9\tan^2\frac{C}{2}+1\right)\ge 64$$

Proposed by Rahim Shahbazov-Azerbaijan

X.76. If $x,y,z,t>0, xyzt=1$ then:

$$\frac{x^2+1}{x^5+3}+\frac{y^2+1}{y^5+1}+\frac{z^2+1}{z^5+1}+\frac{t^2+1}{t^5+1}\le 2$$

Proposed by Rahim Shahbazov-Azerbaijan

X.77. If $a,b,c,d>0$ then:

$$a^4+b^4+c^4+d^4\ge 3abcd\left(\frac{a}{b+c+d}+\frac{b}{c+d+a}+\frac{c}{d+a+b}+\frac{d}{a+b+c}\right)$$

Proposed by Rahim Shahbazov-Azerbaijan

X.78. In ΔABC the following relationship holds:

$$\frac{ab+bc+ca}{2R}\le m_a+m_b+m_c\le\frac{ab+bc+ca}{4r}$$

Proposed by Adil Abdullayev-Azerbaijan

X.79. Solve in $\mathbb{R}$:

$$\sqrt[5]{1-x^3}+\sqrt[7]{1+x^3}=\sqrt[3]{1-x^2}+\sqrt[5]{1+x^2}$$

Proposed by Mokhtar Khassani-Algerie

X.80. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(m_a+m_b)(m_b+m_c)}{\sqrt{(m_a-m_b-m_c)(m_b+m_c-m_a)}}\ge\frac{8r}{R}\sum_{cyc}\sqrt{a}$$

Proposed by Mokhtar Khassani-Algerie

X.81. If $a,b,c>0$ then:

$$\frac{\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\frac{1}{ab}\right)}{\left(\sum_{cyc}\sqrt[3]{a}\right)\left(\sum_{cyc}\sqrt[3]{a^2}\right)}\ge\frac{\left(\sum_{cyc}\frac{1}{\sqrt[3]{a}}\right)\left(\sum_{cyc}\frac{1}{\sqrt[3]{a^2}}\right)}{\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}\frac{1}{a^2b^2}\right)}$$

Proposed by Daniel Sitaru,Tatiana Cristea – Romania

X.82. In ΔABC the following relationship holds:

$$4+\sum_{cyc}\left(\frac{a}{m_a}\right)^2 \le 8\prod_{cyc}\frac{r_a}{h_a}$$

Proposed by Bogdan Fuștei – Romania

X.83. In ΔABC the following relationship holds:

$$(m_a+m_b+m_c)\sqrt{\frac{2R}{r}} \ge \frac{a^2}{r_a-r}+\frac{b^2}{r_b-r}+\frac{c^2}{r_c-r}$$

Proposed by Bogdan Fuștei – Romania

X.84. In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$\sqrt{n_a g_a r_a}+\sqrt{n_b g_b r_b}+\sqrt{n_c g_c r_c} \ge s\sqrt{r}$$

Proposed by Bogdan Fuștei – Romania

X.85. In ΔABC the following relationship holds:

$$\frac{w_b+w_c}{a}+\frac{w_c+w_a}{b}+\frac{w_a+w_b}{c} \le 2\sqrt{6+\frac{3r}{2R}}$$

Proposed by Bogdan Fuștei – Romania

X.86. In ΔABC the following relationship holds:

$$\frac{\cos^2\left(\frac{A-B}{2}\right)}{\tan\frac{C}{2}}+\frac{\cos^2\left(\frac{B-C}{2}\right)}{\tan\frac{A}{2}}+\frac{\cos^2\left(\frac{C-A}{2}\right)}{\tan\frac{B}{2}} \ge 6\sqrt{3}\cdot\frac{r}{R}$$

Proposed by George Apostolopoulos – Greece

X.87. If $a,b,c>0, abc=1$ then:

$$\sqrt[4]{\frac{b^2+c^2}{2a}}+\sqrt[4]{\frac{c^2+a^2}{2b}}+\sqrt[4]{\frac{a^2+b^2}{2c}} \le a+b+c$$

Proposed by George Apostolopoulos – Greece

X.88. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{(r_a^2+r_b^2+r_c^2+2r_ar_b+2r_ar_c)a^2}{r_br_c} \ge 28\sqrt{3}S$$

Proposed by Daniel Sitaru, Anicuța Patricia Bețiu– Romania

X.89. In ΔABC the following relationship holds:

$$\frac{w_a^2}{bc}+\frac{w_b^2}{ca}+\frac{w_c^2}{ab}+\frac{3(a^2+b^2+c^2)}{4(ab+bc+ca)} \le 3$$

Proposed by Rahim Shahbazov-Azerbaijan

X.90. If $a, b, c > 0, a + b + c = abc$ then:

$$(a^2 - 1)(b^2 - 1)(c^2 - 1) \le 8$$

Proposed by Rahim Shahbazov-Azerbaijan

X.91. In ΔABC the following relationship holds:

$$\sum_{cyc}\left(w_a\sqrt{\frac{r_a}{h_a}}\right) \le \frac{3R}{2}\sqrt{1 + \frac{8m_am_bm_c}{h_ah_bh_c}}$$

Proposed by Mokhtar Khassani-Algerie

X.92. Find two complex solutions such that:

$$\sqrt{1 + x^2 + x^4} + \sqrt{1 + x + x^2} = 1 + \sqrt{x}$$

Proposed by Mokhtar Khassani-Algerie

X.93. Let $a, b, c > 0$ prove that:

$$\left(a + b + \frac{1}{ab}\right)^4 + \left(a + c + \frac{1}{ac}\right)^4 + \left(b + c + \frac{1}{bc}\right)^4 \ge 162\sqrt{\frac{6}{a + b + c}}$$

Proposed by Mokhtar Khassani-Algerie

X.94. In ΔABC the following relationship holds:

$$\left(\sum r_ar_b\right)\left(\sum (r_a + r_b)^2 (r_a + r_c)^2\right) \ge \left(\prod (r_a + r_b)^2\right)\left(\sum \cos^2\left(\frac{A}{2}\right)\right)$$

Proposed by Mokhtar Khassani-Algerie

X.95. In ΔABC the following relationship holds:

$$4\left(\sum_{cyc}\frac{r_a}{a}\right)\left(\sum_{cyc}\frac{r_a^2}{r_b + r_c}\right) \ge 9s$$

Proposed by Mokhtar Khassani-Algerie

X.96. If $a, b, c > 0$ then:

$$\frac{1}{a + ab + b} + \frac{1}{b + bc + c} + \frac{1}{c + ca + a} \le \sqrt{\frac{a^2 + b^2 + c^2}{3a^2b^2c^2}}$$

Proposed by Daniel Sitaru, Dumitru Săvulescu – Romania

X.97. If $m, n, p \in \mathbb{N}$ then:

$$3\sqrt{3}\left(\frac{m^3}{(m + 3)!} + \frac{n^5}{(n + 5)!} + \frac{p^7}{(p + 7)!}\right) < \sqrt{(m!)^2 + (n!)^2 + (p!)^2}$$

Proposed by Daniel Sitaru, Sorin Pîrlea – Romania

X.98. If $z_1, z_2, z_3 \in \mathbb{C} - \mathbb{R}$ then:

$$\sum_{cyc} Im\left(\frac{4z_1 + 1}{19z_1 + 5}\right) \cdot (Im\, z_1)^2 \geq \left(\sum_{cyc} Imz_1\right)^3 \cdot \left(\sum_{cyc} |19z_1 + 5|^2\right)^{-1}$$

Proposed by Daniel Sitaru,Dorina Goiceanu– Romania

X.99. In $\Delta ABC, K$ – Lemoine's point the following relationship holds:

$$\frac{[BKC]}{ar_a}\sqrt{1 + \frac{2[BKC]}{ar_a}} + \frac{[CKA]}{br_b}\sqrt{1 + \frac{2[CKA]}{br_b}} + \frac{[AKB]}{cr_c}\sqrt{1 + \frac{2[AKB]}{cr_c}} \geq \frac{\sqrt{3}}{3}$$

Proposed by Daniel Sitaru,Iulia Sanda – Romania

X.100. In ΔABC the following relationship holds (F_n - Fibonacci numbers):

$$\frac{r_a^2 F_{n+2}}{a(bF_n + cF_{n+1})} + \frac{r_b^2 F_{n+2}}{b(cF_n + aF_{n+1})} + \frac{r_c^2 F_{n+2}}{c(aF_n + bF_{n+1})} \geq \left(\frac{3r}{R}\right)^2$$

Proposed by Daniel Sitaru,Nicolae Radu – Romania

X.101. In ΔABC the following relationship holds ($\forall z \in \mathbb{C}$):

$$|z - \cos A - i\sin A| + |z - \cos B - i\sin B| + |z - \cos C - i\sin C| \geq 3(|z| - 1)^2$$

Proposed by Daniel Sitaru,Mihaela Stăncele– Romania

X.102. If $a, b, c, d > 1, abcd = e^4$ then:

$$\frac{\ln\left(\frac{e^2}{a}\right) \cdot \ln\left(\frac{e^2}{b}\right) \cdot \ln\left(\frac{e^2}{c}\right) \cdot \ln\left(\frac{e^2}{d}\right)}{\ln(ab) \cdot \ln(bc) \cdot \ln(cd) \cdot \ln(da)} \leq \frac{1}{16}$$

Proposed by Daniel Sitaru,Carmen Năstase – Romania

X.103. If $a, b, c > 0, (a + b)(b + c)(c + a) = 8$ then:

$$(\sqrt[3]{a} + \sqrt[3]{b})(\sqrt[3]{b} + \sqrt[3]{c})(\sqrt[3]{c} + \sqrt[3]{a}) \leq (a + b)(b + c)(c + a)$$

Proposed by Daniel Sitaru,Cristina Micu – Romania

X.104. In ΔABC the following relationship holds:

$$6 + \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c} = \frac{2R}{r}\sum_{cyc}\frac{h_b h_c}{a^2}$$

Proposed by Bogdan Fuștei – Romania

X.105. In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\sum_{cyc}\frac{r_b + r_c}{a} \geq \frac{1}{2}\sum_{cyc}\frac{b + c}{AI}$$

Proposed by Bogdan Fuștei – Romania

X.106. In $\Delta ABC, g_a$ – Gergonne's cevian the following relationship holds:

$$\sum_{cyc}\left(\frac{w_a - g_a}{a}\right) \ge \frac{1}{2s}\sum_{cyc}(w_a - h_a)$$

Proposed by Bogdan Fuștei – Romania

X.107. In $\Delta ABC, n_a$ –Nagel's cevian the following relationship holds:

$$\sqrt{\frac{n_b n_c}{h_a} + \frac{n_c n_a}{h_b} + \frac{n_a n_b}{h_c}} \le \frac{3\sqrt{2}R}{2}\left(\frac{R}{r} - 1\right)$$

Proposed by Bogdan Fuștei – Romania

X.108. In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$\frac{6s - n_a - n_b - n_c}{r} \ge \sqrt{2}\left(3 + \sum_{cyc}\sqrt{\frac{b+c}{a}} + \sum_{cyc}\sqrt{\frac{2r_a}{h_a}}\right)$$

Proposed by Bogdan Fuștei – Romania

X.109. Fuștei's refinement for Euler's inequality

In $\Delta ABC, n_a$ – Nagel's cevian the following relationship holds:

$$R \ge r\left(1 + \sqrt[3]{\frac{n_a n_b n_c}{h_a h_b h_c}}\right) \ge 2r$$

Proposed by Bogdan Fuștei – Romania

X.110. In ΔABC the following relationship holds:

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\sqrt{4 - \frac{2R}{r}} \le \frac{s}{r}$$

Proposed by Bogdan Fuștei – Romania

X.111. In ΔABC the following relationship holds:

$$\left(s + \sum_{cyc}\sqrt{ab}\right)\left(\sum_{cyc}\frac{1}{2b + 2c - a}\right) \ge \sum_{cyc}\sin A \cdot \sum_{cyc}\tan\frac{A}{2}$$

Proposed by Bogdan Fuștei – Romania

X.112. In $\Delta ABC, n_a$ – Nagel's cevian, g_a – Gergonne's cevian the following relationship holds:

$$\sqrt{2(b^2 + bc + c^2)} \ge \frac{1}{2}|b - c| + \frac{\sqrt{3}}{4}\left(n_a + g_a + \sqrt{2r_b r_c} + 2\sqrt{r r_a}\right)$$

Proposed by Bogdan Fuștei – Romania

X.113. If $x, y, z > 1$ then:

$$3+4\left(\log^2_{xyz^2}\left(\frac{x}{y}\right)+\log^2_{yzx^2}\left(\frac{y}{z}\right)+\log^2_{zxy^2}\left(\frac{z}{x}\right)\right)\le 2\left(\log_{yz}x+\log_{zx}y+\log_{xy}z\right)$$

Proposed by Daniel Sitaru,Dana Cotfasă – Romania

X.114. In ΔABC the following relationship holds:

$$\sum_{cyc} a(\sin 3B-\sin 3C)\le 8s\sum_{cyc}\sin(B-C)$$

Proposed by Daniel Sitaru,Gabriel Tică – Romania

X.115. In ABC triangle, let be the points $D\in(BC), E\in(CA), F\in(AB)$ such that the lines AD, BE, CF are concurrent with the point M, then:

$$\left(\frac{MD^2}{MA^2}+\frac{MC^2}{MB^2}+\frac{MF^2}{MC^2}\right)(a^8+b^8+c^8)\ge 64\cdot S^4$$

where S is the triangle's area.

