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DANIEL SITARU

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Proposed by

Daniel Sitaru-Romania,

Neculai Stanciu-Romania

Cristian Miu-Romania, Marin Chirciu-Romania

Kostas Geronikolas-Greece

Bogdan Fuștei-Romania

Thanasis Gakopoulos-Farsala-Greece

Soumava Chakraborty-Kolkata-India

Nguyen Van Canh-BenTre-Vietnam

George Apostolopoulos-Messolonghi-Greece

Marian Ursărescu-Romania

Ertan Yildirim-Izmir-Turkiye

Vasile Mircea Popa-Romania



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Solutions by

Daniel Sitaru-Romania

Adrian Popa-Romania

George Florin Șerban-Romania

Mohamed Amine Ben Ajiba-Tanger-Morocco

Marian Ursărescu-Romania, Bogdan Fuștei-Romania

Soumava Chakraborty-Kolkata-India

Jose Ferreira Queiroz-Olinda-Brazil

Nguyen Van Canh-BenTre-Vietnam

Soumitra Mandal-Chandar Nagore-India

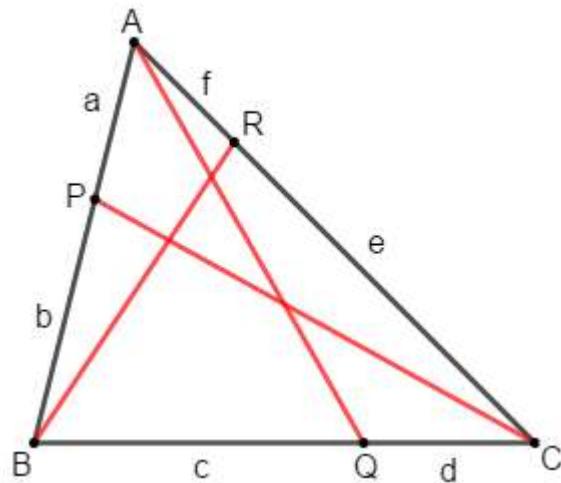
Ertan Yildirim-Izmir-Turkiye

Thanasis Gakopoulos-Farsala-Greece

Avishek Mitra-West Bengal-India

Aggeliki Papaspyropoulou-Greece

2701.



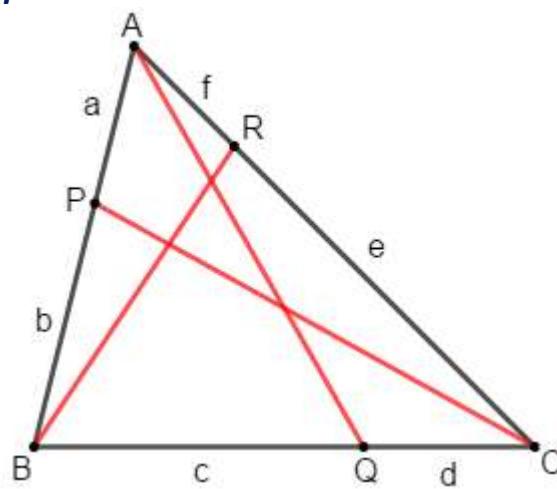
In ΔABC , $P \in (AB)$, $Q \in (BC)$, $R \in (CA)$ such that $AP = a$, $PB = b$,

$BQ = c$, $QC = d$, $CR = e$, $RA = f$. Prove that:

$$\frac{bc}{(a+b)(c+d)} + \frac{de}{(d+c)(e+f)} + \frac{af}{(a+b)(e+f)} < 1$$

Proposed by Neculai Stanciu-Romania

Solution by Adrian Popa-Romania



$$\frac{bc}{(a+b)(c+d)} + \frac{de}{(d+c)(e+f)} + \frac{af}{(a+b)(e+f)} < 1$$

$$\frac{b}{a+b} \cdot \frac{c}{c+d} + \left(1 - \frac{c}{d+c}\right) \frac{e}{e+f} + \left(1 - \frac{b}{a+b}\right) \left(1 - \frac{e}{e+f}\right) < 1$$



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$$\begin{cases} \frac{b}{a+b} = x; x < 1 \\ \frac{c}{c+d} = y; y < 1 \Rightarrow xy + (1-y)z + (1-x)(1-z) < 1 \\ \frac{e}{e+f} = z; z < 1 \end{cases}$$

$$\Leftrightarrow xy + xz < x + yz$$

$$xy + xz < y + xz \stackrel{(?)}{<} x + zy$$

Case I. If $y > x \Rightarrow y - x < z(y - x)$ true, because $y - x, z \in (0, 1)$

Case II. If $y < x \Rightarrow y + xz < x + zy \Rightarrow z(x - y) < x - y$ true, because $x - y, z \in (0, 1)$

2702. If $t > 0$ then in $\Delta ABC, \Delta A'B'C'$ holds:

$$\sum_{cyc} (aa')^t \geq 3^{1-t} \cdot 4^t \cdot (rr'(r+4R)(r'+R'))^{\frac{t}{2}}$$

Proposed by Cristian Miu-Romania

Solution by George Florin Șerban-Romania

$$\sum_{cyc} (aa')^t \stackrel{AM-GM}{\geq} 3^3 \sqrt[3]{\prod_{cyc} (aa')^t} \stackrel{(*)}{\geq} 3^{1-t} \cdot 4^t \cdot (rr'(r+4R)(r'+R'))^{\frac{t}{2}}$$

$$(*) \Leftrightarrow \sqrt[3]{\prod_{cyc} (aa')} \geq \frac{4}{3} \sqrt{rr'(4R+r)(4R'-r')}$$

$$\sqrt[3]{\prod_{cyc} (aa')} = \sqrt[3]{4RRs \cdot 4R'r's'} = \sqrt[3]{16RR'rr'ss'} \stackrel{(**)}{\geq} \frac{4}{3} \sqrt{rr'(4R+r)(4R'-r')}$$

$$256R^2R'^2r^2r'^2s^2s'^2 \geq \frac{2^{12}}{3^6} r^3r'^3(4R+r)^3(4R'+r')^3$$

$$s^2s'^2 \geq \frac{16rr'(4R+r)^3(4R'+r')^3}{3^6R^2R'^2}$$

$$s^2s'^2 \stackrel{Gerretsen}{\geq} (16Rr - 5r^2)(1R'r' - r'^2) = rr'(16R - 5r)(16R' - 5r') \geq$$

$$\geq \frac{2^{12}}{3^6} r^3r'^3(4R+r)^3(4R'+r')^3$$



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$$(16R - 5r)(16R' - 5r') \geq \frac{4(4R + r)^3}{3^3 R^2} \cdot \frac{4(4R' + r')^3}{3^3 R'^2}$$

We prove that:

$$16R - 5r \geq \frac{4(4R + r)^3}{3^3 R^2} \Leftrightarrow 176R^3 - 327R^2r - 48Rr^2 - 4r^3 \geq 0; \left(x = \frac{R}{r} \geq 2 \right) \Rightarrow$$

$(x - 2)(176x^2 + 25x + 2) \geq 0$ which is clearly true from $x \geq 2$.

Therefore,

$$\sum_{cyc} (aa')^t \geq 3^{1-t} \cdot 4^t \cdot (rr'(r + 4R)(r' + R'))^{\frac{t}{2}}$$

2703. In ΔABC the following relationship holds

$$2R \sum \frac{1}{w_a} \cdot \cos \frac{B-C}{2} \geq \frac{13}{4} + \sum \left(\frac{a}{b+c} \right)^2$$

Proposed by Marin Chirciu-Romania

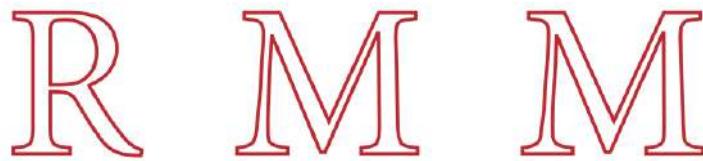
Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} 2R \sum \frac{1}{w_a} \cdot \cos \frac{B-C}{2} &= 2R \sum \frac{b+c}{2bc \cdot \cos \frac{A}{2}} \cdot \frac{b+c}{a} \sin \frac{A}{2} = \frac{R}{abc} \sum (b+c)^2 \cdot \tan \frac{A}{2} \\ &= \frac{1}{4sr} \sum (b+c)^2 \cdot \frac{r}{s-a} = \\ &= \frac{1}{4s} \sum \frac{[(s-a)+s]^2}{s-a} = \frac{1}{4s} \left[\sum (s-a) + \sum 2s + \frac{s}{r} \sum r_a \right] = \frac{7}{4} + \frac{4R+r}{4r} = 2 + \frac{R}{r} \\ &\rightarrow 2R \sum \frac{1}{w_a} \cdot \cos \frac{B-C}{2} = 2 + \frac{R}{r} \quad (1) \end{aligned}$$

$$\begin{aligned} \sum \left(\frac{a}{b+c} \right)^2 &\stackrel{AM-GM}{\leq} \sum \frac{a^2}{4bc} = \frac{1}{4abc} \sum a^3 = \frac{2s(s^2 - 6Rr - 3r^2)}{4 \cdot 4sRr} = \frac{s^2 - 6Rr - 3r^2}{8Rr} \\ &\rightarrow \sum \left(\frac{a}{b+c} \right)^2 \leq \frac{s^2 - 6Rr - 3r^2}{8Rr} \quad (2) \end{aligned}$$

$$(1), (2) \rightarrow \text{It's suffices to prove : } 2 + \frac{R}{r} \geq \frac{13}{4} + \frac{s^2 - 6Rr - 3r^2}{8Rr}$$

$$\begin{aligned} \leftrightarrow 16Rr + 8R^2 &\geq 26Rr + (s^2 - 6Rr - 3r^2) \leftrightarrow (4R^2 + 4Rr + 3r^2 - s^2) + 4R(R - 2r) \\ &\geq 0 \end{aligned}$$



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Which is true from Euler ($R \geq 2r$) and Gerretsen ($4R^2 + 4Rr + 3r^2 \geq s^2$).

$$\text{Therefore, } 2R \sum \frac{1}{w_a} \cdot \cos \frac{B-C}{2} \geq \frac{13}{4} + \sum \left(\frac{a}{b+c} \right)^2$$

2704. In ΔABC the following relationship holds:

$$\sum \frac{w_a}{h_a} \sqrt{r_a} \leq 3 \sqrt{\frac{R}{2} \left(\frac{R^2}{r^2} - \frac{R}{r} + 1 \right)}$$

Proposed by Kostas Geronikolas-Greece

Solution 1 by Marian Ursărescu-Romania

In any ΔABC we have: $\frac{w_a}{h_a} \leq \sqrt{\frac{R}{2r}}$. We must show that:

$$\sqrt{\frac{R}{2r}} \sum_{cyc} \sqrt{r_a} \leq 3 \sqrt{\frac{R}{2r^2} (R^2 - Rr + r^2)} \Leftrightarrow$$

$$\sum_{cyc} \sqrt{r_a} \leq 3 \sqrt{\frac{R^2 - Rr + r^2}{r}} \Leftrightarrow \left(\sum_{cyc} \sqrt{r_a} \right)^2 \leq \frac{9(R^2 - Rr + r^2)}{r}; (1)$$

From Cauchy-Schwarz: $(\sum \sqrt{r_a})^2 \leq 3 \sum_{cyc} r_a$; (2)

From (1),(2) we must show that: $\sum r_a \leq \frac{3(R^2 - Rr + r^2)}{r}$; (3)

But: $\sum r_a = 4R + r$; (4). From (3),(4) we must show that

$$(4R + r)r \leq 3(R^2 - Rr + r^2) \Leftrightarrow 4Rr + r^2 \leq 3R^2 - 3Rr + 3r^2 \Leftrightarrow$$

$$3R^2 - 7Rr + 2r^2 \geq 0 \Leftrightarrow (3R - 1)(R - 2r) \geq 0, \text{ which is true.}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\begin{aligned} w_a &= \frac{2\sqrt{bc}}{b+c} \cdot \sqrt{s(s-a)} \stackrel{AM-GM}{\leq} \sqrt{s(s-a)} \rightarrow w_a \sqrt{r_a} \leq \sqrt{s(s-a)} \sqrt{\frac{sr}{s-a}} = s\sqrt{r} \\ &\rightarrow \sum \frac{w_a}{h_a} \sqrt{r_a} \leq s\sqrt{r} \sum \frac{1}{h_a} = s\sqrt{r} \cdot \frac{1}{r} = \frac{s}{\sqrt{r}} \stackrel{Mitrinovic}{\leq} \frac{3\sqrt{3}R}{2\sqrt{r}} \stackrel{?}{\leq} 3 \sqrt{\frac{R}{2} \left(\frac{R^2}{r^2} - \frac{R}{r} + 1 \right)} \end{aligned}$$



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$$\leftrightarrow \frac{3R}{2r} \leq \frac{R^2}{r^2} - \frac{R}{r} + 1 \leftrightarrow \left(\frac{R}{r} - 2\right)\left(\frac{R}{r} - \frac{1}{2}\right) \geq 0 \text{ which is true from Euler } \left(\frac{R}{r} \geq 2\right).$$

$$\text{Therefore, } \sum \frac{w_a}{h_a} \sqrt{r_a} \leq 3 \sqrt{\frac{R}{2} \left(\frac{R^2}{r^2} - \frac{R}{r} + 1 \right)}.$$

2705. In ΔABC

$$\frac{1}{2} \left(\frac{4R}{r} - \frac{r}{R} - \frac{11}{2} \right) \leq \sum \cot^2 A \leq \frac{1}{2} \left(\frac{2R^2}{r^2} - \frac{3R}{2r} - 3 \right)$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} \sum \cot^2 A &= \sum \frac{\cos^2 A}{\sin^2 A} = \sum \frac{1 - \sin^2 A}{\sin^2 A} = \sum \left(\frac{4R^2}{a^2} - 1 \right) = \frac{4R^2}{(abc)^2} \sum a^2 b^2 - 3 = \\ &= \frac{(s^2 + r^2 + 4Rr)^2 - 16s^2 Rr}{4s^2 r^2} - 3 \stackrel{(i)}{\cong} \frac{s^4 - 2s^2 r(4R + 5r) + r^2(4R + r)^2}{4s^2 r^2} \\ &\rightarrow \sum \cot^2 A \stackrel{?}{\leq} \frac{1}{2} \left(\frac{2R^2}{r^2} - \frac{3R}{2r} - 3 \right) \stackrel{(i)}{\Leftrightarrow} \\ &\frac{s^4 - 2s^2 r(4R + 5r) + r^2(4R + r)^2}{4s^2 r^2} \stackrel{?}{\leq} \frac{4R^2 - 3Rr - 6r^2}{4r^2} \end{aligned}$$

$$\begin{aligned} &\leftrightarrow r^2(4R + r)^2 \leq s^2(4R^2 + 5Rr + 4r^2 - s^2) \\ &\leftrightarrow s^2(4R^2 + 4Rr + 3r^2 - s^2) + s^2 r(R + r) - r^2(4R + r)^2 \geq 0 \end{aligned}$$

$$\leftrightarrow s^2(4R^2 + 4Rr + 3r^2 - s^2) + r(R + r)[s^2 - (16Rr - 5r^2)] + 3r^3(R - 2r) \geq 0$$

Which is true from Gerretsen and Euler \rightarrow $\sum \cot^2 A \leq \frac{1}{2} \left(\frac{2R^2}{r^2} - \frac{3R}{2r} - 3 \right)$

$$\begin{aligned} &\frac{1}{2} \left(\frac{4R}{r} - \frac{r}{R} - \frac{11}{2} \right) \stackrel{?}{\leq} \sum \cot^2 A \stackrel{(i)}{\Leftrightarrow} \frac{8R^2 - 11Rr - 2r^2}{4Rr} \\ &\leq \frac{s^4 - 2s^2 r(4R + 5r) + r^2(4R + r)^2}{4s^2 r^2} \\ &\leftrightarrow Rr^2(4R + r)^2 \geq s^2(16R^2 r - 2r^3 - Rr^2 - Rs^2) \quad (ii) \end{aligned}$$

From Blundon – Gerretsen's inequality and Gerretsen's inequality, we have



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$$s^2(16R^2r - 2r^3 - Rr^2 - Rs^2) \leq \frac{R(4R+r)^2}{2(2R-r)} [16R^2r - 2r^3 - Rr^2 - R(16Rr - 5r^2)]$$

$$= \frac{R(4R+r)^2}{2(2R-r)} (4Rr^2 - 2r^3) = Rr^2(4R+r)^2 \rightarrow (ii) \text{ is true}$$

$$\rightarrow \left[\frac{1}{2} \left(\frac{4R}{r} - \frac{r}{R} - \frac{11}{2} \right) \leq \sum \cot^2 A \right]$$

$$\text{Therefore, } \frac{1}{2} \left(\frac{4R}{r} - \frac{r}{R} - \frac{11}{2} \right) \leq \sum \cot^2 A \leq \frac{1}{2} \left(\frac{2R^2}{r^2} - \frac{3R}{2r} - 3 \right).$$

2706. In ΔABC the following relationship holds:

$$\sum \frac{m_a}{h_a} \geq \frac{\sum \frac{b+c}{a} + \lambda \sum \frac{m_b+m_c}{m_a}}{2(1+\lambda)}, \lambda \geq 0$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} \text{We know that : } m_a &\stackrel{\text{Tereshin}}{\geq} \frac{b^2 + c^2}{4R} = \frac{1}{2} \cdot \frac{bc}{2R} \cdot \frac{b^2 + c^2}{bc} \\ &= \frac{h_a}{2} \left(\frac{b}{c} + \frac{c}{b} \right) \leftrightarrow \frac{b}{c} + \frac{c}{b} \leq \frac{2m_a}{h_a} \quad (1) \end{aligned}$$

m_a, m_b, m_c can be length sides of a triangle with :

$$F_m = \frac{3}{4}F, \overline{m_a} = \frac{3}{4}a, \overline{h_a} = \frac{2F_m}{m_a} = \frac{3F}{2m_a}$$

$$\text{In } \Delta m_a m_b m_c, (1) \leftrightarrow \frac{m_b}{m_c} + \frac{m_c}{m_b} \leq \frac{2\overline{m_a}}{\overline{h_a}} \leftrightarrow$$

$$\frac{m_b}{m_c} + \frac{m_c}{m_b} \leq \frac{3}{2}a \cdot \frac{2m_a}{3F} = \frac{2m_a}{h_a} \rightarrow \frac{m_b}{m_c} + \frac{m_c}{m_b} \leq \frac{2m_a}{h_a} \quad (2)$$

$$(1), (2) \rightarrow \left(\frac{b}{c} + \frac{c}{b} \right) + \lambda \left(\frac{m_b}{m_c} + \frac{m_c}{m_b} \right) \leq 2(1+\lambda) \frac{m_a}{h_a}, \lambda \geq 0 \quad (\text{and analogs})$$

$$\rightarrow \sum \frac{b+c}{a} + \lambda \sum \frac{m_b+m_c}{m_a} = \sum \left[\left(\frac{b}{c} + \frac{c}{b} \right) + \lambda \left(\frac{m_b}{m_c} + \frac{m_c}{m_b} \right) \right] \leq 2(1+\lambda) \sum \frac{m_a}{h_a}$$

$$\text{Therefore, } \sum \frac{m_a}{h_a} \geq \frac{\sum \frac{b+c}{a} + \lambda \sum \frac{m_b+m_c}{m_a}}{2(1+\lambda)}, \lambda \geq 0$$



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2707. In ΔABC the following relationship holds:

$$\frac{4a^2b^2c^2}{(a^2+b^2+c^2)^2} \left(\sum_{cyc} an_a m_a \right)^2 \geq \frac{1}{2} \sum_{cyc} a^2(b^2n_b + c^2n_c - a^2n_a) + \\ + 2F \sqrt{(a^2n_a + b^2n_b + c^2n_c) \prod_{cyc} (a^2n_a + b^2n_b - c^2n_c)}$$

Proposed by Bogdan Fuștei-Romania

Solution by proposer

In ΔABC , N_a – Nagel's point, we have: $AN_a = \frac{a \cdot n_a}{s}$ (and analogs)

If $P \in Int(\Delta ABC)$ and $PA = x, PB = y, PC = z$. Then ax, by, cz can be the sides of a triangle. (Klamkin)

$aN_a A, bN_b B, cN_c C$ can be the sides of a triangle.

a^2n_a, b^2n_b, c^2n_c – can be sides of a triangle.

$F_1 = [A_1B_1C_1]$ are of triangle with sides $a_1 = a^2n_a, b_1 = b^2n_b, c_1 = c^2n_c \Rightarrow$

$4F_1 = \sqrt{(ax+by+cz)(ax+by-cz)(ax-by+cz)(-ax+by+cz)}$, where

$x = an_a, y = bn_b, z = cn_c$.

In ΔABC and $\Delta A_1B_1C_1$, $P \in Int(\Delta ABC)$, the following relationship holds:

$$a_1PA + b_1PB + c_1PC \geq \sqrt{\frac{1}{2} \sum_{cyc} a^2(b_1^2 + c_1^2 - a_1^2) + 8FF_1}; \text{ (Bottema)}$$

Let K – point of intersection of symmedians in ΔABC then, $AK =$

$$\frac{2bcm_a}{a^2+b^2+c^2} \text{ (and analogs).}$$

$$\left(\sum_{cyc} \frac{2a^2bc \cdot m_a n_a}{a^2+b^2+c^2} \right)^2 \geq \frac{1}{2} \sum_{cyc} a^2(b^2n_b + c^2n_c - a^2n_a) + 8FF_1$$

$$4F_1 = \sqrt{(a^2n_a + b^2n_b + c^2n_c) \prod_{cyc} (a^2n_a + b^2n_b - c^2n_c)}$$



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Therefore,

$$\frac{4a^2b^2c^2}{(a^2+b^2+c^2)^2} \left(\sum_{cyc} an_a m_a \right)^2 \geq \frac{1}{2} \sum_{cyc} a^2(b^2n_b + c^2n_c - a^2n_a) + \\ + 2F \sqrt{(a^2n_a + b^2n_b + c^2n_c) \prod_{cyc} (a^2n_a + b^2n_b - c^2n_c)}$$

2708. In ΔABC the following relationship holds:

$$1 \leq \sum \frac{AI^2}{a^2} \leq \left(\frac{R}{2r} \right)^4$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have : } \sum \frac{AI^2}{a^2} \stackrel{\sum x^2 \geq \sum xy}{\geq} \sum \frac{BI}{b} \cdot \frac{CI}{c} = \frac{1}{abc} \sum a \cdot BI \cdot CI \stackrel{\text{Hayashi } P=I}{\geq} \frac{1}{abc} \cdot abc = 1$$

$$\text{Now, we have : } AI^2 = \frac{r^2}{\sin^2 \frac{A}{2}} = \frac{r^2 \cdot bc}{(s-b)(s-c)} = \frac{r^2 \cdot bc(s-a)}{sr^2} = bc - \frac{abc}{s}$$

$$= bc - 4Rr.$$

$$\begin{aligned} & \rightarrow (abc)^2 \sum \frac{AI^2}{a^2} = (abc)^2 \sum \frac{bc - 4Rr}{a^2} = \sum (bc)^3 - 4Rr \sum (bc)^2 = \\ & = \left(\sum bc \right)^3 - 3abc \prod (a+b) - 4Rr \left[\left(\sum bc \right)^2 - 2abc \sum a \right] = \\ & = (s^2 + r^2 + 4Rr)^2 (s^2 + r^2) - 3 \cdot 4Rrs \cdot 2s (s^2 + r^2 + 2Rr) + 4Rr \cdot 2 \cdot 4Rrs \cdot 2s = \\ & = [s^4 + 2rs^2(4R+r) + r^2(4R+r)^2] (s^2 + r^2) - 24Rrs^2(s^2 + r^2 + 2Rr) + 64R^2r^2s^2 \\ & = s^6 - rs^4(16R - 3r) + r^2s^2(32R^2 - 8Rr + 3r^2) + r^4(4R + r)^2 \end{aligned}$$

$$\rightarrow \sum \frac{AI^2}{a^2} = \frac{s^6 - rs^4(16R - 3r) + r^2s^2(32R^2 - 8Rr + 3r^2) + r^4(4R + r)^2}{16R^2r^2s^2} \stackrel{?}{\geq} \left(\frac{R}{2r} \right)^4$$

$$\leftrightarrow r^6(4R + r)^2 \leq s^2[R^6 - 32R^2r^4 + 8Rr^5 - 3r^6 - r^2s^2(s^2 - 16Rr + 3r^2)] (*)$$

$$\text{Let } f(s) = s^2(s^2 - 16Rr + 3r^2) \rightarrow f'(s) = 2s(2s^2 - 16Rr + 3r^2) \stackrel{2s^2 \geq 27Rr}{\geq} 0$$

$\rightarrow f - \text{increasing}$



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*Mitrinovic
Gerretsen*

$$\begin{aligned}
 \rightarrow RHS_{(*)} &\stackrel{\text{Mitrinovic}}{\geq} 27r^2[R^6 - 32R^2r^4 + 8Rr^5 - 3r^6 \\
 &\quad - r^2(4R^2 + 4Rr + 3r^2)(4R^2 - 12Rr + 6r^2)] = \\
 &= 27r^2(R^6 - 16R^4r^2 + 32R^3r^3 - 20R^2r^4 + 20Rr^5 - 21r^6) \stackrel{?}{\geq} r^6(4R + r)^2 \\
 &\Leftrightarrow 27R^6 - 432R^4r^2 + 864R^3r^3 - 556R^2r^4 + 532Rr^5 - 568r^6 \geq 0 \\
 &\Leftrightarrow (R - 2r)\{(R - 2r)[(R - 2r)(27R^3 + 162R^2r + 216Rr^2 + 432r^3) + 740r^4] \\
 &\quad + 36r^5\} \geq 0
 \end{aligned}$$

Which is true from Euler ($R \geq 2r$) $\rightarrow \sum \frac{AI^2}{a^2} \leq \left(\frac{R}{2r}\right)^4$

Therefore, $1 \leq \sum \frac{AI^2}{a^2} \leq \left(\frac{R}{2r}\right)^4$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 AI^2 = bc - 4Rr &\Leftrightarrow \left(\frac{r}{\left(\frac{r}{4R}\right)} \sin \frac{B}{2} \sin \frac{C}{2}\right)^2 \\
 &= 16R^2 \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{B}{2} \cos \frac{C}{2} - 16R^2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \Leftrightarrow \sin \frac{B}{2} \sin \frac{C}{2} \\
 &= \cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{A}{2} \Leftrightarrow \cos \frac{B+C}{2} = \sin \frac{A}{2} \rightarrow \text{true}
 \end{aligned}$$

$$\begin{aligned}
 \therefore AI^2 = bc - 4Rr \text{ and analogs} &\Rightarrow \sum \frac{AI^2}{a^2} = \sum \frac{b^2 c^2 (bc - 4Rr)}{16R^2 r^2 s^2} \\
 &= \frac{1}{16R^2 r^2 s^2} \left(\left(\sum bc \right)^3 - 3 \cdot 4Rrs \cdot 2s(s^2 + 2Rr + r^2) \right. \\
 &\quad \left. - 4Rr \left(\left(\sum bc \right)^2 - 16Rrs^2 \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{16R^2 r^2 s^2} \left((s^2 + 4Rr + r^2)^3 - 3 \cdot 4Rrs \cdot 2s(s^2 + 2Rr + r^2) \right. \\
 &\quad \left. - 4Rr \left((s^2 + 4Rr + r^2)^2 - 16Rrs^2 \right) \right) \leq \left(\frac{R}{2r}\right)^4 \\
 &\Leftrightarrow r^2 s^6 - r^3 (16R - 3r) s^4 - (R^6 - 32R^2 r^4 + 8Rr^5 - 3r^6) s^2 + r^6 (4R + r)^2 \stackrel{(*)}{\geq} 0
 \end{aligned}$$



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Now, Rouche $\Rightarrow s^2 - (m - n) \geq 0$ and $s^2 - (m + n) \leq 0$, where $m = 2R^2 + 10Rr - r^2$ and $n = 2(R - 2r)\sqrt{R^2 - 2Rr} \therefore (s^2 - (m + n))(s^2 - (m - n)) \leq 0$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0 \Rightarrow s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3 \leq 0$$

$$\Rightarrow r^2s^6 - r^2s^4(4R^2 + 20Rr - 2r^2) + r^3(4R + r)^3s^2 \\ \therefore \text{in order to prove } (*), \text{ it suffices to}$$

$$\text{prove : } r^2s^6 - r^3(16R - 3r)s^4 - (R^6 - 32R^2r^4 + 8Rr^5 - 3r^6)s^2 + r^6(4R + r)^2$$

$$\leq r^2s^6 - r^2s^4(4R^2 + 20Rr - 2r^2) + r^3(4R + r)^3s^2$$

$$\Leftrightarrow r^2(4R^2 + 4Rr + r^2)s^4 - (R^6 + 64R^3r^3 + 16R^2r^4 + 20Rr^5 - 2r^6)s^2 + r^6(4R + r)^2 \stackrel{(**)}{\leq} 0$$

$$\text{Now, LHS of } (**) \stackrel{\text{Gerretsen}}{\leq} (r^2(4R^2 + 4Rr + r^2)(4R^2 + 4Rr + 3r^2))$$

$$- (R^6 + 64R^3r^3 + 16R^2r^4 + 20Rr^5 - 2r^6) s^2 + r^6(4R + r)^2 \stackrel{?}{\leq} 0$$

$$\Leftrightarrow (R^6 - 16R^4r^2 + 32R^3r^3 - 16R^2r^4 + 4Rr^5 - 5r^6)s^2 - r^6(4R + r)^2 \stackrel{?}{\leq} 0 \quad (***)$$

$$\text{Now, } R^6 - 16R^4r^2 + 32R^3r^3 - 16R^2r^4 + 4Rr^5 - 5r^6$$

$$= (R - 2r)((R - 2r)(R^2(R^2 - 4r^2) + 4R^3r) + 4r^4) + 3r^5 > 0 \because R - 2r \stackrel{\text{Euler}}{\geq} 0$$

$$\therefore \text{LHS of } (***)) \stackrel{\text{Gerretsen}}{\geq}$$

$$(R^6 - 16R^4r^2 + 32R^3r^3 - 16R^2r^4 + 4Rr^5 - 5r^6)(16Rr - 5r^2) - r^6(4R + r)^2 \stackrel{?}{\geq} 0$$

$$\Leftrightarrow 16t^7 - 5t^6 - 256t^5 + 592t^4 - 416t^3 + 128t^2 - 108t + 24 \stackrel{?}{\geq} 0 \quad (t = \frac{R}{r})$$

$$\Leftrightarrow (t - 2)((t - 2)((t - 2)(16t^4 + 91t^3 + 98t^2 + 216t + 432) + 912) + 84) \stackrel{?}{\geq} 0 \rightarrow \text{true}$$

$$\because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (***)) \Rightarrow (**) \Rightarrow (*) \text{ is true} \Rightarrow \boxed{\sum \frac{AI^2}{a^2} \leq \left(\frac{R}{2r}\right)^4}$$

Again, $\because \forall P$ in the plane of ΔABC , $\sum \frac{AP}{a} \geq \sqrt{3} \therefore \sum \frac{AI}{a} \geq \sqrt{3} \Rightarrow \sum \frac{AI^2}{a^2} \geq \frac{1}{3} \left(\sum \frac{AI}{a}\right)^2$

$$\geq \frac{1}{3} (\sqrt{3})^2 = 1 \therefore \boxed{1 \leq \sum \frac{AI^2}{a^2}} \text{ (QED)}$$

2709. In $\triangle ABC$ the following relationship holds:

$$\frac{R}{2r} \sum n_a \geq \sqrt{\frac{1}{8r^2} \sum a^2(m_b^2 + m_c^2 - m_a^2) + \frac{3}{2}s^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

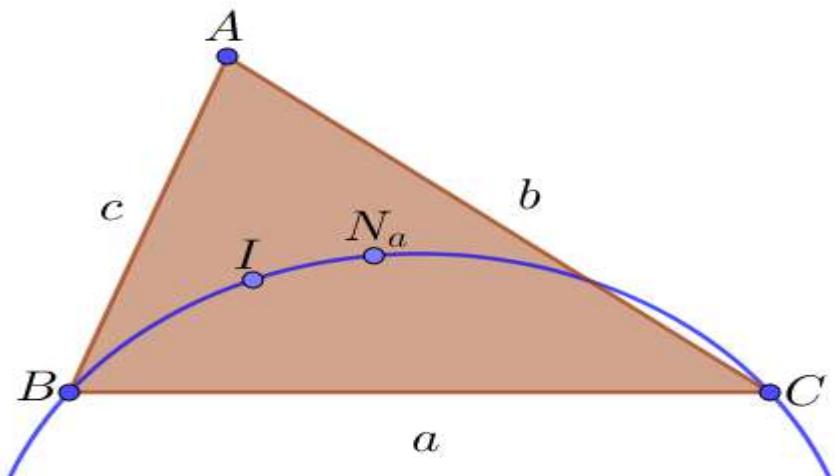
$$\begin{aligned} \frac{R}{2r} \sum n_a &\stackrel{(*)}{\geq} \sqrt{\frac{1}{8r^2} \sum a^2(m_b^2 + m_c^2 - m_a^2) + \frac{3}{2}s^2} \\ 4(m_b^2 + m_c^2 - m_a^2) &= (2c^2 + 2a^2 - b^2) + (2a^2 + 2b^2 - c^2) - (2b^2 + 2c^2 - a^2) \\ &= 5a^2 - b^2 - c^2 \\ \rightarrow 4 \sum a^2(m_b^2 + m_c^2 - m_a^2) &= \sum a^2(5a^2 - b^2 - c^2) = 5 \sum a^4 - 2 \sum a^2b^2 = \\ &= 2 \left(\sum a^2 \right)^2 - 3 \left(2 \sum a^2b^2 - \sum a^4 \right) = 2 \left(\sum a^2 \right)^2 - 3 \cdot 16F^2 \\ \rightarrow RHS_{(*)} &= \sqrt{\frac{1}{8r^2} \cdot \frac{1}{4} \left[2 \left(\sum a^2 \right)^2 - 3 \cdot 16s^2r^2 \right] + \frac{3}{2}s^2} = \sqrt{\frac{1}{16r^2} \left(\sum a^2 \right)^2} = \frac{1}{4r} \sum a^2 \\ \rightarrow (*) \leftrightarrow \frac{R}{2r} \sum n_a &\geq \frac{1}{4r} \sum a^2 \leftrightarrow \sum n_a \geq \sum \frac{b^2 + c^2}{4R} \end{aligned}$$

Tereshin $\sum n_a \geq \sum \frac{b^2 + c^2}{4R}$ (And analogs)

Therefore,
$$\frac{R}{2r} \sum n_a \geq \sqrt{\frac{1}{8r^2} \sum a^2(m_b^2 + m_c^2 - m_a^2) + \frac{3}{2}s^2}.$$

2710.

Prove that: $3a > b + c$.



Proposed by Thanasis Gakopoulos-Farsala-Greece



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Solution by Jose Ferreira Queiroz-Olinda-Brazil

Let M be any point in plane of triangle ABC , so

$$aMA^2 + bMB^2 + cMC^2 = 2s \cdot MI^2 + abc; \quad (1)$$

$$sMN_a^2 = (s-a)MA^2 + (s-b)MB^2 + (s-c)MC^2 + 4r^2s - 4Rrs; \quad (2)$$

Observe that: $MB = MC = MU = MN_a = R_w$; (3)

$$\text{From (1),(3): } MA^2 - R_w^2 = bc$$

From (2),(3): $(s-a)(MA^2 - R_w^2) = 4sRr - 4sr^2$. We get:

$$(s-a)bc = 4Rrs - 4sr^2, sbc - abc = abc - 4sr^2$$

$$bc(a+b+c) = 4abc - 8sr^2, abc + b^2c + bc^2 = 4abc - 8sr^2$$

$$3abc = b^2c + bc^2 + 8sr^2, 3a = b + c + \frac{8sr^2}{bc} \Rightarrow 3a > b + c$$

2711. In ΔABC the following relationship holds:

$$\sum \cos \frac{A}{2} \left(\frac{m_b}{h_b} + \frac{m_c}{h_c} - \frac{m_a}{h_a} \right) \geq \sqrt{2 \sum \cos \frac{A}{2} \cdot \cos \frac{B}{2} - \sum \frac{r_a}{2R}} \cdot \sqrt{2 \sum \frac{m_a m_b}{h_a h_b} - \sum \frac{m_a^2}{h_a^2}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let I and G be the incenter and the centroid of ΔABC , respectively. We know that

$: AI = \frac{r}{\sin \frac{A}{2}}, AG = \frac{2}{3}m_a$. We know that if $P \in \text{Int}(\Delta ABC)$,

then aPA, bPB, cPC can be the sides of a triangle (Klamkin)

$\rightarrow (aIA, bIB, cIC)$ and (aGA, bGB, cGC) can be the sides of a triangle.

Since $aIA = 4Rr \cos \frac{A}{2} \rightarrow \cos \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2}$ can be the sides of a triangle.