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.116. In any ABC triangle having the area F the following inequality holds:

$$e^{a-1}+e^{b-1}+e^{c-1}+\ln(a^a\cdot b^b\cdot c^c)\ge 4\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.117. If $m\in\mathbb{R}_+=[0,\infty)$ and a, b, c are the sides lengths of ABC triangle having the area F, then:

$$\frac{a^{m+2}}{(2a+b+c)^m}+\frac{b^{m+2}}{(a+2b+c)^m}+\frac{c^{m+2}}{(a+b+2c)^m}\ge\frac{\sqrt{3}}{4^{m+1}}F$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.118. Let $A_1B_1C_1, A_2B_2C_2$ be two triangles having the area F_1, F_2 and sides having the lengths a_1, b_1, c_1 respectively a_2, b_2, c_2, then:

$$a_1^2(a_2^2+2b_2c_2)+b_1^2(b_2^2+2c_2a_2)+c_1^2(c_2^2+2a_2b_2)\ge 48F_1F_2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.119. Let $x, y\in\mathbb{R}_+^*=(0,\infty)$ and k_a the length of the common tangent to the circles from the sides $[AB], [AC]$ as diameter included between the intersection points t_b, r_c the analogs of k_a and r the inradii of ABC triangle, then:

$$\frac{\left(xt_a^y+yt_b^y\right)^2}{t_at_b}+\frac{(xt_b^4+yt_c^4)^2}{t_bt_c}+\frac{(xt_c^4+yt_a^4)^2}{t_ct_a}\ge 81(x+y)^2r^6$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.120. In ABC triangle let be the points $D \in (BC), E \in (CA), F \in (AB)$ such that the sides AD, BE, CF are concurrent with the point M, then:

$$\left(\frac{MD^2}{MA^2} + \frac{MC^2}{MB^2} + \frac{MF^2}{MC^2}\right)(a^8 + b^8 + c^8) \geq 64S^4$$

where S is the triangle's area.

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.121. If $m \in \mathbb{N}; x, y, z \in \mathbb{R}_+^* = (0, \infty)$ then in ABC triangle having the area F the following inequality holds:

$$3m + (x^2a^4)^{m+1} + (y^2b^4)^{m+1} + (z^2c^4)^{m+1} \geq \frac{16}{3}(m+1)(xy + yz + zx)F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.122. If $m, u \in \mathbb{R}_+ = [0, \infty), x, y, z \in \mathbb{R}_+ = (0, \infty)$ then in any ABC triangle the following inequality holds:

$$\left(\frac{y+z+u}{x} \cdot a^2\right)^{m+1} + \left(\frac{z+x+u}{y} \cdot b^2\right)^{m+1} + \left(\frac{x+y+u}{z} \cdot c^2\right)^{m+1} \geq$$

$$\geq 2^{2n+2}\left(\sqrt{3}\right)^{1-m}\left(2 + \frac{3}{x+y+z}\right)^{m+1} F^{m+1}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.123. If $m \in \mathbb{R}_+ = [0, \infty);\ x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in any ABC triangle the following inequality holds:

$$\left(\frac{y+z}{x} \cdot a^4\right)^{m+1} + \left(\frac{z+x}{y} \cdot b^4\right)^{m+1} + \left(\frac{x+y}{z}\right)^{m+1} \geq \frac{2^{5m+5}}{3^m} \cdot F^{2m+2}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.124. Let be $m, n \in \mathbb{N}^*, x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and ABC a triangle having the area F, then:

$$\left(3m^2 + (xa^2)^{m^2+1} + (yb^2)^{m^2+1} + (zc^2)^{m^2+1}\right)\left(n^2 + (xa^2 + yb^2 + zc^2)^{n^2+1}\right) \geq$$

$$\geq 64m \cdot n \cdot (xy + yz + zx)F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.125. If $m \in \mathbb{N}; x, y, z \in \mathbb{R}_+^* = (0, \infty)$, then in ABC triangle having the area F, the following inequality holds:

$$m + (xa^2 + yb^2 + zc^2)^{2(m+1)} \geq 16(m+1)(xy + yz + zx)F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.126. Let $M \in Int\ ABC$ where ABC is a triangle having the area F. If x_A, X_B, X_C are the distances from M to the apices A, B, C and d_a, d_b, d_c are the distances from M to the sides BC, CA respectively AB, then:

$$\frac{x_A}{d_a}a^4 + \frac{x_B}{d_b}b^4 + \frac{x_c}{d_c}c^4 \geq 32F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.127. Let M be an interior point in ABC triangle having the area F and $X = AM \cap BC$, $Y = BM \cap CA, Z = CM \cap AB$, then:

$$\frac{MA}{MX}a^4 + \frac{MB}{MY}b^4 + \frac{MC}{MZ}c^4 \geq 32F^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.128. If $x, y, z, t \in \mathbb{R}_+^* = (0, \infty)$ and $x^2 + y^2 + z^2 = t^2$, then:

$$(x + z)(y + t) \leq 2t^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.129. Let $m, n \in \mathbb{R}_+ = [0, \infty); m + n \in \mathbb{R}_+^* = (0, \infty)$ and M an interior point in ABC triangle. If x, y, z are the distances from point M to the sides A, B, C, respectively and u, v, w the distances from point M to the sides BC, CA, AB, then:

$$\frac{m(x + y) + nw}{n(u + v) + mz} + \frac{m(y + z) + nu}{n(v + w) + mx} + \frac{m(z + x) + nv}{n(w + u) + my} \geq \frac{9(2m + n)(u + v + w)}{(m + n)(x + y + z)} - 3$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.130. If $m, p \in \mathbb{N}$ and $A_1A_2 \ldots A_n, n \geq 3$ is a convexe polygon having the area F and the sides having the lengths $A_kA_{k+1} = a_k, k = \overline{1, n}, A_{n+1} = A_1$, then:

$$(m \cdot n)^{p+1} + \sum_{k=1}^{n} a_k^{2(m+1)(p+1)} \geq 4^{p+1} \cdot \frac{(m + 1)^{p+1}}{(n + 1)^p} F^{p+1} \cdot \tan^{p+1} \frac{\pi}{n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.131. Let be $x, y \in \mathbb{R}_+^* = (0, \infty)$ $x \geq y$ and $n \in \mathbb{N}^* - \{1\}$ and $a_k, b_k \in \mathbb{R}_+^*, k = \overline{1, n}$ such that $a_k > a_j, \forall j, h \in \mathbb{N}^*, j > i$, then:

$$\sum_{k=1}^{n-1} \frac{b_k^2}{xa_k - ya_{k+1}} \geq \frac{(\sum_{k=1}^{n-1} b_k)^2}{xa_1 - ya_n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

X.132. Solve for real numbers: $x + 9^{\log_x 27} + x \cdot 9^{\log_x 27} = 279$

Proposed by Marian Ursărescu-Romania

X.133. Let be $A(z_1); B(z_1); C(z_3); z_1, z_2, z_3 \in \mathbb{C} \setminus \{0\}; |z_1| = |z_2| = |z_3|; AB = c;$ $BC = a; CA = b$. If $(b+c)z_B z_C + (c+a)z_C z_A + (a+b)z_A z_B = 0$ then $AB = BC = CA$.

Proposed by Marian Ursărescu-Romania

X.134. Let be $z_A, z_B, z_C \in \mathbb{C}^*$, different in pairs such that $|z_A| = |z_B| = |z_C| = 1$. If $|z_A - z_B - z_C| + |z_B - z_C - z_A| + |z_C - z_A - z_B| = 6$, then ΔABC is an equilateral triangle.

Proposed by Marian Ursărescu-Romania

X.135. Let be $z_1, z_2, z_3 \in \mathbb{C} \setminus \{0\}$ different in pairs:$|z_1| = |z_2| = |z_3| = 1; A(z_1); B(z_2);$ $C(z_3)$.If $|z_1 - z_2 - z_3| + |z_2 - z_1 - z_3| + |z_3 - z_2 - z_1| = 6$ then $AB = BC = CA$.

Proposed by Marian Ursărescu-Romania

X.136. $z_A, z_B, z_C \in \mathbb{C}^*$ –differnt in pairs, $|z_A| = |z_B| = |z_C| = 1, a = BC, b = CA, c = AB$.
Prove that:

$$\left|\prod_{cyc} b(z_A - z_B) + c(z_A - z_C)\right| = (a+b+c)^3 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

X.137. Find all the polynomials $P \in \mathbb{R}[x]$ having the property

$$P(x) = P\left(x + \sqrt{x^2+1}\right), \forall x \in \mathbb{R}$$

Proposed by Marian Ursărescu-Romania

X.138. In any ΔABC the following relationship holds:

$$\sum \frac{(h_b + h_c)(h_a + h_c)}{h_a h_b} \ge 768\left(\frac{r}{R}\right)^6$$

Proposed by Marian Ursărescu-Romania

X.139. In ΔABC the following relationship holds:

$$m_a r_a + m_b r_b + m_c r_c \le \frac{3R}{2r}(2R^2 + r^2)$$

Proposed by Marian Ursărescu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

11-CLASS-STANDARD

XI.1. Inspired by Prof. Daniel Sitaru.Find:

$$\lim_{x\to\infty}\left\{\frac{\sqrt{4x^2+3}+\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}{3\sqrt{(4x^2+3)(x^4+x^2+1)}}\right\}$$

where $\{\cdot\}$ denotes fractional part.

Proposed by Naren Bhandari-Bajura-Nepal

XI.2. Find all functions: $\varphi:\mathbb{R}^*\to\mathbb{R}$ such that:
$\varphi(xy)+xy+m=\varphi(x)+\varphi(y)+x+y,\forall x,y\in\mathbb{R}^*$ and $m=$ constant

Proposed by Mokhtar Khassani-Algerie

XI.3. Let $\alpha,\beta,\gamma>0$. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(\alpha x+\beta\gamma)\cdot f(\gamma x+\beta\gamma)=f^2(\alpha x+\gamma y)+\frac{\alpha}{\beta}xy,\forall x,y\in\mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

XI.4 Let $\{u_n\}_{n\geq 1}$ satisfy:

$$\begin{cases} 0<u_1<1 \\ u_{n+1}=u_n^2-u_n+1, n=1,2,3\dots \end{cases}$$

Find:

$$\Omega=\lim_{n\to\infty}(u_1u_2\dots u_n)$$

Proposed by Nguyen Van Canh-Vietnam

XI.5. Find all functions $\varphi:\mathbb{R}\to[-1,1]$ such that:

$$2018\min\{\varphi(x),y^2\}=2019\min\{\varphi(y),z^2\}+2020\max\{\varphi(z),x^2\},\forall x,y\in\mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

XI.6. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{n+1}{n}H_{n+1}^n-H_n^{n+1}\right)^{\frac{1}{2n}}$$

Proposed by Mohammed Bouras-Morocco

XI.7. Find:

$$\Omega = \lim_{n\to\infty}\left(n^6 \sin\frac{1}{n^3}\tan\frac{1}{n^5}\sum_{1\le k\le l\le n}\sin\left(\frac{k+l}{n}\right)\right)$$

Proposed by Daniel Sitaru,Cristian Moanță – Romania

XI.8. If $a,b,c,d \ge e, e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ then:

$$5\log(ae)\cdot\log(be)\cdot\log(ce)\cdot\log(de) \ge \log(abcde)^{16}$$

Proposed by Daniel Sitaru – Romania

XI.9. Find:

$$\Omega = \lim_{n\to\infty}\left(n\left(\frac{\log\left(1+\frac{\sqrt[n]{e}}{n}\right)^{n+1}}{\log\left(1+\frac{\sqrt[n]{e}}{n+1}\right)^{n}} - 1\right)\right)$$

Proposed by Daniel Sitaru – Romania

XI.10. Find:

$$\Omega = \lim_{n\to\infty}\left(n^{n-2}\left(\frac{2}{3}\right)^2\cdot\left(\frac{3}{5}\right)^3\cdot\ldots\cdot\left(\frac{n}{2n-1}\right)^n\right)$$

Proposed by Daniel Sitaru – Romania

XI.11. If $x,y,z > 0, x+y+z = 1$ then:

$$\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\sqrt{x^x+y^y+z^z} > \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^{xy+xz+yz}$$

Proposed by Mohammed Bouras-Morocco

XI.12. Similar of conjecture Syracuse

$$u_{n+1} = \begin{cases} \frac{u_n}{2}\ if\ u_n\ pair \\ \frac{u_n-1}{4}\in\mathbb{N}\ or\ \frac{u_n+1}{4}\in\mathbb{N}\ if\ u_n\ even\ (we\ choose\ the\ integer\ value) \end{cases}$$

Prove that process will eventually reach the number 1.

Proposed by Mohammed Bouras-Morocco

XI.13. If $a_1, a_2, \ldots, a_n > 0, n\in\mathbb{N}-\{0,1\}, a_1+a_2+\cdots+a_n = n$ then:

$$\frac{n+a_1}{n-a_1}+\frac{n+a_2}{n-a_2}+\cdots+\frac{n+a_n}{n-a_n} \ge n+\frac{2n}{n-1}$$

Proposed by Mohammed Bouras-Morocco

XI.14. Prove that: $\lim_{n\to\infty}\left(\sqrt[3]{ax+b\sqrt[3]{x^2}}-\sqrt[3]{ax-c\sqrt[3]{x^2}}\right) = \frac{b+c}{3\sqrt[3]{a^2}}, a > 0$

Proposed by Mohammed Bouras-Morocco

XI.15. If $a_1, a_2, \dots, a_n > 0, a_1 + a_2 + \dots + a_n = n, p, k \in \mathbb{N}$ then:

$$\sum_{i=1}^{n}\left(\frac{a_i^{p+1}+1}{a_i^{p}+1}\right)^k \ge n$$

Proposed by Marin Chirciu, Daniel Sitaru – Romania

XI.16. If $x_1, x_2, \dots, x_k, n > 0, k \in \mathbb{N} - \{0\}, x_1 + \dots + x_k = kn^2$ then:

$$\frac{1}{2n-\sqrt{x_1}} + \frac{1}{2n-\sqrt{x_2}} + \dots + \frac{1}{2n-\sqrt{x_k}} \ge \frac{k}{n}$$

Proposed by Marin Chirciu – Romania

XI.17. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\left(\sum_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}\right)\left(\sum_{k=1}^{n}\frac{k^3}{3k^2-3nk+n^2}\right)}$$

Proposed by Daniel Sitaru – Romania

XI.18. If $0 \le a, b, c \le 1$ then:

$$27\sum_{cyc}\sin a \cdot \cos^2 c \le \sum_{cyc} b(3-a)^3$$

Proposed by Daniel Sitaru – Romania

XI.19. If $0 < a, b, c \le \frac{\pi}{2}$ then:

$$(1+\cos^2 a)(1+\cos^2 b)(1+\cos^2 c)(\sin a)^{2\sin^2 a}(\sin b)^{2\sin^2 b}(\sin c)^{2\sin^2 c} \ge 1$$

Proposed by Daniel Sitaru – Romania

XI.20. If $m, n \in \mathbb{N} - \{0\}, F_n$ – Fibonacci numbers, L_n - Lucas numbers then:

$$\sqrt[5]{\frac{F_m^2F_n^3L_n^2L_m^3}{F_{m+n}^5}} + \sqrt[5]{\frac{F_m^3F_n^2L_n^3L_m^2}{F_{m+n}^5}} < 2$$

Proposed by Daniel Sitaru – Romania

XI.21 If $a, b, c > 0, a + b + c = 9, n \in \mathbb{N}^*, F_n$ - Fibonacci numbers then:

$$\frac{a^4}{\sin^3(F_{n+2})} + \frac{b^4}{\sin^3(F_n^2)} + \frac{c^4}{\cos^3(F_{n+2}^2)} > 72$$

Proposed by Daniel Sitaru – Romania

XI.22. Let be $m, p \in \mathbb{N}^*$ and $(a_n)_{n\ge} a_n \in \mathbb{R}_+^* = (0,\infty), \lim_{n\to\infty}\frac{a_{n+1}}{a_n n} = a \in \mathbb{R}_+^*$ and

$$b_n = a_1 \sqrt[m+p]{a_{m+p}} \cdot \sqrt[2m+p]{a_{2m+p}} \cdot \ldots \cdot \sqrt[mn+p]{a_{m+p}}, \forall n \in \mathbb{N}^*$$

Find: $\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right)$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

XI.23. Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = \frac{1}{4}$ and $f(5x) - f(x) = x$, for all x.

Proposed by Jalil Hajimir-Canada

XI.24. Solve:

$$\frac{\log_2(x^2+1)}{x+1} + \frac{\log_{x+1} 2}{x^2+1} + \frac{\log_{x^2+1}(x+1)}{2} = \frac{9}{x^2+x+4}$$

Proposed by Jalil Hajimir-Canada

XI.25. Let f be a concave and increasing function on $[a,b]$ and $a \le m \le n \le p < q \le b$. Prove or disprove:

$$\frac{f(q)-f(n)}{q-n} \le \frac{f(p)-f(m)}{p-m}$$

Proposed by Jalil Hajimir-Canada

XI.26. Find:

$$\Omega = \lim_{n\to\infty}\left(n\left(\left((1+\frac{1}{n}\right)^n - e + 1\right) - e^{-\frac{e}{2}}\right)$$

Proposed by Rahim Shahbazov-Azerbaijan

XI.27. If in $ABCD$ – convexe quadrilateral $abcd = 1, a, b, c, d$ – sides then:

$$\left(\sum_{cyc}\mu(A)^{n+2} + \sum_{cyc}\frac{1}{\mu(A)^{n+1}}\right)\left(\sum_{cyc}\frac{1+(bcd)^n}{1+a^n}\right) \ge 4\left(4+\frac{\pi^{n+1}}{2^{n-1}}\right), n \ge 1$$

Proposed by Radu Diaconu – Romania

XI.28. Prove that in any ABC triangle the following inequality holds:

$$\left(\sum_{cyc}\left(A+\frac{1}{A}\right)^2\right)\left(\sum_{cyc}\frac{AH^4+a^4}{h_a^4 \cdot m_a^4}\right) \ge \frac{128(\pi^2+9)^2}{656\pi^2 r^4}$$

with the usual notations in triangle.

Proposed by Radu Diaconu – Romania

XI.29 If $a_1, a_2, \ldots, a_n \ge 0, n \in \mathbb{N}^*$ then:

$$\frac{1}{(1+\sqrt{a_1})^2} + \frac{1}{(1+\sqrt{a_2})^2} + \cdots + \frac{1}{\left(1+\sqrt{a_n}\right)^2} \ge \frac{n^2}{2(a_1+a_2+\cdots+a_n+n)}$$

Proposed by Marin Chirciu – Romania

XI.30. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\sin n}{n^4}\sum_{1\le i<j\le n}\left(\frac{(i+1)(j+1)+\sqrt[4]{n(2i+3j)}+\sqrt[6]{n(3i+4j)}}{(i+1)(j+1)+\sqrt[3]{n(2i+3j)}+\sqrt[5]{n(3i+4j)}}\right)\right)$$

Proposed by Daniel Sitaru – Romania

XI.31. If $x, y, z > 0, xyz = 1$ then:

$$\frac{1}{x^6 - x + 3} + \frac{1}{y^6 - y + 3} + \frac{1}{z^6 - z + 3} \le 1$$

Proposed by Rahim Shahbazov-Azerbaijan

XI.32. Find the limit of the following for all $n \in \mathbb{N}$

$$\lim_{x\to\infty}\frac{((1+x)^{2n+1}+(1-x)^{2n+1})((1+x)^{2n+3}+(1-x)^{2n+3})}{((1+x)^2+(1-x)^2)^{2n+1}}$$

Proposed by Naren Bhandari-Nepal

XI.33. Solve for real numbers:

$$\sin^{\csc^4(2x)} x + \cos^{\sec^4(2x)} x + \tan^{\cot^4(2x)} x = \frac{4\sqrt{2}-1}{4}$$

Proposed by Mokhtar Khassani-Algerie

XI.34. Show that:

$$\lim_{x\to\infty}\left(\frac{\left(1+\frac{1}{x}\right)^{xe^e} - e^{\left(1+\frac{1}{x}\right)^{xe}}}{\left(e^{e+e^e} - e^{e+e^e-1}\right)\left(e-\left(1+\frac{1}{x}\right)^x\right)}\right)^x = \sqrt[4(1-e)]{e^{e^{e+2}+e^2+1-e^e-e}}$$

Proposed by Mokhtar Khassani-Algerie

XI.35 If $a, b, c > 0$ then:

$$\frac{(a^2-ab+b^2)^6}{(a+b)^{12}} + \frac{(b^2-bc+c^2)^6}{(b+c)^{12}} + \frac{(c^2-ca+a^2)^6}{(c+a)^{12}} \ge \frac{3}{4096}$$

Proposed by Daniel Sitaru – Romania

XI.36. Prove that:

$$\frac{\sqrt{\pi(n-1)}}{2n-1} < \prod_{j=0}^{n-1}\frac{2n-2j}{2n-2j+1} < \frac{\sqrt{\pi n}}{2n+1}$$

Proposed by Naren Bhandari-Nepal

XI.37. Find $\Omega = \underbrace{\max(n)}_{n\in\mathbb{N}}$ such that:

$$(x+y+z)^{4n} \ge 3^{4n-1}(xyz)^n(x^n+y^n+z^n), \forall x, y, z > 0$$

Proposed by Rahim Shahbazov-Azerbaijan

XI.38. Find the minimum of:

$$f(x,y,z)=\sqrt{\frac{2x}{y+z}}+\sqrt{\frac{2y}{z+x}}+\sqrt{\frac{zx}{x+y}},x,y,z>0$$

Proposed by Jalil Hajimir-Canada

XI.40. Prove:

$$\tan^{-1}A+\tan^{-1}B+3\tan^{-1}\left(\frac{A+B}{3}\right)\le 2\tan^{-1}\left(\frac{A+B}{2}\right)+4\tan^{-1}\left(\frac{A+B}{4}\right),A,B\ge 0$$

Proposed by Jalil Hajimir-Canada

XI.41. Let x,y and z be positive real numbers such that $x+y+z=3$. Prove:

$$\frac{x}{\sqrt{1+3x^2}}+\frac{y}{\sqrt{1+3y^2}}+\frac{z}{\sqrt{1+3z^2}}\le\frac{3}{2}$$

Proposed by Jalil Hajimir-Canada

XI.42. Solve for natural numbers:

$$13^x+17^y+19^z=2^x+31^y+73^z$$

Proposed by Mokhtar Khassani-Algerie

XI.43. If $x,y,z>0$ and $x+y+z=\frac{3\pi}{4}$ then find the maximum and minimum of:

$$\Omega=\left(\cos^{\sin y}x+\cos^{\sin z}y+\cos^{\sin x}z\right)(\cos(x+y)+\cos(x+y)+\cos(y+z))$$

Proposed by Mokhtar Khassani-Algerie

XI.44. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(xy+x)+f(x+y)+xy=f(xy)+f(x)+f(y),\forall x,y\in\mathbb{R}$$

Proposed by Mokhtar Khassani-Algerie

XI.45. Find the limit

$$\boldsymbol{\Omega}=\lim_{n\to\infty}n^3.\int_0^1\left(\frac{1}{n+cos(\pi x)}+\frac{1}{n}sin(n.ln(x))\right)dx$$

Proposed by Mohammed Bouras-Morocco

XI.46. Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{2\sum_{0\le i<j\le n}\binom{n}{i}\binom{n}{j}+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}}$$

Proposed by Daniel Sitaru – Romania

XI.47. Find:

$$\Omega = \lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(1+\frac{e-1}{n}\log\left(1+\frac{(e-1)k}{n}\right)\right)\right)$$