We know that if m, n, p the sides of a triangle, then $\sqrt{m}, \sqrt{n}, \sqrt{p}$ can be also
the sides of a triangle.

$$\left(\therefore (\sqrt{m} + \sqrt{n})^2 = (m + n) + 2\sqrt{mn} > \sqrt{p}^2 \right)$$

$\rightarrow \sqrt{\cos \frac{A}{2}}, \sqrt{\cos \frac{B}{2}}, \sqrt{\cos \frac{C}{2}}$ can be the sides of a triangle Δ_1 with area F_1
 $= \frac{1}{4} \sqrt{2 \sum \cos \frac{A}{2} \cdot \cos \frac{B}{2} - \sum \cos^2 \frac{A}{2}}$

We have : $\sum \cos^2 \frac{A}{2} = \sum \frac{1 + \cos A}{2} = 2 + \frac{r}{2R} = \frac{4R + r}{2R} = \sum \frac{r_a}{2R} \rightarrow F_1$

$$= \frac{1}{4} \sqrt{2 \sum \cos \frac{A}{2} \cdot \cos \frac{B}{2} - \sum \frac{r_a}{2R}}$$

Since $aGA = \frac{4sr}{3} \cdot \frac{m_a}{h_a} \rightarrow \frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}$ can be the sides of a triangle
 $\rightarrow \sqrt{\frac{m_a}{h_a}}, \sqrt{\frac{m_b}{h_b}}, \sqrt{\frac{m_c}{h_c}}$ can be the sides of a triangle Δ_2 with area F_2

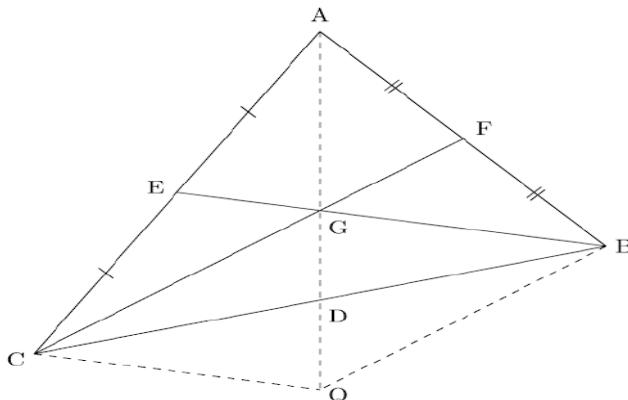
$$= \frac{1}{4} \sqrt{2 \sum \frac{m_a m_b}{h_a h_b} - \sum \frac{m_a^2}{h_a^2}}$$

We know that for any two triangle ΔXYZ and ΔUVW with area S_1 and S_2 respectively, we have : $\sum x^2(v^2 + w^2 - u^2) \geq 16S_1S_2$ (Neuberg – Pedoe)

\rightarrow For the triangles Δ_1 and Δ_2 , we have : $\sum \cos \frac{A}{2} \left(\frac{m_b}{h_b} + \frac{m_c}{h_c} - \frac{m_a}{h_a} \right) \geq 16F_1F_2$
 Therefore,

$$\sum \cos \frac{A}{2} \left(\frac{m_b}{h_b} + \frac{m_c}{h_c} - \frac{m_a}{h_a} \right) \geq \sqrt{2 \sum \cos \frac{A}{2} \cdot \cos \frac{B}{2} - \sum \frac{r_a}{2R}} \cdot \sqrt{2 \sum \frac{m_a m_b}{h_a h_b} - \sum \frac{m_a^2}{h_a^2}}$$

Solution 2 by Soumava Chakraborty-Kolkata-India



Proof :

Via Ptolemy's theorem on quadrilateral ABQC, $AB \cdot CQ + AC \cdot BQ \geq AQ \cdot BC$

$$\Rightarrow c \cdot \frac{2m_b}{3} + b \cdot \frac{2m_c}{3} \geq \frac{4m_a}{3} \cdot a \Rightarrow cm_b + bm_c \geq 2am_a$$

upon squaring

$$\Rightarrow c^2m_b^2 + b^2m_c^2 + 2bcm_bm_c \stackrel{(i)}{\geq} 16a^2m_a^2 - 4c^2m_b^2 - 4b^2m_c^2$$

$$\text{Now, } (2bcm_b + 2cm_c)^2 - 4a^2m_a^2 \stackrel{\text{via (i)}}{=} 4b^2m_b^2 + 4c^2m_c^2 + 12a^2m_a^2$$

$$= 4b^2m_b^2 + 4c^2m_c^2 + 8bcm_bm_c - 4a^2m_a^2 \stackrel{(i)}{\geq} 4b^2m_b^2 + 4c^2m_c^2 + 12a^2m_a^2 - 4c^2m_b^2 - 4b^2m_c^2$$



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$$\begin{aligned}
 &= b^2(4m_b^2 - 4m_c^2) - c^2(4m_b^2 - 4m_c^2) + 12a^2m_a^2 \\
 &= (b^2 - c^2)(2c^2 + 2a^2 - b^2 - 2a^2 - 2b^2 + c^2) + 12a^2m_a^2 \\
 &= 3(a^2 \cdot 4m_a^2 - (b^2 - c^2)^2) \\
 &= 3(a^2(2b^2 + 2c^2 - a^2) - b^4 - c^4 + 2b^2c^2) = 3\left(2 \sum a^2b^2 - \sum a^4\right) = 48F^2 > 0 \\
 &\Rightarrow (2bm_b + 2cm_c)^2 > 4a^2m_a^2 \Rightarrow bm_b + cm_c - am_a > 0 \\
 \Rightarrow &(\sqrt{bm_b + cm_c} - \sqrt{am_a})(\sqrt{bm_b + cm_c} + \sqrt{am_a}) > 0 \Rightarrow \sqrt{bm_b + cm_c} - \sqrt{am_a} > 0 \\
 \Rightarrow &\sqrt{am_a} < \sqrt{bm_b + cm_c} < \sqrt{bm_b} + \sqrt{cm_c} \Rightarrow \sqrt{bm_b} + \sqrt{cm_c} > \sqrt{am_a} \\
 \Rightarrow &\sqrt{\frac{bm_b}{2rs}} + \sqrt{\frac{cm_c}{2rs}} > \sqrt{\frac{am_a}{2rs}} \Rightarrow \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}} > \sqrt{\frac{m_a}{h_a}} \text{ and analogs} \\
 \Rightarrow &\sqrt{\frac{m_a}{h_a}}, \sqrt{\frac{m_b}{h_b}}, \sqrt{\frac{m_c}{h_c}} \text{ form sides of a triangle} \\
 \rightarrow &(1) \text{ with area } F_2 \stackrel{(*)}{=} \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \frac{m_a m_b}{h_a h_b} - \sum_{\text{cyc}} \frac{m_a^2}{h_a^2}}
 \end{aligned}$$

Again, $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} = \frac{1}{2}(1 + \cos A + 1 + \cos B - 1 - \cos C)$

$$\begin{aligned}
 &= \frac{1}{2}\left(2\sin^2 \frac{C}{2} + 2\sin \frac{C}{2} \cos \frac{A-B}{2}\right) > 0 \because 0 < \cos \frac{A-B}{2} \leq 1 \Rightarrow \cos^2 \frac{C}{2} \\
 &< \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} \\
 &< \left(\cos \frac{A}{2} + \cos \frac{B}{2}\right)^2 \Rightarrow \cos \frac{A}{2} + \cos \frac{B}{2} - \cos \frac{C}{2} > 0 \\
 \Rightarrow &\left(\sqrt{\cos \frac{A}{2} + \cos \frac{B}{2}} - \sqrt{\cos \frac{C}{2}}\right)\left(\sqrt{\cos \frac{A}{2} + \cos \frac{B}{2}} + \sqrt{\cos \frac{C}{2}}\right) > 0 \Rightarrow \sqrt{\cos \frac{C}{2}} \\
 &< \sqrt{\cos \frac{A}{2} + \cos \frac{B}{2}} < \sqrt{\cos \frac{A}{2}} + \sqrt{\cos \frac{B}{2}} \\
 \Rightarrow &\sqrt{\cos \frac{A}{2}} + \sqrt{\cos \frac{B}{2}} > \sqrt{\cos \frac{C}{2}} \text{ and analogs} \\
 \Rightarrow &\sqrt{\cos \frac{A}{2}}, \sqrt{\cos \frac{B}{2}}, \sqrt{\cos \frac{C}{2}} \text{ form sides of a triangle} \rightarrow (2) \text{ with area } F_1 \\
 &= \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \cos \frac{A}{2} \cos \frac{B}{2} - \frac{1}{4R} \sum_{\text{cyc}} 4R \cos^2 \frac{A}{2}} \\
 &= \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \cos \frac{A}{2} \cos \frac{B}{2} - \frac{1}{4R} \sum_{\text{cyc}} (r_b + r_c)} = \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \cos \frac{A}{2} \cos \frac{B}{2} - \sum_{\text{cyc}} \frac{r_a}{2R}} \stackrel{(**)}{=} F_1
 \end{aligned}$$

Now, via D. Pedoe (1941) and J. J. B. Neuberg (1891), for any 2 triangles



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with sides a_1, b_1, c_1 and a_2, b_2, c_2 , we have : $\sum_{\text{cyc}} a_1^2(b_2^2 + c_2^2 - a_2^2)$

$\geq 16F_1 F_2$ where area of first

$$\begin{aligned}
 \text{triangle} = F_1 \text{ and that of second one} = F_2 \therefore \text{choosing } a_1 = \sqrt{\cos \frac{A}{2}} \text{ and analogs and } a_2 \\
 = \sqrt{\frac{m_a}{h_a}} \text{ and analogs and making use of axioms (1) and (2), we arrive at :} \\
 \sum_{\text{cyc}} \cos \frac{A}{2} \left(\frac{m_b}{h_b} + \frac{m_c}{h_c} - \frac{m_a}{h_a} \right) \\
 \geq 16F_1 F_2 \stackrel{\text{via } (*), (**)}{\cong} 16 \cdot \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \cos \frac{A}{2} \cos \frac{B}{2} - \sum_{\text{cyc}} \frac{r_a}{2R}} \cdot \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \frac{m_a m_b}{h_a h_b} - \sum_{\text{cyc}} \frac{m_a^2}{h_a^2}} \\
 = \sqrt{2 \sum_{\text{cyc}} \cos \frac{A}{2} \cos \frac{B}{2} - \sum_{\text{cyc}} \frac{r_a}{2R}} \cdot \sqrt{2 \sum_{\text{cyc}} \frac{m_a m_b}{h_a h_b} - \sum_{\text{cyc}} \frac{m_a^2}{h_a^2}} \text{ (QED)}
 \end{aligned}$$

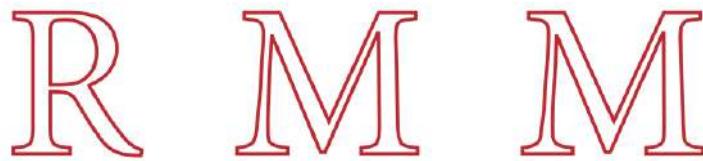
2712. In ΔABC the following relationship holds:

$$\left(\frac{R}{r} - 1\right) \sum m_a \geq \sqrt{\frac{1}{8r^2} \sum m_a^2 (b^2 + c^2 - a^2) + \frac{3}{2}s^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \left(\frac{R}{r} - 1\right) \sum m_a &\stackrel{(*)}{\geq} \sqrt{\frac{1}{8r^2} \sum m_a^2 (b^2 + c^2 - a^2) + \frac{3}{2}s^2} \\
 \frac{1}{8r^2} \sum m_a^2 (b^2 + c^2 - a^2) &= \frac{1}{8r^2} \cdot \frac{1}{4} \sum (2b^2 + 2c^2 - 2a^2)(b^2 + c^2 - a^2) \\
 &= \frac{1}{32r^2} \left(5 \sum a^4 - 2 \sum a^2 b^2 \right) = \\
 &= \frac{1}{32r^2} \left[2 \left(\sum a^2 \right)^2 - 3 \left(2 \sum a^2 b^2 - \sum a^4 \right) \right] = \frac{1}{32r^2} \left[2 \left(\sum a^2 \right)^2 - 3 \cdot 16s^2 r^2 \right] \\
 &= \left(\frac{1}{4r} \sum a^2 \right)^2 - \frac{3}{2}s^2 \\
 \rightarrow (*) \leftrightarrow \left(\frac{R}{r} - 1\right) \sum m_a &\geq \frac{1}{4r} \sum a^2 \leftrightarrow 2(2R - 2r) \sum m_a \geq \sum a^2 \\
 \text{We have : } 2(2R - 2r) \sum m_a &\stackrel{\text{Euler}}{\geq} 2R \sum m_a \stackrel{\text{Tereshin}}{\geq} 2R \sum \frac{b^2 + c^2}{4R} = \sum a^2.
 \end{aligned}$$



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$$\text{Therefore, } \left(\frac{R}{r} - 1\right) \sum m_a \geq \sqrt{\frac{1}{8r^2} \sum m_a^2 (b^2 + c^2 - a^2) + \frac{3}{2} s^2}$$

2713. In ΔABC , v_a, v_b, v_c are cevians through Bevan's point, then:

$$r^2 \sum \frac{AV}{v_a} + R(R - 2r) \geq Rr$$

Proposed by Soumava Chakraborty-Kolkata-India

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let V be the Bevan's point, O the circumcenter and I the incenter of ΔABC .

We know that O is the midpoint of VI, $AO = R$ and $OI = OV = \sqrt{R(R - 2r)}$

$$\text{Also, } AI^2 = \frac{r^2}{\sin^2 \frac{A}{2}} = \frac{r^2 bc}{(s-b)(s-c)} = \frac{bc(s-a)}{s} = bc - \frac{abc}{s} = bc - 4Rr$$

Using Apollonius's theorem in ΔAVI : $4AO^2 = 2AV^2 + 2AI^2 - VI^2$

$$\Leftrightarrow 2AV^2 = 4R^2 - 2(bc - 4Rr) + 4R(R - 2r) \rightarrow AV^2 = 4R^2 - bc \quad (1)$$

Now, let $\{A'\} = (AV) \cap (BC)$ ($\therefore v_a = AA'$)

The barycentric coordinates of V are $(x : y : z)$ with x

$$= a \left(-\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} \right),$$

$$y = b \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) \text{ and } z = c \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right)$$

to A' have the coordinates $(0 : y : z)$ $\rightarrow (y+z)\overrightarrow{AA'} = y\overrightarrow{AB} + z\overrightarrow{AC}$

$$\Leftrightarrow (y+z)^2 v_a^2 = y^2 c^2 + z^2 b^2 + yz \cdot 2\overrightarrow{AB} \cdot \overrightarrow{AC} \\ = y^2 c^2 + z^2 b^2 + yz(b^2 + c^2 - a^2) \quad (*)$$

$$RHS_{(*)} = b^2 c^2 \left[\left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right)^2 + \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right] +$$

$$bc \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right) (b^2 + c^2 - a^2) =$$

$$= 2b^2 c^2 \left[\frac{a^2}{(s-a)^2} + \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right]$$

$$+ bc \left[\frac{a^2}{(s-a)^2} - \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right] (b^2 + c^2 - a^2) =$$

$$= \frac{a^2}{(s-a)^2} \cdot bc[(b+c)^2 - a^2] + bc \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 [a^2 - (b-c)^2]$$

$$= \frac{4sa^2bc}{s-a} + \frac{4s^2bc(b-c)^2}{(s-b)(s-c)} =$$



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$$\begin{aligned}
 &= sbc \cdot \frac{a^2 \cdot 4(s-b)(s-c) + 4s(s-a)(b-c)^2}{sr^2} \\
 &\quad = \frac{bc}{r^2} \{a^2[a^2 - (b-c)^2] + (b-c)^2[(b+c)^2 - a^2]\} = \\
 &= \frac{bc}{r^2} (4a^2bc - 2 \sum a^2b^2 + \sum a^4) = \frac{bc}{r^2} (4a \cdot 4sRr - 16s^2r^2) = \frac{16s(R \cdot abc - sr \cdot bc)}{r} \\
 &\quad = 16s(R \cdot 4Rs - sbc) = \\
 &\stackrel{(1)}{=} 16s^2(4R^2 - bc) \stackrel{(*)}{\Rightarrow} RHS_{(*)} = (4s \cdot AV)^2 \stackrel{(*)}{\Rightarrow} (y+z)v_a = 4s \cdot AV \rightarrow \frac{AV}{v_a} \\
 &\quad = \frac{y+z}{4s} \\
 &y+z = b \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) + c \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right) \\
 &\quad = \frac{a(b+c)}{s-a} - \frac{b(b-c)}{s-b} + \frac{c(b-c)}{s-c} = \\
 &= \frac{a(b+c)}{s-a} - \frac{s(b-c)^2}{(s-b)(s-c)} = \frac{a(b+c)(s-b)(s-c) - s(s-a)(b-c)^2}{sr^2} = \\
 &= \frac{a[s+(s-a)](s-b)(s-c) - s(s-a)(b-c)^2}{sr^2} = \\
 &\quad = \frac{a(s-b)(s-c) + ar^2 - s(b-c)^2 + a(b-c)^2}{r^2} = \\
 &= (s^2 + r^2)a - sa(b+c) + abc + 2sbc - s(b^2 + c^2) + a(b^2 + c^2) - 2abc \\
 &\quad = 4s \cdot r^2 \cdot \frac{AV}{v_a} \\
 &\rightarrow 4s \cdot r^2 \sum \frac{AV}{v_a} = (s^2 + r^2) \sum a - 2s \sum a^2 + (\sum a)(\sum ab) - 6abc \\
 &= 2s(s^2 + r^2) - 4s(s^2 - r^2 - 4Rr) + 2s(s^2 + r^2 + 4Rr) - 6 \cdot 4Rrs = 8sr^2 \\
 &\rightarrow r^2 \sum \frac{AV}{v_a} = 2r^2 \\
 &\rightarrow r^2 \sum \frac{AV}{v_a} + R(R - 2r) = 2r^2 + R^2 - 2Rr = Rr + (R - 2r)(R - r) \stackrel{\text{Euler}}{\geq} Rr. \\
 \text{Therefore, } &r^2 \sum \frac{AV}{v_a} + R(R - 2r) \geq Rr.
 \end{aligned}$$

2714. In ΔABC the following relationship holds:

$$\frac{4Rs}{r^2} \geq \prod \left(\frac{n_b + n_c + h_b + h_c}{s} + \cot B + \cot C \right)$$

Proposed by Bogdan Fuștei-Romania



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Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \frac{n_a}{h_a} &\leq \frac{R}{r} - 1 \Leftrightarrow n_a \leq \frac{2s(R-r)}{a} \Leftrightarrow n_a^2 = s^2 - 2h_ar_a = s^2 \left(1 - 2 \cdot \frac{2r}{a} \cdot \tan \frac{A}{2}\right) \\
 &\leq \frac{4(R-r)^2s^2}{a^2} \\
 \Leftrightarrow (2R\sin A)^2 - 4R\sin \frac{A}{2} \cos \frac{A}{2} \cdot 4r \tan \frac{A}{2} &\leq 4(R-r)^2 \Leftrightarrow (R-r)^2 \\
 &\geq R^2(1 - \cos^2 A) - 2Rr(1 - \cos A) \\
 \Leftrightarrow r^2 &\geq -R^2 \cos^2 A + 2Rr \cos A \Leftrightarrow (R \cos A - r)^2 \geq 0 \text{ which is true} \rightarrow \frac{n_a}{h_a} \\
 &\leq \frac{R}{r} - 1 \rightarrow n_a + h_a \leq \frac{R}{r} h_a = \frac{2sR}{a} \\
 \rightarrow \frac{n_b + n_c + h_b + h_c}{s} + \cot B + \cot C & \\
 &\leq \frac{2R}{b} + \frac{2R}{c} + \frac{c^2 + a^2 - b^2}{2ca \cdot b} \cdot 2R + \frac{a^2 + b^2 - c^2}{2ab \cdot c} \cdot 2R = \\
 &= \frac{2R(b+c)}{bc} + \frac{2R \cdot a}{bc} = \frac{2R \cdot 2s}{bc} = \frac{a}{r} \\
 \rightarrow \prod \left(\frac{n_b + n_c + h_b + h_c}{s} + \cot B + \cot C \right) &\leq \prod \frac{a}{r} = \frac{abc}{r^3} = \frac{4Rs}{r^2}.
 \end{aligned}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Stewart's theorem} \Rightarrow b^2(s - c) + c^2(s - b) &= a n_a^2 + a(s - b)(s - c) \\
 \Rightarrow s(b^2 + c^2) - bc(2s - a) &= a n_a^2 + a(s^2 - s(2s - a) + bc) \\
 \Rightarrow s(b^2 + c^2) - 2sbc &= a n_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = a n_a^2 - as^2 \Rightarrow a n_a^2 \\
 &= as^2 + s(2bccosA - 2bc) = as^2 - 4sbc \sin^2 \frac{A}{2} \\
 &= as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)}
 \end{aligned}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a \left(\frac{2\Delta}{a}\right) \left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_ar_a \therefore n_a^2 \stackrel{(1)}{\cong} s^2 - 2h_ar_a$$

$$\begin{aligned}
 \text{Now, } \frac{n_a}{h_a} &\leq \frac{R}{r} - 1 \Leftrightarrow \frac{R^2}{r^2} - \frac{2R}{r} + 1 \geq \frac{n_a^2}{h_a^2} \stackrel{\text{by (1)}}{\Leftrightarrow} \frac{R^2}{r^2} - \frac{2R}{r} + 1 \geq \frac{s^2 - 2h_ar_a}{h_a^2} = \frac{s^2a^2}{4r^2s^2} - \frac{2r_a}{h_a} \\
 &= \frac{a^2}{4r^2} - \left(\frac{2rs}{s-a}\right) \left(\frac{a}{2rs}\right) = \frac{a^2}{4r^2} - \frac{(a-s)+s}{s-a} = \frac{a^2}{4r^2} + 1 - \frac{s}{s-a}
 \end{aligned}$$



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$$\begin{aligned}
 &= 1 + \frac{a^2(s-a) - 4(sr^2)}{4(s-a)r^2} = 1 + \frac{a^2(s-a) - 4(s-a)(s-b)(s-c)}{4(s-a)r^2} \\
 &= 1 + \frac{a^2 - (a^2 - (b-c)^2)}{4r^2} = 1 + \frac{(b-c)^2}{4r^2} \Leftrightarrow \frac{R^2}{r^2} - \frac{2R}{r} \geq \frac{(b-c)^2}{4r^2} \\
 &\Leftrightarrow \frac{R(R-2r)}{r^2} \geq \frac{b^2 + c^2 - 2bc}{4r^2} \\
 &\Leftrightarrow R - 2r \geq \frac{b^2 + c^2}{4R} - \frac{bc}{2R} \Leftrightarrow R\left(1 - \frac{2r}{R}\right) \geq \frac{4R^2(\sin^2 B + \sin^2 C)}{4R} - \frac{4R^2 \sin B \sin C}{2R} \\
 &\Leftrightarrow 1 - \frac{8R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{R} \geq \sin^2 B + \sin^2 C - 2 \sin B \sin C = (\sin B - \sin C)^2 \\
 &\Leftrightarrow 1 - 4 \sin \frac{A}{2} \left(2 \sin \frac{B}{2} \sin \frac{C}{2}\right) \geq \left(2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}\right)^2 \Leftrightarrow 1 - 4 \sin \frac{A}{2} \left(\cos \frac{B-C}{2} - \cos \frac{B+C}{2}\right) \\
 &\geq 4 \sin^2 \frac{A}{2} \left(1 - \cos^2 \frac{B-C}{2}\right) \\
 &\Leftrightarrow 1 - 4 \sin \frac{A}{2} \cos \frac{B-C}{2} + 4 \sin^2 \frac{A}{2} \geq 4 \sin^2 \frac{A}{2} - 4 \sin^2 \frac{A}{2} \cos^2 \frac{B-C}{2} \\
 &\Leftrightarrow 4 \sin^2 \frac{A}{2} \cos^2 \frac{B-C}{2} - 4 \sin \frac{A}{2} \cos \frac{B-C}{2} + 1 \geq 0 \Leftrightarrow \left(2 \sin \frac{A}{2} \cos \frac{B-C}{2} - 1\right)^2 \\
 &\geq 0 \rightarrow \text{true} \Rightarrow \frac{n_a}{h_a} \leq \frac{R}{r} - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{and analogs} &\Leftrightarrow \frac{n_b + n_c + h_b + h_c}{s} + \cot B + \cot C \leq \frac{\left(\frac{R}{r} - 1\right)(h_b + h_c) + h_b + h_c}{s} + \frac{\sin(B+C)}{\sin B \sin C} \\
 &= \left(\frac{R}{rs}\right) \frac{2rsa(b+c)}{abc} + \frac{4R^2 a^2}{2Rabc} = \frac{2Ra(b+c)}{4Rrs} + \frac{2Ra^2}{4Rrs} = \frac{a(2s)}{2rs} = \frac{a}{r} \text{ and analogs} \\
 &\Rightarrow \prod \left(\frac{n_b + n_c + h_b + h_c}{s} + \cot B + \cot C \right) \leq \prod \frac{a}{r} = \frac{4Rrs}{r^3} \Rightarrow \frac{4Rs}{r^2} \\
 &\geq \prod \left(\frac{n_b + n_c + h_b + h_c}{s} + \cot B + \cot C \right) \text{ (QED)}
 \end{aligned}$$

2715. In ΔABC , v_a, v_b, v_c are cevians through Bevan's point, then:

$$4Rs \sum \frac{a}{b+c} \geq 12Rs - \sum v_a(b+c)$$

Proposed by Soumava Chakraborty-Kolkata-India

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let V be the Bevan's point and $\{A'\} = (AV) \cap (BC)$ ($\because v_a = AA'$)



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The barycentric coordinates of V are (x : y : z)

$$\text{with } x = a \left(-\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} \right),$$

$$y = b \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) \text{ and } z = c \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right)$$

→ A' have the coordinates (0 : y : z) → (y + z)AA' = yAB + zAC

Squaring

$$\Leftrightarrow (y + z)^2 v_a^2 = y^2 c^2 + z^2 b^2 + yz \cdot 2\overrightarrow{AB} \cdot \overrightarrow{AC}$$

$$= y^2 c^2 + z^2 b^2 + yz(b^2 + c^2 - a^2) (*)$$

$$\begin{aligned} RHS_{(*)} &= b^2 c^2 \left[\left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right)^2 + \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right] + \\ &\quad bc \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right) (b^2 + c^2 - a^2) = \\ &= 2b^2 c^2 \left[\frac{a^2}{(s-a)^2} + \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right] \\ &\quad + bc \left[\frac{a^2}{(s-a)^2} - \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right] (b^2 + c^2 - a^2) = \\ &= \frac{a^2}{(s-a)^2} \cdot bc[(b+c)^2 - a^2] + bc \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 [a^2 - (b-c)^2] \\ &= \frac{4sa^2bc}{s-a} + \frac{4s^2bc(b-c)^2}{(s-b)(s-c)} = \\ &= sbc \cdot \frac{a^2 \cdot 4(s-b)(s-c) + 4s(s-a)(b-c)^2}{sr^2} \\ &= \frac{bc}{r^2} \{a^2[a^2 - (b-c)^2] + (b-c)^2[(b+c)^2 - a^2]\} = \end{aligned}$$

$$\begin{aligned} &= \frac{bc}{r^2} (4a^2bc - 2 \sum a^2b^2 + \sum a^4) = \frac{bc}{r^2} (4a \cdot 4sRr - 16s^2r^2) = \frac{16s(R \cdot abc - sr \cdot bc)}{r} \\ &= 16s(R \cdot 4Rs - sbc) = \\ &= 64s^2R^2 \left(1 - \frac{bc}{4R^2} \right) = 64s^2R^2 \left(1 - \frac{s(s-a)}{4R^2 \cos^2 \frac{A}{2}} \right) \\ &= 64s^2R^2 \left(1 - \frac{4s(s-a) \cos^2 \frac{B-C}{2}}{\left(4R \cos \frac{A}{2} \cos \frac{B-C}{2} \right)^2} \right) \geq \end{aligned}$$

$$\stackrel{\cos^2 \frac{B-C}{2} \leq 1}{\geq} 64s^2R^2 \left(1 - \frac{(b+c)^2 - a^2}{(b+c)^2} \right) = \left(8sR \cdot \frac{a}{b+c} \right)^2 \rightarrow RHS_{(*)} \geq \left(8sR \cdot \frac{a}{b+c} \right)^2 \quad (1)$$

$$\begin{aligned} y + z &= b \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) + c \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right) \\ &= \frac{a(b+c)}{s-a} - \frac{b(b-c)}{s-b} + \frac{c(b-c)}{s-c} = \end{aligned}$$



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$$= \frac{a(b+c)}{s-a} - \frac{s(b-c)^2}{(s-b)(s-c)} \leq \frac{a(b+c)}{s-a} \rightarrow LHS_{(*)} \leq \left(\frac{a(b+c)}{s-a} \cdot v_a \right)^2 \quad (2)$$

From (1), (2), we have : $\frac{a(b+c)}{s-a} \cdot v_a \geq 8sR \cdot \frac{a}{b+c} \leftrightarrow v_a(b+c) \geq 8Rs \cdot \frac{s-a}{b+c}$
 $\rightarrow 4Rs \cdot \frac{a}{b+c} + v_a(b+c) \geq 4Rs \left(\frac{a}{b+c} + \frac{-a+b+c}{b+c} \right) = 4Rs$ (And analogs)

Therefore,
$$\sum \left(4Rs \cdot \frac{a}{b+c} + v_a(b+c) \right) \geq 12Rs$$

$$\rightarrow \boxed{4Rs \sum \frac{a}{b+c} \geq 12Rs - \sum v_a(b+c)}.$$

2716. In ΔABC , $x, y, z > 0$, the following relationship holds:

$$\sum \left(\frac{m_a}{h_a} \right)^2 \cdot x \geq \sqrt{\sum xy} \cdot \left[2 \sum \frac{m_a^2 m_b^2}{h_a^2 h_b^2} - \sum \left(\frac{m_a}{h_a} \right)^4 \right]^{\frac{1}{2}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let G be the centroid of ΔABC , we know that : $AG = \frac{2}{3}m_a$.

We know that if

$P \in Int(\Delta ABC)$, then aPA, bPB, cPC can be the sides of a triangle (Klamkin)

$\rightarrow aGA, bGB, cGC$ can be the sides of a triangle.

Since $aGA = \frac{4sr}{3} \cdot \frac{m_a}{h_a} \rightarrow \frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}$ can be the sides of a triangle Δ' with area

$$F' = \frac{1}{4} \sqrt{2 \sum \left(\frac{m_a}{h_a} \right)^2 \left(\frac{m_b}{h_b} \right)^2 - \sum \left(\frac{m_a}{h_a} \right)^4} = \frac{1}{4} \left[2 \sum \frac{m_a^2 m_b^2}{h_a^2 h_b^2} - \sum \left(\frac{m_a}{h_a} \right)^4 \right]^{\frac{1}{2}}$$

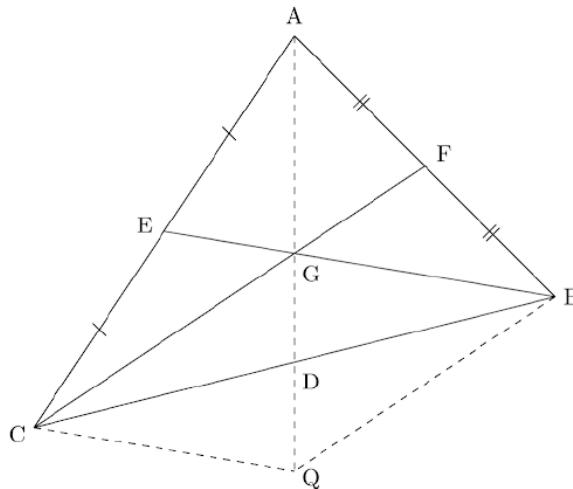
We know that for any ΔUVW with area F , $x, y, z > 0$, then $\sum u^2 \cdot x$

$$\geq 4F \sqrt{\sum xy} \quad (\text{Oppenheim})$$

\rightarrow for the triangle Δ' , we have : $\sum \left(\frac{m_a}{h_a} \right)^2 \cdot x \geq 4F' \cdot \sqrt{\sum xy}$

Therefore, $\sum \left(\frac{m_a}{h_a} \right)^2 \cdot x \geq \sqrt{\sum xy} \cdot \left[2 \sum \frac{m_a^2 m_b^2}{h_a^2 h_b^2} - \sum \left(\frac{m_a}{h_a} \right)^4 \right]^{\frac{1}{2}}$

Solution 2 by Soumava Chakraborty-Kolkata-India



Proof :

Via Ptolemy's theorem on quadrilateral ABQC, $AB \cdot CQ + AC \cdot BQ \geq AQ \cdot BC$

$$\Rightarrow c \cdot \frac{2m_b}{3} + b \cdot \frac{2m_c}{3} \geq \frac{4m_a}{3} \cdot a \Rightarrow cm_b + bm_c \geq 2am_a$$

upon squaring

$$\Rightarrow c^2 m_b^2 + b^2 m_c^2 + 2bcm_b m_c \stackrel{(i)}{\geq} 16a^2 m_a^2 - 4c^2 m_b^2 - 4b^2 m_c^2$$

$$\text{Now, } (2bm_b + 2cm_c)^2 - 4a^2 m_a^2$$

via (i)

$$= 4b^2 m_b^2 + 4c^2 m_c^2 + 8bcm_b m_c - 4a^2 m_a^2 \stackrel{(i)}{\geq} 4b^2 m_b^2 + 4c^2 m_c^2 + 12a^2 m_a^2$$

$$- 4c^2 m_b^2 - 4b^2 m_c^2$$

$$= b^2 (4m_b^2 - 4m_c^2) - c^2 (4m_b^2 - 4m_c^2) + 12a^2 m_a^2$$

$$= (b^2 - c^2)(2c^2 + 2a^2 - b^2 - 2a^2 - 2b^2 + c^2) + 12a^2 m_a^2$$

$$= 3(a^2 \cdot 4m_a^2 - (b^2 - c^2)^2)$$

$$= 3(a^2(2b^2 + 2c^2 - a^2) - b^4 - c^4 + 2b^2 c^2) = 3\left(2 \sum a^2 b^2 - \sum a^4\right) = 48F^2 > 0$$

$$\Rightarrow (2bm_b + 2cm_c)^2 > 4a^2 m_a^2 \Rightarrow bm_b + cm_c - am_a > 0$$



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$$\Rightarrow \frac{bm_b}{2rs} + \frac{cm_c}{2rs} > \frac{am_a}{2rs} \Rightarrow \frac{m_b}{h_b} + \frac{m_c}{h_c} > \frac{m_a}{h_a} \text{ and analogs}$$

$\Rightarrow \frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}$ form sides of a triangle

$$\rightarrow (1) \text{ with area } F^*(\text{say}) \stackrel{(*)}{=} \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \frac{m_a^2 m_b^2}{h_a^2 h_b^2} - \sum_{\text{cyc}} \left(\frac{m_a}{h_a} \right)^4}$$

\therefore via Oppenheim and making use of axiom (1), $\sum_{\text{cyc}} \left(\frac{m_a}{h_a} \right)^2 x$

$$\geq \sqrt{xy + yz + zx} (4F^*) \stackrel{\text{via } (*)}{=} \sqrt{xy + yz + zx} \left[2 \sum_{\text{cyc}} \frac{m_a^2 m_b^2}{h_a^2 h_b^2} - \sum_{\text{cyc}} \left(\frac{m_a}{h_a} \right)^4 \right]^{\frac{1}{2}} \text{ (QED)}$$

2717. In ΔABC the following relationship holds:

$$\frac{s}{r} - \frac{1}{s} \sum (n_a + h_a) \geq \sum \cot A$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{n_a}{h_a} \leq \frac{R}{r} - 1 \leftrightarrow n_a \leq \frac{2s(R-r)}{a} \leftrightarrow n_a^2 = s^2 - 2h_ar_a = s^2 \left(1 - 2 \cdot \frac{2r}{a} \cdot \tan \frac{A}{2} \right)$$

$$\leq \frac{4(R-r)^2 s^2}{a^2}$$

$$\leftrightarrow (2R \sin A)^2 - 4R \sin \frac{A}{2} \cos \frac{A}{2} \cdot 4r \tan \frac{A}{2} \leq 4(R-r)^2 \leftrightarrow (R-r)^2$$

$$\geq R^2(1 - \cos^2 A) - 2Rr(1 - \cos A)$$

$$\leftrightarrow r^2 \geq -R^2 \cos^2 A + 2Rr \cos A \leftrightarrow (R \cos A - r)^2 \geq 0 \text{ which is true} \rightarrow \frac{n_a}{h_a}$$

$$\leq \frac{R}{r} - 1 \rightarrow n_a + h_a \leq \frac{R}{r} h_a = \frac{2sR}{a}$$

$$\rightarrow \frac{s}{r} - \frac{1}{s} \sum (n_a + h_a) \geq \frac{s}{r} - \frac{1}{s} \sum \frac{2sR}{a} = \frac{1}{4sr} \left(\sum a \right)^2 - \frac{1}{4sr} \cdot 2 \sum ab = \frac{1}{4sr} \sum a^2$$

$$= \sum \frac{b^2 + c^2 - a^2}{2bc} \cdot \frac{2R}{a} = \sum \cot A.$$



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$$\text{Therefore, } \frac{s}{r} - \frac{1}{s} \sum (n_a + h_a) \geq \sum \cot A.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Stewart's theorem} &\Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c) \\
 &\Rightarrow s(b^2 + c^2) - bc(2s - a) = an_a^2 + a(s^2 - s(2s - a) + bc) \\
 \Rightarrow s(b^2 + c^2) - 2sbc &= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 \\
 &= as^2 + s(2bcc\cos A - 2bc) = as^2 - 4sbc\sin^2 \frac{A}{2} \\
 &= as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)} \\
 &= as^2 - \frac{4\Delta^2}{s - a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s - a}\right) = as^2 - 2ah_a r_a \therefore n_a^2 \stackrel{(1)}{\cong} s^2 - 2h_a r_a \\
 \text{Now, } \frac{n_a}{h_a} \leq \frac{R}{r} - 1 &\Leftrightarrow \frac{R^2}{r^2} - \frac{2R}{r} + 1 \geq \frac{n_a^2}{h_a^2} \stackrel{\text{by (1)}}{\Leftrightarrow} \frac{R^2}{r^2} - \frac{2R}{r} + 1 \geq \frac{s^2 - 2h_a r_a}{h_a^2} = \frac{s^2 a^2}{4r^2 s^2} - \frac{2r_a}{h_a} \\
 &= \frac{a^2}{4r^2} - \left(\frac{2rs}{s-a}\right)\left(\frac{a}{2rs}\right) = \frac{a^2}{4r^2} - \frac{(a-s)+s}{s-a} = \frac{a^2}{4r^2} + 1 - \frac{s}{s-a} \\
 &= 1 + \frac{a^2(s-a) - 4(sr^2)}{4(s-a)r^2} = 1 + \frac{a^2(s-a) - 4(s-a)(s-b)(s-c)}{4(s-a)r^2} \\
 &= 1 + \frac{a^2 - (a^2 - (b-c)^2)}{4r^2} = 1 + \frac{(b-c)^2}{4r^2} \Leftrightarrow \frac{R^2}{r^2} - \frac{2R}{r} \geq \frac{(b-c)^2}{4r^2} \\
 &\Leftrightarrow \frac{R(R-2r)}{r^2} \geq \frac{b^2 + c^2 - 2bc}{4r^2} \\
 \Leftrightarrow R - 2r \geq \frac{b^2 + c^2}{4R} - \frac{bc}{2R} &\Leftrightarrow R\left(1 - \frac{2r}{R}\right) \geq \frac{4R^2(\sin^2 B + \sin^2 C)}{4R} - \frac{4R^2 \sin B \sin C}{2R} \\
 &\Leftrightarrow 1 - \frac{8R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{R} \geq \sin^2 B + \sin^2 C - 2 \sin B \sin C = (\sin B - \sin C)^2 \\
 \Leftrightarrow 1 - 4 \sin \frac{A}{2} \left(2 \sin \frac{B}{2} \sin \frac{C}{2}\right) &\geq \left(2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}\right)^2 \Leftrightarrow 1 - 4 \sin \frac{A}{2} \left(\cos \frac{B-C}{2} - \cos \frac{B+C}{2}\right) \\
 &\geq 4 \sin^2 \frac{A}{2} \left(1 - \cos^2 \frac{B-C}{2}\right) \\
 \Leftrightarrow 1 - 4 \sin \frac{A}{2} \cos \frac{B-C}{2} + 4 \sin^2 \frac{A}{2} &\geq 4 \sin^2 \frac{A}{2} - 4 \sin^2 \frac{A}{2} \cos^2 \frac{B-C}{2} \\
 &\Leftrightarrow 4 \sin^2 \frac{A}{2} \cos^2 \frac{B-C}{2} - 4 \sin \frac{A}{2} \cos \frac{B-C}{2} + 1 \geq 0 \Leftrightarrow \left(2 \sin \frac{A}{2} \cos \frac{B-C}{2} - 1\right)^2 \\
 &\geq 0 \rightarrow \text{true} \Rightarrow \frac{n_a}{h_a} \leq \frac{R}{r} - 1
 \end{aligned}$$

and analogs $\Leftrightarrow \frac{n_b + n_c + h_b + h_c}{s} + \cot B + \cot C \leq \frac{\left(\frac{R}{r} - 1\right)(h_b + h_c) + h_b + h_c}{s} + \frac{\sin(B+C)}{\sin B \sin C}$

$$\begin{aligned}
 &= \left(\frac{R}{rs}\right) \frac{2rsa(b+c)}{abc} + \frac{4R^2 a^2}{2Rabc} = \frac{2Ra(b+c)}{4Rrs} + \frac{2Ra^2}{4Rrs} = \frac{a(2s)}{2rs} = \frac{a}{r} \text{ and analogs}
 \end{aligned}$$



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$$\begin{aligned}
 & \Rightarrow \sum \left(\frac{n_b + n_c + h_b + h_c}{s} + \cot B + \cot C \right) \leq \sum \frac{a}{r} = \frac{2s}{r} \\
 & \Rightarrow \frac{n_a + n_b + n_c + h_a + h_b + h_c}{s} + 2 \sum \cot A \leq \frac{2s}{r} \\
 & \Rightarrow \frac{s}{r} - \frac{n_a + n_b + n_c + h_a + h_b + h_c}{s} \geq \sum \cot A \quad (\text{QED})
 \end{aligned}$$

2718. Let $\alpha \geq 2$. In any ΔABC the following relationship holds:

$$\left(\frac{R}{2r}\right)^\alpha \sum_{cyc} h_b h_c \geq \sum_{cyc} r_b r_c$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by George Florin Serban-Romania

$$\sum_{cyc} h_b h_c = \sum_{cyc} \frac{2F}{b} \cdot \frac{2F}{c} = 4F^2 \sum_{cyc} \frac{1}{bc} = 4F^2 \sum_{cyc} \frac{a}{abc} = \frac{4F^2 \cdot 2s}{4RF} = \frac{2sF}{R} = \frac{2rs^2}{R}$$

$$\therefore \sum_{cyc} r_b r_c = s^2$$

$$\left(\frac{R}{2r}\right)^\alpha \sum_{cyc} h_b h_c = \left(\frac{R}{2r}\right)^\alpha \frac{2rs^2}{R} \stackrel{(*)}{\geq} \sum_{cyc} r_b r_c = s^2$$

$(*) \Leftrightarrow \left(\frac{R}{2r}\right)^\alpha \cdot \frac{2r}{R} \geq 1 \Leftrightarrow \left(\frac{R}{2r}\right)^\alpha \geq \frac{R}{2r}$ which is true from $R \geq 2r$ (*Euler*) and $\alpha \geq 2$.