Proposed by Daniel Sitaru – Romania

XI.48. Solve for real numbers:

$$\frac{3}{\sqrt[3]{1+x}}+\frac{x}{\sqrt[3]{1+x^3}}=2\sqrt[3]{3}$$

Proposed by Daniel Sitaru – Romania

XI.49. $\Omega_n = \frac{\sqrt[n+1]{\pi^{n+1}+e^{n+1}}}{\sqrt[n]{(n\pi)^n+(ne)^n}}, n \in \mathbb{N}, n \geq 2$. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\cos^2\Omega_n}{\cos^2(2\Omega_n)}\right)^{\frac{1}{\Omega_n^2}}$$

Proposed by Daniel Sitaru – Romania

XI.50.

$$A,B \in M_4(\mathbb{C}), B^3 = I_4, A^3 = AB^2 + BA^2, C = \begin{pmatrix} 28 & 18 & 36 & 723 \\ 120 & 121 & 45 & 891 \\ 330 & 27 & 151 & 210 \\ 450 & 150 & 180 & 181 \end{pmatrix}$$

Prove that:

$$\det\big((CA - CB)(A^2 - B^2)\big) \neq 0$$

Proposed by Daniel Sitaru – Romania

XI.51. If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ and $n \in \mathbb{N}^*$, then:

$$\frac{\tan^{2n+1}x}{\sin y}+\frac{\tan^{2n+1}y}{\sin z}+\frac{\tan^{2n+1}z}{\sin x} > (xy+yz+zx)^n$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.52. Find:

$$\lim_{n\to\infty}\left(\sqrt[3]{n^2}\right)\left(\sqrt[3n+3]{(2n+1)!!}-\sqrt[3n]{(2n-1)!!}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.53. Let $(a_n)_{n\geq 1}$ a sequence of real numbers strictly positive such that:

$$\lim_{n\to\infty}(a_{n+1}-a_n) = a \in \mathbb{R}_+^*$$

Find:

$$\lim_{n\to\infty}\sqrt[3]{n^2}\left(\sqrt[3]{a_{n+1}}-\sqrt[3]{a_n}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.54. Let $(a_n)_{n\geq1}$ be a sequence of real strictly positive numbers such that:

$$\lim_{n\to\infty}\left(e^{H_{n+1}}\cdot\frac{a_{n+1}^2}{\sqrt{(n+1)!\left((2n+1)!!\right)}}-e^{H_n}\cdot\frac{a_n^2}{\sqrt[n]{n!\cdot\left((2n-1)!!\right)}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.55. If $(H_n)_{n\geq1}, H_n = \sum_{k=1}^{n}\frac{1}{k}$ and $(a_n)_{n\geq1}$ is a sequence of real strictly positive numbers such that

$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a \in \mathbb{R}_+^* = (0,\infty)$ and find:

$$\lim_{n\to\infty} e^{-2H_n}\left(\sqrt[n]{a_n}\right)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.56. Find:

$$\lim_{n\to\infty}\left\{\left(45+\sqrt{2019}\right)^n\right\}$$

where $\{x\}$ is the fractionary part of $a \in \mathbb{R}$.

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.57. If $a, b \in \mathbb{R}$, find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)^a\left((n+1)!\right)^b}-\sqrt[n]{n^a(n!)^b}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.58. Let $H_n = \sum_{k=1}^{n}\frac{1}{k}$, find:

$$\lim_{n\to\infty} e^{-H_n}\sum_{k=2}^{n}\left(\frac{(k+1)^2}{\sqrt[k+1]{(2k+1)!!}}-\frac{k^2}{\sqrt[k]{(2k-1)!!}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.59. If $(a_n)_{n\geq1}$ is a sequence of real strictly positive numbers such that:

$$\lim_{n\to\infty}(a_{n+1}-a_n) = a \in \mathbb{R}_+^* = (0,\infty)$$

Find:

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k=1}^{n}\frac{a_k}{\sqrt[k]{(2k-1)!!}}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.60. If $(H_n)_{n\geq 1}, H_n = \sum_{k=1}^{n}\frac{1}{k}$, find:

$$\lim_{n\to\infty} e^{-mH_n}\left(\sqrt[n]{(2n-1)!!}\right)^m$$

where $m \in \mathbb{N}^*$.

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.61. Find:

$$\lim_{n\to\infty}\frac{\sqrt[3]{(n+1)(n+2)\cdot\ldots\cdot(4n)}}{n^3}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.62. Let $(H_n)_{n\geq 1}, H_n = \sum_{k=1}^{n}\frac{1}{k}$. Find:

$$\lim_{n\to\infty} e^{-H_n}\sum_{k=2}^{n}\left(\sqrt[k+1]{(2k+1)!!}-\sqrt[k]{(2k-1)!!}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

XI.63. Let be $A \in M_5(\mathbb{R})$, invertible such that: $\det(A^2+I_5)=0$.Prove that:

$$Tr\,A^* = 1 + \det A\cdot Tr\,A^{-1}$$

Proposed by Marian Ursărescu-Romania

XI.64. Let be $A \in M_4(\mathbb{R})$; $\det A = 1$; $\det(A^2+I_n)=0$. Prove that: $\mathrm{Tr}(A^{-1}) = \mathrm{Tr}\,A$

Proposed by Marian Ursărescu-Romania

XI.65. Let $A \in M_3(\mathbb{R})$ invertible such that: $Tr\,A = Tr\,A^{-1} = 1$. Prove that:

$$\det(A^2+A+I_3) \geq 3\det A$$

Proposed by Marian Ursărescu-Romania

XI.66. If $A \in M_3(\mathbb{R})$; $Tr(A^2)=0$; $\det = 1$ then: $\det(A^2+A+I_3) \geq (Tr\,A)^3$

Proposed by Marian Ursărescu-Romania

XI.67. Let be $A \in M_5(\mathbb{R})$ such that $AA^T = I_5$ and $Tr\,A = Tr\,A^2 = 0$. Find A^{2020}.

Proposed by Marian Ursărescu-Romania

XI.68. If $A, B \in M_4(\Omega)$; $AB = \begin{pmatrix} p & p & p & p \\ 0 & -p & -p & -p \\ 0 & 0 & p & p \\ 0 & 0 & 0 & -p \end{pmatrix}$; $p \in \mathbb{C}, p \neq 0$; $\Omega_1 = BA$;

$$\Omega_2 = (BA)^{-1} \text{ then find: } \Omega = \Omega_1^2 + (p^2\Omega_2^{-1})^2$$

Proposed by Marian Ursărescu-Romania

XI.69. If $A \in M_2(\mathbb{R})$; $Tr\,A = \det A = 1$ then:$\det(A^2+3A+3I_2) \geq 5Tr(A^{-1}) + 3$

Proposed by Marian Ursărescu-Romania

XI.70. If $A \in M_4(\mathbb{Q})$, $\det\left((1-i)A + \sqrt{2}I_4\right) = 0$ then: $\det(A + xI_4) \geq 2x^2, x \in \mathbb{R}$

Proposed by Marian Ursărescu-Romania

XI.71. If $A \in M_6(\mathbb{R})$ such that $det(A^4 + pA^2 + p^2I_6) = det(A^2 + qI_6) = 0, p, q \in \mathbb{R}$ then find: $\Omega = det(A)$.

Proposed by Marian Ursărescu-Romania

XI.72. If $A \in M_2(\mathbb{R})$ such that $det(A^4 + 4I_2) = 0$. Prove that: $(detA)^2 = (trA)^2$.

Proposed by Marian Ursărescu-Romania

XI.73. If $A \in M_n(\mathbb{R})$; $A^3 = 2A^2 + 7A + 4I_n$ then find: $\Omega = \det(A^2 - 3A + 3I_n)$

Proposed by Marian Ursărescu-Romania

XI.74. If $A \in M_3(\mathbb{R}), Tr\ A = \det A = 1$. Prove that: $\det(A^2 + A + I_3) \geq 3Tr\ (A^{-1})$.

Proposed by Marian Ursărescu-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

12-CLASS-STANDARD

RMM Romanian Mathematical Magazine

XII.1. Să se calculeze:

$$a)\ \int_1^e \frac{x^e\left(1-\frac{e}{x}\right)}{e^x\left(1+\frac{x^e}{e^x}\right)^2}dx$$

$$b)\ \int_{-\sqrt{2}/2}^{\sqrt{2}/2} x\ln\left(1+e^{x\sqrt{1-x^2}}\right)dx$$

Proposed by Florică Anastase

XII.2. Să se calculeze:

$$I = \int_0^{\pi} \frac{(x+1)sinx}{3+cos^2x}dx$$

Proposed by Florică Anastase

XII.3. Să se calculeze:

$$I = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{xarctgx}{1+e^{tgx}}dx$$

Proposed by Florică Anastase

XII.4. If $a > 1$ then:

$$\frac{4\log 2}{\pi} + \int_1^{a} \frac{2x\tan^{-1}x - \log(1+x^2)}{(1+x^2)(\tan^{-1}x)^2}dx < \frac{a^2}{\tan^{-1}a}$$

Proposed by Daniel Sitaru – Romania

XII.5. If $a,b,c > 0, a+b+c = 3$ then:

$$\int_0^{\frac{\pi}{2}} a^{\sin x}\,dx + \int_0^{\frac{\pi}{2}} b^{\sin x}\,dx + \int_0^{\frac{\pi}{2}} c^{\sin x}\,dx \leq \frac{3\pi}{2}$$

Proposed by Daniel Sitaru – Romania

XII.6. If $f:[a,b] \to (0,\infty), 0 < a < b, f -$ continuous then:

$$6\int_a^b \left(1-f^7(x)\right)\left(1+f^5(x)\right)dx + 5\int_a^b f^{12}(x)\,dx \geq 5(b-a)$$

Proposed by Daniel Sitaru – Romania

XII.7. Prove that:

$$\int_0^{\frac{1}{2}} \left(\log(1+x)\log\left(\frac{3}{2}+x\right)\right)dx \leq \frac{1}{2}\left(\int_0^1 \log(1+x)\,dx\right)^2$$

Proposed by Daniel Sitaru – Romania

XII.8. Prove that:

$$\int_0^1 \left(\tan^{-1}x + \frac{x}{1+x^2}\right)^2 dx + 4\int_0^1 \frac{1}{(1+x^2)^4} > \frac{(x+2)^2}{16}$$

Proposed by Daniel Sitaru – Romania

XII.9. If $a,b,c>0, a+b+c=9$ then:

$$\int_0^3 e^{x^2}dx+\sum_{cyc}\frac{1}{a}\int_0^a e^{x^2}dx\ge 4\sum_{cyc}\frac{1}{9-a}\int_0^{\sqrt{bc}} e^{x^2}dx$$

Proposed by Daniel Sitaru – Romania

XII.10. Find without softs:

$$\Omega=\int_1^e\left(\frac{e^x(1+\log x-\log^2 x)}{e^{2x}+(x\log x)^2}\right)dx$$

Proposed by Marin Chirciu – Romania

XII.11. Find $x\in\left(0,\frac{\pi}{2}\right)$ such that:

$$7\sin 2x+\frac{32}{\log 4}\int_{-x}^{x}\left(\sin t\cdot\log\left(2^{\sin^3 t}+2^{\cos^3 t}\right)\right)dx=12x$$

Proposed by Daniel Sitaru – Romania

XII.12. Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{n!}\cdot\int_{\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}}^{\frac{\pi^2}{6}} e^{x^2}dx\right)$$

Proposed by Daniel Sitaru – Romania

XII.13. Find without softs:

$$\Omega=\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}}\frac{x}{\sin 2x}dx$$

Proposed by Daniel Sitaru – Romania

XII.14. Find:

$$\Omega=\lim_{n\to\infty}\left(n\cdot\int_{\frac{n^2}{\sqrt[n]{n!}}}^{\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}}\frac{\sqrt[n]{e^x}}{x}dx\right)$$

Proposed by Daniel Sitaru – Romania

XII.15. Find without softs:

$$\Omega=\int_{\frac{1}{e}}^{e}\frac{dx}{(1+x^2)(1+x\log^7 x)}$$

Proposed by Daniel Sitaru – Romania

XII.16 If $a\geq 1$ then:

$$4\left(\sqrt{a}-1\right)^2+\left(\int_1^a\sqrt{1-\frac{1}{x}}dx\right)^2\leq (a-1)^2$$