Therefore,

$$\left(\frac{R}{2r}\right)^\alpha \sum_{cyc} h_b h_c \geq \sum_{cyc} r_b r_c$$

2719. In ΔABC , $x, y, z > 0$,

$$\begin{aligned}
 \omega_1 &= \left[\left(\sum \sqrt{\frac{m_a}{h_a}} \right) \left(\sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}} - \sqrt{\frac{m_a}{h_a}} \right) \right]^{\frac{1}{2}}, \\
 \omega_2 &= \left[\left(\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_c}{h_c}} - \sqrt{\frac{m_b}{h_b}} \right) \left(\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} - \sqrt{\frac{m_c}{h_c}} \right) \right]^{\frac{1}{2}}
 \end{aligned}$$



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$$\text{then : } \sum \frac{m_a}{h_a} \cdot x \geq \sqrt{\sum xy \cdot \omega_1 \cdot \omega_2}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let G be the centroid of ΔABC , we know that :

$$AG = \frac{2}{3}m_a. \text{We know that if}$$

$P \in \text{Int}(\Delta ABC)$, then aPA, bPB, cPC can be the sides of a triangle (Klamkin)
 $\rightarrow aGA, bGB, cGC$ can be the sides of a triangle.

Since $aGA = \frac{4sr}{3} \cdot \frac{m_a}{h_a} \rightarrow \frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}$ can be the sides of a triangle.

We know that if m, n, p the sides of a triangle, then $\sqrt{m}, \sqrt{n}, \sqrt{p}$
can be also the sides of a triangle.

$$\left(\therefore (\sqrt{m} + \sqrt{n})^2 = (m + n) + 2\sqrt{mn} > \sqrt{p}^2 \right)$$

$\rightarrow \sqrt{\frac{m_a}{h_a}}, \sqrt{\frac{m_b}{h_b}}, \sqrt{\frac{m_c}{h_c}}$ can be the sides of a triangle Δ' with area :

$$\begin{aligned} & F' \\ &= \frac{1}{4} \sqrt{\left(\sum \sqrt{\frac{m_a}{h_a}} \right) \left(\sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}} - \sqrt{\frac{m_a}{h_a}} \right) \left(\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_c}{h_c}} - \sqrt{\frac{m_b}{h_b}} \right) \left(\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} - \sqrt{\frac{m_c}{h_c}} \right)} \\ &= \frac{1}{4} \cdot \omega_1 \cdot \omega_2 \end{aligned}$$

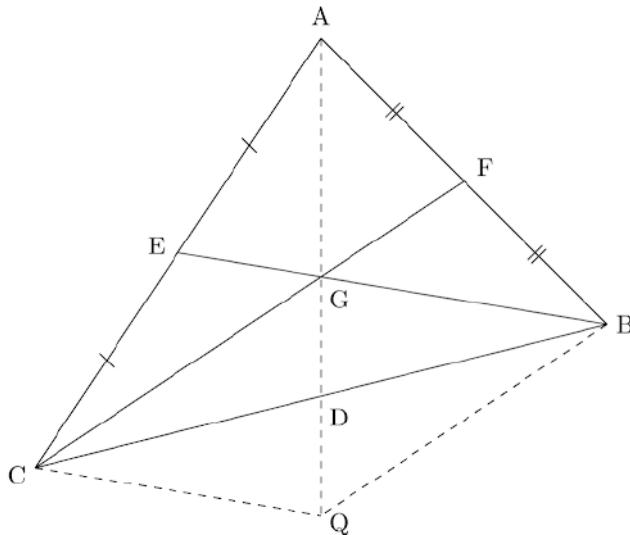
We know that for any ΔUVW with area $F, x, y, z > 0$, then $\sum u^2 \cdot x$

$$\geq 4F \sqrt{\sum xy} \text{ (Oppenheim)}$$

\rightarrow for the triangle Δ' , we have : $\sum \sqrt{\frac{m_a}{h_a}}^2 \cdot x \geq 4F' \cdot \sqrt{\sum xy}$

$$\text{Therefore, } \sum \frac{m_a}{h_a} \cdot x \geq \sqrt{\sum xy \cdot \omega_1 \cdot \omega_2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India



Proof :

Via Ptolemy's theorem on quadrilateral ABQC, $AB \cdot CQ + AC \cdot BQ \geq AQ \cdot BC$

$$\Rightarrow c \cdot \frac{2m_b}{3} + b \cdot \frac{2m_c}{3} \geq \frac{4m_a}{3} \cdot a \Rightarrow cm_b + bm_c \geq 2am_a$$

upon squaring

$$\Rightarrow c^2 m_b^2 + b^2 m_c^2 + 2bcm_b m_c \geq 4a^2 m_a^2 \Rightarrow 8bcm_b m_c \stackrel{(i)}{\geq} 16a^2 m_a^2 - 4c^2 m_b^2 - 4b^2 m_c^2$$

$$\text{Now, } (2bcm_b + 2cm_c)^2 - 4a^2 m_a^2$$

$$= 4b^2 m_b^2 + 4c^2 m_c^2 + 8bcm_b m_c - 4a^2 m_a^2 \stackrel{(i)}{\geq} 4b^2 m_b^2 + 4c^2 m_c^2 + 12a^2 m_a^2$$

$$- 4c^2 m_b^2 - 4b^2 m_c^2$$

$$= b^2(4m_b^2 - 4m_c^2) - c^2(4m_b^2 - 4m_c^2) + 12a^2 m_a^2$$

$$= (b^2 - c^2)(2c^2 + 2a^2 - b^2 - 2a^2 - 2b^2 + c^2) + 12a^2 m_a^2$$

$$= 3(a^2 \cdot 4m_a^2 - (b^2 - c^2)^2)$$

$$= 3(a^2(2b^2 + 2c^2 - a^2) - b^4 - c^4 + 2b^2 c^2) = 3\left(2 \sum a^2 b^2 - \sum a^4\right) = 48F^2 > 0$$

$$\Rightarrow (2bcm_b + 2cm_c)^2 > 4a^2 m_a^2 \Rightarrow bm_b + cm_c - am_a > 0$$

$$\Rightarrow (\sqrt{bm_b + cm_c} - \sqrt{am_a})(\sqrt{bm_b + cm_c} + \sqrt{am_a}) > 0 \Rightarrow \sqrt{bm_b + cm_c} - \sqrt{am_a} > 0$$

$$\Rightarrow \sqrt{am_a} < \sqrt{bm_b + cm_c} < \sqrt{bm_b} + \sqrt{cm_c} \Rightarrow \sqrt{bm_b} + \sqrt{cm_c} > \sqrt{am_a}$$

$$\Rightarrow \sqrt{\frac{bm_b}{2rs}} + \sqrt{\frac{cm_c}{2rs}} > \sqrt{\frac{am_a}{2rs}} \Rightarrow \sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}} > \sqrt{\frac{m_a}{h_a}} \text{ and analogs}$$

$$\Rightarrow \sqrt{\frac{m_a}{h_a}}, \sqrt{\frac{m_b}{h_b}}, \sqrt{\frac{m_c}{h_c}} \text{ form sides of a triangle} \rightarrow (1) \text{ with area } F^* \text{ (say)}$$



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$$\begin{aligned}
 &= \frac{1}{4} \sqrt{\left(\sum_{\text{cyc}} \sqrt{\frac{m_a}{h_a}} \right) \left(\sqrt{\frac{m_b}{h_b}} + \sqrt{\frac{m_c}{h_c}} - \sqrt{\frac{m_a}{h_a}} \right) \left(\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_b}{h_b}} - \sqrt{\frac{m_c}{h_c}} \right) \left(\sqrt{\frac{m_a}{h_a}} + \sqrt{\frac{m_c}{h_c}} - \sqrt{\frac{m_b}{h_b}} \right)} \\
 &\Rightarrow F^* \stackrel{(*)}{=} \frac{\omega_1 \omega_2}{4} \therefore \text{via Oppenheim and making use of axiom (1),} \\
 &\sum_{\text{cyc}} \frac{m_a}{h_a} \cdot x \geq \sqrt{xy + yz + zx} (4F^*) \stackrel{\text{via } (*)}{=} \sqrt{xy + yz + zx} \cdot \omega_1 \cdot \omega_2 \text{ (QED)}
 \end{aligned}$$

2720. In ΔABC , v_a, v_b, v_c are cevians through Bevan's point, then:

$$\sum v_a(b+c) \cos \frac{A}{2} \geq 2s^2$$

Proposed by Soumava Chakraborty-Kolkata-India

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let V be the Bevan's point and $\{A'\} = (AV) \cap (BC)$ ($\therefore v_a = AA'$)

The barycentric coordinates of V are $(x : y : z)$ with x

$$= a \left(-\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} \right),$$

$$y = b \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) \text{ and } z = c \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right)$$

$\rightarrow A'$ have the coordinates $(0 : y : z) \rightarrow (y+z)\overrightarrow{AA'} = y\overrightarrow{AB} + z\overrightarrow{AC}$

$$\stackrel{\text{Squaring}}{\Leftrightarrow} (y+z)^2 v_a^2 = y^2 c^2 + z^2 b^2 + yz \cdot 2\overrightarrow{AB} \cdot \overrightarrow{AC}$$

$$= y^2 c^2 + z^2 b^2 + yz(b^2 + c^2 - a^2) \quad (*)$$

$$RHS_{(*)} = b^2 c^2 \left[\left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right)^2 + \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right] +$$

$$bc \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right) (b^2 + c^2 - a^2) =$$

$$= 2b^2 c^2 \left[\frac{a^2}{(s-a)^2} + \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right]$$

$$+ bc \left[\frac{a^2}{(s-a)^2} - \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right] (b^2 + c^2 - a^2) =$$



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$$= \frac{a^2}{(s-a)^2} \cdot bc[(b+c)^2 - a^2] + bc \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 [a^2 - (b-c)^2]$$

$$= \frac{4sa^2bc}{s-a} + \frac{4s^2bc(b-c)^2}{(s-b)(s-c)} \geq \frac{4sa^2bc}{s-a} \quad (1)$$

$$\begin{aligned} y+z &= b \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) + c \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right) \\ &= \frac{a(b+c)}{s-a} - \frac{b(b-c)}{s-b} + \frac{c(b-c)}{s-c} = \end{aligned}$$

$$= \frac{a(b+c)}{s-a} - \frac{s(b-c)^2}{(s-b)(s-c)} \leq \frac{a(b+c)}{s-a} \rightarrow LHS_{(*)} \leq \left(\frac{a(b+c)}{s-a} \cdot v_a \right)^2 \quad (2)$$

From (1), (2), we have : $\frac{a(b+c)}{s-a} \cdot v_a \geq \sqrt{\frac{4sa^2bc}{s-a}} \leftrightarrow v_a \geq \frac{2\sqrt{bc}}{b+c} \cdot \sqrt{s(s-a)}$
 $= w_a$ (And analogs)

$$\rightarrow \sum v_a(b+c) \cos \frac{A}{2} \geq \sum w_a(b+c) \cos \frac{A}{2} = 2 \sum bc \cos^2 \frac{A}{2} = 2 \sum s(s-a) = 2s^2$$

$$\text{Therefore, } \sum v_a(b+c) \cos \frac{A}{2} \geq 2s^2.$$

2721. In ΔABC the following relationship holds:

$$\sum \frac{1}{\sqrt{\cos \frac{B}{2} \cos \frac{C}{2}}} \leq \sqrt{\frac{2}{3} \sum \frac{1}{\sqrt{\sin \frac{A}{2}}}}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have : } \cos \frac{B}{2} \cos \frac{C}{2} = s \sqrt{\frac{(s-b)(s-c)}{ca \cdot ab}} = \frac{s}{a} \sin \frac{A}{2}$$

$$\text{Also, if } a \geq b \geq c \rightarrow \frac{1}{\sqrt{\sin \frac{A}{2}}} \leq \frac{1}{\sqrt{\sin \frac{B}{2}}} \leq \frac{1}{\sqrt{\sin \frac{C}{2}}} \text{ and } \sqrt{a} \geq \sqrt{b} \geq \sqrt{c}$$



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$$\rightarrow \sum \frac{1}{\sqrt{\cos \frac{B}{2} \cos \frac{C}{2}}} = \frac{1}{\sqrt{s}} \sum \sqrt{a} \cdot \frac{1}{\sqrt{\sin \frac{A}{2}}} \stackrel{\text{Chebyshev}}{\leq} \frac{1}{3\sqrt{s}} \left(\sum \sqrt{a} \right) \left(\sum \frac{1}{\sqrt{\sin \frac{A}{2}}} \right) \leq$$

$$\stackrel{\text{CBS}}{\leq} \frac{1}{3\sqrt{s}} \left(\sqrt{3 \sum a} \right) \left(\sum \frac{1}{\sqrt{\sin \frac{A}{2}}} \right) = \sqrt{\frac{2}{3}} \sum \frac{1}{\sqrt{\sin \frac{A}{2}}}$$

$$\text{Therefore, } \sum \frac{1}{\sqrt{\cos \frac{B}{2} \cos \frac{C}{2}}} \leq \sqrt{\frac{2}{3}} \sum \frac{1}{\sqrt{\sin \frac{A}{2}}}$$

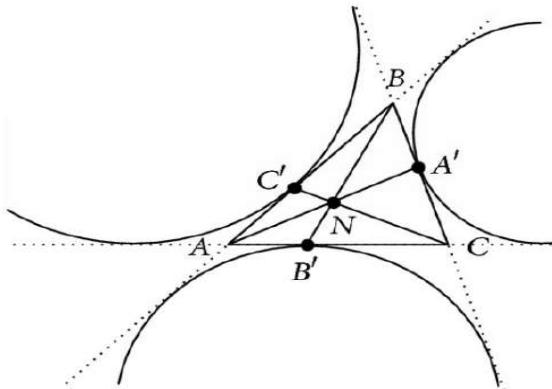
2722. In ΔABC the following relationship holds:

$$xa^2 n_a + yb^2 n_b + zc^2 n_c \geq \sqrt{xy + yz + zx} \sqrt{2 \sum a^2 b^2 n_a n_b - \sum a^4 n_a^2} \quad \forall x, y, z > 0$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

Let m, n, p form sides of a triangle and then $(\sqrt{m} + \sqrt{n})^2 - p = m + n - p + 2\sqrt{mn}$
 $> 2\sqrt{mn} > 0 \Rightarrow \sqrt{m} + \sqrt{n} > \sqrt{p}$ and similarly, $\sqrt{n} + \sqrt{p} > \sqrt{m}$ and
 $\sqrt{p} + \sqrt{m} > \sqrt{n}$
 $\Rightarrow \sqrt{m}, \sqrt{n}, \sqrt{p}$ form sides of a triangle for every m, n, p that form sides of a triangle
 \rightarrow axiom (1)



Let the Nagel point be N . Now, $A'B = s - c, A'C = s - b, B'A = s - c, B'C = s - a, C'A$

$$= s - b, C'B = s - a \therefore \frac{AN}{A'N} \stackrel{\text{van Aubel}}{\asymp} \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{a}{s-a} \Rightarrow \frac{A'N}{AN} + 1 = \frac{s-a}{a} + 1$$

$$\Rightarrow \frac{n_a}{AN} = \frac{s}{a} \Rightarrow a \cdot AN = \frac{a^2 n_a}{s} \text{ and analogs} \Rightarrow \frac{a^2 n_a + b^2 n_b - c^2 n_c}{s} = a \cdot AN + b \cdot BN - c \cdot CN > 0$$

$\because a \cdot AM + b \cdot BM - c \cdot CM \text{ and cyclic analogs} \stackrel{\text{Klamkin}}{\gtrsim} 0 \forall \text{ interior points } M \text{ and}$

choosing $M \equiv N$ which is justifiable since N is an interior point

$$\therefore a^2 n_a + b^2 n_b - c^2 n_c \text{ and cyclic analogs} > 0$$

$\Rightarrow a^2 n_a, b^2 n_b, c^2 n_c \text{ form sides of a triangle}$

via axiom (1)

$$\stackrel{\text{axiom (1)}}{\Rightarrow} a \sqrt{n_a}, b \sqrt{n_b}, c \sqrt{n_c} \text{ form sides of a triangle} \rightarrow \text{axiom (2)}$$

Now, via generalised Ionescu – Weitzenbock's inequality, we have : $\forall x, y, z$

$$> 0 \text{ and } \forall \alpha, \beta, \gamma \text{ that form sides of a triangle, } x\alpha^2 + y\beta^2 + z\gamma^2 \geq 4\sqrt{xy + yz + zx}. [PQR]$$

(where PQR is the triangle formed by α, β, γ) $= \sqrt{xy + yz + zx} \sqrt{16[PQR]^2}$

$$= \sqrt{xy + yz + zx} \sqrt{2 \sum \alpha^2 \beta^2 - \sum \alpha^4} \Rightarrow x\alpha^2 + y\beta^2 + z\gamma^2 \geq \sqrt{xy + yz + zx} \sqrt{2 \sum \alpha^2 \beta^2 - \sum \alpha^4}$$

and choosing $\alpha = a \sqrt{n_a}, \beta = b \sqrt{n_b}, \gamma$

$= c \sqrt{n_c}$, which is justifiable as is evident from axiom (2), we have $\forall x, y, z > 0$:

$$x\alpha^2 n_a + y\beta^2 n_b + z\gamma^2 n_c \geq \sqrt{xy + yz + zx} \sqrt{2 \sum a^2 b^2 n_a n_b - \sum a^4 n_a^2} \quad (\text{QED})$$

2723. Let $\alpha \geq 1$. In any ΔABC the following relationship holds:



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$$\left(\frac{R}{2r}\right)^{\alpha} \sum_{cyc} w_a^2 \geq \sum_{cyc} r_b r_c$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by George Florin Șerban-Romania

$$\sum_{cyc} h_b h_c = \sum_{cyc} \frac{2F}{b} \cdot \frac{2F}{c} = 4F^2 \sum_{cyc} \frac{1}{bc} = 4F^2 \sum_{cyc} \frac{a}{abc} = \frac{4F^2 \cdot 2s}{4RF} = \frac{2sF}{R} = \frac{2rs^2}{R}$$

$$\therefore \sum_{cyc} r_b r_c = s^2$$

$$\left(\frac{R}{2r}\right)^{\alpha} \sum_{cyc} h_b h_c = \left(\frac{R}{2r}\right)^{\alpha} \frac{2rs^2}{R} \stackrel{(*)}{\geq} \sum_{cyc} r_b r_c = s^2$$

$(*) \Leftrightarrow \left(\frac{R}{2r}\right)^{\alpha} \cdot \frac{2r}{R} \geq 1 \Leftrightarrow \left(\frac{R}{2r}\right)^{\alpha} \geq \frac{R}{2r}$ which is true from $R \geq 2r$ (Euler) and $\alpha \geq 1$.

$$\left(\frac{R}{2r}\right)^{\alpha} \sum_{cyc} w_a^2 \geq \left(\frac{R}{2r}\right)^{\alpha} \sum_{cyc} w_a w_b \geq \left(\frac{R}{2r}\right)^{\alpha} \sum_{cyc} h_a h_b \geq \sum_{cyc} r_b r_c$$

Therefore,

$$\left(\frac{R}{2r}\right)^{\alpha} \sum_{cyc} w_a^2 \geq \sum_{cyc} r_b r_c$$

2724.

In any ΔABC , $\sum n_a(r_b + r_c) \sqrt{\frac{r_a}{h_a}} \geq \frac{s^2}{3} \sum \sqrt{\frac{2m_a + n_a + h_a}{h_a}}$

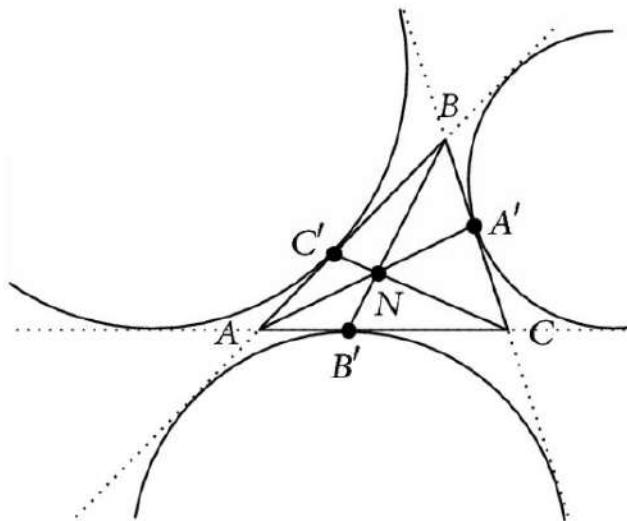
Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2} \therefore r_b + r_c \stackrel{(1)}{=} 4R \cos^2 \frac{A}{2}$$

$$\begin{aligned} \text{Stewart's theorem} &\Rightarrow b^2(s - c) + c^2(s - b) = a n_a^2 + a(s - b)(s - c) \\ &\Rightarrow s(b^2 + c^2) - bc(2s - a) = a n_a^2 + a(s^2 - s(2s - a) + bc) \\ &\Rightarrow s(b^2 + c^2) - 2sbc \end{aligned}$$

$$\begin{aligned}
 &= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bcc\cos A - 2bc) \\
 &\quad = as^2 - 4sbc\sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2 - \frac{4\Delta^2}{s-a} \\
 &= as^2 - 2a \left(\frac{2\Delta}{a}\right) \left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_a r_a \Rightarrow n_a^2 = s^2 - 2h_a r_a \Rightarrow a^2 n_a^2 \leq 4(R-r)^2 s^2 \\
 &\quad \Leftrightarrow a^2(s^2 - 2h_a r_a) \leq 4(R-r)^2 s^2 \\
 &\quad \Leftrightarrow (4R^2 \sin^2 A)s^2 - 4rs \left(4R \sin \frac{A}{2} \cos \frac{A}{2}\right) \left(s \tan \frac{A}{2}\right) \\
 &\leq 4(R^2 - 2Rr + r^2)s^2 \Leftrightarrow R^2(1 - \sin^2 A) - 2Rr \left(1 - 2\sin^2 \frac{A}{2}\right) + r^2 \geq 0 \\
 &\quad \Leftrightarrow R^2 \cos^2 A - 2Rr \cos A + r^2 \geq 0 \Leftrightarrow (R \cos A - r)^2 \geq 0 \rightarrow \text{true} \therefore an_a \\
 &\quad \leq 2Rs - 2rs \\
 &\Rightarrow \frac{n_a}{h_a} \leq \frac{2Rs}{a \left(\frac{2rs}{a}\right)} - \frac{2rs}{a \left(\frac{2rs}{a}\right)} \Rightarrow \frac{n_a}{h_a} \leq \frac{R}{r} - 1 \text{ and } \because \text{via Panaitopol, } \frac{2m_a}{h_a} \leq \frac{R}{r} \therefore \frac{2m_a}{h_a} + \frac{n_a}{h_a} + 1 \\
 &\quad \leq \frac{R}{r} + \frac{R}{r} - 1 + 1 \Rightarrow \sqrt{\frac{2m_a + n_a + h_a}{h_a}} \leq \sqrt{\frac{2R}{r}} \text{ and analogs} \\
 &\Rightarrow \frac{s^2}{3} \sum \sqrt{\frac{2m_a + n_a + h_a}{h_a}} \leq \left(\frac{s^2}{3}\right) 3 \sqrt{\frac{2R}{r}} \therefore \frac{s^2}{3} \sum \sqrt{\frac{2m_a + n_a + h_a}{h_a}} \stackrel{(*)}{\leq} s^2 \sqrt{\frac{2R}{r}}
 \end{aligned}$$



Let the Nagel point be N. Now, $A'B = s - c$, $A'C = s - b$, $B'A = s - c$, $B'C = s - a$, $C'A =$

$$\begin{aligned}
 &= s - b, C'B = s - a \therefore \frac{AN}{A'N} \stackrel{\text{van Aubel}}{=} \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{a}{s-a} \Rightarrow \frac{A'N}{AN} + 1 \\
 &= \frac{s-a}{a} + 1
 \end{aligned}$$



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$$\begin{aligned}
 \Rightarrow \frac{n_a}{AN} = \frac{s}{a} \Rightarrow n_a = \frac{sAN}{a} \Rightarrow n_a(r_b + r_c) \sqrt{\frac{r_a}{h_a}} &\stackrel{\text{via (i)}}{=} \frac{sAN}{a} \left(4R \cos^2 \frac{A}{2} \right) \sqrt{\frac{s \tan \frac{A}{2} \left(4R \sin \frac{A}{2} \cos \frac{A}{2} \right)}{2rs}} \\
 &= \sqrt{\frac{2R}{r}} \left(\frac{4R \sin \frac{A}{2} \cos^2 \frac{A}{2}}{4R \sin \frac{A}{2} \cos \frac{A}{2}} \right) sAN = s \sqrt{\frac{2R}{r} AN \cdot \cos \frac{A}{2}} \\
 \Rightarrow n_a(r_b + r_c) \sqrt{\frac{r_a}{h_a}} &= s \sqrt{\frac{2R}{r} AN \cdot \cos \frac{A}{2}} \text{ and analogs} \Rightarrow \sum n_a(r_b + r_c) \sqrt{\frac{r_a}{h_a}} \\
 &= s \sqrt{\frac{2R}{r} \sum AN \cdot \cos \frac{A}{2}} \stackrel{?}{\geq} s^2 \sqrt{\frac{2R}{r}} \Leftrightarrow \sum AN \cdot \cos \frac{A}{2} \stackrel{?}{\geq} s \\
 &\Leftrightarrow \sum a \cdot AN \cdot \frac{\cos \frac{A}{2}}{4R \cos \frac{A}{2} \sin \frac{A}{2}} \geq s \\
 \Leftrightarrow \sum a \cdot AN \cdot AI &\geq abc \Leftrightarrow \frac{AN \cdot AI}{bc} + \frac{BN \cdot BI}{ca} + \frac{CN \cdot CI}{ab} \stackrel{(**)}{\geq} 1
 \end{aligned}$$

Now, Via Xiao – Guang Chu and Zhi

– Hua Zhang, for any 2 arbitrary interior points P and M, if x, y, z are the barycentric coordinates of point P or M, then :
 $(x + y + z)(x \cdot PA \cdot MA + y \cdot PB \cdot MB + z \cdot PC \cdot MC) \geq yza^2 + zxb^2 + xyc^2$ and choosing M $\equiv I$ (I is an interior point) and as the point corresponding to which the barycentric coordinates are evaluated, then $\because x = ap_1, y = bp_2, z = cp_3$ where p_1, p_2, p_3 are the perpendicular distances from M to BC, CA, AB respectively, we get :

$$\begin{aligned}
 x = ar, y = br, z = cr (\because M \equiv I) \text{ and so, using these substitutions, we arrive at} \\
 &: (ar + br + cr)(ar \cdot PA \cdot IA + br \cdot PB \cdot IB + cr \cdot PC \cdot IC) \\
 &\geq br \cdot cra^2 + cr \cdot arb^2 + ar \cdot brc^2 \\
 \Rightarrow 2s \left(\frac{PA \cdot IA}{bc} + \frac{PB \cdot IB}{ca} + \frac{PC \cdot IC}{ab} \right) &\geq \frac{a^2bc + b^2ca + c^2ab}{abc} \\
 \Rightarrow \frac{PA \cdot IA}{bc} + \frac{PB \cdot IB}{ca} + \frac{PC \cdot IC}{ab} &\stackrel{(***)}{\geq} 1 \text{ and choosing P} \\
 \equiv N (N \text{ is an interior point}) \text{ in (***)}, we get :
 \end{aligned}$$

$$\begin{aligned}
 \frac{AN \cdot AI}{bc} + \frac{BN \cdot BI}{ca} + \frac{CN \cdot CI}{ab} \geq 1 \Rightarrow (**) \text{ is true} \Rightarrow \sum n_a(r_b + r_c) \sqrt{\frac{r_a}{h_a}} \\
 \geq s^2 \sqrt{\frac{2R}{r}} \stackrel{(*)}{\geq} \frac{s^2}{3} \sum \sqrt{\frac{2m_a + n_a + h_a}{h_a}} \text{ (QED)}
 \end{aligned}$$

2725. In $\triangle ABC$ the following relationship holds:



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$$\sum w_a^2 + \frac{R}{2r} \sum m_a^2 \geq 2s^2$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 w_a^2 + \frac{R}{2r} \cdot m_a^2 &\stackrel{\text{Euler}}{\geq} w_a^2 + m_a^2 \stackrel{\text{AM-GM}}{\geq} 2w_a m_a \stackrel{\text{Lascu}}{\geq} 2 \cdot \frac{2bc}{b+c} \cos \frac{A}{2} \cdot \frac{b+c}{2} \cos \frac{A}{2} \\
 &= 2bc \cos^2 \frac{A}{2} = 2s(s-a) \\
 \rightarrow \sum \left(w_a^2 + \frac{R}{2r} \cdot m_a^2 \right) &\geq 2s \sum (s-a) = 2s^2
 \end{aligned}$$

$$\text{Therefore, } \sum w_a^2 + \frac{R}{2r} \sum m_a^2 \geq 2s^2.$$

2726. In ΔABC , v_a, v_b, v_c are cevians through Bevan's point, then :

$$\sum \frac{v_a}{s-a} \geq \frac{27R}{2s}$$

Proposed by Soumava Chakraborty-Kolkata-India

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let V be the Bevan's point and $\{A'\} = (AV) \cap (BC)$ ($\therefore v_a = AA'$)

The barycentric coordinates of V are $(x : y : z)$ with

$$\begin{aligned}
 x &= a \left(-\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} \right), \\
 y &= b \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) \text{ and } z = c \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right) \\
 \rightarrow A' \text{ have the coordinates } (0 : y : z) &\rightarrow (y+z)\overrightarrow{AA'} = y\overrightarrow{AB} + z\overrightarrow{AC}
 \end{aligned}$$

squaring

$$\begin{aligned}
 \Leftrightarrow (y+z)^2 v_a^2 &= y^2 c^2 + z^2 b^2 + yz \cdot 2\overrightarrow{AB} \cdot \overrightarrow{AC} \\
 &= y^2 c^2 + z^2 b^2 + yz(b^2 + c^2 - a^2) \quad (*)
 \end{aligned}$$

$$RHS_{(*)} = b^2 c^2 \left[\left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right)^2 + \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right] +$$



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$$\begin{aligned}
 & bc \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right) (b^2 + c^2 - a^2) = \\
 & = 2b^2c^2 \left[\frac{a^2}{(s-a)^2} + \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right] \\
 & \quad + bc \left[\frac{a^2}{(s-a)^2} - \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 \right] (b^2 + c^2 - a^2) = \\
 & = \frac{a^2}{(s-a)^2} \cdot bc[(b+c)^2 - a^2] + bc \left(\frac{b}{s-b} - \frac{c}{s-c} \right)^2 [a^2 - (b-c)^2] \\
 & = \frac{4sa^2bc}{s-a} + \frac{4s^2bc(b-c)^2}{(s-b)(s-c)} = \\
 & = sbc \cdot \frac{a^2 \cdot 4(s-b)(s-c) + 4s(s-a)(b-c)^2}{sr^2} \\
 & = \frac{bc}{r^2} \{a^2[a^2 - (b-c)^2] + (b-c)^2[(b+c)^2 - a^2]\} = \\
 & = \frac{bc}{r^2} (4a^2bc - 2 \sum a^2b^2 + \sum a^4) = \frac{bc}{r^2} (4a \cdot 4sRr - 16s^2r^2) = \frac{16s(R \cdot abc - sr \cdot bc)}{r} \\
 & = 16s(R \cdot 4Rs - sbc) = \\
 & = 64s^2R^2 \left(1 - \frac{bc}{4R^2} \right) = 64s^2R^2 \left(1 - \frac{s(s-a)}{4R^2 \cos^2 \frac{A}{2}} \right) \\
 & = 64s^2R^2 \left(1 - \frac{4s(s-a) \cos^2 \frac{B-C}{2}}{\left(4R \cos \frac{A}{2} \cos \frac{B-C}{2} \right)^2} \right) \geq \\
 & \stackrel{\cos^2 \frac{B-C}{2} \leq 1}{\geq} 64s^2R^2 \left(1 - \frac{(b+c)^2 - a^2}{(b+c)^2} \right) = \left(8sR \cdot \frac{a}{b+c} \right)^2 \rightarrow RHS_{(*)} \geq \left(8sR \cdot \frac{a}{b+c} \right)^2 \quad (1) \\
 & y+z = b \left(\frac{a}{s-a} - \frac{b}{s-b} + \frac{c}{s-c} \right) + c \left(\frac{a}{s-a} + \frac{b}{s-b} - \frac{c}{s-c} \right) \\
 & = \frac{a(b+c)}{s-a} - \frac{b(b-c)}{s-b} + \frac{c(b-c)}{s-c} = \\
 & = \frac{a(b+c)}{s-a} - \frac{s(b-c)^2}{(s-b)(s-c)} \leq \frac{a(b+c)}{s-a} \rightarrow LHS_{(*)} \leq \left(\frac{a(b+c)}{s-a} \cdot v_a \right)^2 \quad (2) \\
 & \text{From (1), (2), we have : } \frac{a(b+c)}{s-a} \cdot v_a \geq 8sR \cdot \frac{a}{b+c} \leftrightarrow \frac{v_a}{s-a} \\
 & \geq \frac{8sR}{(b+c)^2} \text{ (And analogs)}
 \end{aligned}$$



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$$\text{Therefore, } \sum \frac{v_a}{s-a} \geq 8sR \sum \frac{1}{(b+c)^2} \stackrel{\text{H\"older}}{\geq} 8sR \cdot \frac{3^3}{[\sum(b+c)]^2} = \frac{27R}{2s}.$$

2727. In ΔABC the following relationship holds:

$$\begin{aligned} & \sum \cos^2 \frac{A}{2} \left(\frac{\sqrt{n_b}}{h_b} + \frac{\sqrt{n_c}}{h_c} - \frac{\sqrt{n_a}}{h_a} \right) \geq \\ & \sqrt{2 \sum \cos^2 \frac{A}{2} \cdot \cos^2 \frac{B}{2} - \sum \cos^4 \frac{A}{2}} \cdot \sqrt{2 \sum \frac{\sqrt{n_a n_b}}{h_a h_b} - \sum \frac{n_a}{h_a^2}} \end{aligned}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let I and N be the incenter and the Nagel's point of ΔABC , respectively.

$$\text{We know that : } AI = \frac{r}{\sin \frac{A}{2}}.$$

Let AA' , BB' , CC' be the Nagel cevians. We know that : $AB' = s - c$, $AC' = s - b$, $BC' = CB' = s - a$

From Van Aubel's theorem, we have : $\frac{AN}{A'N} = \frac{AC'}{C'B} + \frac{AB'}{B'C} = \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{a}{s-a}$
 $\rightarrow \frac{n_a}{AN} = 1 + \frac{A'N}{AN} = 1 + \frac{s-a}{a} = \frac{s}{a} \rightarrow AN = \frac{a \cdot n_a}{s}$ (and analogs).

We know that if

$P \in \text{Int}(\Delta ABC)$, then aPA , bPB , cPC can be the sides of a triangle (Klamkin)
 $\rightarrow (aIA, bIB, cIC)$ and (aNA, bNB, cNC) can be the sides of a triangle.

Since $aIA = 4Rr \cos \frac{A}{2}$
 $\rightarrow \cos \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2}$ can be the sides of a triangle Δ_1 with area :

$$F_1 = \frac{1}{4} \sqrt{2 \sum \cos^2 \frac{A}{2} \cdot \cos^2 \frac{B}{2} - \sum \cos^4 \frac{A}{2}}.$$

Also, since $aNA = a \cdot \frac{a \cdot n_a}{s} = 4sr^2 \cdot \frac{n_a}{h_a^2}$
 $\rightarrow \frac{n_a}{h_a^2}, \frac{n_b}{h_b^2}, \frac{n_c}{h_c^2}$ can be the sides of a triangle.

We know that if m, n, p the sides of a triangle, then $\sqrt{m}, \sqrt{n}, \sqrt{p}$ can be also the sides of a triangle.

$$\left(\because (\sqrt{m} + \sqrt{n})^2 = (m+n) + 2\sqrt{mn} > \sqrt{p}^2 \right)$$

$\rightarrow \sqrt{\frac{\sqrt{n_a}}{h_a}}, \sqrt{\frac{\sqrt{n_b}}{h_b}}, \sqrt{\frac{\sqrt{n_c}}{h_c}}$ can be the sides of a triangle Δ_2 with area : F_2

$$= \frac{1}{4} \sqrt{2 \sum \frac{\sqrt{n_a n_b}}{h_a h_b} - \sum \frac{n_a}{h_a^2}}$$

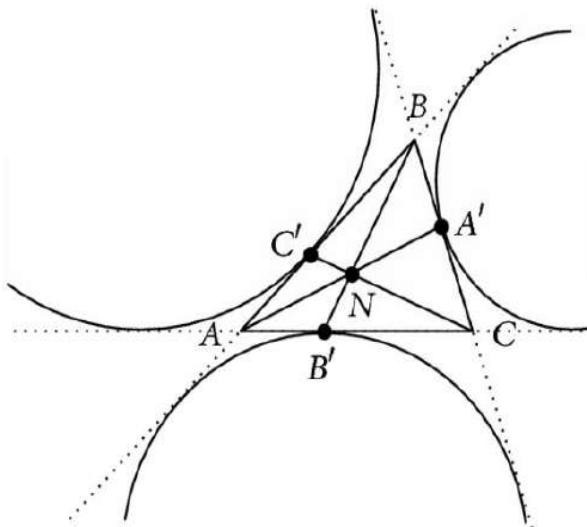
We know that for any two triangle ΔXYZ and ΔUVW with area S_1 and S_2 respectively, we have : $\sum x^2(v^2 + w^2 - u^2) \geq 16S_1S_2$ (Neuberg – Pedoe)

For Δ_1 and Δ_2 , we have : $\sum \cos^2 \frac{A}{2} \left(\frac{\sqrt{n_b}}{h_b} + \frac{\sqrt{n_c}}{h_c} - \frac{\sqrt{n_a}}{h_a} \right) \geq 16F_1F_2$

Therefore, $\sum \cos^2 \frac{A}{2} \left(\frac{\sqrt{n_b}}{h_b} + \frac{\sqrt{n_c}}{h_c} - \frac{\sqrt{n_a}}{h_a} \right)$

$$\geq \sqrt{2 \sum \cos^2 \frac{A}{2} \cdot \cos^2 \frac{B}{2} - \sum \cos^4 \frac{A}{2}} \cdot \sqrt{2 \sum \frac{\sqrt{n_a n_b}}{h_a h_b} - \sum \frac{n_a}{h_a^2}}$$

Solution 2 by Soumava Chakraborty-Kolkata-India



Let the Nagel point be N . Now, $A'B = s - c$, $A'C = s - b$, $B'A = s - c$, $B'C = s - a$, $C'A = s - b$, $C'B = s - a$.

$$\begin{aligned} & \therefore \frac{AN}{A'N} \stackrel{\text{Van Aubel}}{\cong} \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{a}{s-a} \Rightarrow \frac{A'N}{AN} + 1 \\ & = \frac{s-a}{a} + 1 \end{aligned}$$

$$\Rightarrow \frac{n_a}{AN} = \frac{s}{a} \Rightarrow a \cdot AN = \frac{a^2 n_a}{s} \text{ and analogs} \Rightarrow \frac{a^2 n_a + b^2 n_b - c^2 n_c}{s} = a \cdot AN + b \cdot BN - c \cdot CN > 0$$

$\therefore a \cdot AM + b \cdot BM - c \cdot CM$ and cyclic analogs $\stackrel{\text{Klamkin}}{>} 0 \forall$ interior points M and choosing $M \equiv N$ which is justifiable since N is an interior point $\therefore a^2 n_a + b^2 n_b - c^2 n_c > 0 \Rightarrow (\sqrt{a^2 n_a + b^2 n_b} + c \sqrt{n_c})(\sqrt{a^2 n_a + b^2 n_b} - c \sqrt{n_c}) > 0$



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$$\begin{aligned}
 \Rightarrow \sqrt{a^2 n_a + b^2 n_b} > c \sqrt{n_c} \Rightarrow c \sqrt{n_c} < \sqrt{a^2 n_a + b^2 n_b} < a \sqrt{n_a} + b \sqrt{n_b} \Rightarrow \frac{a \sqrt{n_a} + b \sqrt{n_b} - c \sqrt{n_c}}{2rs} \\
 > 0 \Rightarrow \frac{\sqrt{n_a}}{h_a} + \frac{\sqrt{n_b}}{h_b} - \frac{\sqrt{n_c}}{h_c} > 0 \\
 \Rightarrow \left(\sqrt{\frac{n_a}{h_a}} + \sqrt{\frac{n_b}{h_b}} + \sqrt{\frac{n_c}{h_c}} \right) \left(\sqrt{\frac{n_a}{h_a}} + \sqrt{\frac{n_b}{h_b}} - \sqrt{\frac{n_c}{h_c}} \right) > 0 \Rightarrow \sqrt{\frac{n_c}{h_c}} < \sqrt{\frac{n_a}{h_a}} + \sqrt{\frac{n_b}{h_b}} \\
 < \sqrt{\frac{n_a}{h_a}} + \sqrt{\frac{n_b}{h_b}} \Rightarrow \sqrt{\frac{n_a}{h_a}} + \sqrt{\frac{n_b}{h_b}} - \sqrt{\frac{n_c}{h_c}} > 0 \\
 \Rightarrow \sqrt{\frac{n_a}{h_a}}, \sqrt{\frac{n_b}{h_b}}, \sqrt{\frac{n_c}{h_c}} \text{ form sides}
 \end{aligned}$$

of a triangle \rightarrow (1) with area $F_2 \stackrel{(*)}{=} \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \frac{\sqrt{n_b n_c}}{h_b h_c} - \sum_{\text{cyc}} \frac{n_a}{h_a^2}}$

$$\begin{aligned}
 \text{Again, } \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} &= \frac{1}{2}(1 + \cos A + 1 + \cos B - 1 - \cos C) \\
 &= \frac{1}{2}\left(2 \sin^2 \frac{C}{2} + 2 \sin \frac{C}{2} \cos \frac{A-B}{2}\right) > 0 \because 0 < \cos \frac{A-B}{2} \leq 1 \Rightarrow \cos^2 \frac{C}{2} \\
 &< \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} \\
 < \left(\cos \frac{A}{2} + \cos \frac{B}{2}\right)^2 &\Rightarrow \cos \frac{A}{2} + \cos \frac{B}{2} - \cos \frac{C}{2} > 0 \\
 \Rightarrow \cos \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2} &\text{ form sides of a triangle} \\
 \rightarrow (2) \text{ with area } F_1 \stackrel{(**)}{=} \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} - \sum_{\text{cyc}} \cos^4 \frac{A}{2}}
 \end{aligned}$$

Now, via D. Pedoe (1941) and J. J. B. Neuberg (1891), for any 2 triangles with sides a_1, b_1, c_1 and a_2, b_2, c_2 ,
 $\therefore \sum_{\text{cyc}} a_1^2(b_2^2 + c_2^2 - a_2^2) \geq 16F_1F_2$ where area of first

$$\begin{aligned}
 \text{triangle} = F_1 \text{ and that of second one} = F_2 \therefore \text{choosing } a_1 = \cos \frac{A}{2} \text{ and analogs and } a_2 \\
 = \sqrt{\frac{n_a}{h_a}} \text{ and analogs and making use of axioms (1) and (2), we arrive at} \\
 \sum_{\text{cyc}} \cos^2 \frac{A}{2} \left(\frac{\sqrt{n_b}}{h_b} + \frac{\sqrt{n_c}}{h_c} - \frac{\sqrt{n_a}}{h_a} \right) \\
 \geq 16F_1F_2 \stackrel{\text{via } (*), (**)}{=} 16 \cdot \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} - \sum_{\text{cyc}} \cos^4 \frac{A}{2}} \cdot \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \frac{\sqrt{n_b n_c}}{h_b h_c} - \sum_{\text{cyc}} \frac{n_a}{h_a^2}}
 \end{aligned}$$



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$$= \sqrt{2 \sum_{\text{cyc}} \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} - \sum_{\text{cyc}} \cos^4 \frac{A}{2}} \cdot \sqrt{2 \sum_{\text{cyc}} \frac{\sqrt{n_b n_c}}{h_b h_c} - \sum_{\text{cyc}} \frac{n_a}{h_a^2}} \quad (\text{QED})$$

2728.

$$\text{In any } \Delta ABC, \sum w_a^2 + \frac{\alpha p^2(R - 2r)}{r} \geq \sum m_a^2 \quad \forall \alpha \geq 5$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 m_a^2 - w_a^2 &= \frac{(b-c)^2 + 4s(s-a) - \frac{16bc(s-a)}{(b+c)^2}}{4} = \frac{(b-c)^2 + 4s(s-a)\left(1 - \frac{4bc}{(b+c)^2}\right)}{4} \\
 &= \frac{(b-c)^2 \left(1 + \frac{4s(s-a)}{(b+c)^2}\right)}{4} = \frac{(b-c)^2 \left(1 + \frac{(b+c)^2 - a^2}{(b+c)^2}\right)}{4} \\
 &= \frac{(b-c)^2 \left(2 - \frac{a^2}{(b+c)^2}\right)}{4} = \frac{(b-c)^2}{2} - \frac{a^2(b-c)^2}{4(b+c)^2} \stackrel{-(b-c)^2 \leq 0}{\leq} \frac{(b-c)^2}{2} \therefore m_a^2 - w_a^2 \\
 &\leq \frac{(b-c)^2}{2} \text{ and analogs} \Rightarrow \sum m_a^2 - \sum w_a^2 \leq \sum \frac{(b-c)^2}{2} = \sum a^2 - \sum ab \\
 &= s^2 - 12Rr - 3r^2 \stackrel{?}{\leq} \frac{5s^2(R-2r)}{r} \Leftrightarrow r(s^2 - 12Rr - 3r^2) \stackrel{?}{\leq} 5s^2(R-2r) \text{ and} \\
 &\quad \therefore r(s^2 - 12Rr - 3r^2) \stackrel{\text{Gerretsen}}{\leq} r(4R^2 - 8Rr) \text{ and } 5s^2(R-2r) \stackrel{\text{Gerretsen}}{\geq} 5(16Rr - 5r^2)(R-2r) \\
 &\therefore \text{in order to prove (i), it suffices to prove : } 5(16Rr - 5r^2)(R-2r) \geq r(4R^2 - 8Rr) \\
 &\quad \Leftrightarrow 76R^2 - 177Rr + 50r^2 \geq 0 \Leftrightarrow (R-2r)(76R-25r) \geq 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \\
 \Rightarrow (i) \text{ is true} &\Rightarrow \sum m_a^2 - \sum w_a^2 \stackrel{5 \leq \alpha}{\leq} \frac{5s^2(R-2r)}{r} \Rightarrow \sum w_a^2 + \frac{\alpha p^2(R-2r)}{r} \\
 &\geq \sum m_a^2 \quad \forall \alpha \geq 5 \quad (\text{QED})
 \end{aligned}$$

2729. In $\Delta ABC, x, y, z > 0$, then:



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$$\sum \frac{n_a}{h_a^2} \cdot x \geq \sqrt{\sum xy} \cdot \left[2 \sum \frac{n_a n_b}{h_a^2 h_b^2} - \sum \frac{n_a^2}{h_a^4} \right]^{\frac{1}{2}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let N be the Nagel's point of ΔABC and AA' , BB' , CC' be the Nagel cevians.

We know that : $AB' = s - c$, $AC' = s - b$, $BC' = CB' = s - a$

From Van Aubel's theorem, we have : $\frac{AN}{A'N} = \frac{AC'}{C'B} + \frac{AB'}{B'C} = \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{a}{s-a}$
 $\rightarrow \frac{n_a}{AN} = 1 + \frac{A'N}{AN} = 1 + \frac{s-a}{a} = \frac{s}{a} \rightarrow AN = \frac{a \cdot n_a}{s}$ (and analogs).
We know that if

*$P \in Int(\Delta ABC)$, then aPA, bPB, cPC can be the sides of a triangle (Klamkin)
 $\rightarrow aNA, bNB, cNC$ can be the sides of a triangle.*

Since $aNA = a \cdot \frac{a \cdot n_a}{s} = 4sr^2 \cdot \frac{n_a}{h_a^2} \rightarrow \frac{n_a}{h_a^2}, \frac{n_b}{h_b^2}, \frac{n_c}{h_c^2}$ can be the sides of a triangle.

*We know that if m, n, p the sides of a triangle,
then $\sqrt{m}, \sqrt{n}, \sqrt{p}$ can be also the sides of a triangle.*

$$\left(\therefore (\sqrt{m} + \sqrt{n})^2 = (m + n) + 2\sqrt{mn} > \sqrt{p}^2 \right)$$

$\rightarrow \frac{\sqrt{n_a}}{h_a}, \frac{\sqrt{n_b}}{h_b}, \frac{\sqrt{n_c}}{h_c}$ can be the sides of a triangle Δ' with area : F'

$$= \frac{1}{4} \left[2 \sum \frac{n_a n_b}{h_a^2 h_b^2} - \sum \frac{n_a^2}{h_a^4} \right]^{\frac{1}{2}}$$

We know that for any ΔUVW with area F, $x, y, z > 0$, then $\sum u^2 \cdot x \geq 4F \sqrt{\sum xy}$ (Oppenheim)

\rightarrow for the triangle Δ' , we have : $\sum \left(\frac{\sqrt{n_a}}{h_a} \right)^2 \cdot x \geq 4F' \cdot \sqrt{\sum xy}$

Therefore, $\sum \frac{n_a}{h_a^2} \cdot x \geq \sqrt{\sum xy} \cdot \left[2 \sum \frac{n_a n_b}{h_a^2 h_b^2} - \sum \frac{n_a^2}{h_a^4} \right]^{\frac{1}{2}}$

2730. In ΔABC , $\alpha \geq 6$, prove that:



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$$\sum w_a^2 + \frac{\alpha(R^3 - 8r^3)}{r} \geq s^2$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 s(s-a) - w_a^2 &= s(s-a) - \frac{4bc \cdot s(s-a)}{(b+c)^2} = \frac{(b-c)^2 \cdot s(s-a)}{(b+c)^2} \\
 &= \frac{(b-c)^2 \cdot [(b+c)^2 - a^2]}{4(b+c)^2} \leq \frac{(b-c)^2}{4} \\
 \rightarrow \sum [s(s-a) - w_a^2] &\leq \sum \frac{(b-c)^2}{4} = \frac{1}{2} \left(\sum a^2 - \sum bc \right) \\
 &= \frac{s^2 - 3r^2 - 12Rr}{2} \stackrel{Gerretsen}{\leq} \frac{4R^2 - 8Rr}{2} \\
 \rightarrow s^2 - \sum w_a^2 &\leq 2R(R-2r) \stackrel{?}{\leq} \frac{6(R^3 - 8r^3)}{r} \leftrightarrow (R-2r)(3R^2 + 5Rr + 12r^2) \geq 0
 \end{aligned}$$

Which is true from Euler. Therefore, $\sum w_a^2 + \frac{\alpha(R^3 - 8r^3)}{r} \geq s^2, \forall \alpha \geq 6.$

2731.

$$\text{In any } \Delta ABC, \sum w_a^2 + \frac{\alpha p^2(R-2r)}{r} \geq \frac{R}{2r} \sum m_a^2 \quad \forall \alpha \geq 5$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India



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$$\begin{aligned}
 m_a^2 - w_a^2 &= \frac{(b-c)^2 + 4s(s-a) - \frac{16bc(s-a)}{(b+c)^2}}{4} = \frac{(b-c)^2 + 4s(s-a)\left(1 - \frac{4bc}{(b+c)^2}\right)}{4} \\
 &= \frac{(b-c)^2\left(1 + \frac{4s(s-a)}{(b+c)^2}\right)}{4} = \frac{(b-c)^2\left(1 + \frac{(b+c)^2 - a^2}{(b+c)^2}\right)}{4} \\
 &= \frac{(b-c)^2\left(2 - \frac{a^2}{(b+c)^2}\right)}{4} = \frac{(b-c)^2}{2} - \frac{a^2(b-c)^2}{4(b+c)^2} \stackrel{-(b-c)^2 \leq 0}{\leq} \frac{(b-c)^2}{2} \therefore m_a^2 - w_a^2 \\
 &\leq \frac{(b-c)^2}{2} \text{ and analogs} \Rightarrow \sum m_a^2 - \sum w_a^2 \leq \sum \frac{(b-c)^2}{2} = \sum a^2 - \sum ab \\
 \\
 &= s^2 - 12Rr - 3r^2 \Rightarrow \frac{R}{2r} \sum m_a^2 - \sum w_a^2 = \left(\frac{R}{2r} - 1\right) \sum m_a^2 + \sum m_a^2 - \sum w_a^2 \\
 &\leq \frac{3}{2} \left(\frac{R-2r}{2r}\right) (s^2 - 4Rr - r^2) + s^2 - 12Rr - 3r^2 \stackrel{?}{\leq} \frac{5s^2(R-2r)}{r} \\
 \Leftrightarrow 20(R-2r)s^2 - 4r(s^2 - 12Rr - 3r^2) - 3(R-2r)(s^2 - 4Rr - r^2) &\stackrel{?}{\geq} 0 \\
 \Leftrightarrow (17R - 38r)s^2 + r(12R^2 + 27Rr + 6r^2) &\stackrel{?}{\geq} 0 \quad (*) \\
 \end{aligned}$$

Case 1 $17R - 38r \geq 0$ and then, LHS of $(*) \geq r(12R^2 + 27Rr + 6r^2) > 0 \Rightarrow (*)$ is true

Case 2 $17R - 38r < 0$ and then, LHS of $(*)$

$$\begin{aligned}
 &= -(38r - 17R)s^2 + r(12R^2 + 27Rr + 6r^2) \stackrel{\text{Gerretsen}}{\geq} \\
 &\quad - (38r - 17R)(4R^2 + 4Rr + 3r^2) + r(12R^2 + 27Rr + 6r^2) \stackrel{?}{\geq} 0 \\
 \Leftrightarrow 34t^3 - 36t^2 - 37t - 54 &\stackrel{?}{\geq} 0 \quad \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(34t^2 + 32t + 27) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \\
 \Rightarrow (*) \text{ is true and } \therefore \text{combining both cases, } (*) \text{ is true in all triangles}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \text{in all triangles, } \frac{R}{2r} \sum m_a^2 - \sum w_a^2 &\leq \frac{5s^2(R-2r)}{r} \stackrel{5 \leq \alpha}{\leq} \frac{\alpha s^2(R-2r)}{r} \\
 \therefore \text{in any } \Delta ABC, \sum w_a^2 + \frac{\alpha p^2(R-2r)}{r} &\geq \frac{R}{2r} \sum m_a^2 \quad \forall \alpha \geq 5 \quad (\text{QED})
 \end{aligned}$$

2732. In ΔABC , $P \in \text{Int}(\Delta ABC)$. Prove that:



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$$\sum \sqrt{n_b + n_c} \cdot PA \geq \sqrt{\sum a^2 n_a + 4Fs}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$(\sqrt{n_b + n_c} + \sqrt{n_c + n_a})^2 = (n_a + n_b) + 2n_c + 2\sqrt{n_b + n_c} \cdot \sqrt{n_c + n_a} > \sqrt{n_a + n_b}^2 \\ \rightarrow a_1 = \sqrt{n_b + n_c}, b_1 = \sqrt{n_c + n_a},$$

$c_1 = \sqrt{n_a + n_b}$ can be the sides of a triangle $\Delta A_1 B_1 C_1$,

with area F_1 such that : $16F_1^2 = 2 \sum \sqrt{n_b + n_c}^2 \cdot \sqrt{n_c + n_a}^2 - \sum \sqrt{n_b + n_c}^4 =$
 $= 2 \sum (n_c^2 + n_a n_b + n_b n_c + n_c n_a) - \sum (n_b^2 + 2n_b n_c + n_c^2) = 4 \sum n_a n_b \rightarrow$
 $2F_1 = \sqrt{\sum n_a n_b}$

Now, let N be the Nagel's point and AA' , BB' , CC' be the Nagel cevians.

We know that : $AB' = s - c, AC' = s - b, BC' = CB' = s - a$.

From Van Aubel's theorem, we have : $\frac{AN}{A'N} = \frac{AC'}{C'B} + \frac{AB'}{B'C} = \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{a}{s-a}$
 $\rightarrow \frac{n_a}{AN} = 1 + \frac{A'N}{AN} = 1 + \frac{s-a}{a} = \frac{s}{a} \rightarrow AN = \frac{a \cdot n_a}{s}$ (and analogs).

→ From Hayashi's inequality, we have : $\sum a \cdot BN \cdot CN \geq abc$ ($P = N$)

$$\leftrightarrow \sum a \cdot \frac{b \cdot n_b}{s} \cdot \frac{c \cdot n_c}{s} \geq abc \leftrightarrow \sum n_a n_b \geq s^2 \rightarrow 2F_1 = \sqrt{\sum n_a n_b} \geq s$$

We know that for any triangles ΔABC and $\Delta A_1 B_1 C_1$, $P \in \text{Int}(\Delta ABC)$, we have :

$$\sum a_1 \cdot PA \geq \sqrt{\frac{1}{2} \sum a^2 (b_1^2 + c_1^2 - a_1^2) + 8FF_1} \quad (\text{Bottema}) \\ \leftrightarrow \sum \sqrt{n_b + n_c} \cdot PA \\ \geq \sqrt{\frac{1}{2} \sum a^2 (\sqrt{n_c + n_a}^2 + \sqrt{n_a + n_b}^2 - \sqrt{n_b + n_c}^2) + 8FF_1} \stackrel{2F_1 \geq s}{\geq} \sqrt{\sum a^2 n_a + 4Fs}$$

Therefore, $\sum \sqrt{n_b + n_c} \cdot PA \geq \sqrt{\sum a^2 n_a + 4Fs}$.

2733. In ΔABC the following relationship holds:

$$3F \leq \sum_{cyc} h_b h_c \tan \frac{A}{2} \leq 2F \left(2 - \frac{r}{R} \right)$$



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Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$\begin{aligned}
 \sum_{cyc} h_b h_c \tan \frac{A}{2} &= \sum_{cyc} \frac{4F^2}{bc} \cdot \tan \frac{A}{2} = \frac{4F^2}{abc} \sum_{cyc} a \tan \frac{A}{2} = \frac{4F^2}{4FR} \sum_{cyc} a \tan \frac{A}{2} = \\
 &= \frac{F}{R} \cdot \sum_{cyc} a \tan \frac{A}{2}, \text{ but } \sum_{cyc} a \tan \frac{A}{2} = 2(2R - r), \text{ then} \\
 \sum_{cyc} h_b h_c \tan \frac{A}{2} &= \frac{F}{R} \cdot 2(2R - r)
 \end{aligned}$$

We must show that

$$3F \leq \frac{F}{R} \cdot 2(2R - r) \leq 2F \left(2 - \frac{r}{R}\right); (1)$$

For LHS: $3R \leq 4R - 2r \Leftrightarrow 2r \leq R$ (Euler).

For RHS: $\frac{2R-r}{R} \leq \frac{2R-r}{R}$ true.

2734. If in $\Delta ABC, P \in Int(\Delta ABC), x, y, z > 0$ then prove that:

$$\sum \sqrt{y+z} \cdot PA \geq \sqrt{\sum x \cdot a^2 + 4F\sqrt{xy+yz+zx}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have : $(\sqrt{x+y} + \sqrt{y+z})^2 = (z+x) + 2y + 2\sqrt{x+y} \cdot \sqrt{y+z} > \sqrt{z+x}^2$

→ For any $x, y, z > 0, a_1 = \sqrt{y+z}, b_1 = \sqrt{z+x}, c_1$

$= \sqrt{x+y}$ can be the sides of a triangle $\Delta A_1 B_1 C_1$,

with area F_1 such that : $16F_1^2 = 2 \sum \sqrt{x+y}^2 \cdot \sqrt{y+z}^2 - \sum \sqrt{x+y}^4 =$

$$= 2 \sum (y^2 + xy + yz + zx) - \sum (x^2 + 2xy + y^2) = 4 \sum xy \rightarrow 2F_1 = \sqrt{\sum xy}$$

We know that for any triangles ΔABC and $\Delta A_1 B_1 C_1, P \in Int(\Delta ABC)$, we have :

$$\sum a_1 \cdot PA \geq \sqrt{\frac{1}{2} \sum a^2 (b_1^2 + c_1^2 - a_1^2) + 8FF_1} \quad (\text{Bottema})$$



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$$\leftrightarrow \sum \sqrt{y+z} \cdot PA \geq \sqrt{\frac{1}{2} \sum a^2 (\sqrt{z+x}^2 + \sqrt{x+y}^2 - \sqrt{y+z}^2) + 4F\sqrt{xy+yz+zx}}$$

$$\text{Therefore, } \sum \sqrt{y+z} \cdot PA \geq \sqrt{\sum x \cdot a^2 + 4F\sqrt{xy+yz+zx}}.$$

2735. If $\Delta ABC \sim \Delta A'B'C'$ then:

$$4(m_a m_{a'} + m_b m_{b'} + m_c m_{c'}) \geq 9\sqrt[3]{abca'b'c'}$$

Proposed by Daniel Sitaru-Romania

Solution by Nguyen Van Canh-BenTre-Vietnam

$$\text{Since: } \Delta ABC \sim \Delta A'B'C' \rightarrow \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{m_a}{m_{a'}} = \frac{m_b}{m_{b'}} = \frac{m_c}{m_{c'}} = k (\because k > 0)$$

Therefore, we can write:

$$\begin{aligned} & 4(m_a m_{a'} + m_b m_{b'} + m_c m_{c'}) \geq 9\sqrt[3]{abca'b'c'} \\ \leftrightarrow & 4 \left(\frac{m_a^2}{k} + \frac{m_b^2}{k} + \frac{m_c^2}{k} \right) \geq 9 \cdot \sqrt[3]{abc \cdot \frac{abc}{k^3}} \leftrightarrow 4(m_a^2 + m_b^2 + m_c^2) \geq 9\sqrt[3]{(abc)^2} \\ \leftrightarrow & 4 \cdot \frac{3}{4} (a^2 + b^2 + c^2) \geq 9\sqrt[3]{(abc)^2} \leftrightarrow a^2 + b^2 + c^2 \geq 3\sqrt[3]{(abc)^2} \end{aligned}$$

Which is true by AM-GM. Proved.

$$\text{Equality} \leftrightarrow k = 1 \leftrightarrow a = b = c = a' = b' = c'$$

2736. In ΔABC the following relationship holds:

$$3 \sum r_b r_c \tan \frac{A}{2} \leq \sum r_b r_c \cot \frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have: } 3 \sum r_b r_c \tan \frac{A}{2} = 3 \sum r_b r_c \frac{r_a}{s} = 3 \cdot 3sr = 9sr.$$

$$\text{And: } \sum r_b r_c \cot \frac{A}{2} = r_a r_b r_c \sum \frac{\cot \frac{A}{2}}{r_a} = s^2 r \sum \frac{\cot \frac{A}{2}}{s \tan \frac{A}{2}} = sr \sum \cot^2 \frac{A}{2} \geq$$



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$$\sum_{\geq}^{\Sigma x^2 \geq \Sigma xy} sr \sum \cot \frac{A}{2} \cot \frac{B}{2} \stackrel{CBS}{\geq} sr \cdot \frac{3^2}{\sum \tan \frac{A}{2} \tan \frac{B}{2}} = 9sr = 3 \sum r_b r_c \tan \frac{A}{2}$$

Therefore, $3 \sum r_b r_c \tan \frac{A}{2} \leq \sum r_b r_c \cot \frac{A}{2}$.

2737. In ΔABC the following relationship holds:

$$3 \sum (b+c) \tan \frac{A}{2} \leq \sum (b+c) \cot \frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$3 \sum (b+c) \tan \frac{A}{2} \stackrel{(*)}{\leq} \sum (b+c) \cot \frac{A}{2}$$

$$LHS_{(*)} = 3 \sum (b+c) \frac{r_a}{s} = 3 \sum (b+c) \frac{r}{s-a} = 3r \sum \frac{s+(s-a)}{s-a} = 3 \sum r_a + 3r \sum 1$$

$$= 12(R+r)$$

$$RHS_{(*)} = \sum (b+c) \frac{s}{r_a} = \sum (b+c) \frac{s-a}{r} = \frac{s}{r} \sum (b+c) - \frac{1}{r} \sum a(b+c)$$

$$= \frac{s}{r} \cdot 4s - \frac{2}{r} (s^2 + r^2 + 4Rr) =$$

$$= \frac{2(s^2 - r^2 - 4Rr)}{r} \stackrel{Gerretsen}{\geq} \frac{2(12Rr - 6r^2)}{r} = 12(2R - r) \stackrel{?}{\geq} LHS_{(*)} \leftrightarrow R$$

$$\geq 2r \text{ (Euler)}$$

$$Therefore, \quad 3 \sum (b+c) \tan \frac{A}{2} \leq \sum (b+c) \cot \frac{A}{2}.$$

2738. In ΔABC the following relationship holds:

$$\sum m_a^2 r_a^2 \leq \frac{1}{3} \left(\sum r_a^2 \right)^2 \leq \frac{1}{3} \left(\sum w_a^2 + \alpha(R^2 - 4r^2) \right)^2, \forall \alpha \geq 30$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam



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Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

If $a \geq b \geq c \rightarrow m_a \leq m_b \leq m_c$ and $r_a \geq r_b \geq r_c$

$$\begin{aligned}
 & \text{Using Chebyshev} \\
 & \stackrel{\text{?}}{\Rightarrow} m_a^2 r_a^2 \leq \frac{1}{3} \left(\sum m_a^2 \right) \left(\sum r_a^2 \right) \stackrel{?}{\geq} \frac{1}{3} \left(\sum r_a^2 \right)^2 \leftrightarrow \sum m_a^2 \\
 & \leq \sum r_a^2 \\
 & \leftrightarrow \frac{3}{2} (s^2 - r^2 - 4Rr) \leq (4R + r)^2 - 2s^2 \leftrightarrow 7s^2 \leq 32R^2 + 28Rr + 5r^2 \\
 & \leftrightarrow 7(4R^2 + 4Rr + 3r^2 - s^2) + 4(R - 2r)(R + 2r) \geq 0
 \end{aligned}$$

Which is true from Euler and Gerretsen $\rightarrow \boxed{\sum m_a^2 r_a^2 \leq \frac{1}{3} \left(\sum r_a^2 \right)^2}$

$$\begin{aligned}
 & \text{Also, we have : } \sum w_a^2 \stackrel{w_a \geq h_a}{\stackrel{?}{\geq}} \sum h_a^2 \stackrel{\text{Hölder}}{\stackrel{?}{\geq}} \frac{3^3}{\left(\sum \frac{1}{h_a} \right)^2} = 27r^2. \\
 & \rightarrow \sum w_a^2 + \alpha(R^2 - 4r^2) \stackrel{\alpha \geq 30}{\stackrel{?}{\geq}} 27r^2 + 30(R^2 - 4r^2) = 30R^2 - 93r^2 \stackrel{?}{\geq} \sum r_a^2 \\
 & = (4R + r)^2 - 2s^2 \\
 & \leftrightarrow 12R^2 + 2s^2 \geq 8Rr + 94r^2 \leftrightarrow 2(s^2 - 16Rr + 5r^2) + 2(R - 2r)(7R + 26r) \geq 0
 \end{aligned}$$

Which is true from Gerretsen and Euler

$$\rightarrow \boxed{\sum w_a^2 + \alpha(R^2 - 4r^2) \geq \sum r_a^2, \forall \alpha \geq 30}$$

$$\text{Therefore, } \sum m_a^2 r_a^2 \leq \frac{1}{3} \left(\sum r_a^2 \right)^2 \leq \frac{1}{3} \left(\sum w_a^2 + \alpha(R^2 - 4r^2) \right)^2, \forall \alpha \geq 30.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 AI^2 &= bc - 4Rr \Leftrightarrow \left(\frac{r}{(\frac{r}{4R})} \sin \frac{B}{2} \sin \frac{C}{2} \right)^2 \\
 &= 16R^2 \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{B}{2} \cos \frac{C}{2} - 16R^2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \Leftrightarrow \sin \frac{B}{2} \sin \frac{C}{2} \\
 &= \cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{A}{2} \Leftrightarrow \cos \frac{B+C}{2} = \sin \frac{A}{2} \rightarrow \text{true} \\
 \therefore AI^2 &\stackrel{(i)}{\stackrel{?}{\equiv}} bc - 4Rr \text{ and } AI + BI + CI \stackrel{A-G}{\stackrel{?}{\geq}} 3^3 \sqrt[3]{\prod \frac{r}{\sin \frac{A}{2}}} = 3r \sqrt[3]{\frac{4R}{r}} \stackrel{\text{Euler}}{\stackrel{?}{\geq}} 6r \Rightarrow AI + BI + CI \stackrel{(ii)}{\stackrel{?}{\geq}} 6r
 \end{aligned}$$



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$$\begin{aligned}
 \text{Now, } AI + r \leq w_a &\Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \leq \frac{2abccos \frac{A}{2}}{a(b+c)} \Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \leq \frac{8Rr \cos \frac{A}{2}}{4R(b+c) \sin \frac{A}{2} \cos \frac{A}{2}} \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \\
 &\leq \frac{a+b+c}{(b+c)\sin \frac{A}{2}} \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \leq + \frac{a}{(b+c)\sin \frac{A}{2}} + \frac{1}{\sin \frac{A}{2}} \Leftrightarrow (b+c)\sin \frac{A}{2} \leq a \\
 &\Leftrightarrow 4R \cos \frac{A}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \leq 4R \sin \frac{A}{2} \cos \frac{A}{2} \Leftrightarrow \cos \frac{B-C}{2} \leq 1 \left(\because \cos \frac{B-C}{2} > 0 \right) \rightarrow \text{true}
 \end{aligned}$$

$$\begin{aligned}
 \therefore AI + r \leq w_a &\Rightarrow w_a^2 \geq AI^2 + 2rAI + r^2 \text{ and analogs} \quad \stackrel{\text{summing up}}{\Rightarrow} \sum w_a^2 \geq \\
 &\sum AI^2 + 2r \sum AI + 3r^2 \quad \stackrel{\text{using (ii) and (i) and analogs}}{\leq} \sum bc - 12Rr + 12r^2 + 3r^2 \\
 &= s^2 + 4Rr + r^2 - 12Rr + 15r^2 = s^2 - 8Rr + 16r^2 \Rightarrow \sum r_a^2 - \sum w_a^2 \\
 &\leq (4R+r)^2 - 2s^2 - s^2 + 8Rr - 16r^2 \\
 &= 16R^2 + 16Rr - 15r^2 - 3s^2 \quad \stackrel{\text{Gerretsen}}{\leq} \quad 16R^2 + 16Rr - 15r^2 - 48Rr + 15r^2 \\
 &= 16R^2 - 32Rr \quad \stackrel{\text{Euler}}{\leq} \quad 16R^2 - 64r^2 = 16(R^2 - 4r^2) \\
 \therefore \sum r_a^2 - \sum w_a^2 &\leq 16(R^2 - 4r^2) \leq 30(R^2 - 4r^2) \quad \stackrel{30 \leq \alpha}{\leq} \quad \alpha(R^2 - 4r^2) \Rightarrow \sum r_a^2 \\
 &\leq \sum w_a^2 + \alpha(R^2 - 4r^2) \Rightarrow \frac{1}{3} \left(\sum r_a^2 \right)^2 \leq \frac{1}{3} \left(\sum w_a^2 + \alpha(R^2 - 4r^2) \right)^2 \quad \forall \alpha \geq 30
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } R - 2r &\geq \frac{b^2 + c^2}{4R} - \frac{bc}{2R} \Leftrightarrow R \left(1 - \frac{2r}{R} \right) \geq \frac{4R^2(\sin^2 B + \sin^2 C)}{4R} - \frac{4R^2 \sin B \sin C}{2R} \\
 &\Leftrightarrow 1 - \frac{8R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{R} \geq \sin^2 B + \sin^2 C - 2 \sin B \sin C = (\sin B - \sin C)^2 \\
 &\Leftrightarrow 1 - 4 \sin \frac{A}{2} \left(2 \sin \frac{B}{2} \sin \frac{C}{2} \right) \geq \left(2 \cos \frac{B+C}{2} \sin \frac{B-C}{2} \right)^2 \Leftrightarrow 1 - 4 \sin \frac{A}{2} \left(\cos \frac{B-C}{2} - \cos \frac{B+C}{2} \right) \\
 &\geq 4 \sin^2 \frac{A}{2} \left(1 - \cos^2 \frac{B-C}{2} \right) \\
 &\Leftrightarrow 1 - 4 \sin \frac{A}{2} \cos \frac{B-C}{2} + 4 \sin^2 \frac{A}{2} \geq 4 \sin^2 \frac{A}{2} - 4 \sin^2 \frac{A}{2} \cos^2 \frac{B-C}{2} \\
 &\Leftrightarrow 4 \sin^2 \frac{A}{2} \cos^2 \frac{B-C}{2} - 4 \sin \frac{A}{2} \cos \frac{B-C}{2} + 1 \geq 0 \Leftrightarrow \left(2 \sin \frac{A}{2} \cos \frac{B-C}{2} - 1 \right)^2 \\
 &\geq 0 \rightarrow \text{true} \Rightarrow
 \end{aligned}$$

$$\begin{aligned}
 (b-c)^2 &\stackrel{(I)}{\leq} 4R(R-2r) \\
 4 \sum m_a^2 r_a^2 &= \sum ((b-c)^2 + 4s(s-a)) r_a^2 \stackrel{\text{via (I)}}{\leq} 4R(R-2r) \sum r_a^2 \\
 &+ 4s \sum \left((s-a) \frac{r^2 s^2}{(s-a)^2} \right) \stackrel{?}{\leq} \frac{4}{3} \left(\sum r_a^2 \right)^2 \\
 &\Leftrightarrow \left(\sum r_a^2 \right) ((4R+r)^2 - 2s^2 - 3R(R-2r)) \stackrel{?}{\leq} 3(4R+r)rs^2
 \end{aligned}$$



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$$\Leftrightarrow (16R^2 + 8Rr + r^2 - 2s^2)(13R^2 + 14Rr + r^2 - 2s^2) \stackrel{?}{\underset{\substack{\geq \\ (*)}}{\underset{\substack{\geq \\ (1)}}{\underset{\substack{\geq \\ (2)}}{\geq}}} 3(4R + r)rs^2$$

Now, $(16R^2 + 8Rr + r^2 - 2s^2)(13R^2 + 14Rr + r^2 - 2s^2) \stackrel{\text{Gerretsen}}{\underset{(1)}{\geq}} (16R^2 + 8Rr + r^2 - 2(4R^2 + 4Rr + 3r^2))(13R^2 + 14Rr + r^2 - 2(4R^2 + 4Rr + 3r^2))$ and

$$3(4R + r)rs^2 \stackrel{\text{Gerretsen}}{\underset{(2)}{\geq}} 3r(4R + r)(4R^2 + 4Rr + 3r^2) \therefore (1), (2)$$

\Rightarrow in order to prove (*), it suffices to prove :

$$(16R^2 + 8Rr + r^2 - 2(4R^2 + 4Rr + 3r^2))(13R^2 + 14Rr + r^2 - 2(4R^2 + 4Rr + 3r^2)) \geq 3r(4R + r)(4R^2 + 4Rr + 3r^2) \Leftrightarrow 40t^4 - 125t^2 - 78t + 16 \geq 0 \quad (t = \frac{R}{r})$$

$$\Leftrightarrow (t-2)(40t^3 + 80t^2 + 31t + 4t(t-2)) \geq 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true} \Rightarrow \sum m_a^2 r_a^2 \leq \frac{1}{3} \left(\sum r_a^2 \right)^2 \text{ (QED)}$$