Proposed by Daniel Sitaru – Romania

XII.17. Prove:

$$\frac{\pi}{4}<\int_0^\pi e^{\sin x+\cos x-2}\,dx<\frac{\pi}{2}$$

Proposed by Jalil Hajimir-Canada

XII.18. Find:

$$\int\left(x^2+x+1-\frac{8}{x^4}-\frac{4}{x^3}-\frac{2}{x^2}\right)6^{\left(x+\frac{2}{x}\right)}dx$$

Proposed by Jalil Hajimir-Canada

XII.19. Prove:

$$\frac{\pi}{16}<\int_0^1\sqrt{\frac{x(1-x)}{\sin\pi x+\cos\pi x+2}}dx<\frac{\pi}{8}$$

Proposed by Jalil Hajimir-Canada

XII.20. Let f be continuous on $[0,1]$. If $af(b)+bf(a)\leq 2;\forall a,b\in[0,1]$, prove:

$$\int_0^1 f(x)\,dx\leq\frac{\pi}{2}$$

Proposed by Jalil Hajimir-Canada

XII.21. Prove:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left((\sec x)^{2\sec^2 x-1}+(\csc x)^{2\csc x^2-1}\right)dx>\frac{\pi}{\sqrt{3}}$$

Proposed by Jalil Hajimir-Canada

XII.22.

$$\int \sec x\,(\sec x+\tan x)^n\, n^{(\sec x+\tan x)^n}dx=?$$

Proposed by Jalil Hajimir-Canada

XII.23. If $f:[a,b]\to\left(0,\frac{\pi}{2}\right)$, f – continuous, $a\le b$ then:

$$\int_a^b \sin f(x)\,dx+\frac{1}{2}\int_a^b \tan f(y)\,dy+\int_a^b \cos f(t)\,dt+\frac{1}{2}\int_a^b \cot f(z)\,dz\ge\left(\sqrt{2}+1\right)(b-a)$$

Proposed by Daniel Sitaru – Romania

XII.24. Prove that:

$$\frac{\pi+4}{\pi-4}+\int_1^a \frac{(\tan^{-1}x)^2}{(x-\tan^{-1}x)^2}dx>\frac{1+\sin a\cdot\tan^{-1}a}{\tan^{-1}a-a},\ a>1$$

Proposed by Daniel Sitaru – Romania

XII.25. If $a,b,c\in(0,1)$, $a+b+c=1$ then:

$$\int_0^{\sqrt[4]{a}}\left(\frac{x^3+x^2+1}{x-1}\right)^2dx+\int_0^{\sqrt[4]{b}}\left(\frac{x^3+x^2+1}{x-1}\right)^2dx+\int_0^{\sqrt[4]{c}}\left(\frac{x^3+x^2+1}{x-1}\right)^2dx>1$$

Proposed by Daniel Sitaru – Romania

XII.26. Prove without softs:

$$\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}}\frac{(1+\sin x)(1+x\sin x)(1+x\sin x\cos x)}{\sin x\cos x\,(1+x\cos x)}dx>\pi$$

Proposed by Jalil Hajimir-Canada

XII.27. Let $f\in C^1[0,1]$ and $m\le[f'(x)]\le M;\forall x\in[0,1]$

$$\text{Prove: }\frac{1}{10}m^2\le\int_0^1 f^2(x)\,dx-\left(\int_0^1 f(x)\,dx\right)^2\le\frac{1}{12}M^2$$

Proposed by Jalil Hajimir-Canada, Dinu Șerbănescu – Romania

XII.28. Prove that

$$\int_0^1\frac{\left(ln(1+x)\right)^{\boldsymbol{n}}}{1+x}dx=\left(\int_0^1 arctan(1-x+x^2)\,dx\right)^{\boldsymbol{n+1}}.\int_0^1 x^{\boldsymbol{n-1}}.\sqrt[n]{1-x^{\boldsymbol{n}}}\,dx,\ n\in\mathbb{N}-\{0\}$$

Proposed by Mohammed Bouras-Morocco

XII.29. If $f:[a,b]\to[0,\infty), a<b, f$ – continuous, then:

$$\int_a^b \left(\sqrt[3]{f(x)}+\sqrt[3]{x}\right)^3 dx + 2\int_a^b \sqrt{xf(x)}\,dx \ge 8\int_a^b xf(x)\,dx + \int_a^b f(x)\,dx + \frac{(b-a)^2}{2}$$

Proposed by Daniel Sitaru – Romania

XII.30. If $f:[a,b]\to(0,\infty);\ a<b; f$ continuous, then:

$$3(b-a)\int_a^b f^2(x)\,dx + (b-a)^2 \ge 2(b-a)\int_a^b f(x)\,dx + 2\left(\int_a^b f(x)\,dx\right)^2$$

Proposed by Daniel Sitaru – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.1. The Lucas numbers are defined as following: $L_0=2, L_1=1$ and $L_{n+2}=L_{n+1}+L_n$

then show that:

$$\sum_{n}^{\infty}\frac{L_n}{n(2n+1)5^n} = 4-2\sqrt{\frac{10}{1+\sqrt{5}}}\arctan\left(\sqrt{\frac{1+\sqrt{5}}{10}}\right)-$$

$$-(5+\sqrt{5})\sqrt{\frac{2}{1+\sqrt{5}}}\arctan\left(\sqrt{\frac{2}{5(1+\sqrt{5})}}\right)+\log\left(\frac{25}{19}\right)$$

Proposed by Mokhtar Khassani-Algerie

U.2. General Version of Prof. Dan Sitaru's limit. If $b\in\mathbb{N}$ and

$\phi(b) = \lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(\int_0^1 e^{\frac{x^{k^b}}{n}}dx\right)\right)^n$ then prove

$$\lim_{k\to\infty}\frac{\phi(1)}{k}\left(\sum_{k=1}^{\infty}\log(\phi(2))\right) = \frac{e^{\gamma}(\pi e^{2\pi}+\pi-e^{2\pi}+1)}{2e(e^{2\pi}-1)}$$

Proposed by Naren Bhandari-Bajura-Nepal

U.3. Generalized version for Prof. Dan Sitaru's problem

Find:

$$\phi(k) = \lim_{n\to\infty}\frac{1}{n^{k+1}}\left(\sum_{1\le k_1<k_2<\cdots<k_m\le n}(-1)^{k_1+k_2+,,,+k_m}\prod_{i=1}^{m}k_i\right)$$

Proposed by Naren Bhandari-Bajura-Nepal

U.4. A modified old problem of Prof. Dan Sitaru. Prove that:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{m=1}^{n}\sum_{k=1}^{m}\big((H_k-\log k)(H_{m-k+1}-\log(m-k+1))\big) = \gamma^2$$

where γ is Euler's Mascheroni constant

Proposed by Naren Bhandari-Bajura-Nepal

U.5. Bounding of Prof. Dan Sitaru's limit by Naren. Prove that:

$$\sqrt{\frac{5e}{\tilde{n}}} < \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k(k+1)}{n(n+1)(n+2)}\exp\left(\frac{k(k+1)(2k+1)}{n(n+1)(n+2)}\right) < \sqrt{\frac{5e}{\widetilde{N}}}$$

where $1\le\widetilde{N}\le 11$ and $12\le\tilde{n}\in\mathbb{N}<8$; notation $\exp(x) = e^x$

Proposed by Naren Bhandari-Bajura-Nepal

U.6. Find:

$$\Omega = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{(-1)^r}{2^{2n}(2n+1)(2n+2+r)}\binom{2n}{n}\right)$$

Proposed by Naren Bhandari-Bajura-Nepal

U.7. Evaluate the following sum in a closed form:

$$\sum_{k}^{\infty}\left(\frac{1}{6k+1}+\frac{1}{6k+2}+\frac{1}{6k+3}-\frac{1}{6k+4}-\frac{1}{6k+5}-\frac{1}{6k+6}\right)$$

Proposed by Prem Kumar-India

U.8. Evaluate the following sum:

$$\sum_{k=0}^{\infty}\left(\frac{1}{(4k+1)!}+\frac{1}{(4k+2)}-\frac{1}{(4k+3)!}-\frac{1}{(4k+4)!}\right)$$

Proposed by Prem Kumar-India

U.9. Let $\pi(x)$ denotes the prime counting function and p_n denotes the n^{th} prime number. Find:

$$\Omega = \lim_{n\to\infty} p_n^{\frac{\pi(n)}{n}}$$

Proposed by Prem Kumar-India

U.10. Integrate:

$$I = \int_0^{\frac{1}{2}} \frac{x(1-x)}{\sin(\pi x)}dx$$

Proposed by Prem Kumar-India

U.11. Find the closed form:

$$\Omega = \sum_n^{\infty} \frac{\left\{\frac{1}{n^4+n^2+1}+\frac{1}{8}\right\}}{n^2+1}, \{x\} = x-[x], [*] \text{ - great integer function}$$

Proposed by Mokhtar Khassani-Algerie

U.12. Evaluate the integral in a closed form:

$$M = \int_0^1 \frac{\ln(1+x^2)\, Li_2(x)}{1+x^2}dx$$

Proposed by Mokhtar Khassani-Algerie

U.13. If $a_1 = 3$ and $a_{n+1} + 2 = a_n^2$ then find: $M = \lim_{n\to\infty} n\left(3-\sqrt{5}-\sum_k^n \frac{2}{\prod_{j=1}^k a_j}\right)$

Proposed by Mokhtar Khassani-Algerie

U.14. Evaluate:

$$\lim_{n\to\infty} n\left(1-\sum_{k=1}^{n}\left(\frac{1}{k}\right)^{\frac{2020}{2019}}\right)$$

Proposed by Mokhtar Khassani-Algerie

U.15. Find all functions $\phi(t)$ such that:

$$\begin{cases} \phi(t) = \int_1^{\tau} \left(\frac{\phi(x)}{x} + e^{-\frac{\phi(x)}{x}}\right) dx \\ \phi\left(\frac{1}{\tau}\right) = \frac{1}{t^2}\int_1^{\tau} \left(x\phi\left(\frac{1}{x}\right) - e^{-x\phi\left(\frac{1}{x}\right)}\right) dx \end{cases}$$

Proposed by Mohammed Bouras-Morocco

U.16. Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{2}H_n + \log\left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right)\right)$$

Proposed by Daniel Sitaru – Romania

U.17. Let $\phi(x) = \underbrace{x^{x^{\cdot^{\cdot^{x}}}}}_{for\ "n"\ times}$, $n \in \mathbb{N} - \{0\}$. Prove that:

$$\phi_n\left(\sqrt[n]{n + \frac{1}{n}}\right) + \phi_n\left(\sqrt[n]{n - \frac{1}{n}}\right) \geq 2\phi_n\left(\sqrt[n]{n}\right)$$

Proposed by Mohammed Bouras-Morocco

U.18. Let $\varphi_n(m) = \underbrace{\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}\ \cdots\ \cfrac{}{1+\cfrac{1}{1+m}}}_{for\ "n"\ times\ division}$; $\varphi_1(m) = \frac{1}{m}, \varphi_2(m) = \frac{m}{m+1}; n, m \in \mathbb{N} - \{0\}$

Prove that: $\varphi_n(m) = \frac{m\cdot F_{n-1}+F_{n-2}}{m\cdot F_n+F_{n-1}}$. Then $\varphi_n(i) = \sqrt{1 - \varphi_{2n}\left(1 - \frac{F_{2n-1}}{F_{2n}}\right)} + i\prod_k^{2n-1} \varphi_k(1)$

F_n – Fibonacci number

Proposed by Mohammed Bouras-Morocco

U.19. Let $A \geq 4$ numbers pair, (P_i, P_i') prime numbers

$$A = P_1 + P_1' = P_2 + P_2' = \cdots = P_n + P_n' \ (n \text{ solution})$$

Prove that: $\begin{cases} A > \sum_{i=1}^{n} \sqrt{P_i \cdot P_i'} \ if\ n \leq 2 \\ A < \sum_{i=1}^{n} \sqrt{P_i \cdot P_i'} \ if\ n \geq 3 \end{cases}$

Proposed by Mohammed Bouras-Morocco

U.20. Let $\phi_n(a) = \underbrace{\sqrt{a + \sqrt{a + \sqrt{a + \cdots + \sqrt{a}}}}}_{for\ "n"\ times\ "a"}$, $a > 0$

Prove that: $\lim_{n\to\infty}\left(\frac{\sqrt{a+\phi_n(a)\sqrt{a-\phi_n(a)}}}{\phi_n(a)+\sqrt{a-\phi_n(a)}}\times\frac{\sqrt{a-\phi_n(a)\sqrt{a-\phi_n(a)}}}{\phi_n(a)-\sqrt{a-\phi_n(a)}}\right)=\frac{1}{2}$