2739. In any ΔABC holds:

$$\frac{(na + b)(nb + c)(nc + a)}{\sqrt{(a+b)^3(b+c)^3(c+a)^3}} \leq \left(\frac{n+1}{4} \right)^3 \cdot \frac{1}{r \sqrt{p}} \quad \forall n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$n \in \mathbb{N} \Rightarrow na + b, nb + c, nc + a > 0$$

$$\begin{aligned} (na + b)(nb + c)(nc + a) &\stackrel{A-G}{\leq} \frac{(na + b + nb + c + nc + a)^3}{27} = \frac{8s^3 \cdot (n+1)^3}{27} \\ &\Rightarrow \frac{(na + b)(nb + c)(nc + a)}{\sqrt{(a+b)^3(b+c)^3(c+a)^3}} \leq \frac{8s^3 \cdot (n+1)^3}{27} \cdot \frac{1}{\sqrt{(a+b)^3(b+c)^3(c+a)^3}} \\ &\stackrel{\text{Cesaro}}{\leq} \frac{8s^3 \cdot (n+1)^3}{27} \cdot \frac{1}{\sqrt{(a+b)(b+c)(c+a)}} \cdot \frac{1}{\sqrt{64a^2b^2c^2}} \\ &= \frac{8s^3 \cdot (n+1)^3}{27} \cdot \frac{1}{\sqrt{2s(s^2 + 2Rr + r^2)}} \cdot \frac{1}{\sqrt{2^{10}R^2r^2s^2}} \stackrel{?}{\leq} \left(\frac{n+1}{4} \right)^3 \cdot \frac{1}{r \sqrt{s}} \\ &\Leftrightarrow \frac{2^6 s^6}{729 \cdot 2s(s^2 + 2Rr + r^2) \cdot 2^{10}R^2r^2s^2} \stackrel{?}{\leq} \frac{1}{2^{12}r^2s} \\ &\Leftrightarrow 729R^2(s^2 + 2Rr + r^2) \stackrel{?}{\geq} 128s^4 \text{ and } 128s^4 \stackrel{\text{Gerretsen}}{\leq} \end{aligned}$$



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$$\begin{aligned}
 128(4R^2 + 4Rr + 3r^2)s^2 &\stackrel{?}{\leq} 729R^2(s^2 + 2Rr + r^2) \\
 \Leftrightarrow (217R^2 - 512Rr - 384r^2)s^2 + 729(R + 2r)R^2r &\stackrel{?}{\geq} 0 \quad \text{(**) (□)}
 \end{aligned}$$

Case 1 $217R^2 - 512Rr - 384r^2 \geq 0$ and then LHS of $(**)$ $\geq 729(R + 2r)R^2r > 0 \Rightarrow (**)$

$$\Rightarrow (*) \text{ is true} \Rightarrow \frac{(na+b)(nb+c)(nc+a)}{\sqrt{(a+b)^3(b+c)^3(c+a)^3}} < \left(\frac{n+1}{4}\right)^3 \cdot \frac{1}{r\sqrt{s}}$$

Case 2 $217R^2 - 512Rr - 384r^2 < 0$ and then LHS of $(**)$

$$= -(-(217R^2 - 512Rr - 384r^2))s^2 + 729(R + 2r)R^2r$$

$$\begin{aligned}
 \stackrel{\text{Gerretsen}}{\geq} & -(-(217R^2 - 512Rr - 384r^2))(4R^2 + 4Rr + 3r^2) + 729(R + 2r)R^2r \stackrel{?}{\geq} 0 \\
 \Leftrightarrow 868t^4 + 278t^3 - 2204t^2 - 3072t - 1152 &\stackrel{?}{\geq} 0 \quad \left(t = \frac{R}{r}\right)
 \end{aligned}$$

$$\Leftrightarrow (t-2)(868t^3 + 2014t^2 + 1824t + 576) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\Rightarrow \frac{(na+b)(nb+c)(nc+a)}{\sqrt{(a+b)^3(b+c)^3(c+a)^3}} \leq \left(\frac{n+1}{4}\right)^3 \cdot \frac{1}{r\sqrt{s}} \therefore \text{combining cases (1), (2),}$$

$$\text{in any } \triangle ABC, \frac{(na+b)(nb+c)(nc+a)}{\sqrt{(a+b)^3(b+c)^3(c+a)^3}} \leq \left(\frac{n+1}{4}\right)^3 \cdot \frac{1}{r\sqrt{p}} \forall n \in N \text{ (QED)}$$

2740. In any $\triangle ABC$ holds:

$$\sum \sin(w_a^2) \leq \sum m_a^2 \leq \sum w_a^2 + \alpha(R^2 - 4r^2) \forall \alpha \geq 25$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Let } f(x) = \sin x - x \forall x \in [0, \infty) \therefore f''(x) = \cos x - 1 \leq 0 \Rightarrow f(x) \text{ is } \downarrow \text{ on } [0, \infty) \Rightarrow f(x) \leq f(0) \\
 = 0 \Rightarrow \sin x \leq x \forall x \in [0, \infty) \Rightarrow \forall x > 0, \sin x < x \therefore \sin(w_a^2) < w_a^2
 \end{aligned}$$

$$\Rightarrow \sum \sin(w_a^2) < \sum w_a^2 \leq \sum m_a^2 \therefore \sum \sin(w_a^2)$$

$$< \sum m_a^2 \text{ which can be written as } \boxed{\sum \sin(w_a^2) \leq \sum m_a^2}$$



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$$\begin{aligned}
 \text{Now, } m_a^2 - w_a^2 &= \frac{(b-c)^2 + 4s(s-a) - \frac{16bc(s-a)}{(b+c)^2}}{4} = \frac{(b-c)^2 + 4s(s-a)\left(1 - \frac{4bc}{(b+c)^2}\right)}{4} \\
 &= \frac{(b-c)^2\left(1 + \frac{4s(s-a)}{(b+c)^2}\right)}{4} = \frac{(b-c)^2\left(1 + \frac{(b+c)^2 - a^2}{(b+c)^2}\right)}{4} \\
 &= \frac{(b-c)^2\left(2 - \frac{a^2}{(b+c)^2}\right)}{4} = \frac{(b-c)^2}{2} - \frac{a^2(b-c)^2 - (b-c)^2 \leq 0}{4(b+c)^2} \stackrel{?}{\leq} \frac{(b-c)^2}{2} \therefore m_a^2 - w_a^2 \\
 &\leq \frac{(b-c)^2}{2} \text{ and analogs} \Rightarrow \sum m_a^2 - \sum w_a^2 \leq \sum \frac{(b-c)^2}{2} = \sum a^2 - \sum ab \\
 &= s^2 - 12Rr - 3r^2 \stackrel{\text{Gerretsen}}{\leq} 4R^2 - 8Rr \stackrel{\text{Gerretsen}}{\leq} 4R^2 - 16r^2 \\
 &\Rightarrow \boxed{\sum m_a^2 - \sum w_a^2 \leq 4(R^2 - 4r^2)} \leq 25(R^2 - 4r^2) \stackrel{25 \leq \alpha}{\leq} \alpha(R^2 - 4r^2)
 \end{aligned}$$

$$\Rightarrow \sum m_a^2 \leq \sum w_a^2 + \alpha(R^2 - 4r^2) \forall \alpha \geq 25 \text{ (QED)}$$

2741. In ΔABC , $\alpha \geq 2$ the following relationship holds:

$$\left(\frac{R}{2r}\right)^\alpha \sum m_a^2 \geq \sum n_a^2$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that :

$$\begin{aligned}
 n_a^2 &= s^2 - 2r_a h_a = s^2 - 4sr^2 \cdot \frac{a + (s-a)}{a(s-a)} = s^2 - 4rr_a - \frac{r \cdot bc}{R} \text{ (And analogs)} \\
 \rightarrow \sum n_a^2 &= 3s^2 - 4r(4R+r) - \frac{r(s^2 + r^2 + 4Rr)}{R} \\
 &= \frac{(3R-r)s^2 - r(4R+r)^2}{R} \stackrel{?}{\geq} \left(\frac{R}{2r}\right)^2 \sum m_a^2 = \\
 &= \left(\frac{R}{2r}\right)^2 \cdot \frac{3(s^2 - r^2 - 4Rr)}{2} \leftrightarrow 8r^2(3R-r)s^2 - 8r^3(4R+r)^2 \\
 &\leq 3R^3s^2 - 3R^3r^2 - 12R^4r
 \end{aligned}$$



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$$\Leftrightarrow -12R^4r - 3R^3r^2 + 128R^2r^3 + 64Rr^4 + 8r^5 \geq (-3R^3 + 24Rr^2 - 8r^3)s^2 \quad (*)$$

If $-3R^3 + 24Rr^2 - 8r^3 \geq 0 \rightarrow$

$$RHS_{(*)} \leq (-3R^3 + 24Rr^2 - 8r^3)(4R^2 + 4Rr + 3r^2) =$$

$$= -12R^5 - 12R^4r + 87R^3r^2 + 64R^2r^3 + 40Rr^4 - 24r^5 \stackrel{?}{\geq} LHS_{(*)}$$

$$\Leftrightarrow 6R^5 - 45R^3r^2 + 32R^2r^3 + 12Rr^4 + 16r^5 \geq 0$$

$$\Leftrightarrow (R - 2r)[(R - 2r)(6R^3 + 24R^2r + 27Rr^2 + 44r^3) + 80r^4] \geq 0$$

Which is true from Euler ($R \geq 2r$) \rightarrow () is true.*

$$\text{If } -3R^3 + 24Rr^2 - 8r^3 \leq 0 \rightarrow RHS_{(*)} \leq (-3R^3 + 24Rr^2 - 8r^3)(16R - 5r)r =$$

$$= r(-48R^4 + 15R^3r + 384R^2r^2 - 248Rr^3 + 40r^4) \stackrel{?}{\leq} LHS_{(*)}$$

$$\Leftrightarrow 36R^4 - 18R^3r - 256R^2r^2 + 312Rr^3 - 32r^4 \geq 0$$

$$\Leftrightarrow (R - 2r)[(R - 2r)(36R^2 + 126Rr + 104r^2) + 224r^3] \geq 0$$

Which is true from Euler ($R \geq 2r$) \rightarrow () is true.*

$$\text{Therefore, } \sum n_a^2 \leq \left(\frac{R}{2r}\right)^2 \sum m_a^2 \leq \left(\frac{R}{2r}\right)^\alpha \sum m_a^2, \forall \alpha \geq 2.$$

2742. Let $\alpha \geq 1$. In ΔABC the following relationship holds:

$$\left(\frac{R}{2r}\right)^\alpha (ab + bc + ca) \geq a^2 + b^2 + c^2 \geq ab + bc + ca + \frac{r(R - 2r)}{R - r}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

$$\sum a = 2(s^2 - r^2 - 4Rr), \sum ab = s^2 + r^2 + 4Rr$$

$$\sum a^2 \geq \sum ab + \frac{r^2(R - 2r)}{R - r} \Leftrightarrow 2(s^2 - r^2 - 4Rr) \geq s^2 + r^2 + 4Rr + \frac{r^2(R - 2r)}{R - r}$$

$$\Leftrightarrow s^2 - 3r^2 - 12Rr \geq \frac{r^2(R - 2r)}{R - r} \Leftrightarrow$$

$$s^2 - 3r^2 - 12Rr \geq 16Rr - 5r^2 - 3r^2 - 12Rr \geq \frac{r^2(R - 2r)}{R - r} \Leftrightarrow$$

$$4r(R - 2r) \geq \frac{r^2(R - 2r)}{R - r} \Leftrightarrow r(R - 2r) \left(\frac{4R - 5r}{R - r}\right) \geq 0, \text{ which is true, because } R \geq 2r \Leftrightarrow$$

$$4R - 5r \geq 3r > 0$$



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$$\begin{aligned} \left(\frac{R}{2r}\right)^\alpha \sum ab &\geq \sum a^2 \Leftrightarrow \left(\frac{R}{2r}\right)^\alpha \geq \frac{a^2 + b^2 + c^2}{ab + bc + ca} = \frac{2(s^2 - r^2 - 4Rr)}{s^2 + r^2 + 4Rr} \\ &\Leftrightarrow s^2(R^\alpha - (2r)^\alpha) + r^2(R^\alpha + 2(2r)^\alpha) + 4Rr(R^\alpha + 2(2r)^\alpha) \geq 0. \end{aligned}$$

So, above statement is true for $\alpha = 1$, for $\alpha = 2$ the statement is also true.

Let us assume it is for $\alpha = m$, we need to prove its true for $\alpha = m + 1$.

$$\begin{aligned} s^2(R^{m+1} - (2r)^{m+1} + r^2(R^{m+1} + 2(2r)^{m+1}) + 4Rr(R^{m+1} + 2(2r)^{m+1})) &= \\ &= s^2(R - 2r)(R^m + R^{m-1}(2r) + \dots + R(2r)^{m-1} + (2r)^m) + \\ &+ r^2(R^{m+1} + 2(2r)^{m+1}) + 4Rr(R^{m+1} + 2(2r)^{m+1}) \geq 0, \text{ since } R \geq 2r. \end{aligned}$$

Hence proved for $\alpha = m + 1$. Therefore,

$$\left(\frac{R}{2r}\right)^\alpha (ab + bc + ca) \geq a^2 + b^2 + c^2 \geq ab + bc + ca + \frac{r(R - 2r)}{R - r}$$

2743. In ΔABC , I –incenter, R' –circumradius of

ΔDEF , $AD \cap BE \cap CF = \{I\}$, the following relationship holds

$$\sqrt[3]{\frac{2R'}{R}} \leq \frac{R}{2r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have : $R' = \frac{DE \cdot EF \cdot FD}{4 \cdot [DEF]}$, where $[DEF]$ be the area of ΔDEF .

We know that : $AE = \frac{bc}{a+c}$ and $AF = \frac{bc}{a+b} \rightarrow [AEF] = \frac{1}{2} \cdot AE \cdot AF \cdot \sin A$
 $= \frac{bc \cdot F}{(a+b)(a+c)}$ ($\because F = [\Delta ABC]$)

Similarly, $[BFD] = \frac{ca \cdot F}{(b+c)(b+a)}$ and $[CDE] = \frac{ab \cdot F}{(c+a)(c+b)} \rightarrow [DEF]$
 $= F - ([AEF] + [BFD] + [CDE]) =$
 $= F \cdot \left(1 - \frac{bc}{(a+b)(a+c)} - \frac{ca}{(b+c)(b+a)} - \frac{ab}{(c+a)(c+b)}\right)$
 $= \frac{2abc}{(a+b)(b+c)(c+a)} \cdot F$



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Using now the Law of cosines in ΔAEF : $EF^2 = AE^2 + AF^2 - 2 \cdot AE \cdot AF \cdot \cos A =$

$$\begin{aligned}
 &= \frac{b^2 c^2}{(a+c)^2} + \frac{b^2 c^2}{(a+b)^2} - \frac{bc(b^2 + c^2 - a^2)}{(a+b)(a+c)} \\
 &= b^2 c^2 \left(\frac{1}{a+c} - \frac{1}{a+b} \right)^2 - \frac{bc(b-c)^2}{(a+b)(a+c)} + \frac{a^2 bc}{(a+b)(a+c)} = \\
 &= -\frac{bc(b-c)^2}{(a+b)^2(a+c)^2} [(a+b)(a+c) - bc] + \frac{a^2 bc}{(a+b)(a+c)} \leq \frac{a^2 bc}{(a+b)(a+c)}
 \end{aligned}$$

Similarly, we have : $FD^2 \leq \frac{ab^2 c}{(b+c)(b+a)}$ and $DE^2 = \frac{abc^2}{(c+a)(c+b)}$

$$\rightarrow (DE \cdot EF \cdot FD)^2 \leq \left(\frac{a^2 b^2 c^2}{(a+b)(b+c)(c+a)} \right)^2$$

$$\rightarrow R' = \frac{DE \cdot EF \cdot FD}{4 \cdot [DEF]} \leq \frac{1}{4} \cdot \frac{a^2 b^2 c^2}{(a+b)(b+c)(c+a)} \cdot \frac{(a+b)(b+c)(c+a)}{2abc \cdot F} = \frac{abc}{8F} = \frac{R}{2}$$

$$\text{Therefore, } \sqrt[3]{\frac{2R'}{R}} \stackrel{\text{Euler}}{\leq} \frac{R}{2r}.$$

2744. In ΔABC the following relationship holds:

$$s\sqrt{3} + w_a - m_a \geq \sqrt{\left(\sum w_a w_b \right) \left(\sum \frac{w_a + w_b}{w_b + w_c} \right)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(\sqrt{w_a + w_b} + \sqrt{w_b + w_c})^2 = (w_c + w_a) + 2w_b + 2\sqrt{w_a + w_b} \cdot \sqrt{w_b + w_c} \geq \sqrt{w_c + w_a}^2$$

$$\rightarrow \sqrt{w_a + w_b} + \sqrt{w_b + w_c} \geq \sqrt{w_c + w_a} \quad (\text{And analogs})$$

$$\rightarrow \sqrt{w_a + w_b}, \sqrt{w_b + w_c}, \sqrt{w_c + w_a}$$

can be the sides of a triangle Δ' with area F' such that :

$$\begin{aligned}
 16F'^2 &= 2 \sum \sqrt{w_a + w_b}^2 \sqrt{w_b + w_c}^2 - \sum \sqrt{w_a + w_b}^4 = 4 \sum w_a w_b \rightarrow 2F' \\
 &= \sqrt{\sum w_a w_b}
 \end{aligned}$$



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From Lessel – Pelling inequality, we have : $s\sqrt{3} \geq m_a + w_b + w_c$

$$\rightarrow s\sqrt{3} + w_a - m_a \geq \sum w_a$$

$$\begin{aligned} \rightarrow \text{It's suffices to prove : } \sum w_a &\geq 2F' \sqrt{\sum \frac{w_a + w_b}{w_b + w_c}} \leftrightarrow \sum \sqrt{w_a + w_b}^2 \\ &\geq 4F' \sqrt{\sum \frac{\sqrt{w_a + w_b}^2}{\sqrt{w_b + w_c}^2}} \end{aligned}$$

$$\rightarrow \text{It's suffices to prove that : } \sum a^2 \geq 4F \sqrt{\sum \frac{a^2}{b^2}}, \forall \Delta ABC.$$

We know that : $\forall u, v, w > 0, \sum u \cdot a^2 \geq 4F \sqrt{\sum uv}$ (Oppenheim)

$$\text{Let } u = \frac{b^2}{a^2}, v = \frac{c^2}{b^2}, w = \frac{a^2}{c^2} \rightarrow \sum \frac{b^2}{a^2} \cdot a^2 \geq 4F \sqrt{\sum \frac{b^2}{a^2} \cdot \frac{c^2}{b^2}} \leftrightarrow \sum a^2 \geq 4F \sqrt{\sum \frac{a^2}{b^2}}$$

$$\text{Therefore, } s\sqrt{3} + w_a - m_a \geq \sqrt{\left(\sum w_a w_b\right) \left(\sum \frac{w_a + w_b}{w_b + w_c}\right)}$$

2745. For $x, y, z, u, v, w > 0$. Prove that:

$$x(v+w) + y(w+u) + z(u+v) \geq 2 \sqrt{\left(\sum_{x,y,z} xy\right) \left(\sum_{u,v,w} uv\right)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} (\sqrt{v+w} + \sqrt{w+u})^2 &= (u+v) + 2w + 2\sqrt{v+w} \cdot \sqrt{w+u} \geq \sqrt{u+v}^2 \\ \rightarrow \sqrt{v+w} + \sqrt{w+u} &\geq \sqrt{u+v} \text{ (And analogs)} \\ \rightarrow \sqrt{v+w}, \sqrt{w+u}, \sqrt{u+v} \end{aligned}$$

can be the sides of a triangle Δ with area F such that :



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$$16F^2 = 2 \sum \sqrt{v+w}^2 \sqrt{w+u}^2 - \sum \sqrt{v+w}^4 = 4 \sum uv \rightarrow F = \frac{1}{2} \sqrt{\sum uv}$$

Using Oppenheim's inequality in triangle Δ :

$$\sum x \cdot \sqrt{v+w}^2 \geq 4F \cdot \sqrt{\sum xy}, \forall x, y, z > 0$$

$$\leftrightarrow \sum x(v+w) \geq 4 \cdot \frac{1}{2} \sqrt{\sum uv} \cdot \sqrt{\sum xy}$$

$$\text{Therefore, } x(v+w) + y(w+u) + z(u+v) \geq 2 \sqrt{\left(\sum_{x,y,z} xy\right)\left(\sum_{u,v,w} uv\right)}.$$

2746. In ΔABC the following relationship holds:

$$n_a \geq \sqrt{r \left[(4R+r) \left(\sum \frac{a}{b} \right) - 2(2r_a + h_a) \right]}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$n_a \stackrel{(*)}{\geq} \sqrt{r \left[(4R+r) \left(\sum \frac{a}{b} \right) - 2(2r_a + h_a) \right]}$$

Since ABC is a triangle $\rightarrow b+c \geq a$ (And analogs) $\rightarrow (\sqrt{b} + \sqrt{c})^2$

$$= b+c + 2\sqrt{bc} \geq a = \sqrt{a^2} \text{ (And analogs)}$$

$\rightarrow \sqrt{a}, \sqrt{b}, \sqrt{c}$ can be the sides of a triangle Δ' with area F' such that :

$$\begin{aligned} 16F'^2 &= 2 \sum \sqrt{a}^2 \sqrt{b}^2 - \sum \sqrt{a}^4 = 2 \sum ab - \sum a^2 \\ &= 2(s^2 + r^2 + 4Rr) - 2(s^2 - r^2 - 4Rr) \\ \rightarrow 16F'^2 &= 4r(4R+r) \rightarrow F' = \frac{1}{2} \sqrt{r(4R+r)} \end{aligned}$$

Using Oppenheim's inequality in triangle Δ' :

$$\sum x \cdot \sqrt{a^2} \geq 4F' \cdot \sqrt{\sum xy}, \forall x, y, z > 0$$

$$\text{Let } x = \frac{b}{a}, y = \frac{c}{b}, z = \frac{a}{c} \rightarrow \sum \frac{b}{a} \cdot a \geq 4 \cdot \frac{1}{2} \sqrt{r(4R+r)} \cdot \sqrt{\sum \frac{b}{a} \cdot \frac{c}{b}} \leftrightarrow$$

$$\sum a \geq 2 \sqrt{r(4R+r) \sum \frac{a}{b}}$$

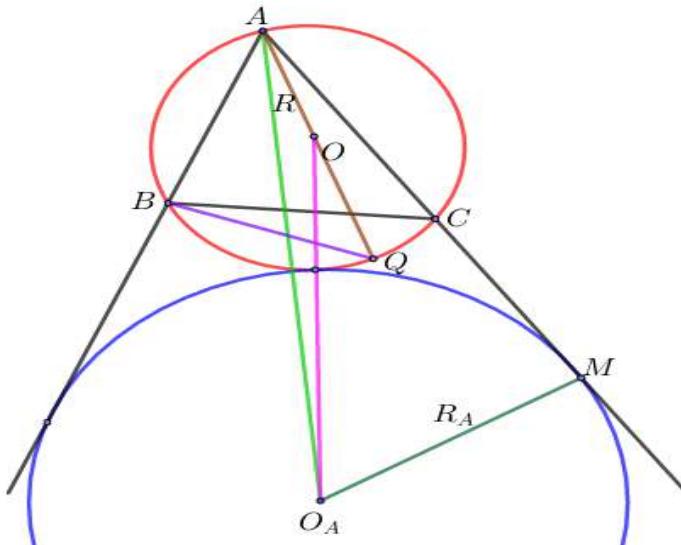
$$\rightarrow r(4R+r) \sum \frac{a}{b} \leq s^2$$

$$\rightarrow RHS_{(*)} \leq \sqrt{s^2 - 2r(2r_a + h_a)} = \sqrt{s^2 - 2r \cdot 2sr \left(\frac{1}{s-a} + \frac{1}{a} \right)} = \sqrt{s^2 - 2 \cdot \frac{sr}{s-a} \cdot \frac{2sr}{a}}$$

$$= \sqrt{s^2 - 2r_a h_a} = n_a.$$

$$\text{Therefore, } n_a \geq \sqrt{r \left[(4R+r) \left(\sum \frac{a}{b} \right) - 2(2r_a + h_a) \right]}.$$

2747.



In ΔABC let R_A – be the radius of circle tangent simultaneous to AB, AC and external tangent to circumcircle of ΔABC . Prove that:

$$\frac{R_A R_B}{r_a r_b} + \frac{R_B R_C}{r_b r_c} + \frac{R_C R_A}{r_c r_a} \geq \frac{64r^2}{3R^2}$$



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Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

O_A – tangent of $AB, AC \Rightarrow O, O_A$ – tangent in $T \Rightarrow O, T, O_A$ – collinear and

$$OO_A \equiv OT + TO_A = R + R_A$$

$$\text{In } \Delta OO_A M, (\hat{M} = 90^\circ): \sin \widehat{O_A OM} = \frac{O_A M}{AO_A} \Rightarrow \sin \frac{A}{2} = \frac{R_A}{AO_A} \Rightarrow AO_A = \frac{R_A}{\sin \frac{A}{2}}$$

$\widehat{O_A AO} = \widehat{BAO} - \widehat{BAO_A}$, AQ – diameter, then $\widehat{ABQ} = 90^\circ \Rightarrow$

$$\widehat{BAO} = 90^\circ - \widehat{ABQ} = 90^\circ - \frac{\widehat{AB}}{2} = 90^\circ - \widehat{C} \Rightarrow \widehat{O_A AO} = 90^\circ - \widehat{C} - \frac{\widehat{A}}{2} = \frac{\widehat{B} - \widehat{C}}{2}$$

From Law of cosines in $\Delta AO_A O$: $OO_A^2 = AO_A^2 - 2OO_A \cdot AO \cdot \cos \widehat{OAO_A}$

$$(R + R_A)^2 = \frac{R_A^2}{\sin^2 \frac{A}{2}} + R^2 - 2 \cdot \frac{R_A}{\sin \frac{A}{2}} \cdot R \cdot \cos \left(\frac{B - C}{2} \right)$$

$$R^2 + 2RR_A + R_A^2 = \frac{R_A^2}{\sin^2 \frac{A}{2}} + R^2 - \frac{2RR_A}{\sin \frac{A}{2}} \cdot \cos \left(\frac{B - C}{2} \right)$$

$$\frac{R_A^2}{\sin^2 \frac{A}{2}} - R_A^2 = 2RR_A + \frac{2RR_A}{\sin \frac{A}{2}} \cdot \cos \left(\frac{B - C}{2} \right) \Rightarrow$$

$$R_A - R_A \cdot \sin^2 \frac{A}{2} = 2R \cdot \sin^2 \frac{A}{2} + 2R \cdot \sin \frac{A}{2} \cdot \cos \left(\frac{B - C}{2} \right) \Rightarrow$$

$$R_A \cdot \cos^2 \frac{A}{2} = 2R \cdot \sin \frac{A}{2} \left(\sin \frac{A}{2} + \cos \left(\frac{B - C}{2} \right) \right)$$

$$\sin \frac{A}{2} + \cos \left(\frac{B - C}{2} \right) = \sin \frac{\pi - (B + C)}{2} + \cos \left(\frac{B - C}{2} \right) = \cos \left(\frac{B + C}{2} \right) + \cos \left(\frac{B - C}{2} \right) =$$

$$= 2 \cos \frac{\frac{B+C}{2} + \frac{B-C}{2}}{2} \cos \frac{\frac{B+C}{2} - \frac{B-C}{2}}{2} = 2 \cos \frac{B}{2} \cos \frac{C}{2}$$

Hence,

$$R_A \cos^2 \frac{A}{2} = 4R \cdot \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \Rightarrow R_A = \frac{4R \cdot \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{\cos^2 \frac{A}{2}} = \frac{r_a}{\cos^2 \frac{A}{2}}$$

Therefore, $R_A = \frac{r_a}{\cos^2 \frac{A}{2}}, R_b = \frac{r_b}{\cos^2 \frac{B}{2}}, R_c = \frac{r_c}{\cos^2 \frac{C}{2}}$



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$$\begin{aligned}
 \sum_{cyc} \frac{R_A R_B}{r_a r_b} &= \sum_{cyc} \frac{1}{\cos^2 \frac{A}{2} \cos^2 \frac{B}{2}} = \frac{\sum \cos^2 \frac{A}{2}}{\prod \cos^2 \frac{A}{2}} = \frac{\frac{4R+r}{2R}}{\prod \frac{s(s-a)}{bc}} = \frac{\frac{4R+r}{2R}}{\frac{s^2 F^2}{16R^2 F^2}} = \\
 &= \frac{8R(4R+r)}{s^2} \geq \frac{32R^2 + 8Rr}{s^2} \stackrel{(*)}{\geq} \frac{64r^2}{3R^2} \Leftrightarrow \\
 (*) \Leftrightarrow \frac{(32R^2 + 8Rr) \cdot 4}{27R^2} &\geq \frac{64r^2}{3R^2} \Leftrightarrow 96R^2 + 24Rr \geq 432r^2 \\
 96R^2 + 24Rr &\geq 96 \cdot (2r)^2 + 24 \cdot 2r \cdot r = 324r^2 + 48r^2 = 432r^2
 \end{aligned}$$

2748. In ΔABC the following relationship holds:

$$\sum \frac{1}{2 \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2}} \leq \frac{1}{6\sqrt{3}} \left(\frac{5R}{r} + \frac{r}{R} + 3 \right)$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \sum \frac{1}{2 \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2}} &\stackrel{r_a=s \tan \frac{A}{2}}{\cong} \sum \frac{s}{2r_a + r_b + r_c} = s \sum \frac{1}{(r_a + r_b) + (r_a + r_c)} \\
 &\leq \frac{s}{4} \sum \left(\frac{1}{r_a + r_b} + \frac{1}{r_a + r_c} \right) = \\
 &= \frac{s}{2} \sum \frac{1}{r_a + r_b} = \frac{s}{2} \cdot \frac{\sum (r_a + r_b)(r_a + r_c)}{\prod (r_a + r_b)} = \frac{s}{2} \cdot \frac{(\sum r_a)^2 + \sum r_a r_b}{4Rs^2} = \frac{(4R+r)^2 + s^2}{8Rs} \leq \\
 &\stackrel{\substack{\text{Gerretsen} \\ \text{Mitrić}}}{{\cong}} \frac{(4R+r)^2 + 4R^2 + 4Rr + 3r^2}{8R \cdot 3\sqrt{3}r} = \frac{5R^2 + 3Rr + r^2}{6\sqrt{3}Rr} = \frac{1}{6\sqrt{3}} \left(\frac{5R}{r} + \frac{r}{R} + 3 \right) \\
 \text{Therefore, } &\sum \frac{1}{2 \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2}} \leq \frac{1}{6\sqrt{3}} \left(\frac{5R}{r} + \frac{r}{R} + 3 \right).
 \end{aligned}$$

Solution 2 by Nguyen Van Canh-Ben Tre-Vietnam

$$\begin{aligned}
 \bullet \quad \sum \left(\cot \frac{A}{2} \right)^2 &= \frac{p^2 - 8Rr - 2r^2}{r^2} \\
 \bullet \quad \sum \cot^5 \frac{A}{2} \cot \frac{B}{2} &= \sum \frac{\left(\left(\cot \frac{A}{2} \right)^2 \right)^2}{\frac{1}{\cot \frac{A}{2} \cot \frac{B}{2}}} \stackrel{C-S}{\geq} \frac{\left(\sum \left(\cot \frac{A}{2} \right)^2 \right)^2}{\sum \tan \frac{A}{2} \tan \frac{B}{2}} = \frac{\left(\sum \left(\cot \frac{A}{2} \right)^2 \right)^2}{1} = \left(\frac{p^2 - 8Rr - 2r^2}{r^2} \right)^2
 \end{aligned}$$



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$$\stackrel{\text{Blundon's inequality}}{\leq} \left(\frac{2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} - 8Rr - 2r^2}{r^2} \right)^2$$

$$= \left(\frac{2R^2 + 2Rr - 3r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr}}{r^2} \right)^2 \stackrel{(*)}{\geq} \frac{27}{2} \left(\frac{4R^2}{r^2} - \frac{5R}{r} \right);$$

$$(*) \Leftrightarrow 2 \left(2R^2 + 2Rr - 3r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \right)^2 \geq 27(4R^2 - 5Rr^2);$$

$$\Leftrightarrow 2 \left(2x^2 + 2x - 3 - 2(x - 2)\sqrt{x^2 - 2x} \right)^2 \geq 27(4x^2 - 5x); \left(\therefore x = \frac{R}{r} \geq 2 \right)$$

$$\Leftrightarrow 16x^4 - 32x^3 - 28x^2 + 47x + 18 \geq 8(x - 2)(2x^2 + 2x - 3)\sqrt{x^2 - 2x};$$

$$\Leftrightarrow (x - 2)(16x^3 - 28x - 9) \geq 8(x - 2)(2x^2 + 2x - 3)\sqrt{x^2 - 2x};$$

$$\Leftrightarrow (x - 2) \left(16x^3 - 28x - 9 - 8(2x^2 + 2x - 3)\sqrt{x^2 - 2x} \right) \geq 0;$$

Because: $x \geq 2 \rightarrow x - 2 \geq 0$. We need to prove that:

$$16x^3 - 28x - 9 - 8(2x^2 + 2x - 3)\sqrt{x^2 - 2x} > 0;$$

$$\Leftrightarrow 16x^3 - 28x - 9 > 8(2x^2 + 2x - 3)\sqrt{x^2 - 2x};$$

$$\Leftrightarrow (16x^3 - 28x - 9)^2 > \left(8(2x^2 + 2x - 3)\sqrt{x^2 - 2x} \right)^2;$$

$$\left(\therefore 16x^3 - 28x - 9 = x(16x^2 - 28) - 9 \stackrel{x \geq 2}{\geq} 2(64 - 28) - 9 = 63 > 0 \right)$$

$$\Leftrightarrow 256x^6 - 896x^4 - 288x^3 + 784x^2 + 504x + 81$$

$$> 64(x^2 - 2x)(2x^2 + 2x - 3);$$

$$\Leftrightarrow 256x^6 - 896x^4 - 288x^3 + 784x^2 + 504x + 81$$

$$> 256x^6 - 1536x^4 + 256x^3 + 2112x^2 - 1152x;$$

$$\Leftrightarrow 640x^4 - 544x^3 - 1328x^2 + 1656x + 81 > 0;$$

$$\Leftrightarrow 16x^2(40x^2 - 34x - 83) + 1656x + 81 > 0;$$

Which is true since:

$$x \geq 2 \rightarrow 16x^2(40x^2 - 34x - 83) + 1656x + 81$$

$$= 16x^2(x(40x - 34) - 83) + 1656x + 81$$

$$\geq 16 \cdot 2^2 \cdot (2(40 \cdot 2 - 34) - 83) + 1656 \cdot 2 + 81 = 3969 > 0.$$

Therefore, (*) is true. Proved.



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2749. If $x, y, z > 0, xy + yz + zx = 3$ then in acute ΔABC the following relationship holds:

$$\frac{\tan^4 A \cdot \tan^4 B}{x^3 y^3} + \frac{\tan^4 B \cdot \tan^4 C}{y^3 z^3} + \frac{\tan^4 C \cdot \tan^4 A}{z^3 x^3} \geq 243$$

Proposed by Daniel Sitaru-Romania

Solution by George Florin Serban-Romania

Let $f: \left(0, \frac{\pi}{2}\right) \rightarrow \mathbb{R}, f(x) = \tan x$ then $f'(x) = \frac{1}{\cos^2 x}, f''(x) = \frac{2 \sin x}{\cos^3 x} > 0 \Rightarrow f$ -convexe.

Hence,

$$\begin{aligned} \tan\left(\frac{A+B+C}{3}\right) &\leq \frac{1}{3}(\tan A + \tan B + \tan C) \Leftrightarrow \\ \sqrt{3} &\leq \frac{1}{3} \tan A \tan B \tan C \Leftrightarrow \tan A \tan B \tan C \geq 3\sqrt{3} \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{cyc} \frac{\tan^4 A \cdot \tan^4 B}{x^3 y^3} &= \sum_{cyc} \frac{(\tan A \cdot \tan B)^4}{(xy)^3} \stackrel{\text{Radon}}{\geq} \frac{(\sum \tan A \cdot \tan B)^4}{(\sum xy)^3} \stackrel{\text{AM-GM}}{\geq} \\ &\geq 3 \left(\sqrt[3]{\tan A \tan B \tan C} \right)^8 \geq 3 \left(\sqrt[3]{3\sqrt{3}} \right)^8 = 243 \end{aligned}$$

Therefore,

$$\frac{\tan^4 A \cdot \tan^4 B}{x^3 y^3} + \frac{\tan^4 B \cdot \tan^4 C}{y^3 z^3} + \frac{\tan^4 C \cdot \tan^4 A}{z^3 x^3} \geq 243$$

Equality holds for $x = y = z = 1, a = b = c$.

2750. In ΔABC the following relationship holds:

$$\frac{c(h_a^2 + h_b^2)}{r_a^2 + r_b^2} + \frac{a(h_b^2 + h_c^2)}{r_b^2 + r_c^2} + \frac{b(h_c^2 + h_a^2)}{r_c^2 + r_a^2} \leq \frac{3\sqrt{3}R(R-r)}{r}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Marian Ursărescu-Romania

$$\begin{aligned} \sum_{cyc} \frac{c(h_a^2 + h_b^2)}{r_a^2 + r_b^2} &\leq \sum_{cyc} \frac{c(w_a^2 + w_b^2)}{2r_a r_b} \leq \sum_{cyc} \frac{c(s(s-a) + s(s-b))}{2 \cdot \frac{F^2}{(s-a)(s-b)}} = \\ &= \frac{1}{2} \sum_{cyc} \frac{sc^2}{s(s-c)} = \frac{1}{2} \sum_{cyc} \frac{c^2}{s-c} = \frac{1}{4} \cdot \frac{4s(R-r)}{r} = \frac{2s(R-r)}{r} \end{aligned}$$



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We must show that:

$$\frac{2s(R-r)}{r} \leq \frac{3\sqrt{3}R(R-r)}{r} \Leftrightarrow s \leq \frac{3\sqrt{3}}{2}R \quad (\text{Mitrinovic})$$

2751. K –Lemoine's point in }ABC. Prove that:

$$\frac{m_a}{AK \cdot \sin A} + \frac{m_b}{BK \cdot \sin B} + \frac{m_c}{CK \cdot \sin C} \geq 3\sqrt{3}$$

Proposed by Daniel Sitaru-Romania

Solution by George Florin Ţerban-Romania

$$\text{Lemma. } AK = \frac{(b^2+c^2)s_a}{a^2+b^2+c^2} = \frac{(b^2+c^2)\frac{2bcm_a}{b^2+c^2}}{a^2+b^2+c^2} = \frac{2bcm_a}{a^2+b^2+c^2}$$

Now, we have:

$$\begin{aligned} \sum_{cyc} \frac{m_a}{AK \cdot \sin A} &= \sum_{cyc} \frac{m_a(a^2 + b^2 + c^2)}{2bcm_a \cdot \sin A} = \frac{1}{2}(a^2 + b^2 + c^2) \sum_{cyc} \frac{1}{bc \cdot \frac{a}{2R}} = \\ &= \frac{1}{2}(a^2 + b^2 + c^2) \cdot \sum_{cyc} \frac{2R}{abc} = R(a^2 + b^2 + c^2) \cdot \frac{3}{abc} = \\ &= \frac{3R}{4RF}(a^2 + b^2 + c^2) = \frac{3}{4F}(a^2 + b^2 + c^2) \stackrel{\text{Ionescu-Wetzenbock}}{\geq} \\ &\geq \frac{3}{4F} \cdot 4F\sqrt{3} = 3\sqrt{3} \end{aligned}$$

Therefore,

$$\frac{m_a}{AK \cdot \sin A} + \frac{m_b}{BK \cdot \sin B} + \frac{m_c}{CK \cdot \sin C} \geq 3\sqrt{3}$$

2752. In }ABC the following relationship holds:

$$\sum a^2 \geq \frac{4}{3} \sum m_a m_b + \frac{4}{3\sqrt{3}} \sum |(m_a - m_b)(m_a - m_c)|$$

Proposed Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*) \Leftrightarrow \frac{4}{3} \sum m_a^2 \geq \frac{4}{3} \sum m_a m_b + \frac{4}{3\sqrt{3}} \sum |(m_a - m_b)(m_a - m_c)|$$



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$$\leftrightarrow \sqrt{3} \sum (m_a - m_b)^2 \geq 2 \sum |(m_a - m_b)(m_a - m_c)| \quad (**)$$

*Since $(**)$ is symmetrical on variable m_a, m_b, m_c , WLOG we may assume that*

$$m_a \geq m_b \geq m_c$$

Let $x = m_a - m_b \geq 0, y = m_b - m_c \geq 0 \rightarrow m_a - m_c = x + y$

$$\rightarrow (**) \leftrightarrow \sqrt{3}(x^2 + y^2 + (x+y)^2) \geq 2(x(x+y) + xy + y(x+y))$$

$$\leftrightarrow (\sqrt{3} - 1)(x^2 + y^2) \geq (3 - \sqrt{3})xy$$

$$\leftrightarrow x^2 + y^2 \geq \sqrt{3}xy \leftrightarrow \left(x - \frac{\sqrt{3}}{2}y\right)^2 + \frac{y^2}{4} \geq 0 \text{ which is true} \rightarrow (**) \text{ is true.}$$

$$\text{Therefore, } \sum a^2 \geq \frac{4}{3} \sum m_a m_b + \frac{4}{3\sqrt{3}} \sum |(m_a - m_b)(m_a - m_c)|.$$

2753. In ΔABC the following relationship holds:

$$\frac{c(h_a^2 + h_b^2)}{r_a^2 + r_b^2} + \frac{a(h_b^2 + h_c^2)}{r_b^2 + r_c^2} + \frac{b(h_c^2 + h_a^2)}{r_c^2 + r_a^2} \leq \frac{3\sqrt{3}R(R-r)}{r}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Marian Ursărescu-Romania

$$\begin{aligned} \sum_{cyc} \frac{c(h_a^2 + h_b^2)}{r_a^2 + r_b^2} &\leq \sum_{cyc} \frac{c(w_a^2 + w_b^2)}{2r_a r_b} \leq \sum_{cyc} \frac{c(s(s-a) + s(s-b))}{2 \cdot \frac{F^2}{(s-a)(s-b)}} = \\ &= \frac{1}{2} \sum_{cyc} \frac{sc^2}{s(s-c)} = \frac{1}{2} \sum_{cyc} \frac{c^2}{s-c} = \frac{1}{4} \cdot \frac{4s(R-r)}{r} = \frac{2s(R-r)}{r} \end{aligned}$$

We must show that:

$$\frac{2s(R-r)}{r} \leq \frac{3\sqrt{3}R(R-r)}{r} \Leftrightarrow s \leq \frac{3\sqrt{3}}{2}R \text{ (Mitrinovic)}$$

2754. In ΔABC the following relationship holds:

$$\sum m_a \sqrt{a(b+c-a)} \geq \sqrt{\frac{9}{8} \sum a(b+c-a)(b^2 + c^2 - a^2) + 18F^2}$$

Proposed by Bogdan Fuștei-Romania



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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \sum m_a \sqrt{a(b+c-a)} &\stackrel{(*)}{\geq} \sqrt{\frac{9}{8} \sum a(b+c-a)(b^2+c^2-a^2) + 18F^2} \\
 RHS_{(*)}^2 &= \frac{9}{8} \sum (ab^3 + abc^2 - a^3b + ab^2c + ac^3 - ca^3 - a^2b^2 - c^2a^2 + a^4) + 18F^2 \\
 &= \\
 &= \frac{9}{8} \cdot \left(2abc \sum a - \left(2 \sum a^2b^2 - \sum a^4 \right) \right) + 18F^2 = \frac{9}{8} \cdot (16s^2Rr - 16F^2) + 18F^2 \\
 &= 18s^2Rr.
 \end{aligned}$$

→ We need to prove that : $\sum m_a \sqrt{a(b+c-a)} \geq \sqrt{18s^2Rr} = 3s\sqrt{2Rr}$ (**)

We have : $a(b+c-a) = 4R \sin \frac{A}{2} \cos \frac{A}{2} \cdot 8R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 8Rr \cos^2 \frac{A}{2}$

$$\begin{aligned}
 \rightarrow \sum m_a \sqrt{a(b+c-a)} &= 2\sqrt{2Rr} \sum m_a \cdot \cos \frac{A}{2} \stackrel{\text{Lascu}}{\geq} 2\sqrt{2Rr} \sum \frac{b+c}{2} \cos \frac{A}{2} \cdot \cos \frac{A}{2} \\
 &= \frac{1}{2} \sqrt{2Rr} \sum (b+c)(1 + \cos A) = \\
 &= \frac{1}{2} \sqrt{2Rr} \left(4s + \sum (b+c) \cos A \right) = \frac{1}{2} \sqrt{2Rr} \left(4s + \sum (c \cos B + b \cos C) \right) = \\
 &= 2s\sqrt{2Rr} + \frac{1}{2} \sqrt{2Rr} \cdot \sum \left(\frac{c^2 + a^2 - b^2}{2a} + \frac{a^2 + b^2 - c^2}{2a} \right) = 2s\sqrt{2Rr} + \frac{1}{2} \sqrt{2Rr} \cdot \sum a \\
 &= 3s\sqrt{2Rr} \rightarrow \text{(**) is true.}
 \end{aligned}$$

Therefore, $\sum m_a \sqrt{a(b+c-a)} \geq \sqrt{\frac{9}{8} \sum a(b+c-a)(b^2+c^2-a^2) + 18F^2}.$

2755. In ΔABC the following relationship holds:

$$2 \sum ab \geq \sum a^2 + 4F \sqrt{\sum \frac{a(s-a)}{b(s-b)}}$$

Proposed by Bogdan Fuștei-Romania



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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that if

P ∈ Int(ΔABC), then aPA, bPB, cPC can be the sides of a triangle (Klamkin)
→ aIA, bIB, cIC can be the sides of a triangle, where I be the incenter of ΔABC.

Since aIA = 2√Rr√a(s-a)

→ √a(s-a), √b(s-b), √c(s-c) can be the sides of a triangle Δ₁

with area F₁ such that : 4 · 16F₁²

$$\begin{aligned}
 &= 4 \left(2 \sum \sqrt{a(s-a)}^2 \sqrt{b(s-b)}^2 - \sum \sqrt{a(s-a)}^4 \right) = \\
 &= 2 \sum ab(-a+b+c)(a-b+c) - \sum (-a^2 + ab + ac)^2 \\
 &= 2 \sum ab(c^2 + 2ab - a^2 - b^2) - \\
 &- \sum (a^4 + a^2b^2 + a^2c^2 + 2a^2bc - 2a^3b - 2a^3c) = \\
 &2 \sum a^2b^2 - \sum a^4 = 16F^2 \rightarrow F_1 = \frac{1}{2}F.
 \end{aligned}$$

We know that if ΔUVW a triangle with area F' then : ∑ x.u²

$$\geq 4F' \cdot \sqrt{\sum xy}, \forall x, y, z > 0 \text{ (Oppenheim)}$$

If we take : x = $\frac{v^2}{u^2}$, y = $\frac{w^2}{v^2}$, z = $\frac{u^2}{w^2}$ → $\sum u^2 \geq 4F' \cdot \sqrt{\sum \frac{u^2}{v^2}}$, ∀ΔUVW

→ For the triangle Δ₁: $\sum \sqrt{a(s-a)}^2 \geq 4 \cdot \frac{1}{2}F \cdot \sqrt{\sum \frac{\sqrt{a(s-a)}}{\sqrt{b(s-b)}}^2}$

$$\Leftrightarrow \sum a(-a+b+c) \geq 4F \cdot \sqrt{\sum \frac{a(s-a)}{b(s-b)}}$$

Therefore, 2 ∑ ab ≥ ∑ a² + 4F √{ ∑ a(s-a) / b(s-b) }.



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2756. $\sum xy > 0, x + y, y + z, z + x > 0$. In ΔABC holds:

$$\sum a(b+c)x \geq 8F \sqrt{\sum xy}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that if

$P \in \text{Int}(\Delta ABC)$, then aPA, bPB, cPC can be the sides of a triangle (Klamkin)
 $\rightarrow aIA, bIB, cIC$ can be the sides of a triangle, where I be the incenter of ΔABC .

Since $aIA = 2\sqrt{Rr}\sqrt{a(s-a)}$

$\rightarrow \sqrt{a(s-a)}, \sqrt{b(s-b)}, \sqrt{c(s-c)}$ can be the sides of a triangle Δ_1

with area F_1 such that : $4 \cdot 16F_1^2$

$$= 4 \left(2 \sum \sqrt{a(s-a)}^2 \sqrt{b(s-b)}^2 - \sum \sqrt{a(s-a)}^4 \right) =$$

$$= 2 \sum ab(-a+b+c)(a-b+c) - \sum (-a^2 + ab + ac)^2$$

$$= 2 \sum ab(c^2 + 2ab - a^2 - b^2) -$$

$$- \sum (a^4 + a^2b^2 + a^2c^2 + 2a^2bc - 2a^3b - 2a^3c) = 2 \sum a^2b^2 - \sum a^4 = 16F^2$$

$$\rightarrow 2F_1 = F.$$

$$\begin{aligned} \text{We have : } (\sqrt{x+y} + \sqrt{y+z})^2 &= (x+z) + 2y + 2\sqrt{x+y} \cdot \sqrt{y+z} \stackrel{?}{\geq} \sqrt{z+x}^2 \\ \Leftrightarrow \sqrt{y^2 + \sum xy} &\geq -y \end{aligned}$$

$$\begin{aligned} \text{Which is true because } \sqrt{y^2 + \sum xy} &\stackrel{\Sigma xy > 0}{\geq} |y| \geq -y \rightarrow \sqrt{x+y} + \sqrt{y+z} \\ &\geq \sqrt{z+x} \text{ (And analogs)} \end{aligned}$$

$\rightarrow \sqrt{x+y}, \sqrt{y+z}, \sqrt{z+x}$ can be the sides of a triangle Δ_2 with area F_2 such that

: $16F_2^2 =$

$$2 \sum \sqrt{x+y}^2 \cdot \sqrt{y+z}^2 - \sum \sqrt{x+y}^4$$

$$= 2 \sum (y^2 + xy + yz + zx) - \sum (x^2 + 2xy + y^2) = 4 \sum xy \rightarrow 2F_2$$

$$= \sqrt{\sum xy}$$

We know that for any two triangle ΔXYZ and ΔUVW with area S_1 and S_2 respectively,

we have : $\sum x^2(v^2 + w^2 - u^2) \geq 16S_1S_2$ (Neuberg – Pedoe) (**)



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→ Using (**) for the triangles Δ_1 and Δ_2 , we have

$$\begin{aligned} & : \sum \sqrt{a(s-a)^2} (\sqrt{x+y^2} + \sqrt{z+x^2} - \sqrt{y+z^2}) \geq 4F \sqrt{\sum xy} \\ & \Leftrightarrow \sum a(b+c-a) \cdot x \geq 4F \sqrt{\sum xy} \quad (1) \end{aligned}$$

→ Using now (**) for the triangles ΔABC and Δ_2 , we have

$$\begin{aligned} & : \sum a^2 (\sqrt{x+y^2} + \sqrt{z+x^2} - \sqrt{y+z^2}) \geq 8F \sqrt{\sum xy} \\ & \Leftrightarrow \sum a^2 \cdot x \geq 4F \sqrt{\sum xy} \quad (2) \\ (1) + (2) \rightarrow & \sum a(b+c)x \geq 8F \sqrt{\sum xy}. \end{aligned}$$

2757. $\Sigma xy > 0, x+y, y+z, z+x > 0$. In ΔABC holds:

$$\sum ax(b+c-a) \geq 4F \sqrt{\sum xy}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that if

$P \in Int(\Delta ABC)$, then aPA, bPB, cPC can be the sides of a triangle (Klamkin)

→ aIA, bIB, cIC can be the sides of a triangle, where I be the incenter of ΔABC .

Since $aIA = 2\sqrt{Rr}\sqrt{a(s-a)}$

→ $\sqrt{a(s-a)}, \sqrt{b(s-b)}, \sqrt{c(s-c)}$ can be the sides of a triangle Δ_1

with area F_1 such that : $4 \cdot 16F_1^2$

$$\begin{aligned} & = 4 \left(2 \sum \sqrt{a(s-a)^2} \sqrt{b(s-b)^2} - \sum \sqrt{a(s-a)^4} \right) = \\ & = 2 \sum ab(-a+b+c)(a-b+c) - \sum (-a^2 + ab + ac)^2 \\ & = 2 \sum ab(c^2 + 2ab - a^2 - b^2) - \\ & - \sum (a^4 + a^2b^2 + a^2c^2 + 2a^2bc - 2a^3b - 2a^3c) = 2 \sum a^2b^2 - \sum a^4 = 16F^2 \\ \rightarrow & 2F_1 = F. \end{aligned}$$

We have :



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$$(\sqrt{x+y} + \sqrt{y+z})^2 = (x+z) + 2y + 2\sqrt{x+y}\sqrt{y+z} \stackrel{?}{\geq} \sqrt{z+x}^2 \leftrightarrow \\ \sqrt{y^2 + \sum xy} \geq -y$$

Which is true because

$$\sqrt{y^2 + \sum xy} \stackrel{\Sigma xy > 0}{\geq} |y| \geq -y \rightarrow \sqrt{x+y} + \sqrt{y+z} \geq \sqrt{z+x} \text{ (And analogs)}$$

$\rightarrow \sqrt{x+y}, \sqrt{y+z}, \sqrt{z+x}$ can be the sides of a triangle Δ_2 with area F_2 such that

$$: 16F_2^2 =$$

$$2 \sum \sqrt{x+y} \cdot \sqrt{y+z} - \sum \sqrt{x+y}^4 \\ = 2 \sum (y^2 + xy + yz + zx) - \sum (x^2 + 2xy + y^2) = 4 \sum xy \rightarrow \\ 2F_2 = \sqrt{\sum xy}$$

We know that for any two triangle ΔXYZ and ΔUVW with area S_1 and S_2 respectively,

$$\text{we have : } \sum x^2(v^2 + w^2 - u^2) \geq 16S_1S_2 \text{ (Neuberg - Pedoe) (**)}$$

\rightarrow Using (**) for the triangles Δ_1 and Δ_2 , we have

$$: \sum \sqrt{a(s-a)}^2 (\sqrt{x+y}^2 + \sqrt{z+x}^2 - \sqrt{y+z}^2) \geq 4F \sqrt{\sum xy}$$

$$\text{Therefore, } \sum ax(b+c-a) \geq 4F \sqrt{\sum xy}.$$

2758. In ΔABC the following relationship holds:

$$\frac{1}{2F} \sum ab \geq \sqrt{\sum \left(\frac{a}{b}\right)^2} + \sqrt{\sum \frac{a(s-a)}{b(s-b)}}$$

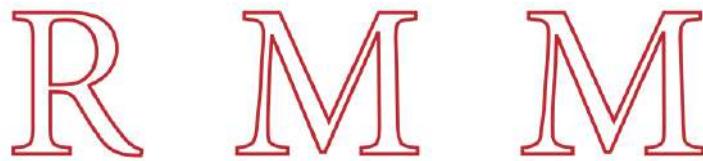
Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that if

$P \in \text{Int}(\Delta ABC)$, then aPA, bPB, cPC can be the sides of a triangle (Klamkin)

$\rightarrow aIA, bIB, cIC$ can be the sides of a triangle, where I be the incenter of ΔABC .



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Since $aIA = 2\sqrt{Rr}\sqrt{a(s-a)}$

$\rightarrow \sqrt{a(s-a)}, \sqrt{b(s-b)}, \sqrt{c(s-c)}$ can be the sides of a triangle Δ_1

with area F_1 such that : $4 \cdot 16F_1^2$

$$\begin{aligned}
 &= 4 \left(2 \sum \sqrt{a(s-a)}^2 \sqrt{b(s-b)}^2 - \sum \sqrt{a(s-a)}^4 \right) = \\
 &= 2 \sum ab(-a+b+c)(a-b+c) - \sum (-a^2 + ab + ac)^2 = \\
 &= 2 \sum ab(c^2 + 2ab - a^2 - b^2) - \sum (a^4 + a^2b^2 + a^2c^2 + 2a^2bc - 2a^3b - 2a^3c) \\
 &= 2 \sum a^2b^2 - \sum a^4 = 16F^2 \rightarrow F_1 = \frac{1}{2}F.
 \end{aligned}$$

We know that if ΔUVW a triangle with area F' then : $\sum x \cdot u^2$

$$\geq 4F' \cdot \sqrt{\sum xy}, \forall x, y, z > 0 \text{ (Oppenheim)}$$

If we take : $x = \frac{v^2}{u^2}, y = \frac{w^2}{v^2}, z = \frac{u^2}{w^2} \rightarrow \forall \Delta UVW, \sum u^2 \geq 4F' \cdot \sqrt{\sum \frac{u^2}{v^2}}$ ()*

Using () for Δ_1 , we get : $\sum \sqrt{a(s-a)}^2 \geq 4 \cdot \frac{1}{2}F \cdot \sqrt{\sum \frac{\sqrt{a(s-a)}^2}{\sqrt{b(s-b)}^2}}$*

$$\leftrightarrow \sum a(-a+b+c) \geq 4F \cdot \sqrt{\sum \frac{a(s-a)}{b(s-b)}}$$

$$\rightarrow \frac{1}{4F} \left(2 \sum ab - \sum a^2 \right) \geq \sqrt{\sum \frac{a(s-a)}{b(s-b)}} \quad (1)$$

Using () for ΔABC , we get : $\frac{1}{4F} \sum a^2 \geq \sqrt{\sum \left(\frac{a}{b} \right)^2}$ (2)*

$$(1) + (2) \rightarrow \frac{1}{2F} \sum ab \geq \sqrt{\sum \left(\frac{a}{b} \right)^2} + \sqrt{\sum \frac{a(s-a)}{b(s-b)}}.$$



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2759. $\Sigma xy > 0, x + y, y + z, z + x > 0$. In ΔABC holds:

$$\sum m_a(m_b + m_c)x \geq 6F \sqrt{\sum xy}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that m_a, m_b, m_c can be the sides of a triangle with area $\frac{3}{4}F$

\rightarrow It's suffices to prove : $\sum a(b+c)x \geq 8F \sqrt{\sum xy}, \forall \Delta ABC$ (*)

We know that if

$P \in Int(\Delta ABC)$, then aPA, bPB, cPC can be the sides of a triangle (Klamkin)

$\rightarrow aIA, bIB, cIC$ can be the sides of a triangle, where I be the incenter of ΔABC .

Since $aIA = 2\sqrt{Rr}\sqrt{a(s-a)}$

$\rightarrow \sqrt{a(s-a)}, \sqrt{b(s-b)}, \sqrt{c(s-c)}$ can be the sides of a triangle Δ_1

with area F_1 such that : $4 \cdot 16F_1^2$

$$\begin{aligned}
 &= 4 \left(2 \sum \sqrt{a(s-a)}^2 \sqrt{b(s-b)}^2 - \sum \sqrt{a(s-a)}^4 \right) = \\
 &= 2 \sum ab(-a+b+c)(a-b+c) - \sum (-a^2 + ab + ac)^2 \\
 &= 2 \sum ab(c^2 + 2ab - a^2 - b^2) - \\
 &- \sum (a^4 + a^2b^2 + a^2c^2 + 2a^2bc - 2a^3b - 2a^3c) = 2 \sum a^2b^2 - \sum a^4 = 16F^2 \\
 &\rightarrow 2F_1 = F.
 \end{aligned}$$

We have : $(\sqrt{x+y} + \sqrt{y+z})^2 = (x+z) + 2y + 2\sqrt{x+y} \cdot \sqrt{y+z} \stackrel{?}{\geq} \sqrt{z+x}^2$

$$\Leftrightarrow \sqrt{y^2 + \sum xy} \geq -y$$

Which is true because $\sqrt{y^2 + \sum xy} \stackrel{\Sigma xy > 0}{\geq} |y| \geq -y \rightarrow \sqrt{x+y} + \sqrt{y+z} \geq \sqrt{z+x}$ (And analogs)



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→ $\sqrt{x+y}, \sqrt{y+z}, \sqrt{z+x}$ can be the sides of a triangle Δ_2 with area F_2 such that

$$\therefore 16F_2^2 =$$

$$\begin{aligned} & 2 \sum \sqrt{x+y}^2 \cdot \sqrt{y+z}^2 - \sum \sqrt{x+y}^4 \\ &= 2 \sum (y^2 + xy + yz + zx) - \sum (x^2 + 2xy + y^2) = 4 \sum xy \rightarrow 2F_2 \\ &= \sqrt{\sum xy} \end{aligned}$$

We know that for any two triangle ΔXYZ and ΔUVW with area S_1 and S_2 respectively,

$$\text{we have : } \sum x^2(v^2 + w^2 - u^2) \geq 16S_1S_2 \text{ (Neuberg - Pedoe) (**)}$$

→ Using (**) for the triangles Δ_1 and Δ_2 , we have

$$\begin{aligned} & \therefore \sum \sqrt{a(s-a)}^2 (\sqrt{x+y}^2 + \sqrt{z+x}^2 - \sqrt{y+z}^2) \geq 4F \sqrt{\sum xy} \\ & \leftrightarrow \sum a(b+c-a) \cdot x \geq 4F \sqrt{\sum xy} \quad (1) \end{aligned}$$

→ Using now (**) for the triangles ΔABC and Δ_2 , we have

$$\begin{aligned} & \therefore \sum a^2 (\sqrt{x+y}^2 + \sqrt{z+x}^2 - \sqrt{y+z}^2) \geq 8F \sqrt{\sum xy} \\ & \leftrightarrow \sum a^2 \cdot x \geq 4F \sqrt{\sum xy} \quad (2) \end{aligned}$$

$$\begin{aligned} (1) + (2) \rightarrow & \sum a(b+c)x \geq 8F \sqrt{\sum xy} \\ \rightarrow & (*) \text{ is true. Therefore, } \sum m_a(m_b + m_c)x \geq 6F \sqrt{\sum xy}. \end{aligned}$$

2760. In ΔABC the following relationship holds:

$$\left(\frac{R}{2r}\right)^2 \cdot abc \geq \frac{a^3 + b^3 + c^3}{3} \geq abc + \frac{r^2(R-2r)}{3}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Ertan Yildirim-Izmir-Turkiye

$$\therefore \sum a^3 = 2s(s^2 - 3r^2 - 6Rr)$$



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$$\left(\frac{R}{2r}\right)^2 \cdot abc \geq \frac{a^3 + b^3 + c^3}{3} \Leftrightarrow$$

$$\frac{R^2}{4r^2} \cdot 4Rrs \cdot 3 \geq 2s(s^2 - 3r^2 - 6Rr) \Leftrightarrow 3R^3 \geq 2r(s^2 - 3r^2 - 6Rr)$$

$$\Rightarrow 2s^2r \leq 3R^3 + 6r^3 + 12Rr^2$$

$$2r \cdot s^2 \stackrel{\text{Gerretsen}}{\leq} 2r(4R^2 + 4Rr + 3r^2) \stackrel{(1)}{\leq} 3R^3 + 6r^3 + 12Rr^2$$

$$(1) \Leftrightarrow 8R^2r + 8Rr^2 + 6r^3 \leq 3R^3 + 6r^3 + 12Rr^2$$

$$\Leftrightarrow 3R^3 + 4Rr^2 - 8R^2r \geq 0 \Leftrightarrow (3R - 2r)(R - 2r) \geq 0 \text{ true from } R \geq 2r (\text{Euler})$$

Now,

$$\frac{a^3 + b^3 + c^3}{3} \geq abc + \frac{r^2(R - 2r)}{3} \Leftrightarrow a^3 + b^3 + c^3 - 3abc \geq r^2(R - 2r) \Leftrightarrow$$

$$2s(s^2 - 3r^2 - 6Rr) - 3 \cdot 4Rrs \stackrel{(2)}{\geq} r^2(R - 2r) \Leftrightarrow$$

$$2s(s^2 - 3r^2 - 6Rr) - 12Rrs = 2s(s^2 - 3r^2 - 12Rr) \geq$$

$$\geq 2s(4R^2 + 4Rr + 3r^2 - 3r^2 - 12Rr) = 2s(4R^2 - 8Rr) =$$

$$= 2s \cdot 4R(R - 2r) \stackrel{(3)}{\geq} r^2(R - 2r)$$

$$(3) \Leftrightarrow 2s \cdot 4R(R - 2r) - r^2(R - 2r) \geq 0 \Leftrightarrow (R - 2r)(8Rs - r^2) \geq 0$$

$$\Leftrightarrow (R - 2r)(8Rs - r^2) \geq (R - 2r)(16sr - r^2) \geq 0$$

$$\Leftrightarrow r(R - 2r)(16s - r) \geq 0 \text{ true from } R \geq 2r (\text{Euler})$$

2761. If $xyz(x + y + z) > 0$, in ΔABC the following relationship holds:

$$\frac{1}{2} \left| \sum ayz \right| \geq \sqrt{r \left(\sum r_a \right) \cdot xyz \left(\sum x \right)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since ABC is a triangle $\rightarrow b + c \geq a$ (And analogs) \rightarrow

$$(\sqrt{b} + \sqrt{c})^2 = b + c + 2\sqrt{bc} \geq a = \sqrt{a}^2$$

$$\rightarrow \sqrt{b} + \sqrt{c} \geq \sqrt{a} \text{ (And analogs)}$$

$\rightarrow \sqrt{a}, \sqrt{b}, \sqrt{c}$ can be the sides of a triangle Δ_1 with area F_1 such that :



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$$\begin{aligned} 16F_1^2 &= 2 \sum \sqrt{a}^2 \sqrt{b}^2 - \sum \sqrt{a}^4 = 2 \sum ab - \sum a^2 = 4r(4R+r) \rightarrow F_1 \\ &= \frac{1}{2} \sqrt{r(4R+r)} \end{aligned}$$

Also, we have : $(xy + yz)(yz + zx) = xyz(x + y + z) + y^2z^2 > 0$

→ $xy + yz, yz + zx$ have the same sign.

Similarly, $yz + zx, zx + xy$ have the same sign

→ $xy + yz, yz + zx, zx + xy$ have the same sign.

If $xy + yz, yz + zx, zx + xy > 0$, we pose $u = \sqrt{xy + yz}, v = \sqrt{yz + zx}, w$

$$= \sqrt{zx + xy}$$

$$\begin{aligned} (u + v)^2 &= xy + 2yz + zx + 2\sqrt{(xy + yz)(yz + zx)} \\ &= w^2 + 2yz + 2\sqrt{xyz(x + y + z) + y^2z^2} > \end{aligned}$$

$xyz(x+y+z) > 0$

$$\stackrel{x,y,z > 0}{\geq} w^2 + 2(yz + |yz|) \geq w^2 \rightarrow u + v > w \text{ (And analogs)}$$

→ u, v, w can be the sides of a triangle Δ_2 with area F_2 such that : $16F_2^2$

$$= 2 \sum u^2 v^2 - \sum u^4 =$$

$$= 2 \sum [xyz(x + y + z) + y^2z^2] - \sum [x^2y^2 + 2xy^2z + y^2z^2] = 4xyz(x + y + z) \rightarrow F_2$$

$$= \frac{1}{2} \sqrt{xyz(x + y + z)}$$

From Neuberg – Pedoe's inequality in the triangles Δ_1 and Δ_2 :

$$\sum \sqrt{a}^2 \cdot (u^2 + v^2 - w^2) \geq 16F_1 F_2$$

$$\leftrightarrow \sum a[(xy + yz) + (yz + zx) - (zx + xy)] \geq 16 \cdot \frac{1}{2} \sqrt{r(4R+r)} \cdot \frac{1}{2} \sqrt{xyz(x + y + z)}$$

$$\leftrightarrow \frac{1}{2} \sum ayz \geq \sqrt{r \left(\sum r_a \right) \cdot xyz \left(\sum x \right)} \quad (1)$$

If $xy + yz, yz + zx, zx + xy < 0$, we pose $u = \sqrt{-(xy + yz)}, v = \sqrt{-(yz + zx)}, w$

$$= \sqrt{-(zx + xy)}$$

$$\begin{aligned} (u + v)^2 &= -xy - 2yz - zx + 2\sqrt{(xy + yz)(yz + zx)} \\ &= w^2 - 2yz + 2\sqrt{xyz(x + y + z) + y^2z^2} > \end{aligned}$$



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$xyz(x+y+z) > 0$

$$\stackrel{x+y+z}{\geq} w^2 + 2(|yz| - yz) \geq w^2 \rightarrow u+v > w \text{ (And analogs)}$$

$\rightarrow u, v, w$ can be the sides of a triangle Δ_3 with area F_3 such that : $16F_3^2$

$$= 2 \sum u^2v^2 - \sum u^4 =$$

$$= 2 \sum [xyz(x+y+z) + y^2z^2] - \sum [x^2y^2 + 2xy^2z + y^2z^2] = 4xyz(x+y+z) \rightarrow F_2$$

$$= \frac{1}{2} \sqrt{xyz(x+y+z)}$$

From Neuberg – Pedoe's inequality in the triangles Δ_1 and Δ_3

$$: \sum \sqrt{a}^2 \cdot (u^2 + v^2 - w^2) \geq 16F_1F_3$$

$$\leftrightarrow \sum a[-(xy+yz)-(yz+zx)+(zx+xy)] \geq 16 \cdot \frac{1}{2} \sqrt{r(4R+r)} \cdot \frac{1}{2} \sqrt{xyz(x+y+z)}$$

$$\leftrightarrow -\frac{1}{2} \sum ayz \geq \sqrt{r(\sum r_a) \cdot xyz(\sum x)} \quad (2)$$

$$(1), (2) \rightarrow \frac{1}{2} |\sum ayz| \geq \sqrt{r(\sum r_a) \cdot xyz(\sum x)}.$$

2762. If $xyz(x+y+z) > 0$, in ΔABC holds:

$$\left| \sum yz \sqrt{\frac{m_a}{h_a}} \right| \geq \sqrt{\left(2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a} \right) (x+y+z) xyz}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that if $P \in \text{Int}(\Delta ABC)$, then aPA, bPB, cPC
can be the sides of a triangle (Klamkin)

$\rightarrow aGA, bGB, cGC$ can be the sides of a triangle, where G be the centroid of ΔABC .

Since $aGA = \frac{4sr}{3} \cdot \frac{m_a}{h_a} \rightarrow \frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}$ can be the sides of a triangle

We know that if m, n, p the sides of a triangle, then $\sqrt[4]{m}, \sqrt[4]{n}, \sqrt[4]{p}$
can be also the sides of a triangle.

$$(\therefore m+n \geq p \rightarrow (\sqrt[4]{m} + \sqrt[4]{n})^4 = (m+n) + 4\sqrt[4]{m^3n} + 4\sqrt[4]{mn^3} + 6\sqrt{mn} > \sqrt[4]{p^4})$$



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$\rightarrow \sqrt[4]{\frac{m_a}{h_a}}, \sqrt[4]{\frac{m_b}{h_b}}, \sqrt[4]{\frac{m_c}{h_c}}$ can be the sides of a triangle Δ_1 with area :

$$F_1 = \frac{1}{4} \sqrt{2 \sum \sqrt[4]{\frac{m_a^2}{h_a}} \sqrt[4]{\frac{m_b^2}{h_b}} - \sum \sqrt[4]{\frac{m_a^4}{h_a}}} = \frac{1}{4} \sqrt{2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a}}$$

Also, we have : $(xy + yz)(yz + zx) = xyz(x + y + z) + y^2 z^2 > 0$

$\rightarrow xy + yz, yz + zx$ have the same sign.

Similarly, $yz + zx, zx + xy$ have the same sign

$\rightarrow xy + yz, yz + zx, zx + xy$ have the same sign.

If $xy + yz, yz + zx, zx + xy > 0$, we pose $u = \sqrt{xy + yz}, v = \sqrt{yz + zx}, w = \sqrt{zx + xy}$

$$(u + v)^2 = xy + 2yz + zx + 2\sqrt{(xy + yz)(yz + zx)} \\ = w^2 + 2yz + 2\sqrt{xyz(x + y + z) + y^2 z^2} >$$

$xyz(x+y+z) > 0$

$$\stackrel{?}{>} w^2 + 2(yz + |yz|) \geq w^2 \rightarrow u + v > w \text{ (And analogs)}$$

$\rightarrow u, v, w$ can be the sides of a triangle Δ_2 with area F_2 such that : $16F_2^2$

$$= 2 \sum u^2 v^2 - \sum u^4 =$$

$$= 2 \sum [xyz(x + y + z) + y^2 z^2] - \sum [x^2 y^2 + 2xy^2 z + y^2 z^2] = 4xyz(x + y + z) \rightarrow F_2 \\ = \frac{1}{2} \sqrt{xyz(x + y + z)}$$

From Neuberg – Pedoe's inequality in the triangles Δ_1 and Δ_2 :

$$\sum \sqrt[4]{\frac{m_a}{h_a}}^2 \cdot (u^2 + v^2 - w^2) \geq 16F_1 F_2$$

$$\leftrightarrow \sum \sqrt{\frac{m_a}{h_a}} \cdot [(xy + yz) + (yz + zx) - (zx + xy)]$$

$$\geq 16 \cdot \frac{1}{4} \sqrt{2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a}} \cdot \frac{1}{2} \sqrt{xyz(x + y + z)}$$

$$\leftrightarrow \sum yz \sqrt{\frac{m_a}{h_a}} \geq \sqrt{\left(2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a} \right) (x + y + z) xyz} \quad (1)$$

If $xy + yz, yz + zx, zx + xy < 0$, let $u = \sqrt{-(xy + yz)}, v = \sqrt{-(yz + zx)}, w = \sqrt{-(zx + xy)}$



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$$\begin{aligned}
 (\mathbf{u} + \mathbf{v})^2 &= -xy - 2yz - zx + 2\sqrt{(xy + yz)(yz + zx)} \\
 &= w^2 - 2yz + 2\sqrt{xyz(x + y + z) + y^2z^2} > \\
 &\stackrel{xyz(x+y+z)>0}{\geq} w^2 + 2(|yz| - yz) \geq w^2 \rightarrow u + v > w \text{ (And analogs)} \\
 \rightarrow u, v, w \text{ can be the sides of a triangle } \Delta_3 \text{ with area } F_3 \text{ such that : } 16F_3^2 \\
 &= 2 \sum u^2v^2 - \sum u^4 = \\
 &= 2 \sum [xyz(x + y + z) + y^2z^2] - \sum [x^2y^2 + 2xy^2z + y^2z^2] = 4xyz(x + y + z) \rightarrow F_2 \\
 &= \frac{1}{2}\sqrt{xyz(x + y + z)}
 \end{aligned}$$

From Neuberg – Pedoe's inequality in the triangles Δ_1 and Δ_3

$$\begin{aligned}
 &: \sum \sqrt[4]{\frac{m_a}{h_a}}^2 \cdot (u^2 + v^2 - w^2) \geq 16F_1F_3 \\
 \Leftrightarrow &\sum \sqrt{\frac{m_a}{h_a}} \cdot [-(xy + yz) - (yz + zx) + (zx + xy)] \\
 &\geq 16 \cdot \frac{1}{4} \sqrt{2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a} \cdot \frac{1}{2} \sqrt{xyz(x + y + z)}} \\
 \Leftrightarrow &- \sum yz \sqrt{\frac{m_a}{h_a}} \geq \sqrt{\left(2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a}\right)(x + y + z)xyz} \quad (2) \\
 (1), (2) \rightarrow &\left| \sum yz \sqrt{\frac{m_a}{h_a}} \right| \geq \sqrt{\left(2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a}\right)(x + y + z)xyz}
 \end{aligned}$$

2763. In ΔABC the following relationship holds:

$$\sum a^2 \geq 4F \sqrt{\sum \left(\frac{m_a}{m_b}\right)^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum a^2 \stackrel{(*)}{\geq} 4F \sqrt{\sum \left(\frac{m_a}{m_b}\right)^2}$$



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We know that m_a, m_b, m_c can be the sides of a triangle with area $F' = \frac{3}{4}F$.

$$(*) \leftrightarrow \frac{4}{3} \sum m_a^2 \geq \frac{16}{3} F'. \sqrt{\sum \left(\frac{m_a}{m_b}\right)^2} \leftrightarrow \sum m_a^2 \geq 4F'. \sqrt{\sum \left(\frac{m_a}{m_b}\right)^2}$$

We know that if ΔUVW a triangle with area S then : $\sum x \cdot u^2$

$$\geq 4S \cdot \sqrt{\sum xy}, \forall x, y, z > 0 \text{ (Oppenheim)}$$

If we take : $x = \frac{v^2}{u^2}, y = \frac{w^2}{v^2}, z = \frac{u^2}{w^2} \rightarrow \forall \Delta UVW, \sum u^2 \geq 4S \cdot \sqrt{\sum \frac{u^2}{v^2}}$ (1)

Using (1) for $\Delta m_a m_b m_c$, we get : $\sum m_a^2 \geq 4F'. \sqrt{\sum \left(\frac{m_a}{m_b}\right)^2} \rightarrow (*) \text{ is true.}$

$$\text{Therefore, } \sum a^2 \geq 4F \sqrt{\sum \left(\frac{m_a}{m_b}\right)^2}.$$

2764. In ΔABC the following relationship holds:

$$\left(\sum m_a\right) \sqrt{r_a r_b r_c} \geq \left(\sum r_a\right) \sqrt{h_a h_b h_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum m_a\right) \sqrt{r_a r_b r_c} \geq \left(\sum r_a\right) \sqrt{h_a h_b h_c} \leftrightarrow \left(\sum m_a\right) \sqrt{s^2 r} \geq (4R + r) \sqrt{\frac{2s^2 r^2}{R}} \leftrightarrow$$

$$\sum m_a \geq \sqrt{\frac{2r}{R}} (4R + r)$$

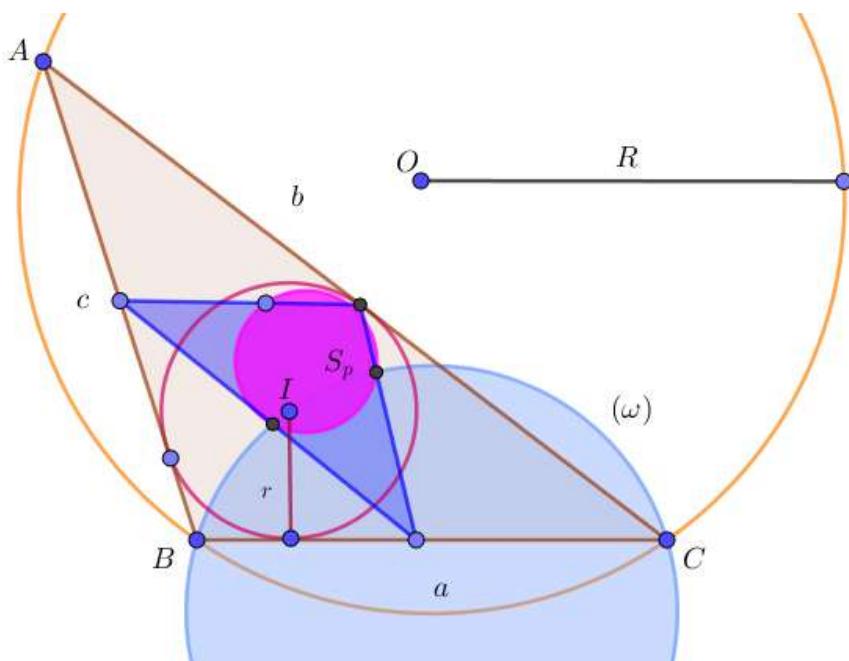
$$\text{We have : } \sum m_a \stackrel{\text{Lascu}}{\geq} \sum \frac{b+c}{2} \cos \frac{A}{2}$$

$$= \sum \frac{(b+c-a)+a}{2} \cos \frac{A}{2} \stackrel{\text{AM-GM}}{\geq} \sum \sqrt{a(b+c-a)} \cdot \sqrt{\frac{s(s-a)}{bc}} =$$

$$\begin{aligned}
 &= \sum (s-a) \sqrt{\frac{2sa}{bc}} = \sum (s-a) \sqrt{\frac{2sa^2}{4Rrs}} = \frac{1}{\sqrt{2Rr}} \sum a(s-a) = \frac{2r(4R+r)}{\sqrt{2Rr}} \\
 &= \sqrt{\frac{2r}{R}} (4R+r).
 \end{aligned}$$

Therefore, $(\sum m_a) \sqrt{r_a r_b r_c} \geq (\sum r_a) \sqrt{h_a h_b h_c}$.