Proposed by Mohammed Bouras-Morocco

U.21. Let: $\phi_n = \int_0^{+\infty}\frac{1}{1+x+x^2+\cdots+x^n}dx$

Prove that: $\phi_{13}+\phi_6 = \frac{2\pi}{7}\left(\sin\left(\frac{\pi}{7}\right)+\cos\left(\frac{3\pi}{14}\right)\right)$

Proposed by Mohammed Bouras-Morocco

U.22. Prove the relation

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin^3(2x)\log(\log(\tan(x)))\,dx = \frac{1}{6}\left(2\log(\pi)-\frac{7\zeta(3)}{\pi^2}-2\gamma-4\log(2)\right)$$

$$\int_0^1\int_0^1\frac{\log\left(\frac{1}{x}\right)-\log\left(\frac{1}{y}\right)}{\log\left(\log\left(\frac{1}{x}\right)\right)-\log\left(\log\left(\frac{1}{y}\right)\right)}dx\,dy = \int_0^1\int_0^1\frac{\log\left(\frac{y}{x}\right)}{\log(-\log(x))-\log(-\log(y))}dy\,dx$$
$$=\frac{7\zeta(3)}{\pi^2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.23. Prove that:

$$\int_0^\infty\frac{e^{-x}}{e^{2x}+1}\log^2\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{3}-\frac{\pi^3}{16}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.24. Evaluate in a closed – form

$$\int_0^\infty\frac{\cos(\sqrt{x})}{e^{2\pi\sqrt{x}}-1}dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.25. Prove the sum

$$1-\frac{1}{4}\left(\frac{2}{3}\right)^3+\frac{1}{7}\left(\frac{2\times 5}{3\times 6}\right)^3-\frac{1}{10}\left(\frac{2\times 5\times 8}{3\times 6\times 9}\right)^3+\cdots = \frac{3\sqrt{3}}{4(2\pi)^5}\Gamma\left(\frac{1}{3}\right)^9$$

Proposed by Srinivasa Raghava-AIRMC-India

U.26. For $n>1$, prove the inequality:

$$\frac{1}{\pi(n+1)^2} < \int_{\pi n}^{\pi(n+1)} \frac{1-\cos(x)}{x^2} dx < \frac{1}{\pi n^2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.27. Evaluate the sum:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4(-1)^{m-1}}{2m-1} \left(\frac{1}{F_{2n+1}}\right)^{2m-1}$$

F_k – Fibonacci number

Proposed by Srinivasa Raghava-AIRMC-India

U.28. Solve for β

$$\int_{-\infty}^{\infty} \frac{(\beta + x\coth(2\pi x))^2}{\cosh^2(\pi x)} dx = 0$$

Proposed by Srinivasa Raghava-AIRMC-India

U.29. If for any complex number $n, Re(n) > 0$, $\theta(n) = \int_{e^{-x}}^{\infty} e^{-nx^2} dx$

then show that

$$\int_{-\infty}^{\infty} \theta(n) e^{-nx} dx = \frac{1}{2} n^{-\frac{n}{2}-\frac{3}{2}} \Gamma\left(\frac{n+1}{2}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.30. Let, $S(x) = \int_0^x \frac{(x^2+y+1)^2}{(x^2-y+1)^3} dy$

then evaluate the integral in a closed – form

$$\int_{-\infty}^{\infty} \frac{S(x)}{x} dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.31. Evaluate the sum

$$\sum_{m=0}^{n-1} \left(\frac{\sqrt{5} + 5 - 4\sin^2\left(\frac{\pi n}{n}\right)}{2\left(5 - 4\sin^2\left(\frac{\pi m}{n}\right)\right)} - \frac{\phi^2 + \cos\left(\frac{2\pi m}{n}\right)}{3 + 2\cos\left(\frac{2\pi m}{n}\right)} \right)$$

ϕ – Golden Ratio

Proposed by Srinivasa Raghava-AIRMC-India

U.32. If

$$\alpha = \frac{4\pi}{3} - \int_0^1 \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sqrt{y}\tan^2\left(\frac{x}{2}\right) + 1}} dy$$

then prove that: $9\alpha(9\alpha+64)=2176$

Proposed by Srinivasa Raghava-AIRMC-India

U.33. If

$$S(n)=1^n+2^n+3^n+4^n+5^n+6^n+7^n+8^n+9^n$$
$$S(24n+19)\equiv 0\ (mod\ 3^3\times 5^2)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.34.

$$\frac{107811}{3},\frac{110778111}{3},\frac{111077781111}{3},\frac{111107777811111}{3},\frac{111110777778111111}{3},\dots$$

Prove that all the numbers in the above sequence are perfect cubes.

Proposed by Srinivasa Raghava-AIRMC-India

U.35. Prove this sharp inequality:

$$\sum_{k=0}^{\infty}\frac{1}{k!+k!!}>e\pi\sum_{k=0}^{\infty}\frac{(-1)^k}{k!+k!!}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.36. For $n\geq 0$

$$\Lambda(n)=\int_0^{\infty}\frac{(1+x)e^{-nx}}{\sqrt{1+\cosh(x)}}dx$$

then compute the integral in a closed – form

$$\int_0^{\infty}\Lambda(n)\,e^{-n}dn$$

Proposed by Srinivasa Raghava-AIRMC-India

U.37. Find without softs:

$$\Omega=\int_0^{\infty}\left(5^{-3x^2+9x-7}-\left[5^{-3x^2+9x-7}\right]\right)dx,\ [*]-\text{great integer function}$$

Proposed by Jalil Hajimir-Canada

U.38. Find without softs:

$$\Omega=\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\sin(x+y)\csc\left(x+y+\frac{\pi}{4}\right)dx\,dy$$

Proposed by Jalil Hajimir-Canada

U.39. If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \int_a^b \left(\frac{(x+y+z)(xy+yz+zx)}{xyz}\right) dx\, dy\, dz \leq \frac{(2a^2+5ab+2b^2)(b-a)^3}{ab}$$

Proposed by Daniel Sitaru – Romania

U.40. If $f: \mathbb{R} \to (0,\infty), f$ – continuous, $a, b \in \mathbb{R}, a \leq b$ then:

$$\int_a^b \int_a^b \left(\log\left(\frac{(1+f(x))(1+f(y))}{\left(1+\frac{f(x)+f(y)}{2}\right)^2}\right)\right) dx\, dy \leq (b-a)\int_a^b f^2(x)\, dx - \left(\int_a^b f(x)\, dx\right)^2$$

Proposed by Daniel Sitaru – Romania

U.41.

$$\Omega(a) = \lim_{b\to\infty}\left(\sum_{n=1}^{\infty} \frac{n(n+1)(n+2)\cdot\ldots\cdot(n+a-1)}{(-b)^{n-1}}\right), a \in \mathbb{N} - \{0,1\}$$

Find:

$$\Omega = \sum_{a=2}^{\infty} \frac{1}{\Omega(a)}$$

Proposed by Daniel Sitaru – Romania

U.42.

$$\Omega_k(m) = 2\lim_{x\to 0}\left(\frac{1-(\cos kx)^{\frac{1}{k^{m+2}}}}{x^2}\right), k, m \in \mathbb{N}^*$$

Find a closed form for:

$$\Omega = \left(\sum_{k=1}^{\infty} \Omega_k(2)\right)\left(\sum_{k=1}^{\infty} \Omega_k(3)\right)$$

Proposed by Daniel Sitaru – Romania

U.43. If $1 < a \leq b$ then:

$$\log\left(\frac{\sqrt{b}\cdot\Gamma(b)}{\sqrt{a}\cdot\Gamma(a)}\right) \leq \int_a^b \log x\, dx \leq \log\left(\frac{b\cdot\Gamma(b)}{a\cdot\Gamma(a)}\right)$$

Proposed by Daniel Sitaru – Romania

U.44. If $0 \leq a \leq b$ then:

$$\int_a^b \int_a^b \frac{dx\,dy}{(x+y)^4} \leq \frac{(b-a)^2(a^2+ab+b^2)}{48a^3b^3}$$

Proposed by Daniel Sitaru – Romania

U.45. Inspired by Seyran Ibrahimov

Show that:

$$\int_1^\infty \frac{1}{1-\cos x + x^2}\,dx > \frac{\pi}{4}$$

Proposed by Naren Bhandari-Nepal

U.46. Find:

$$\Omega = \sum_{n=0}^{\infty}\left(\frac{1}{625n^4+1250n^3+875n^2+250n+24}\right)$$

Proposed by Naren Bhandari-Nepal

U.47. Prove the following inequality

$$\int_1^\infty \frac{dx}{1+x^2} < \int_1^\infty \frac{dx}{1+\cos x + x^2} < \int_1^\infty \frac{dx}{1-\cos x + x^2}$$

Proposed by Naren Bhandari-Nepal

U.48.

$$\Phi(k) = \lim_{n\to\infty}\left(2+\frac{3^2}{2}+\frac{4^3}{3}+\cdots+\frac{(n+k)^n}{n^{n-1}}\right)$$

$$\Phi(k') = \lim_{n\to\infty}\frac{\sqrt{(n-k')!}}{\left(1+\sqrt{1^{k'}}\right)\left(1+\sqrt{2^{k'}}\right)\ldots\left(1+\sqrt{n^{k'}}\right)}$$

where $k > 0$ and $1 \leq k' < n$. Find $\Phi(k)+\Phi(k')$

Proposed by Naren Bhandari-Nepal

U.49. Let $(a_n)_{n=1}^{\infty}$ be a sequence and let $s_k = a_1 + \cdots + a_k = \sum_{n=1}^{k} a_n$ be it's k^{th}partial sum. The sequence (a_n) is called Cesàro summable, with Cesàro sum $A \in R, A$, as it as n tends to infinity the arithmetic mean of its first n partial sums $s_1, s_2, \ldots, s_n$ tends to A:

$\lim_{n\to\infty}\frac{1}{n}\sum_k^n s_k = A$. It is well known that

$G = 1-1+1-1+1-1+\cdots$

Is Grandi Series which is divergent in nature.

However, the series G is Cesàro summable giving Cesàro sum $\frac{1}{2}$. Now if we define a divergent series $C = 1 + 4 + 9 + 16 + \cdots$

It is true that series C is also Cesàro summable?.If yes, find the sum.

Proposed by Naren Bhandari-Nepal

U.50. $n \in \mathbb{N} - \{0\}$, fixed, $x_1, x_2, \dots, x_n$ – are different in pairs. Find the greatest value of $M \in \mathbb{N}, n \leq M$ such that:

$$4074341 < \sum_{n=1}^{M} \sqrt{x_n + \sqrt{x_n + \sqrt{x_n + \cdots}}} < 4074343$$

Proposed by Naren Bhandari-Nepal

U.51. Show that:

$$\int_0^1 \frac{1 + x + \{x\}^2 + \left\{\frac{1}{1+x}\right\}}{1 + \{x\} + \left\{\frac{1}{1+x}\right\} + \left\{\frac{2}{1+x}\right\} + \left\{\frac{3}{1+x}\right\}} dx =$$

$$= \frac{8}{\sqrt{5}} \arctan\left(\frac{1}{2\sqrt{5}}\right) - \frac{8}{\sqrt{5}} \arctan\left(\frac{1}{\sqrt{5}}\right) - \frac{19}{\sqrt{15}} \arctan\left(\frac{1}{\sqrt{15}}\right) + \log\left(\frac{7\sqrt{105}}{64\sqrt{2}}\right) + 3$$

$\{.\}$: the fractional part function

Proposed by Mokhtar Khassani-Algerie

U.52. Find a closed form:

$$\Omega = \int_0^1 \left(\frac{\log^4 x}{\sqrt{1 - x^2}}\right) dx$$

Proposed by Naren Bhandari-Nepal

U.53. Generalized version of Kays Tomy summation.Show that:

$$\sum_{n=1}^{\infty} \left(\int_0^1 \int_0^2 \int_0^3 \dots \int_0^n \left\{ \sum_{k=1}^{n} x_k \right\} dx_1 dx_2 \dots dx_n \right)^{-1} = 2(e - 1)$$

Notation: $\{x\}$ represents fractional part of x and $e \approx 2.718281 \dots$ is Euler's number.