2765.



I –incenter, S_p –Spieker's point center. Prove that:

$$r^2 \leq \frac{a[a^3 + (b+c)(b-c)^2]}{4[(b+c)^2 - a^2]}$$

Proposed by Thanasis Gakopoulos-Farsala-Greece

Solution 1 by Jose Ferreira Queiroz-Olinda-Brazil

Let M –be any point in plane of triangle ABC , so

$$2s \cdot MI^2 = a \cdot MA^2 + b \cdot MB^2 + c \cdot MC^2 - abc; \quad (1)$$

$$\begin{aligned}
 8s \cdot MS_p^2 &= 2(b+c) \cdot MA^2 + 2(c+a) \cdot MB^2 + 2(a+b) \cdot MC^2 \\
 &\quad - (a^3 + b^3 + c^3 + abc); \quad (2)
 \end{aligned}$$



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Observe that: $MB = MC = MI - MS_p = R_\omega$; (3)

From (1) and (3): $MA^2 - R_\omega^2 = bc$

From (2) and (3): $2(b + c)R_\omega^2 = 2(b + c) \cdot MA^2 - (a^3 + b^3 + c^3 + abc)$

It follows that:

$$a^3 + b^3 + c^3 + abc = 2bc(b + c)$$

$$a^3 + b^3 + c^3 - b^2c - bc^2 = b^2c + bc^2 - abc$$

$$a[a^3 + (b + c)(b^2 - bc + c^2) - bc(b + c)] = abc(b + c - a)$$

$$abc(b + c - a) = a[a^3 + (b + c)(b - c)^2]$$

$$2Rr(b + c - a)(b + c + a) = a[a^3 + (b + c)(b - c)^2]$$

$$R = \frac{a[a^3 + (b + c)(b - c)^2]}{4[(b + c)^2 - a^2]} \geq 2r; \text{(Euler)}$$

Therefore,

$$r^2 \leq \frac{a[a^3 + (b + c)(b - c)^2]}{4[(b + c)^2 - a^2]}$$

Solution 2 by proposer

Plagiogonal system: $BC \equiv Bx, BA \equiv By$

$$B, C, I \in (\omega) \Rightarrow (\omega): x^2 + y^2 + \frac{a^2 - b^2 + c^2}{ac} \cdot xy - ax + (b - c)y = 0; \quad (1)$$

$$S_p(S_{p_1}, S_{p_2}); S_{p_1} = \frac{a(a + b)}{2(a + b + c)}, S_{p_2} = \frac{c(b + c)}{2(a + b + c)}$$

From (1) and $x = S_{p_1}, y = S_{p_2}$ **it follows:**

$$a^3 + (b + c)(b - c)^2 = bc(-a + b + c)$$

$$a^3 + (b + c)(b - c)^2 = 2R \cdot h_a(-a + b + c)$$

$$a^3 + (b + c)(b - c)^2 = 2R \cdot \frac{2[ABC]}{a} (-a + b + c) \Rightarrow$$

$$R = \frac{a^3 + (b + c)(b - c)^2}{4[ABC](-a + b + c)} \cdot a \Rightarrow \frac{R}{r} = \frac{a[a^3 + (b + c)(b - c)^2]}{4[(b + c)^2 - a^2]} \geq 2$$

Therefore,

$$r^2 \leq \frac{a[a^3 + (b + c)(b - c)^2]}{4[(b + c)^2 - a^2]}$$



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2766. In any ΔABC , $(a + b)(b + c)(c + a) \geq 8abc + \frac{r^4(R - 2r)}{R^2 + 4r^2}$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a + b)(b + c)(c + a) - 8abc = 2s(s^2 + 2Rr + r^2) - 32Rrs = 2s(s^2 - 14Rr + r^2)$$

$$\geq \frac{r^4(R - 2r)}{R^2 + 4r^2} \Leftrightarrow 4s^2(s^2 - 14Rr + r^2)^2 \stackrel{(*)}{\geq} \frac{r^8(R - 2r)^2}{(R^2 + 4r^2)^2}$$

Now, LHS of (*) $\stackrel{\text{Gerretsen}}{\leq} 4s^2(16Rr - 5r^2 - 14Rr + r^2)^2$

$$= 16s^2r^2(R - 2r)^2 \stackrel{?}{\geq} \frac{r^8(R - 2r)^2}{(R^2 + 4r^2)^2} \Leftrightarrow 16s^2 \stackrel{?}{>} \frac{r^6}{(R^2 + 4r^2)^2}$$

$$\Leftrightarrow 16s^2(R^2 + 4r^2)^2 \stackrel{?}{>} r^6 \rightarrow \text{true}$$

$$\begin{aligned} & \because 16s^2(R^2 + 4r^2)^2 \stackrel{\text{Mitrinovic + Euler}}{\geq} 27r^2(8r^2)^2 > r^6 \Rightarrow (*) \text{ is true} \therefore (a + b)(b + c)(c + a) \\ & \geq 8abc + \frac{r^4(R - 2r)}{R^2 + 4r^2} \quad (\text{QED}) \end{aligned}$$

2767. If $xyz(x + y + z) > 0$, in ΔABC holds:

$$\sqrt{\frac{R}{2r} \left| \sum yz \right|} \geq \sqrt{\left(2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a} \right) xyz(x + y + z)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that if $P \in \text{Int}(\Delta ABC)$, then aPA, bPB, cPC can be the sides of triangle (Klamkin)

$\rightarrow aGA, bGB, cGC$ can be the sides of a triangle, where G is the centroid of ΔABC .

Since $aGA = \frac{4sr}{3} \cdot \frac{m_a}{h_a} \rightarrow \frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}$ can be the sides of triangle.

Also, we know that if m, n, p the sides of a triangle, then $\sqrt[4]{m}, \sqrt[4]{n}, \sqrt[4]{p}$ can be also the sides of a triangle.

$$\left(\therefore m + n \geq p \rightarrow (\sqrt[4]{m} + \sqrt[4]{n})^4 = (m + n) + 4\sqrt[4]{m^3n} + 4\sqrt[4]{mn^3} + 6\sqrt{mn} > \sqrt[4]{p^4} \right)$$

$\rightarrow \sqrt[4]{\frac{m_a}{h_a}}, \sqrt[4]{\frac{m_b}{h_b}}, \sqrt[4]{\frac{m_c}{h_c}}$ can be the sides of a triangle Δ_1 with area :

$$F_1 = \frac{1}{4} \sqrt{2 \sum \sqrt[4]{\frac{m_a}{h_a}}^2 \sqrt[4]{\frac{m_b}{h_b}}^2 - \sum \sqrt[4]{\frac{m_a}{h_a}}^4} = \frac{1}{4} \sqrt{2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a}}$$

Using Ionescu Weitzenbock in Δ_1 : $\sum \sqrt[4]{\frac{m_a}{h_a}}^2 \geq 4\sqrt{3} \cdot \frac{1}{4} \sqrt{2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a}}$

$$\rightarrow \sqrt{\frac{R}{2r}} = \frac{1}{3} \sum \sqrt{\frac{R}{2r}} \stackrel{\text{Panaitopol}}{\geq} \frac{1}{3} \sum \sqrt{\frac{m_a}{h_a}} \geq \frac{1}{3} \cdot \sqrt{3} \sqrt{2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a}} \quad (1)$$

Now, we have : $|\sum xy| \stackrel{?}{\geq} \sqrt{3xyz(x+y+z)} \leftrightarrow (\sum xy)^2 \geq 3xyz \sum x$

$$\leftrightarrow \sum x^2 y^2 - xyz \sum x \geq 0$$

$\leftrightarrow \sum x^2(y-z)^2 \geq 0$ which is true $\rightarrow |\sum xy| \geq \sqrt{3xyz(x+y+z)}$ (2)

$$(1), (2) \rightarrow \sqrt{\frac{R}{2r}} |\sum yz| \geq \sqrt{\left(2 \sum \sqrt{\frac{m_a m_b}{h_a h_b}} - \sum \frac{m_a}{h_a}\right) xyz(x+y+z)}.$$

2768. In ΔABC the following relationship holds:

$$a) \quad \frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{2(2R-r)}{R+2r}$$

$$b) \quad \sum_{cyc} a^4 + abc \sum_{cyc} a \geq \sum_{cyc} ab(a^2 + b^2) + r^3(R - 2r)$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Avishek Mitra-West Bengal-India

$$\begin{aligned} \sum_{cyc} \frac{a}{b+c} &= \left(\sum_{cyc} \frac{a}{b+c} \right) + 3 - 3 = \left(\sum_{cyc} a \right) \left(\sum_{cyc} \frac{1}{a+b} \right) - 3 \\ &= 2s \cdot \frac{\sum(b+c)(c+a)}{\prod(a+b)} - 3 = 2s \cdot \frac{\sum(bc+ca+ab+c^2)}{2abc + \sum ab(a+b)} - 3 = \end{aligned}$$



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$$\begin{aligned}
 &= 2s \cdot \frac{\sum c^2 + 3\sum ab}{2abc + \sum ab(2s - c)} - 3 = 2s \cdot \frac{2s^2 - 8Rr - 2r^2 + 3s^2 + 3r^2 + 12Rr}{2s(\sum ab) - abc} - 3 = \\
 &= 2s \cdot \frac{5s^2 + 4Rr + r^2}{2s(s^2 + r^2 + 4Rr) - 4Rrs} - 3 = 2s \cdot \frac{5s^2 + 4Rr + r^2}{2s(s^2 + r^2 + 2Rr)} - 3 = \\
 &= \frac{5s^2 + 4Rr + r^2 - 3s^2 - 3r^2 - 6Rr}{s^2 + r^2 + 2Rr} = \frac{2s^2 - 2Rr - 2r^2}{s^2 + r^2 + 2Rr}
 \end{aligned}$$

Need to show:

$$\begin{aligned}
 \frac{2(s^2 - Rr - r^2)}{s^2 + r^2 + 2Rr} &\leq \frac{2(2R - r)}{R + 2r} \Leftrightarrow \\
 (R + 2r)(s^2 - Rr - r^2) &\leq (2R - r)(s^2 + r^2 + 2Rr) \\
 Rs^2 + 5R^2s + 3Rr^2 + r^3 &\geq 3rs^2
 \end{aligned}$$

But $4R^2 + 4Rr + 3r^2 \geq s^2 \geq 16Rr - 5r^2$ (Gerretsen)

Need to show:

$$\begin{aligned}
 R(16Rr - 5r^2) + 5R^2r + 3rr^2 + r^3 &\geq 3r(4R^2 + 4Rr + 3r^2) \\
 21R^2r - 2Rr^2 + r^3 &\geq 12R^2r + 12Rr^2 + 9r^3
 \end{aligned}$$

$9R^2r - 14Rr^2 - 8r^3 \geq 0 \Leftrightarrow r(R - 2r)(9R + 4r) \geq 0$, true from $R \geq 2r$ (Euler)

$$\sum_{cyc} \frac{a}{b+c} = \sum_{cyc} \frac{a^2}{ab+ac} \stackrel{\text{Bergstrom}}{\geq} \frac{(\sum a)^2}{2\sum ab} \geq \frac{3\sum ab}{2\sum ab} = \frac{3}{2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \sum \frac{a}{b+c} &= \sum \frac{a - b - c + b + c}{b + c} = -2 \sum \frac{s - a}{b + c} + 3 \\
 &= -2 \sum \frac{(s - a)^2}{(b + c)(s - a)} \\
 + 3 &\stackrel{\text{Bergstrom}}{\geq} \frac{-2(s - a + s - b + s - c)^2}{(2s - a)(s - a) + (2s - b)(s - b) + (2s - c)(s - c)} + 3 \\
 &= \frac{-2s^2}{6s^2 + 2(s^2 - 4Rr - r^2) - 3s(2s)} + 3 = 3 - \frac{s^2}{s^2 - 4Rr - r^2} \leq \frac{2(2R - r)}{R + 2r} \\
 &\Leftrightarrow \frac{2(2R - r)(s^2 - 4Rr - r^2) + (R + 2r)s^2}{(R + 2r)(s^2 - 4Rr - r^2)} \geq 3 \\
 &\Leftrightarrow (2R - 6r)s^2 - r(4R^2 - 31Rr - 8r^2) \geq 0
 \end{aligned}$$



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$$\Leftrightarrow (2R - 4r)s^2 - 2rs^2 - r(4R^2 - 31Rr - 8r^2) \stackrel{(*)}{\geq} 0 \text{ and}$$

$$\therefore (2R - 4r)s^2 - 2rs^2 - r(4R^2 - 31Rr - 8r^2)$$

Gerretsen

$$\stackrel{\leq}{\Leftrightarrow} (2R - 4r)(16Rr - 5r^2) - 2r(4R^2 + 4Rr + 3r^2) - r(4R^2 - 31Rr - 8r^2)$$

∴ in order to prove (*), it suffices to prove :

$$(2R - 4r)(16Rr - 5r^2) - 2r(4R^2 + 4Rr + 3r^2) - r(4R^2 - 31Rr - 8r^2) \geq 0$$

$$\Leftrightarrow 20R^2 - 51Rr + 22r^2 \geq 0 \Leftrightarrow (R - 2r)(20R - 11r) \geq 0 \rightarrow \text{true}$$

$$\therefore R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{2(2R-r)}{R+2r} \text{ and via Nesbitt, } \frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

$$\therefore \left[\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{2(2R-r)}{R+2r} \right]$$

$$\begin{aligned} & \sum a^4 + abc \sum a - \sum ab(a^2 + b^2) \\ &= 2 \sum a^2 b^2 - 16r^2 s^2 + abc(2s) - \sum ab \left(\sum a^2 - c^2 \right) \\ &= 2 \left(\sum ab \right)^2 - 4abc(2s) - 16r^2 s^2 + 2abc(2s) - \left(\sum ab \right) \left(\sum a^2 \right) \\ &= \left(\sum ab \right) \left(2 \sum ab - \sum a^2 \right) - 2abc(2s) - 16r^2 s^2 \\ &= 2(s^2 + 4Rr + r^2)(s^2 + 4Rr + r^2 - s^2 + 4Rr + r^2) - 16Rrs^2 - 16r^2 s^2 \\ &\geq r^3(R - 2r) \end{aligned}$$

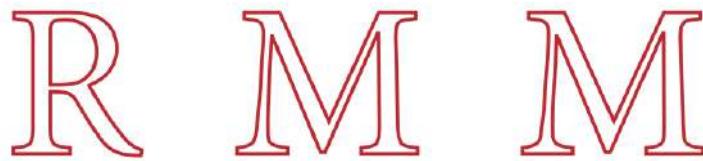
$$\Leftrightarrow 4(4R + r)^2 - r(R - 2r) \stackrel{(**)}{\geq} 12s^2 \text{ and } \because 12s^2 \stackrel{\text{Gerretsen}}{\leq} 12(4R^2 + 4Rr + 3r^2)$$

∴ in order to prove (), it suffices to prove :**

$$4(4R + r)^2 - r(R - 2r) - 12(4R^2 + 4Rr + 3r^2) \geq 0 \Leftrightarrow 16R^2 - 17Rr - 30r^2 \geq 0$$

$$\Leftrightarrow (R - 2r)(16R + 15r) \geq 0 \rightarrow \text{true} \quad \therefore R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow (**) \text{ is true}$$

$$\therefore \left[\sum a^4 + abc \sum a \geq \sum ab(a^2 + b^2) + r^3(R - 2r) \right] (\text{QED})$$



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2769. If P is any point in plane of ΔABC , then the following relationship holds:

$$\sum AP\sqrt{m_a(m_b + m_c - m_a)} \geq \sqrt{\frac{1}{2} \sum m_a(m_b + m_c - m_a)(b^2 + c^2 - a^2) + 6F^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that if $P \in \text{Int}(\Delta ABC)$,

then aPA, bPB, cPC can be the sides of a triangle (Klamkin)

→ aIA, bIB, cIC can be the sides of a triangle, where I is the incenter of ΔABC .

Since $aIA = \sqrt{2Rr}\sqrt{a(b+c-a)}$

→ $\sqrt{a(b+c-a)}, \sqrt{b(c+a-b)}, \sqrt{c(a+b-c)}$ can be the sides of triangle.

→ Since m_a, m_b, m_c can be the sides of triangle → $a_1 = \sqrt{m_a(m_b + m_c - m_a)}, b_1 = \sqrt{m_b(m_c + m_a - m_b)}$,

$c_1 = \sqrt{m_c(m_a + m_b - m_c)}$ can be the sides of a triangle Δ_1 with area F_1

such that :

$$16F_1^2 = 2 \sum \sqrt{m_a(m_b + m_c - m_a)}^2 \sqrt{m_b(m_c + m_a - m_b)}^2 - \sum \sqrt{m_a(m_b + m_c - m_a)}^4 =$$

$$= 2 \sum m_a m_b (m_c^2 + 2m_a m_b - m_a^2 - m_b^2) - \sum (m_a^4 + m_a^2 m_b^2 + m_a^2 m_c^2 + 2m_a^2 m_b m_c - 2m_a^3 m_b - 2m_a^3 m_c)$$

$$= 2 \sum m_a^2 m_b^2 - \sum m_a^4 = 16[\Delta m_a m_b m_c]^2 = 16 \left(\frac{3}{4} F \right)^2 = 9F^2 \rightarrow F_1 = \frac{3}{4} F.$$

Also, we know that for any triangles ΔABC and $\Delta_1, P \in \text{Int}(\Delta ABC)$, we have :

$$\sum a_1 \cdot AP \geq \sqrt{\frac{1}{2} \sum a_1^2 (b^2 + c^2 - a^2) + 8FF_1} \quad (\text{Bottema})$$



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$$\text{Therefore, } \sum AP\sqrt{m_a(m_b + m_c - m_a)} \\ \geq \sqrt{\frac{1}{2} \sum m_a(m_b + m_c - m_a)(b^2 + c^2 - a^2) + 6F^2}.$$

2770. If P is any point in plane of ΔABC and

$$xyz(x+y+z) > 0, x(y+z), y(z+x), z(x+y) > 0.$$

then prove that:

$$\sum \sqrt{x(y+z)}.AP \geq \sqrt{\frac{1}{2} \sum x(y+z)(b^2 + c^2 - a^2) + 4F\sqrt{xyz(x+y+z)}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} \text{We have : } & \left(\sqrt{x(y+z)} + \sqrt{y(z+x)} \right)^2 \\ &= (zx + yz) + 2xy + 2\sqrt{x(y+z)} \cdot \sqrt{y(z+x)} \stackrel{?}{\geq} \sqrt{z(x+y)}^2 \\ &\Leftrightarrow \sqrt{x^2y^2 + xyz \sum x} \geq -xy \text{ which is true because} \\ & \sqrt{x^2y^2 + xyz \sum x} \stackrel{xyz \sum x > 0}{\geq} |xy| \geq -xy \\ &\rightarrow \sqrt{x(y+z)} + \sqrt{y(z+x)} \geq \sqrt{z(x+y)} \text{ (And analogs)} \\ &\rightarrow a_1 = \sqrt{x(y+z)}, b_1 = \sqrt{y(z+x)}, c_1 = \sqrt{z(x+y)} \end{aligned}$$

can be the sides of a triangle Δ_1 with area F_1

$$\begin{aligned} \text{such that : } & 16F_1^2 = 2 \sum \sqrt{x(y+z)}^2 \cdot \sqrt{y(z+x)}^2 - \sum \sqrt{x(y+z)}^4 = \\ &= 2 \sum (x^2y^2 + xyz(x+y+z)) - \sum (x^2y^2 + 2x^2yz + x^2z^2) = 4xyz \sum x \rightarrow \\ & F_1 = \frac{1}{2} \sqrt{xyz \sum x}. \end{aligned}$$

Also, we know that for any triangles ΔABC and $\Delta_1, P \in \text{Int}(\Delta ABC)$, we have :



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$$\sum a_1 \cdot AP \geq \sqrt{\frac{1}{2} \sum a_1^2 (b^2 + c^2 - a^2) + 8F F_1} \quad (\text{Bottema})$$

Therefore, $\sum \sqrt{x(y+z)} \cdot AP \geq \sqrt{\frac{1}{2} \sum x(y+z)(b^2 + c^2 - a^2) + 4F\sqrt{xyz(x+y+z)}}$

2771. In ΔABC the following relationship holds:

$$\sum h_a \geq \sqrt{\left(\sum h_b h_c\right) \left(\sum \frac{h_a + h_b}{h_b + h_c}\right)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x = h_a$, $y = h_b$ and $z = h_c$.

We have :

$$\begin{aligned} (\sqrt{x+y} + \sqrt{y+z})^2 &= (x+z) + 2y + 2\sqrt{x+y} \cdot \sqrt{y+z} \geq z+x = \sqrt{z+x}^2 \\ \rightarrow \sqrt{x+y} + \sqrt{y+z} &\geq \sqrt{z+x} \quad (\text{And analogs}) \\ \rightarrow \sqrt{x+y}, \sqrt{y+z}, \sqrt{z+x} \end{aligned}$$

can be the sides of a triangle Δ with area F such that :

$$\begin{aligned} 16F^2 &= 2 \sum \sqrt{x+y}^2 \cdot \sqrt{y+z}^2 - \sum \sqrt{x+y}^4 = \\ 2 \sum (y^2 + xy + yz + zx) - \sum (x^2 + 2xy + y^2) &= 4 \sum xy \rightarrow F = \frac{1}{2} \sqrt{\sum xy} \end{aligned}$$

Also, we know that if ΔUVW a triangle with area S then : $\sum_{\substack{\alpha, \beta, \gamma \\ u, v, w}} \alpha \cdot u^2$

$$\geq 4S \cdot \sqrt{\sum_{u,v,w} uv}, \forall \alpha, \beta, \gamma > 0 \quad (\text{Oppenheim})$$

If we take : $\alpha = \frac{v^2}{u^2}, \beta = \frac{w^2}{v^2}, \gamma = \frac{u^2}{w^2} \rightarrow \forall \Delta UVW, \sum u^2 \geq 4S \cdot \sqrt{\sum \frac{u^2}{v^2}}$ (*)

Using (*) for triangle Δ , we get :



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$$\sum \sqrt{x+y}^2 \geq 2\sqrt{\sum xy} \cdot \sqrt{\sum \frac{\sqrt{x+y}^2}{\sqrt{y+z}^2}} \Leftrightarrow \sum x \geq \sqrt{\left(\sum xy\right)\left(\sum \frac{x+y}{y+z}\right)}.$$

$$\text{Therefore, } \sum h_a \geq \sqrt{\left(\sum h_b h_c\right)\left(\sum \frac{h_a + h_b}{h_b + h_c}\right)}.$$

2772. Let P be point in plane of ΔABC . Prove that:

$$\sum AP\sqrt{a(b+c-a)} \geq \sqrt{\frac{1}{2} \sum a(b+c-a)(b^2+c^2-a^2) + 8F^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We know that if

$P \in \text{Int}(\Delta ABC)$, then aPA, bPB, cPC can be the sides of a triangle (Klamkin)
 $\rightarrow aIA, bIB, cIC$ can be the sides of a triangle, where I is the incenter of ΔABC .

$$\begin{aligned} & \text{Since } aIA = \sqrt{2Rr}\sqrt{a(b+c-a)} \rightarrow \\ & a_1 = \sqrt{a(b+c-a)}, b_1 = \sqrt{b(c+a-b)}, c_1 = \sqrt{c(a+b-c)} \text{ can be the sides} \\ & \text{of a triangle } \Delta_1 \text{with area } F_1 \text{ such that : } 16F_1^2 \\ & = 2 \sum \sqrt{a(b+c-a)}^2 \sqrt{b(c+a-b)}^2 - \sum \sqrt{a(b+c-a)}^4 = \\ & = 2 \sum ab(c^2 + 2ab - a^2 - b^2) - \sum (a^4 + a^2b^2 + a^2c^2 + 2a^2bc - 2a^3b - 2a^3c) \\ & = 2 \sum a^2b^2 - \sum a^4 = 16F^2 \end{aligned}$$

Also, we know that for any triangles ΔABC and $\Delta_1, P \in \text{Int}(\Delta ABC)$, we have :

$$\sum a_1 \cdot AP \geq \sqrt{\frac{1}{2} \sum a_1^2(b^2+c^2-a^2) + 8FF_1} \quad (\text{Bottema})$$

$$\text{Therefore, } \sum AP\sqrt{a(b+c-a)} \geq \sqrt{\frac{1}{2} \sum a(b+c-a)(b^2+c^2-a^2) + 8F^2}.$$



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2773. In ΔABC the following relationship holds:

$$\sum_{cyc} a^3 + 3abc \geq \sum_{cyc} ab(a+b) + \frac{r^2(R-2r)}{5}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by George Florin Șerban-Romania

$$\begin{aligned} \sum a^3 &= 2s(s^2 - 3r^2 - 6Rr) \\ \sum ab(a+b) &= 2s(s^2 + r^2 - 2Rr) \\ \sum_{cyc} a^3 + 3abc &\geq \sum_{cyc} ab(a+b) + \frac{r^2(R-2r)}{5} \\ \Leftrightarrow 2s^3 - 6r^2s - 12Rrs + 12Rrs &\geq 2s^3 + 2r^2s - 4Rrs + \frac{r^2(R-2r)}{5} \\ \Leftrightarrow 8r^2s - 4Rrs + \frac{r^2(R-2r)}{5} &\leq 0 \\ \Leftrightarrow 4rs(R-2r) &\geq \frac{r^2(R-2r)}{5} \Leftrightarrow (R-2r)\left(4sr - \frac{r^2}{5}\right) \geq 0 \end{aligned}$$

From $R \geq 2r$ (*Euler*) remains to prove that: $4sr \geq \frac{r^2}{5} \Leftrightarrow s \geq \frac{r}{20}$

$$s \stackrel{\text{Mitrić}}{\geq} 3\sqrt{3}r > \frac{r}{20} \Leftrightarrow 60\sqrt{3} > 1 \text{ true.}$$

Therefore,

$$\sum_{cyc} a^3 + 3abc \geq \sum_{cyc} ab(a+b) + \frac{r^2(R-2r)}{5}$$

Solution 2 by Ertan Yıldırım-Izmir-Turkiye

$$\because \sum a^3 = 2(s^3 - 3r^2s - 6Rrs)$$

$$\sum ab = s^2 + r^2 + 4Rr$$

$$\sum_{cyc} a^3 + 3abc \geq \sum_{cyc} ab(a+b) + \frac{r^2(R-2r)}{5}$$

$$\sum_{cyc} a^3 + 3abc - \sum_{cyc} ab(a+b) \geq \frac{r^2(R-2r)}{5}$$



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$$\begin{aligned}
 Lhs &= 2(s^3 - 3r^2s - 6Rrs) + 3 \cdot 4Rrs - ab(2s - c) - ac(2s - b) - bc(2s - a) = \\
 &= 2s^3 - 6r^2s - 12Rrs + 12Rrs - 2s(ab + bc + ca) + 3abc = \\
 &= 2s^3 - 6sr^2 - 2s(s^2 + r^2 + 4Rr) + 12Rrs = \\
 &= 2s^3 = 6r^2s - 2s^3 - 2r^2s - 8Rrs + 12Rrs = 4Rrs - 8r^2s \\
 &\quad 4Rrs - 8r^2s \stackrel{(1)}{\geq} \frac{r^2(R - 2r)}{5}
 \end{aligned}$$

(1) $\Leftrightarrow 20Rrs - 40r^2s - r^2(R - 2r) \geq 0 \Leftrightarrow r(R - 2r)(20s - r) \geq 0$ true from $R \geq 2r$ (Euler).

Therefore,

$$\sum_{cyc} a^3 + 3abc \geq \sum_{cyc} ab(a + b) + \frac{r^2(R - 2r)}{5}$$

2774. In ΔABC the following relationship holds:

$$\sum \frac{1}{2 \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}} \leq \frac{1}{6\sqrt{3}} \left(\frac{R}{r} + \frac{r}{R} + 2 \right)$$

Proposed by Marian Ursărescu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 & \sum \frac{1}{2 \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}} \\
 &= \sum \frac{1}{(\cot \frac{A}{2} + \cot \frac{B}{2}) + (\cot \frac{A}{2} + \cot \frac{C}{2})} \stackrel{CBS}{\leq} \frac{1}{4} \left(\frac{1}{\cot \frac{A}{2} + \cot \frac{B}{2}} + \frac{1}{\cot \frac{A}{2} + \cot \frac{C}{2}} \right) = \\
 &= \frac{1}{2} \sum \frac{1}{\cot \frac{A}{2} + \cot \frac{B}{2}} \stackrel{s=r_a \cdot \cot \frac{A}{2}}{\cong} \frac{1}{2} \sum \frac{1}{s \left(\frac{1}{r_a} + \frac{1}{r_b} \right)} = \frac{1}{2} \sum \frac{r}{(s-a)+(s-b)} = \\
 &= \frac{r}{2} \sum \frac{1}{c} = \frac{r}{2} \cdot \frac{s^2 + r^2 + 4Rr}{4Rsr} \leq \\
 &\stackrel{\substack{Gerretsen \\ Mitrinovic}}{\leq} \frac{4R^2 + 8Rr + 4r^2}{8R \cdot 3\sqrt{3}r} = \frac{1}{6\sqrt{3}} \left(\frac{R}{r} + \frac{r}{R} + 2 \right)
 \end{aligned}$$



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$$\text{Therefore, } \sum \frac{1}{2 \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}} \leq \frac{1}{6\sqrt{3}} \left(\frac{R}{r} + \frac{r}{R} + 2 \right).$$

2775. Prove that for any acute triangle ABC :

$$1 < \frac{\sqrt{\cos A} + \sqrt{\cos B} + \sqrt{\cos C}}{\cos A + \cos B + \cos C} \leq \sqrt{2}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} \text{We have : } \sum \sqrt{\cos A} &= \sum \sqrt{\frac{\sin 2A}{2 \sin A}} \stackrel{CBS}{\geq} \sqrt{\frac{1}{2} \left(\sum \sin 2A \right) \left(\sum \frac{1}{\sin A} \right)} \\ &= \sqrt{\frac{1}{2} \cdot 4 \prod \sin A \cdot \left(\sum \frac{1}{\sin A} \right)} = \\ &= \sqrt{2 \sum \sin A \sin B} = \sqrt{2 \sum \frac{ab}{4R^2}} = \sqrt{\frac{s^2 + r^2 + 4Rr}{2R^2}} \stackrel{?}{\geq} \sqrt{2} \sum \cos A = \sqrt{2} \left(1 + \frac{r}{R} \right) \\ \leftrightarrow \frac{s^2 + r^2 + 4Rr}{2R^2} &\leq \frac{2(R+r)^2}{R^2} \leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2 \text{ (Gerretsen's inequality)} \\ \rightarrow \frac{\sqrt{\cos A} + \sqrt{\cos B} + \sqrt{\cos C}}{\cos A + \cos B + \cos C} &\leq \sqrt{2} \\ \text{Since } 0 \leq \cos A < 1 \rightarrow \cos A &< \sqrt{\cos A} \text{ (And analogs)} \rightarrow \sum \cos A < \sum \sqrt{\cos A} \\ \rightarrow 1 &< \frac{\sqrt{\cos A} + \sqrt{\cos B} + \sqrt{\cos C}}{\cos A + \cos B + \cos C} \\ \text{Therefore, } 1 &< \frac{\sqrt{\cos A} + \sqrt{\cos B} + \sqrt{\cos C}}{\cos A + \cos B + \cos C} \leq \sqrt{2}. \end{aligned}$$

2776. In ΔABC , prove that:

$$\sum \frac{a^2}{b+c} \geq \frac{a+b+c}{2} + \frac{r^2(R-2r)}{3(R^2+r^2)}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Wlog, we may assume that : $a \geq b \geq c \rightarrow a^2 \geq b^2 \geq c^2$ and

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$$



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$$\begin{aligned}
 & \stackrel{\text{Using Chebyshev}}{\Rightarrow} \sum \frac{a^2}{b+c} - \frac{a+b+c}{2} \\
 & \geq \frac{1}{3} \left(\sum a^2 \right) \left(\sum \frac{1}{b+c} \right) - \frac{1}{2} \sum a \stackrel{\text{CBS}}{\geq} \frac{1}{3} \left(\sum a^2 \right) \cdot \frac{3^2}{\sum (b+c)} - \frac{1}{2} \sum a = \\
 & = \frac{3(a^2 + b^2 + c^2)}{2(a+b+c)} - \frac{a+b+c}{2} = \frac{(a^2 + b^2 + c^2) - (ab + bc + ca)}{a+b+c} = \frac{s^2 - 3r^2 - 12Rr}{2s} \\
 & \stackrel{\substack{\text{Gerretsen} \\ \text{Mitrinovic}}}{\geq} \frac{4r(R-2r)}{3\sqrt{3}R} = \frac{8r^2(R-2r)}{3\sqrt{3}\cdot 2Rr} \stackrel{\text{AM-GM}}{\geq} \frac{8}{\sqrt{3}} \cdot \frac{r^2(R-2r)}{3(R^2+r^2)} \geq \frac{r^2(R-2r)}{3(R^2+r^2)}. \\
 & \text{Therefore, } \sum \frac{a^2}{b+c} \geq \frac{a+b+c}{2} + \frac{r^2(R-2r)}{3(R^2+r^2)}.
 \end{aligned}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \sum \frac{a^2}{b+c} &= \sum \frac{a^4}{a^2b+a^2c} \stackrel{\text{Bergstrom}}{\geq} \frac{(a^2+b^2+c^2)^2}{ab(a+b)+bc(b+c)+ca(c+a)} \\
 &= \frac{4(s^2-4Rr-r^2)^2}{2s(s^2+4Rr+r^2)-12Rrs} = \frac{2(s^2-4Rr-r^2)^2}{s(s^2-2Rr+r^2)} \\
 &\Rightarrow \sum \frac{a^2}{b+c} - \frac{a+b+c}{2} \\
 &\geq \frac{2(s^2-4Rr-r^2)^2}{s(s^2-2Rr+r^2)} - s = \frac{2(s^2-4Rr-r^2)^2-s^2(s^2-2Rr+r^2)}{s(s^2-2Rr+r^2)} \\
 &= \frac{s^4-(14Rr+5r^2)s^2+2r^2(4R+r)^2}{s(s^2-2Rr+r^2)} \stackrel{\text{Trucht}}{\geq} \frac{s^4-(14Rr+5r^2)s^2+6r^2s^2}{s(s^2-2Rr+r^2)} \\
 &= \frac{s(s^2-14Rr+r^2)}{s^2-2Rr+r^2} \\
 &\stackrel{\text{Gerretsen}}{\geq} \frac{s(2Rr-4r^2)}{s^2-2Rr+r^2} \stackrel{?}{\geq} \frac{r^2(R^2-4r^2)}{R^3+2r^3+Rr^2} \stackrel{\because R-2r \geq 0 \text{ via Euler}}{\Leftrightarrow} \frac{2s}{s^2-2Rr+r^2} \stackrel{?}{\geq} \frac{r(R+2r)}{R^3+2r^3+Rr^2} \\
 &\Leftrightarrow 4s^2(R^3+2r^3+Rr^2)^2 \stackrel{?}{\geq} \underset{(i)}{r^2(R+2r)^2(s^2-2Rr+r^2)^2} \\
 &\text{Now, RHS of (i) } \stackrel{\text{Gerretsen}}{\leq} r^2(R+2r)^2(s^2-2Rr+r^2)(4R^2+2Rr \\
 &\quad + 4r^2) \stackrel{?}{\geq} 4s^2(R^3+2r^3+Rr^2)^2
 \end{aligned}$$



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$$\Leftrightarrow \left(2(R^3 + 2r^3 + Rr^2)^2 - r^2(R + 2r)^2(2R^2 + Rr + 2r^2) \right) s^2$$

$$+ (2Rr - r^2)(2R^2 + Rr + 2r^2)r^2(R + 2r)^2 \stackrel{?}{\geq} 0$$

$$\Leftrightarrow R \left((R - 2r)(2R^4 + 4R^3r + 10R^2r^2 + 19Rr^3 + 26r^4) + 48r^5 \right) s^2$$

$$+ (2r(R - 2r) + 3r^2)(2R^2 + Rr + 2r^2)r^2(R + 2r)^2 \stackrel{?}{\geq} 0 \rightarrow \text{true}$$

$$\therefore R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow \text{(i) is true}$$

$$\therefore \sum \frac{a^2}{b+c} - \frac{a+b+c}{2}$$

$$\geq \frac{r^2(R^2 - 4r^2)}{R^3 + 2r^3 + Rr^2} \stackrel{?}{\geq} \frac{r^2(R - 2r)}{3(R^2 + r^2)} \stackrel{R-2r \geq 0}{\Leftrightarrow} \frac{R + 2r}{R^3 + 2r^3 + Rr^2} \stackrel{?}{\geq} \frac{1}{3(R^2 + r^2)}$$

$$\Leftrightarrow 3(R + 2r)(R^2 + r^2) - R^3 - 2r^3 - Rr^2 \stackrel{?}{\geq} 0$$

$$\Leftrightarrow R^3 + 2r^3 + 3R^2r + Rr^2 \stackrel{?}{\geq} 0$$

$$\rightarrow \text{true} \therefore \sum \frac{a^2}{b+c} - \frac{a+b+c}{2} \geq \frac{r^2(R - 2r)}{3(R^2 + r^2)} \Rightarrow \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}$$

$$\geq \frac{a+b+c}{2} + \frac{r^2(R - 2r)}{3(R^2 + r^2)} \quad (\text{QED})$$

2777. In ΔABC the following relationship holds:

$$\sum \cot^5 \frac{A}{2} \cdot \cot \frac{B}{2} \geq \frac{27}{2} \left(\frac{4R^2}{r^2} - \frac{5R}{r} \right)$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } x = \cot \frac{A}{2} = \frac{s}{r_a}, y = \cot \frac{B}{2} = \frac{s}{r_b}, z = \cot \frac{C}{2} = \frac{s}{r_c}.$$

$$\sum \cot^5 \frac{A}{2} \cdot \cot \frac{B}{2} = \sum x^5 y = xyz \sum \frac{x^4}{z} = xyz \sum \frac{x^6}{x^2 z} \stackrel{\text{Bergstrom}}{\geq} xyz \cdot \frac{(x^3 + y^3 + z^3)^2}{x^2 z + y^2 x + z^2 y}$$



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$$\begin{aligned}
 & \stackrel{\text{Rearrangement}}{\geq} xyz \cdot \frac{(x^3 + y^3 + z^3)^2}{x^3 + y^3 + z^3} = xyz \sum x^3 = \frac{s}{r} \cdot \sum \left(\frac{s}{r_a}\right)^3 \\
 & = \frac{s^4}{r} \left[\left(\sum \frac{1}{r_a} \right)^3 - 3 \prod \left(\frac{1}{r_a} + \frac{1}{r_b} \right) \right] = \\
 & = \frac{s^4}{r} \left(\frac{1}{r^3} - 3 \cdot \frac{4Rs^2}{(s^2r)^2} \right) = \frac{s^2(s^2 - 12Rr)}{r^4} \stackrel{\substack{\text{Gerretsen} \\ \text{Coșniță-Turtoiu}}}{\geq} \frac{27Rr}{2} \cdot \frac{(16Rr - 5r^2) - 12Rr}{r^4} \\
 & = \frac{27}{2} \cdot \frac{R(4R - 5r)}{r^2} \\
 & \text{Therefore, } \sum \cot^5 \frac{A}{2} \cdot \cot \frac{B}{2} \geq \frac{27}{2} \left(\frac{4R^2}{r^2} - \frac{5R}{r} \right).
 \end{aligned}$$

Solution 2 by Nguyen Van Canh-BenTre-Vietnam

$$\begin{aligned}
 \sum_{cyc} \cot^2 \frac{A}{2} &= \frac{s^2 - 8Rr - 2r^2}{r^2} \\
 \sum_{cyc} \cot^5 \frac{A}{2} \cot \frac{B}{2} &= \sum_{cyc} \frac{\left(\cot^2 \frac{A}{2}\right)^2}{\tan \frac{A}{2} \tan \frac{B}{2}} \stackrel{\text{CBS}}{\geq} \frac{\left(\sum \cot^2 \frac{A}{2}\right)^2}{\sum \tan \frac{A}{2} \tan \frac{B}{2}} = \left(\sum_{cyc} \cot^2 \frac{A}{2}\right)^2 = \\
 &= \left(\frac{s^2 - 8Rr - 2r^2}{r^2}\right)^2 \stackrel{\text{Blundon}}{\geq} \left(\frac{2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} - 8Rr - 2r^2}{r^2}\right)^2 \\
 &= \left(\frac{2R^2 + 2Rr - 3r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr}}{r^2}\right)^2 \stackrel{(*)}{\geq} \frac{27}{2} \left(\frac{4R^2}{r^2} - \frac{5R}{r}\right)
 \end{aligned}$$

$$(*) \Leftrightarrow 2 \left(2R^2 + 2Rr - 3r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \right)^2 \geq 27(4R^2 - 5Rr^2)$$

$$16x^4 - 32x^3 - 28x^2 + 47x + 18 \geq 8(x - 2)(2x^2 + 2x - 3)\sqrt{x^2 - 2x}; \left(x = \frac{R}{r} \geq 2\right)$$

$$(x - 2) \left(16x^3 - 28x - 9 - 8(2x^2 + 2x - 3)\sqrt{x^2 - 2x} \right) \geq 0$$

$$\because x \geq 2 \Rightarrow x - 2 \geq 0$$

$$16x^3 - 28x - 9 - 8(2x^2 + 2x - 3)\sqrt{x^2 - 2x} > 0 \Leftrightarrow$$

$$16x^3 - 28x - 9 > 8(2x^2 + 2x - 3)\sqrt{x^2 - 2x}$$

$$(16x^3 - 28x - 9)^2 > \left(8(2x^2 + 2x - 3)\sqrt{x^2 - 2x}\right)^2$$



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$$\begin{aligned}
 16x^3 - 28x - 9 &= x(16x^2 - 28) - 9 \stackrel{(x \geq 2)}{\geq} 2(64 - 28) - 9 = 63 > 0 \\
 256x^6 - 896x^4 - 288x^3 + 784x^2 + 504x + 81 &> \\
 &> 64(x^2 - 2x)(2x^2 + 2x - 3)
 \end{aligned}$$

$16x^2(40x^2 - 34x - 83) + 1656x + 81 > 0$, which is true $\forall x \geq 2 \Rightarrow (*)$ true.