Proposed by Naren Bhandari-Nepal

U.54. Prove that:

$$\prod_{m=1}^{\infty} \prod_{s=1}^{m} \left(\lim_{n\to\infty} \frac{1}{\sqrt[n+1]{n}} \left(\prod_{m=1}^{n} k^{k^n} \right)^{\frac{1}{n^{s+1} m^2}} \right) = \frac{e^3}{e^{\zeta(2) + \frac{7}{4}\zeta(4)}}$$

Proposed by Naren Bhandari-Nepal

U.55. Inspired by Mokhtar Khassani. Without Software, show that:

$$\left\lfloor 10^4 \int_{\frac{1}{10^4}}^{\infty} \frac{e^{-x}\ln(1+x^2)\tan\left(1+10^4x^{10^4}\right)}{1+x^{10^4}}dx \right\rfloor = 2019$$

where $\lfloor . \rfloor$ denotes floor function.

Proposed by Naren Bhandari-Nepal

U.56. Prove that for all $|k| > 1$

$$2\sum_{k=2}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{k^n(k^nm+k^mn)} = \zeta(2)+2\zeta(3)+\zeta(4)$$

where $\zeta(.)$ denotes Riemann Zeta Function

Proposed by Naren Bhandari-Nepal

U.57. Find the closed form:

$$\Omega = \sum_{n=0}^{\infty}\frac{H_n^{(2)}}{n(n+1)\cdot 8^n}$$

Proposed by Mokhtar Khassani-Algerie

U.58. Compute:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{H_{n+1}H_{2n+3}\log(n+1)\binom{2n+3}{n+1}} - \sqrt[n]{H_nH_{2n+1}\log(n)\binom{2n+1}{n}}\right)$$

Proposed by Mokhtar Khassani-Algerie

U.59. Find:

$$\lim_{n\to\infty}\left(e+1-\left(\zeta(2)-\sum_{k=2}^{n}\frac{1}{k^2}\right)^n\right)^n$$

Proposed by Mokhtar Khassani-Algerie

U.60. Find:

$$\lim_{n\to\infty} n^3\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{(1+ij)\binom{5n}{i+j}}$$

Proposed by Mokhtar Khassani-Algerie

U.61. Show that:

$$\int_0^1 x^2\log(2+x)\log(2-x)\,dx = \frac{2}{27}\left(72Li_2\left(\frac{1}{4}\right)+144\log^2 2-54\log 3-6\pi^2+31\right)$$

Proposed by Mokhtar Khassani-Algerie

U.62. Show that:

$$\int_0^\infty \frac{\log x}{1+e^{2x}+e^{4x}+e^{6x}}dx = \frac{\pi}{16}\log\left(\frac{\Gamma^4\left(\frac{1}{4}\right)}{2\pi^3}\right)-\frac{11}{16}\log^2 2$$

Proposed by Mokhtar Khassani-Algerie

U.63. Find:

$$\Omega = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{k\cdot(2n-2k-1)!!}{(k+1)!\cdot(2+2n-2k)!!}\right)$$

Proposed by Daniel Sitaru – Romania

U.64. Find without softs (without Wolfram, MathCad, Maple, MathLab, Derive, etc)

$$\Omega = \int_0^\infty \left(\frac{\sin(2016!\,(mod\ 2017)\cdot x)}{e^{2\pi x}-1}\right)dx$$

Proposed by Naren Bhandari-Nepal

U.65. Prove:

$$\int_a^{2a}\int_a^{2b}\int_a^{2c}\sqrt{\frac{e^x+e^y+e^z+3\sqrt[3]{e^{x+y+z}}}{\sqrt{e^{x+y}}+\sqrt{e^{y+z}}+\sqrt{e^{z+x}}}}\,dx\,dy\,dz \geq \sqrt{2}abc$$

Proposed by Jalil Hajimir-Canada

U.66. Let $0 < a \leq b$. Prove:

$$\int_a^{2a}\int_a^{2b}\frac{xy\sin\sqrt{xy}}{(x+y)\sin\left(\frac{2x}{x+y}\right)}dx\,dy \geq \frac{2}{9}ab\sqrt{ab}$$

Proposed by Jalil Hajimir-Canada

U.67. Prove that:

$$\sum_{n}^{\infty}\frac{\exp(-\pi n)}{n(n+1)(2n+1)(2n)!!} =$$

$$= Ei\left(\frac{\exp(-\pi)}{2}\right)-2\exp\left(\frac{\pi}{2}\right)\sqrt{2\pi}\,\mathrm{erfi}\left(\frac{\exp\left(-\frac{\pi}{2}\right)}{\sqrt{2}}\right)-\gamma-\log\left(\frac{\exp(-\pi)}{2}\right)+$$

$$+2\exp\left(\frac{\exp(-\pi)}{2+\pi}\right)-2\exp(\pi)+3$$

γ: Euler- Mascheroni constant, Ei: exponential integral erfi: imaginary error function

Proposed by Mokhtar Khassani-Algerie

U.68. If: $\Psi(x) = \sum_{n=0}^{\infty} \frac{4^n x^{2n+3}}{(2n+1)(2n+3)(n+1)\binom{2n}{n}}$ for $|2x| < 1$

then show that:

$$\int_0^1 (\Psi(x) + \log(1 + x^2))\, x^2 dx = \frac{27\zeta(2)}{64} - \frac{\pi}{6} + \frac{\log 2}{3} - \frac{13}{72}$$

Proposed by Mokhtar Khassani-Algerie

U.69. Find:

$$M = \int_0^{\infty} \frac{\arctan(x^4)}{1 + x^3 + x^6} dx$$

Proposed by Mokhtar Khassani-Algerie

U.70. If: $\Omega = \lim_{n\to\infty} n^2 \left(\frac{\pi}{20} + \frac{\log 2}{10} - \frac{1}{20} - \int_0^1 x^4 \cot^{-1} x \cdot e^{-\frac{x^n\sqrt{x^n - x^{2n}}}{n}} dx\right)$

then show that: $\sum_n^{\infty} \frac{F_n}{n} \Omega^n = -\frac{2}{\sqrt{5}} \coth^{-1}\left(\frac{\pi^2 - 64}{\sqrt{5\pi^2}}\right)$

F_n: is the n^{th} Fibonacci number

Proposed by Mokhtar Khassani-Algerie

U.71. Prove that:

$${}_4F_3\left(1, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{5}{2}, \frac{11}{2}, \frac{7}{2}, -1\right) = \frac{3(1260G - 105\pi - 754)}{224}$$

G: catalan's constant

Proposed by Mokhtar Khassani-Algerie

U.72. Find:

$$\Omega = \int_0^{\frac{\pi}{2}} \left(\sec\left(\frac{x}{2}\right) \cdot \sqrt[4]{\csc(2x)} \cdot \log^2(\tan x)\right) dx$$

Proposed by Mokhtar Khassani-Algerie

U.73. Show that:

$$\int_0^1 \{\log(1 + x)\} \log\{1 + x\}\, dx = 2(1 - \log 2) - \frac{\pi^2}{12}$$

$\{.\}$: is the fractional part function

Proposed by Mokhtar Khassani-Algerie

U.74. Show that:

$$\int_0^{\frac{3\pi}{4}} \left\{\frac{1}{3} + \sin(2x)\right\} dx = \frac{5}{6}\arcsin\left(\frac{2}{3}\right) - \frac{\pi}{3} + \frac{\sqrt{5}}{6} + \frac{1}{2}$$

$\{.\}$: the fractional part function

Proposed by Mokhtar Khassani-Algerie

U.75. Prove that:

$$\int_0^1 \frac{x\log^2(1+x^2)\log(1-x^2)}{1+x^2} dx = Li_4\left(\frac{1}{2}\right) + \zeta(3)\log(2) - \zeta(4) - \frac{\zeta(2)}{2}\log^2 2 + \frac{\log^4 2}{6}$$

Proposed by Mokhtar Khassani-Algerie

U.76. If $0 < a \le b$ then:

$$\int_a^b \int_a^b \sqrt{\left(1+\frac{1}{x^4}\right)\left(1+\frac{1}{y^4}\right)} dxdy \ge \frac{2(b-a)^2}{ab}$$

Proposed by Daniel Sitaru – Romania

U.77. Find a closed form:

$$\Omega = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}^n, \Omega \in M_2(\mathbb{R})$$

Proposed by Daniel Sitaru – Romania

U.78. Find a closed form:

$$\Omega = \prod_{n=1}^{\infty} \left(\frac{n^{\frac{1}{n+1}}}{2}\right)$$

Proposed by Daniel Sitaru – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-WINTER 2021

PROBLEMS FOR JUNIORS

JP.331. In acute ΔABC with the lengths $BC = a, CA = b, AB = c$. Prove that:

$$\frac{a(b+c-a)}{b^2+c^2-a^2}+\frac{b(c+a-b)}{c^2+a^2-b^2}+\frac{c(a+b-c)}{a^2+b^2-c^2}\geq 3$$

Proposed by Hoang Le Nhat Tung–Vietnam

JP.332. If $x_i > 1, \forall i = \overline{1,n}; n \in \mathbb{N}, n \geq 3$ then prove:

$$\frac{\log x_2}{\log^2(x_1^2x_2)}+\frac{\log x_3}{\log^2(x_1^2x_2^2x_3)}+\cdots+\frac{\log x_n}{\log^2(x_1^2x_2^2\ldots x_{n-1}^2x_n)}\leq\frac{\log\sqrt[4]{x_2x_3\ldots x_n}}{\log x_1\cdot\log(x_1x_2x_3\ldots x_n)}$$

Proposed by Florică Anastase-Romania

JP.333. In ΔABC the following relationship holds:

$$\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}\leq\sqrt{(ab+bc+ca)\left(2+\frac{r}{2R}\right)}$$

Proposed by Nguyen Viet Hung -Vietnam

JP.334. In ΔABC the following relationship holds:

$$\sqrt{\frac{a+b}{a-b+c}}+\sqrt{\frac{b+c}{b-c+a}}+\sqrt{\frac{c+a}{c-a+b}}\leq\frac{3R}{\sqrt{2r}}$$

Proposed by Nguyen Viet Hung -Vietnam

JP.335 If $a, b, c > 0$ such that $ab + bc + ca \leq 3$ then prove:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\ge\frac{15}{4}$$

Proposed by Nguyen Viet Hung -Vietnam

JP.336 For all positive integers $n>3$ prove that:

$$\frac{\sqrt{2n+1}-1}{2}<\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}<\frac{\sqrt{2n}}{2}$$

Proposed by Nguyen Hung Viet -Vietnam

JP.337. If $a_i,b_i\in(0,1);p,q\in\mathbb{N}^*,n\ge 2$ then prove:

$$\sum_{i=1}^{n}log_{a_i}\sqrt[n]{\frac{2a_i^{2p}\cdot b_i^{2q}}{a_i^{2p}+b_i^{2q}}}+\sum_{i=1}^{n}log_{b_i}\sqrt[n]{\frac{2a_i^{2q}\cdot b_i^{2p}}{a_i^{2q}+b_i^{2p}}}\ge\left(\sqrt{p}+\sqrt{q}\right)^2$$

Proposed by Florică Anastase-Romania

JP.338 In ΔABC, P, Q$\in \mathrm{Int}(\Delta ABC)$ such that:

$$\beta\overrightarrow{AB}+\gamma\overrightarrow{BP}+\overrightarrow{PC}=0 \text{ and } \overrightarrow{AQ}+\alpha\overrightarrow{QB}+\overrightarrow{BC}=0,\alpha,\beta,\gamma\in\mathbb{R},\alpha,\gamma\neq 1$$

Prove that A, P, Q are collinear if and only if $\alpha+\gamma=\beta+1$.

Proposed by Florică Anastase-Romania

JP.339 Solve in real numbers the system:

$$\begin{cases}11(x^4-y^4)+4xy(x^2+y^2)+x=0\\2(x^4-y^4)-22xy(x^2+y^2)+y=0\end{cases}$$

Proposed by Florică Anastase-Romania

JP.340 Prove that :

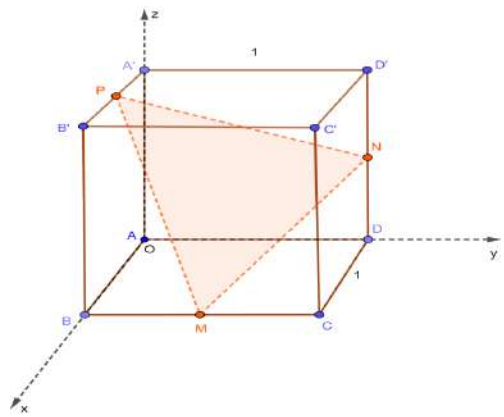
$$sin10^{\circ}=\frac{1}{4}-\frac{\sqrt{3}}{4}tan10^{\circ}+\frac{1}{4}tan^2 10^{\circ}-\frac{\sqrt{3}}{4}tan^3 10^{\circ}$$

Proposed by Pedro Henrique O. Pantoja -Brazil

JP.341 Find all positive integers n such that: $N=\frac{2^{2n}-n^2-1}{n!}$

Proposed by Pedro Henrique O. Pantoja -Brazil

JP.342. Let be $ABCDA'B'C'D'$ cube with length side 1 and $M \in BC, N \in DD', P \in A'B'$. Find minimum perimeter of ΔMNP.



Proposed by Florentin Vișescu-Romania

JP.343 In acute ΔABC, g_a −Gergonne's cevian the following relationship holds:

$$max\{g_a^2 \cdot cosA, g_b^2 \cdot cosB, g_c^2 \cdot cosC\} \ge r^2\left(1+\frac{r}{R}\right)\left(\frac{43}{9}-\frac{8R}{9r}\right)$$

Proposed by Radu Diaconu-Romania

JP.344. Let a, b, c be positive real numbers such that $ab+bc+ca=3$. Prove that:

$$(3a^5-3a+2b^3+34)(3b^5-3b+2c^3+34)(3c^5-3c+2a+34) \ge 6^6$$

Proposed by Hoang Le Nhat Tung -Vietnam

JP.345. If $a, b, c \in \mathbb{C}; |a|=|b|=|c|=3$ then:

$$\sum_{cyc}|a+3|+3\sum_{cyc}|a^2+1|+\sum_{cyc}|a^3+3| \ge 18$$

Proposed by Daniel Sitaru – Romania

PROBLEMS FOR SENIORS

SP.331. If ΔABC has inradius r, circumradius R, sides lengths $a=BC, b=AC, c=AB$, and altitudes h_a, h_b, h_c from the vertices A, B, C, respectively, then:

$$\frac{9r^2}{R} \leq \frac{c}{b+c} \cdot h_a + \frac{a}{c+a} \cdot h_b + \frac{b}{a+b} \cdot h_c \leq \frac{9R}{4}$$

Proposed by George Apostolopoulos-Greece

SP.322. Let a, b, c be the lengths of the sides of a triangle ABC with inradius r and circumradius R. Prove that:

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \leq \frac{3\sqrt{6}R}{4r}\sqrt{R^2 - 2r^2}$$

Proposed by George Apostolopoulos-Greece

SP.333. Let $x, y, z > 0$ be positive real numbers such that $x + y + z = 3$.

Find the maximum value of expression:

$$P = \frac{x}{2\sqrt{y} + \sqrt{z}} + \frac{y}{2\sqrt{z} + \sqrt{x}} + \frac{z}{2\sqrt{x} + \sqrt{y}} + \frac{(x+y)(y+z)(z+x)}{16}$$

Proposed by Hoang Le Nhat Tung -Vietnam

SP.334. Let x, y, z be a positive real numbers such that $x + y + z = 1$. Prove that:

$$(3x^2 + 1)(3y^2 + 1)(3z^2 + 1) \geq 27(xy + z)(yz + x)(zx + y)$$

Proposed by Hoang Le Nhat Tung -Vietnam

SP.335. Let $x, y, z > 0$ positive real numbers such that

$$\left(\sqrt{x^3} + \sqrt{y^3}\right)\left(\sqrt{y^3} + \sqrt{z^3}\right)\left(\sqrt{z^3} + \sqrt{x^3}\right) = 8$$

Prove that: $x + y + z \geq \sqrt[3]{xyz(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2)}$

Proposed by Hong Le Nhat Tung -Vietnam

SP.336. Let x, y, z be a positive real numbers such that $(x^6 + y^6)(y^6 + z^6)(z^6 + x^6) = 8$

Prove that: $(3x^2 - 4xy + 3y^2)(3y^2 - 4yz + 3z^2)(\,3z^2 - 4zx + 3x^2) \geq 9$

Proposed by Hoang Le Nhat Tung -Vietnam

SP.337. Let $x, y, z > 0$.

1) If $xy + yz + zx \le 3(2\sqrt{3} - 3)$ then $\sqrt{\frac{xy+yz+zx}{3}} + 1 \le \sqrt[3]{(x+1)(y+1)(z+1)}$.

2) If $xy + yz + zx > 3(2\sqrt{3} - 3)$ then

$$\sqrt{xy + yz + zx} + 1 < \sqrt{(x+1)(y+1)(z+1)}.$$

Proposed by Florentin Vișescu – Romania

SP.338. If $t \in [0, 2\pi); n \in \mathbb{N}$ then:

$$|1 + \cos nt + i \sin nt| + |1 + \cos 2nt + i \sin 2nt| + |1 + \cos 3nt + i \sin 3nt| \ge 2$$

Proposed by Daniel Sitaru – Romania

SP.339. Solve for real numbers:

$$\sqrt{x^3 - 2x^2 + 2x} + 3\sqrt[3]{x^2 - x + 1} + 2\sqrt[4]{4x - 3x^4} = \frac{x^4 - 3x^3}{2} + 7$$

Proposed by Hoang Le Nhat Tung -Vietnam

SP.340. Find all pairs of integers (x, y) such that $x^4 - 2x^2 - y^2 - 5y - 3 = 0$

Proposed by George Apostolopoulos-Greece

SP.341. Let a, b, c be positive real numbers such that $abc + ab + bc + ca = 4$. Find the maximum value of expression:

$$T = \frac{1}{\sqrt{2a^5 + b^3 - 2a^2 + 26}} + \frac{1}{\sqrt{2b^5 + c^3 - 2b^2 + 26}} + \frac{1}{\sqrt{2c^5 + a^3 - 2c^2 + 26}}$$

Proposed by Hoang Le Nhat-Tung -Vietnam

SP.342. Let a, b, c be positive real numbers such that $a + b + c + 1 = 4abc$. Find the minimum value of expression:

$$S = \frac{1}{\sqrt[3]{2a^5 - 2a^3 + b^2 + 26}} + \frac{1}{\sqrt[3]{2b^5 - 2b^3 + c^2 + 26}} + \frac{1}{\sqrt[3]{2c^5 - 2c^3 + a^2 + 26}}$$

Proposed by Hoang Le Nhat-Tung -Vietnam

SP.343. If $a, b, c \in \mathbb{C}; |a| = |b| = |c| = 5$ then:

$$\sum_{cyc}|a+5|+5\sum_{cyc}|a^{10}+1|+\sum_{cyc}|a^{11}+5|\geq 30$$

Proposed by Daniel Sitaru – Romania

SP.344. If $n\in\mathbb{N}, n\geq 2$ prove that:

$$\frac{n}{n+2}+\int_0^1\left(tan^{-1}(x^n)\right)^2dx\geq 2\int_0^1 tan^{-1}(x^n)\sqrt[n]{tan^{-1}x}dx$$

Proposed by Florică Anastase–Romania

SP.345. Prove that in any triangle ABC,

$$\left(\frac{b+c-a}{a}\right)^2+\left(\frac{c+a-b}{b}\right)^2+\left(\frac{a+b-c}{c}\right)^2+\frac{8r}{R}\geq 7$$

Proposed by Nguyen Viet Hung – Vietnam

UNDERGRADUATE PROBLEMS

UP.331. If $a,b,c\in(0,1), n\in\mathbb{N}, n\geq 2$ then prove:

$$\sum_{cyc}\left(1-\sqrt[n]{sina}\right)\geq\sum_{cyc}\frac{1-sinasinb}{2n+1-sinasinb}$$

Proposed by Florică Anastase-Romania

UP.332. Let $(x_n)_{n\geq 1},(y_n)_{n\geq 1}$ be sequences of positive real numbers such that:

$$x_1>1, x_{n+1}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}};\ y_1>0, y_{n+1}=\frac{(n+1)n^n y_n}{y_n^n+n^n(n-1)}$$

Find: $\lim_{n\to\infty}\left(\frac{x_n+y_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}}$

Proposed by Florică Anastase-Romania

UP.333. If $x_p=a_p+ib_p, p=\overline{1,4}$ are roots of the equation:

$$x^4-2(k+1)x^3+2(k+1)^2x^2-2(k^2+1)(k+1)x+(k^2+1)^2=0, k\in R^*.$$

Then prove:

$$\sum_{p=1}^{4} arctg\frac{a_p}{|b_p|} = \pi + 2\left(\frac{k-|k|}{k}\right)arctgk.$$

Proposed by Florentin Vişescu-Romania

UP.334. Let be $n \in N^*$ si $A_n \in M_{8n}(Q)$, such that

$$det(A_n^4 + 2A_n^2(1-n^2-k) + (1+n^2+k)^2 I_{8n}) = 0, \forall k = \overline{1,2n}.$$ **Then find:**

$$\lim_{n\to\infty} det\left(\frac{1}{n}A_n\right).$$

Proposed by Florentin Vişescu-Romania

UP.335. If $a, b, c \in \left(0, \frac{\pi}{2}\right)$, $a + b + c = \pi$ and

$$I(n) = \sum_{i=1}^{n}\int_{i}^{i+1}\frac{dx}{(ae^{tana}x^2 + be^{tanb}x + ce^{tanc})(e^c tancx^2 + e^b tanbx + e^a tana)}$$

Then find maximum value of expression:

$$\Omega = \prod_{k=1}^{2020} I(k)$$

Proposed by Florică Anastase-Romania

UP.336. If $0 < a < b < \frac{\pi}{2}$ then prove:

$$\frac{3(b-a)\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b) - sin4(a+b)}} < 3\int_a^b \frac{dx}{\sqrt[3]{1-cos4x}} < cot2a - cot2b + \frac{\pi}{4}$$

Proposed by Florică Anastase-Romania

UP.337. If $0 < a \le b$ then:

$$\int_a^b\int_a^b\int_a^b \frac{yzdxdydz}{3x^2 + 2y^2 + z^2} \le \frac{(b-a)^2(b+a)}{12}\cdot log\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru-Romania

UP.338. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 12$. Prove that:

$$\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ba}\right)\left(\frac{a^2}{\sqrt{a^3+1}} + \frac{b^2}{\sqrt{b^3+1}} + \frac{c^2}{\sqrt{c^3+1}}\right) \ge 12$$

Proposed by George Apostolopoulos

UP.339. Prove that for any positive real numbers a, b, c:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{1}{2}(a+b+c)\geq\frac{9(a^2+b^2+c^2)}{2(a+b+c)}$$

Proposed by Nguyen Viet Hung – Vietnam

UP.340. If $0<a\leq b; f:[a,b]\to[1,\infty); f$ continuous, then:

$$3(b-a)^2\int_a^b f(x)\,dx\leq 2(b-a)^3+\left(\int_a^b f(x)\,dx\right)^3$$

Proposed by Daniel Sitaru – Romania

UP.341. Find:

$$\Omega=\lim_{n\to\infty}\left(\int_0^{\frac{\pi}{3}}\frac{\sin x}{\cos x\,(1+\cos^n x)}dx\right)$$

Proposed by Daniel Sitaru – Romania

UP.342. Prove that if $0<a\leq b$ then:

$$\left(\int_a^b\frac{\log x}{x}dx\right)^2\geq\left(\int_{\frac{a+b}{2}}^b\frac{\log x}{x}dx+\int_{\sqrt{ab}}^b\frac{\log x}{x}dx\right)\left(\int_a^{\frac{a+b}{2}}\frac{\log x}{x}dx+\int_a^{\sqrt{ab}}\frac{\log x}{x}dx\right)$$

Proposed by Daniel Sitaru – Romania

UP.343. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that:

$$2(a^4+b^4+c^4)-(a^3+b^3+c^3)\geq 3abc$$

Proposed by George Apostolopoulos -Greece

UP.344. Let a, b, c be non-negative real numbers, no two of which are zero. Prove that:

$$\frac{a}{a^2+2(b+c)^2}+\frac{b}{b^2+2(c+a)^2}+\frac{c}{c^2+2(a+b)^2}\geq\frac{1}{a+b+c}$$

Proposed by Nguyen Viet Hung – Vietnam

UP.345. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that:

$$(a+b)(b+c)(c+a)-2abc\leq 6$$

Proposed by George Apostolopoulos –Greece

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13	FLAVIU CRISTIAN VERDE-ROMANIA	53	NGUYEN VAN CANH-VIETNAM
14	IOAN SERDEAN-ROMANIA	54	PEDRO PANTOJA-BRAZIL
15	CARMEN VICTORITA CHIRFOT-ROMANIA	55	GEORGE APOSTOLOPOULOS-GREECE
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