2778. In $\Delta ABC, x, y, z > 0$ the following relationship holds:

$$x \cos \frac{A}{2} + y \cos \frac{B}{2} + z \cos \frac{C}{2} \leq \frac{1}{2}(xy + yz + zx) \sqrt{\frac{x+y+z}{xyz}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$x \cos \frac{A}{2} + y \cos \frac{B}{2} + z \cos \frac{C}{2} \stackrel{(*)}{\geq} \frac{1}{2}(xy + yz + zx) \sqrt{\frac{x+y+z}{xyz}}$$

$$\text{We have : } \sum x \cos \frac{A}{2} =$$

$$= \sum \sqrt{x} \cdot \sqrt{x} \cos \frac{A}{2} \stackrel{CBS}{\geq} \sqrt{\left(\sum x\right) \left(\sum x \cos^2 \frac{A}{2}\right)} \stackrel{?}{\geq} \frac{1}{2}(xy + yz + zx) \sqrt{\frac{x+y+z}{xyz}}$$

$$\begin{aligned}
 \leftrightarrow 2xyz \sum x \cdot 2 \cos^2 \frac{A}{2} &\leq (xy + yz + zx)^2 \leftrightarrow 2xyz \sum x(1 + \cos A) \\
 &\leq (xy + yz + zx)^2
 \end{aligned}$$

$$\leftrightarrow 2xyz \sum x \cos A \leq \sum x^2 y^2$$

$$\leftrightarrow 2x^2 yz \cos[\pi - (B + C)] + 2xy^2 z \cos B + 2xyz^2 \cos C \leq \sum x^2 y^2$$

$$\begin{aligned}
 \leftrightarrow 2x^2 yz (\sin B \sin C - \cos B \cos C) + 2xy^2 z \cos B + 2xyz^2 \cos C &\leq (xy)^2 (\sin^2 B + \cos^2 B) + (yz)^2 + (zx)^2 (\sin^2 C + \cos^2 C) \\
 &\leq (xy)^2 + (yz)^2 + (zx)^2
 \end{aligned}$$

$$\begin{aligned}
 \leftrightarrow 0 &\leq (xy \sin B - zx \sin C)^2 + (xy \cos B + zx \cos C)^2 - 2yz(xy \cos B + zx \cos C) \\
 &\quad + (yz)^2
 \end{aligned}$$

$\leftrightarrow 0 \leq (xy \sin B - zx \sin C)^2 + [(xy \cos B + zx \cos C) - yz]^2$ which is true.



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$$\text{Therefore, } x \cos \frac{A}{2} + y \cos \frac{B}{2} + z \cos \frac{C}{2} \leq \frac{1}{2} (xy + yz + zx) \sqrt{\frac{x+y+z}{xyz}}.$$

2779. Let P be point in plane of ΔABC and $t > 1$, then holds:

$$(PA \cdot PB)^t + (PB \cdot PC)^t + (PC \cdot PA)^t \geq \frac{(abc)^t}{\left(a^{\frac{t}{t-1}} + b^{\frac{t}{t-1}} + c^{\frac{t}{t-1}}\right)^{t-1}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder, we have :

$$\left(\sum_{cyc} (PB \cdot PC)^t\right) \left(\sum_{cyc} a^{\frac{t}{t-1}}\right)^{t-1} \geq \left(\sum_{cyc} \sqrt[t]{(PB \cdot PC)^t \cdot \left(a^{\frac{t}{t-1}}\right)^{t-1}}\right)^t = \left(\sum_{cyc} a \cdot PB \cdot PC\right)^t$$

By Hayashi's inequality, we have :

$$\begin{aligned} \sum_{cyc} a \cdot PB \cdot PC &\geq abc \\ \rightarrow \left(\sum_{cyc} (PB \cdot PC)^t\right) \left(\sum_{cyc} a^{\frac{t}{t-1}}\right)^{t-1} &\geq (abc)^t \end{aligned}$$

$$\text{Therefore, } (PA \cdot PB)^t + (PB \cdot PC)^t + (PC \cdot PA)^t \geq \frac{(abc)^t}{\left(a^{\frac{t}{t-1}} + b^{\frac{t}{t-1}} + c^{\frac{t}{t-1}}\right)^{t-1}}.$$

2780. In ΔABC the following relationship holds:

$$\begin{aligned} &\left\{2 \left(\cos^2 \frac{B}{7} + \cos^2 \frac{C}{7}\right) - \cos^2 \frac{A}{7}\right\} \prod_{cyc} \left(\cos \frac{B}{7} + \cos \frac{C}{7} - \cos \frac{A}{7}\right) \leq \\ &\leq \cos^2 \frac{B}{7} \cdot \cos^2 \frac{C}{7} \left(\cos \frac{A}{7} + \cos \frac{B}{7} + \cos \frac{C}{7}\right) \end{aligned}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let's prove that $\cos \frac{A}{7}, \cos \frac{B}{7}, \cos \frac{C}{7}$ can be the sides of triangle.



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Since $0 < A, B, C < \pi \rightarrow 0 < \frac{A}{7}, \frac{B}{7}, \frac{C}{7} < \frac{\pi}{7} < \frac{\pi}{3}$ and x

$\rightarrow \cos x$ is a decreasing function on $(0, \frac{\pi}{2})$

$$\begin{aligned} \rightarrow \cos \frac{A}{7}, \cos \frac{B}{7}, \cos \frac{C}{7} &> \cos \frac{\pi}{3} = \frac{1}{2} \rightarrow \cos \frac{A}{7} + \cos \frac{B}{7} > \frac{1}{2} + \frac{1}{2} \\ &= 1 \stackrel{1 \geq \cos x, \forall x}{\triangleright} \cos \frac{C}{7} \quad (\text{And analogs}) \end{aligned}$$

$\rightarrow \cos \frac{A}{7}, \cos \frac{B}{7}, \cos \frac{C}{7}$ can be the sides of triangle.

\rightarrow It suffices to prove that

$$\begin{aligned} : \{2(b^2 + c^2) - a^2\} \prod_{cyc} (b + c - a) &\stackrel{(*)}{\leq} b^2 c^2 (a + b + c), \forall \Delta ABC \\ (*) \leftrightarrow 4m_a^2 \cdot 8sr^2 &\leq \frac{(4Rrs)^2}{a^2} \cdot 2s \leftrightarrow m_a^2 \leq \left(\frac{Rs}{a}\right)^2 \leftrightarrow m_a \\ &\leq \frac{Rs}{a} \quad (\text{Panaitopol's inequality}) \rightarrow (*) \text{ is true.} \end{aligned}$$

$$\begin{aligned} \text{Therefore, } \{2\left(\cos^2 \frac{B}{7} + \cos^2 \frac{C}{7}\right) - \cos^2 \frac{A}{7}\} \prod_{cyc} \left(\cos \frac{B}{7} + \cos \frac{C}{7} - \cos \frac{A}{7}\right) \\ \leq \cos^2 \frac{B}{7} \cdot \cos^2 \frac{C}{7} \left(\cos \frac{A}{7} + \cos \frac{B}{7} + \cos \frac{C}{7}\right) \end{aligned}$$

2781. If x, y, z are real numbers, then in ΔABC holds:

$$x^2 + y^2 + z^2 \geq \left(yz \sin \frac{A}{2} + zx \sin \frac{B}{2} + xy \sin \frac{C}{2}\right) \sec \frac{\pi}{3}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} x^2 + y^2 + z^2 &\stackrel{(*)}{\geq} \left(yz \sin \frac{A}{2} + zx \sin \frac{B}{2} + xy \sin \frac{C}{2}\right) \sec \frac{\pi}{3} \\ (*) \leftrightarrow x^2 + y^2 + z^2 &\geq 2yz \sin \left(\frac{\pi}{2} - \frac{B+C}{2}\right) + 2zx \sin \frac{B}{2} + 2xy \sin \frac{C}{2} \\ \leftrightarrow x^2 + y^2 + z^2 &\geq 2yz \cos \left(\frac{B}{2} + \frac{C}{2}\right) + 2zx \sin \frac{B}{2} + 2xy \sin \frac{C}{2} \end{aligned}$$



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$$\begin{aligned}
 &\leftrightarrow x^2 + y^2 \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) + z^2 \left(\cos^2 \frac{B}{2} + \sin^2 \frac{B}{2} \right) \\
 &\qquad\qquad\qquad \geq 2yz \left(\cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2} \right) + 2zx \sin \frac{B}{2} + 2xy \sin \frac{C}{2} \\
 &\leftrightarrow x^2 - 2x \left(y \sin \frac{C}{2} + z \sin \frac{B}{2} \right) + \left(y \sin \frac{C}{2} + z \sin \frac{B}{2} \right)^2 + \left(y \cos \frac{C}{2} - z \cos \frac{B}{2} \right)^2 \geq 0 \\
 &\leftrightarrow \left[x - \left(y \sin \frac{C}{2} + z \sin \frac{B}{2} \right) \right]^2 + \left(y \cos \frac{C}{2} - z \cos \frac{B}{2} \right)^2 \geq 0 \text{ which is true.} \\
 &\text{Therefore, } x^2 + y^2 + z^2 \geq \left(yz \sin \frac{A}{2} + zx \sin \frac{B}{2} + xy \sin \frac{C}{2} \right) \sec \frac{\pi}{3}.
 \end{aligned}$$

2782. In ΔABC , prove that:

$$\sum \frac{a^3}{b^2 + c^2} \geq \frac{a + b + c}{2} + \frac{r^2(R^2 - 4r^2)}{R^3 + Rr^2 + 2r^3}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \sum \frac{a^3}{b^2 + c^2} &= \sum \frac{a^6}{a(a^2b^2 + c^2a^2)} \stackrel{\text{Hölder}}{\leq} \frac{(a^2 + b^2 + c^2)^3}{(a + b + c) \cdot 2(a^2b^2 + b^2c^2 + c^2a^2)} \geq \\
 &\stackrel{(\Sigma x)^2 \geq 3\Sigma xy}{\geq} \frac{(a^2 + b^2 + c^2) \cdot 3(a^2b^2 + b^2c^2 + c^2a^2)}{2(a + b + c)(a^2b^2 + b^2c^2 + c^2a^2)} = \frac{3(a^2 + b^2 + c^2)}{2(a + b + c)} \\
 &\rightarrow \sum \frac{a^3}{b^2 + c^2} - \frac{a + b + c}{2} \geq \frac{3(a^2 + b^2 + c^2)}{2(a + b + c)} - \frac{a + b + c}{2} \\
 &\qquad\qquad\qquad = \frac{(a^2 + b^2 + c^2) - (ab + bc + ca)}{a + b + c} = \frac{s^2 - 3r^2 - 12Rr}{2s} \geq \\
 &\stackrel{\text{Gerretsen}}{\geq} \frac{4r(R - 2r)}{3\sqrt{3}R} \geq \frac{4r(R - 2r)}{3 \cdot 2R} \stackrel{?}{\geq} \frac{r^2(R^2 - 4r^2)}{R^3 + Rr^2 + 2r^3} \leftrightarrow 2(R^3 + Rr^2 + 2r^3) \\
 &\qquad\qquad\qquad \geq 3R(R + 2r) \\
 &\leftrightarrow (R - 2r)(2R^2 + r(R - 2r)) \\
 &\qquad\qquad\qquad \geq 0 \text{ which is true from Euler's inequality } (R \geq 2r).
 \end{aligned}$$

$$\text{Therefore, } \sum \frac{a^3}{b^2 + c^2} \geq \frac{a + b + c}{2} + \frac{r^2(R^2 - 4r^2)}{R^3 + Rr^2 + 2r^3}.$$



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Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \sum \frac{a^3}{b^2 + c^2} &= \sum \frac{a^4}{ab^2 + ac^2} \stackrel{\text{Bergstrom}}{\geq} \frac{(a^2 + b^2 + c^2)^2}{ab(a+b) + bc(b+c) + ca(c+a)} \\
 &= \frac{4(s^2 - 4Rr - r^2)^2}{2s(s^2 + 4Rr + r^2) - 12Rrs} = \frac{2(s^2 - 4Rr - r^2)^2}{s(s^2 - 2Rr + r^2)} \\
 &\Rightarrow \sum \frac{a^3}{b^2 + c^2} - \frac{a+b+c}{2} \\
 &\geq \frac{2(s^2 - 4Rr - r^2)^2}{s(s^2 - 2Rr + r^2)} - s = \frac{2(s^2 - 4Rr - r^2)^2 - s^2(s^2 - 2Rr + r^2)}{s(s^2 - 2Rr + r^2)} \\
 &= \frac{s^4 - (14Rr + 5r^2)s^2 + 2r^2(4R + r)^2}{s(s^2 - 2Rr + r^2)} \stackrel{\text{Trucht}}{\geq} \frac{s^4 - (14Rr + 5r^2)s^2 + 6r^2s^2}{s(s^2 - 2Rr + r^2)} \\
 &= \frac{s(s^2 - 14Rr + r^2)}{s^2 - 2Rr + r^2} \\
 &\stackrel{\text{Gerretsen}}{\geq} \frac{s(2Rr - 4r^2)}{s^2 - 2Rr + r^2} \stackrel{?}{\geq} \frac{r^2(R^2 - 4r^2)}{R^3 + 2r^3 + Rr^2} \stackrel{\because R - 2r \geq 0 \text{ via Euler}}{\Leftrightarrow} \frac{2s}{s^2 - 2Rr + r^2} \stackrel{?}{>} \frac{r(R + 2r)}{R^3 + 2r^3 + Rr^2} \\
 &\Leftrightarrow 4s^2(R^3 + 2r^3 + Rr^2)^2 \stackrel{?}{\geq} \underset{(i)}{r^2(R + 2r)^2(s^2 - 2Rr + r^2)^2} \\
 &\text{Now, RHS of (i)} \stackrel{\text{Gerretsen}}{\leq} r^2(R + 2r)^2(s^2 - 2Rr + r^2)(4R^2 + 2Rr \\
 &\quad + 4r^2) \stackrel{?}{\geq} 4s^2(R^3 + 2r^3 + Rr^2)^2 \\
 &\Leftrightarrow \left(2(R^3 + 2r^3 + Rr^2)^2 - r^2(R + 2r)^2(2R^2 + Rr + 2r^2) \right) s^2 \\
 &\quad + (2Rr - r^2)(2R^2 + Rr + 2r^2)r^2(R + 2r)^2 \stackrel{?}{>} 0 \\
 &\Leftrightarrow R \left((R - 2r)(2R^4 + 4R^3r + 10R^2r^2 + 19Rr^3 + 26r^4) + 48r^5 \right) s^2 \\
 &\quad + (2r(R - 2r) + 3r^2)(2R^2 + Rr + 2r^2)r^2(R + 2r)^2 \stackrel{?}{>} 0 \rightarrow \text{true}
 \end{aligned}$$

Euler
∴ R ≥ 2r ⇒ (i) is true

$$\begin{aligned}
 \therefore \sum \frac{a^3}{b^2 + c^2} - \frac{a+b+c}{2} &\geq \frac{r^2(R^2 - 4r^2)}{R^3 + 2r^3 + Rr^2} \Rightarrow \sum \frac{a^3}{b^2 + c^2} \\
 &\geq \frac{a+b+c}{2} + \frac{r^2(R^2 - 4r^2)}{R^3 + 2r^3 + Rr^2} \quad (\text{QED})
 \end{aligned}$$



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2783. If $M \in R^3, x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{MA^2}{yz} + \frac{MB^2}{zx} + \frac{MC^2}{xy} \geq \left(\frac{a+b+c}{x+y+z} \right)^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \text{We have : } & (x\overrightarrow{MA} + y\overrightarrow{MB} + z\overrightarrow{MC})^2 \geq 0 \leftrightarrow \sum x^2 MA^2 + \sum yz \cdot 2\overrightarrow{MB} \cdot \overrightarrow{MC} \geq 0 \\
 \leftrightarrow & \sum x^2 MA^2 + \sum yz \cdot (MB^2 + MC^2 - BC^2) \geq 0 \leftrightarrow \\
 & \sum (x^2 + xy + xz) MA^2 \geq \sum yza^2 \\
 \leftrightarrow & (\sum x)(\sum x \cdot MA^2) \geq \sum yza^2 \leftrightarrow (\sum x) \cdot xyz \sum \frac{MA^2}{yz} \geq xyz \sum \frac{a^2}{x} \\
 \leftrightarrow & (\sum x)^2 \cdot \sum \frac{MA^2}{yz} \geq (\sum x) \left(\sum \frac{a^2}{x} \right) \stackrel{CBS}{\geq} (\sum a)^2 \\
 \text{Therefore, } & \frac{MA^2}{yz} + \frac{MB^2}{zx} + \frac{MC^2}{xy} \geq \left(\frac{a+b+c}{x+y+z} \right)^2.
 \end{aligned}$$

2784. For any arbitrary point P in the plane of ΔABC ,

$$AP \cdot \cos \frac{A}{2} + BP \cdot \cos \frac{B}{2} + CP \cdot \cos \frac{C}{2} \geq s$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} &= \frac{1}{2}(1 + \cos A + 1 + \cos B - 1 - \cos C) \\
 &= \frac{1}{2} \left(2\sin^2 \frac{C}{2} + 2\sin \frac{C}{2} \cos \frac{A-B}{2} \right) > 0 \because 0 < \cos \frac{A-B}{2} \leq 1 \Rightarrow \cos^2 \frac{C}{2} \\
 &< \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} \\
 < \left(\cos \frac{A}{2} + \cos \frac{B}{2} \right)^2 &\Rightarrow \cos \frac{A}{2} + \cos \frac{B}{2} - \cos \frac{C}{2} > 0 \text{ and analogs} \\
 \Rightarrow \cos \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2} &\text{ form sides of a triangle} \rightarrow (1) \text{ with area } F_1 \\
 = \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} - \sum_{\text{cyc}} \cos^4 \frac{A}{2}}
 \end{aligned}$$



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$$\begin{aligned}
 &= \frac{1}{4} \sqrt{4 \sum_{\text{cyc}} \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} - \left(\sum_{\text{cyc}} \cos^2 \frac{A}{2} \right)^2} \\
 &= \frac{1}{4} \sqrt{4 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} \sum_{\text{cyc}} \sec^2 \frac{A}{2} - \frac{1}{4} \left(\sum_{\text{cyc}} (1 + \cos A) \right)^2} \\
 &= \frac{1}{4} \sqrt{\left(\frac{s^2}{4R^2} \right) \frac{(4R+r)^2 + s^2}{s^2} - \frac{(4R+r)^2}{4R^2}} = \frac{s}{8R} \Rightarrow 8F_1 \stackrel{(i)}{\cong} \frac{s}{R}
 \end{aligned}$$

Via Bottema, for any arbitrary point P in the plane of $\triangle ABC$ and for any arbitrary $\triangle A'B'C'$, $(AP.a' + BP.b' + CP.c')^2$

$$\begin{aligned}
 &\geq \frac{1}{2} (a^2(b'^2 + c'^2 - a'^2) + b^2(c'^2 + a'^2 - b'^2) + c^2(a'^2 + b'^2 - c'^2)) \\
 &\quad + 8[ABC][A'B'C'] \therefore \text{via (1) and choosing } a' = \cos \frac{A}{2}, b' = \cos \frac{B}{2}, c' = \cos \frac{C}{2}, \\
 &\left(AP \cdot \cos \frac{A}{2} + BP \cdot \cos \frac{B}{2} + CP \cdot \cos \frac{C}{2} \right)^2 \\
 &\geq \frac{1}{2} \sum_{\text{cyc}} a^2 \left(\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} - \cos^2 \frac{A}{2} \right) + 8rsF_1 \stackrel{\text{via (i)}}{\cong} \frac{1}{2} \sum_{\text{cyc}} a^2 \left(\frac{s(s-b)}{ca} + \frac{s(s-c)}{ab} - \frac{s(s-a)}{bc} \right) \\
 &\quad + rs \left(\frac{s}{R} \right) \\
 &= \frac{s}{8Rrs} \left(2s \sum_{\text{cyc}} a^2(s-a) - \sum_{\text{cyc}} a^2(b^2 + c^2 - a^2) \right) + \frac{rs^2}{R} \\
 &\quad = \frac{1}{8Rr} \left(4s^2(s^2 - 4Rr - r^2) - 4s^2(s^2 - 6Rr - 3r^2) - 2 \sum_{\text{cyc}} a^2 b c \cos A \right) + \frac{rs^2}{R} \\
 &= \frac{1}{8Rr} \left(4s^2(2Rr + 2r^2) - 8R^2rs \sum_{\text{cyc}} \sin 2A \right) + \frac{rs^2}{R} \\
 &\quad = \frac{1}{8Rr} \left(4s^2(2Rr + 2r^2) - 8R^2rs(2\sin C \cos(A-B) - 2\sin C \cos(A+B)) \right) + \frac{rs^2}{R} \\
 &= \frac{1}{8Rr} \left(4s^2(2Rr + 2r^2) - 8R^2rs \cdot 4 \prod_{\text{cyc}} \sin A \right) + \frac{rs^2}{R} \\
 &\quad = \frac{1}{8Rr} \left(4s^2(2Rr + 2r^2) - 8R^2rs \cdot 4 \left(\frac{4Rrs}{8R^3} \right) \right) + \frac{rs^2}{R} \\
 &\quad = \frac{8Rrs^2 + 8r^2s^2 - 16r^2s^2 + 8r^2s^2}{8Rr} = s^2 \\
 \Rightarrow AP \cdot \cos \frac{A}{2} + BP \cdot \cos \frac{B}{2} + CP \cdot \cos \frac{C}{2} &\geq s \text{ for any arbitrary point P in the plane of } \triangle ABC \text{ (QED)}
 \end{aligned}$$



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2785. In ΔABC , prove that:

$$\sum \frac{a^3}{b+c} \geq \frac{(a+b+c)^2}{6} + \frac{r^2(R-2r)}{R+r}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Wlog, we may assume that : $a \geq b \geq c \rightarrow a^3 \geq b^3 \geq c^3$ and

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$$

$$\begin{aligned} & \stackrel{\text{Using Chebyshev}}{\Rightarrow} \sum \frac{a^3}{b+c} - \frac{(a+b+c)^2}{6} \\ & \geq \frac{1}{3} \left(\sum a^3 \right) \left(\sum \frac{1}{b+c} \right) - \frac{1}{6} \left(\sum a \right)^2 \stackrel{\text{CBS}}{\geq} \frac{1}{3} \left(\sum a^3 \right) \cdot \frac{3^2}{\sum (b+c)} - \frac{1}{6} \left(\sum a \right)^2 = \\ & = \frac{3(a^3 + b^3 + c^3)}{2(a+b+c)} - \frac{(a+b+c)^2}{6} = \frac{9(a^3 + b^3 + c^3) - (a+b+c)^3}{6(a+b+c)} \\ & = \frac{9(s^2 - 3r^2 - 6Rr) - 4s^2}{6} = \\ & = \frac{5s^2 - 27r^2 - 54Rr}{6} \stackrel{\text{Gerretsen}}{\geq} \\ & \frac{26r(R-2r)}{6} = \frac{13r^2(R-2r)}{3r} \stackrel{\text{Euler}}{\geq} \frac{13r^2(R-2r)}{R+r} \geq \frac{r^2(R-2r)}{R+r}. \\ & \text{Therefore, } \sum \frac{a^3}{b+c} \geq \frac{(a+b+c)^2}{6} + \frac{r^2(R-2r)}{R+r}. \end{aligned}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum \frac{a^3}{b+c} &= \sum \frac{a^4}{ab+ac} \stackrel{\text{Bergstrom}}{\geq} \frac{(a^2+b^2+c^2)^2}{2(ab+bc+ca)} = \frac{2(s^2-4Rr-r^2)^2}{s^2+4Rr+r^2} \\ &\Rightarrow \sum \frac{a^3}{b+c} - \frac{(a+b+c)^2}{6} \geq \frac{2(s^2-4Rr-r^2)^2}{s^2+4Rr+r^2} - \frac{2s^2}{3} \\ &= \frac{6(s^2-4Rr-r^2)^2 - 2s^2(s^2+4Rr+r^2)}{3(s^2+4Rr+r^2)} \\ &= \frac{2(2s^4 - (28Rr+7r^2)s^2 + 3r^2(4R+r)^2)}{3(s^2+4Rr+r^2)} \stackrel{\text{Trucht}}{\geq} \frac{2(2s^4 - (28Rr+7r^2)s^2 + 9r^2s^2)}{3(s^2+4Rr+r^2)} \\ &= \frac{2s^2(2s^2 - 28Rr + 2r^2)}{3(s^2+4Rr+r^2)} \end{aligned}$$



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$$\begin{aligned} \stackrel{\text{Gerretsen}}{\geq} \frac{2s^2(4Rr - 8r^2)}{3(s^2 + 4Rr + r^2)} &\stackrel{?}{\geq} \frac{r^2(R - 2r)}{R + r} \because R - 2r \geq 0 \text{ via Euler} \\ &\Leftrightarrow \stackrel{?}{\Rightarrow} 8(R + r)s^2 \stackrel{?}{\geq} 3r(s^2 + 4Rr + r^2) \\ &\Leftrightarrow (8R + 5r)s^2 \stackrel{?}{\geq} \stackrel{(i)}{3r(4Rr + r^2)} \end{aligned}$$

$$\begin{aligned} \text{Now, RHS of (i)} &\stackrel{\text{Gerretsen}}{\geq} (8R + 5r)(16Rr - 5r^2) \stackrel{?}{\geq} \stackrel{\text{Euler}}{3r(4Rr + r^2)} \\ &\Leftrightarrow 128R^2 + 14Rr + 14r(R - 2r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{?}{\geq} 2r \Rightarrow (i) \text{ is true} \\ &\therefore \sum \frac{a^3}{b+c} - \frac{(a+b+c)^2}{6} \geq \frac{r^2(R-2r)}{R+r} \Rightarrow \sum \frac{a^3}{b+c} \\ &\geq \frac{(a+b+c)^2}{6} + \frac{r^2(R-2r)}{R+r} \quad (\text{QED}) \end{aligned}$$

2786. In ΔABC the following relationship holds:

$$\prod_{cyc} \left(\cot^2 \frac{A}{2} + \cot \frac{A}{2} \cot \frac{B}{2} + \cot^2 \frac{B}{2} \right) \geq \left(\frac{s}{r} \right)^4$$

Proposed by Marian Ursărescu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Lemma : } \forall x, y > 0, x^2 + xy + y^2 \stackrel{(*)}{\geq} \frac{3}{4}(x+y)^2.$$

Proof : $(*) \Leftrightarrow 4(x^2 + xy + y^2) \geq 3(x^2 + 2xy + y^2) \Leftrightarrow (x-y)^2 \geq 0$ which is true
 $\rightarrow (*)$ is true.

$$\begin{aligned} \text{Also, we know that : } \cot \frac{A}{2} &= \frac{s}{r_a}, \prod_{cyc} (r_a + r_b) = 4Rs^2 \text{ and } r_a r_b r_c = s^2 r. \\ \rightarrow \prod_{cyc} \left(\cot^2 \frac{A}{2} + \cot \frac{A}{2} \cot \frac{B}{2} + \cot^2 \frac{B}{2} \right) &\stackrel{(*)}{\geq} \prod_{cyc} \left[\frac{3}{4} \left(\cot \frac{A}{2} + \cot \frac{B}{2} \right)^2 \right] \\ &= \frac{27}{64} \left(\prod_{cyc} \left(\frac{s}{r_a} + \frac{s}{r_b} \right) \right)^2 = \\ &= \frac{27}{64} \left(s^3 \prod_{cyc} \frac{r_a + r_b}{r_a r_b} \right)^2 = \frac{27}{64} \left(s^3 \cdot \frac{4Rs^2}{(s^2 r)^2} \right)^2 = \frac{27}{4} \left(\frac{Rs}{r^2} \right)^2 \stackrel{\text{Mitrićević}}{\geq} \end{aligned}$$

$$\frac{27}{4} \left(\frac{2s \cdot s}{3\sqrt{3} \cdot r^2} \right)^2 = \left(\frac{s}{r} \right)^4.$$

Therefore, $\prod_{cyc} \left(\cot^2 \frac{A}{2} + \cot \frac{A}{2} \cot \frac{B}{2} + \cot^2 \frac{B}{2} \right) \geq \left(\frac{s}{r} \right)^4.$

2787. If $M \in R^3, x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{MA^2}{x} + \frac{MB^2}{y} + \frac{MC^2}{z} \geq xyz \left(\frac{a+b+c}{xy+yz+zx} \right)^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We have : (yz\overrightarrow{MA} + zx\overrightarrow{MB} + xy\overrightarrow{MC})^2 \geq 0 \leftrightarrow$$

$$\sum y^2 z^2 MA^2 + \sum zx \cdot xy \cdot 2\overrightarrow{MB} \cdot \overrightarrow{MC} \geq 0$$

$$\leftrightarrow \sum y^2 z^2 MA^2 + xyz \sum x(MB^2 + MC^2 - BC^2) \geq 0$$

$$\leftrightarrow \sum (y^2 z^2 + xy^2 z + xyz^2) MA^2 \geq xyz \sum xa^2$$

$$\leftrightarrow \left(\sum xy \right) \left(\sum yz \cdot MA^2 \right) \geq xyz \sum xa^2 \leftrightarrow \left(\sum xy \right) \cdot xyz \sum \frac{MA^2}{x} \geq (xyz)^2 \sum \frac{a^2}{yz}$$

$$\leftrightarrow \left(\sum xy \right)^2 \cdot \sum \frac{MA^2}{x} \geq xyz \cdot \left(\sum yz \right) \left(\sum \frac{a^2}{yz} \right) \stackrel{CBS}{\geq} xyz \left(\sum a \right)^2$$

$$Therefore, \quad \frac{MA^2}{x} + \frac{MB^2}{y} + \frac{MC^2}{z} \geq xyz \left(\frac{a+b+c}{xy+yz+zx} \right)^2.$$

2788. In ΔABC the following relationship holds:

$$\frac{a^3}{m_a^2} + \frac{b^3}{m_b^2} + \frac{c^3}{m_c^2} \geq \frac{8s}{3}$$

Proposed by Marin Chirciu-Romania



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Solution 1 by George Florin Ţerban-Romania

$$\sum_{cyc} \frac{a^3}{m_a^2} = \sum_{cyc} \frac{a^4}{am_a^2} \stackrel{\text{Holder}}{\geq} \frac{(a+b+c)^4}{3^2 \sum am_a^2} = \frac{32s^3}{9(s^2 + 5r^2 + 2Rr)} \stackrel{(1)}{\geq} \frac{8s}{3}$$

$$(1) \Leftrightarrow \frac{4s^2}{3(s^2 + 5r^2 + 2Rr)} \geq 1 \Leftrightarrow 4s^2 \geq 3s^2 + 15r^2 + 6Rr \\ \Leftrightarrow s^2 \geq 15r^2 + 6Rr; (2)$$

$$\text{But } s^2 \geq 16Rr - 5r^2 \text{ (Gerretsen); (3)}$$

From (2), (3) remains to prove that

$$16Rr - 5r^2 \geq 15r^2 + 6Rr \Leftrightarrow 10Rr \geq 20r^2 \Leftrightarrow R \geq 2r \text{ (Euler).}$$

Solution 2 by Marian Ursărescu-Romania

Lemma. In any triangle ABC , we have:

$$m_a \leq \frac{a^2 + b^2 + c^2}{2\sqrt{3}a}; (*)$$

Proof. Because $4m_a^2 + 3a^2 \geq 4\sqrt{3}a \cdot m_a$ then $2b^2 + 2c^2 - a^2 + 3a^2 \geq 4\sqrt{3}a \cdot m_a \Leftrightarrow$

(*) is true. Now, we have:

$$\frac{1}{m_a^2} \geq \frac{12a^2}{(a^2 + b^2 + c^2)^2} \Rightarrow \frac{a^3}{m_a^2} \geq \frac{12a^5}{(a^2 + b^2 + c^2)^2}$$

We must show that:

$$\frac{12}{(a^2 + b^2 + c^2)^2} (a^5 + b^5 + c^5) \geq \frac{4(a+b+c)}{3} \Leftrightarrow \\ 9(a^5 + b^5 + c^5) \geq (a+b+c)(a^2 + b^2 + c^2)^2; (1)$$

WLOG, let $a < b < c$ then $a^2 \leq b^2 \leq c^2$ and hence, $a^3 \leq b^3 \leq c^3$.

From Chebyshev's Inequality, we have:

$$\left. \begin{aligned} (a+b+c)(a^2 + b^2 + c^2) &\leq 3(a^3 + b^3 + c^3) \\ (a^3 + b^3 + c^3)(a^2 + b^2 + c^2) &\leq 3(a^5 + b^5 + c^5) \end{aligned} \right\} \Rightarrow \\ (a+b+c)(a^2 + b^2 + c^2) \leq 9(a^5 + b^5 + c^5) \Rightarrow (1) \text{ its true.}$$



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2789. In ΔABC the following relationship holds:

$$\frac{(h_a + h_b + h_c)^2}{a^2 + b^2 + c^2} \leq 2 + \left(\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\begin{aligned} \frac{(h_a + h_b + h_c)^2}{a^2 + b^2 + c^2} &= \frac{\left(2F \sum \frac{1}{a}\right)^2}{\sum a} = \frac{4F^2 \cdot \frac{(\sum ab)^2}{(abc)^2}}{\sum a^2} = \frac{4F^2 \cdot \frac{(\sum ab)^2}{16R^2 F^2}}{\sum a^2} \leq \\ &\leq \frac{1}{4R^2} \cdot \frac{(\sum a^2)^2}{\sum a^2} = \frac{1}{4R^2} \sum a^2 = \frac{2(s^2 - 4Rr - r^2)}{4R^2} \stackrel{\text{Gerretsen}}{\leq} \\ &\stackrel{\text{Gerretsen}}{\leq} \frac{2(4R^2 + 4Rr + 3r^2 - 4Rr - r^2)}{4R^2} = \frac{4R^2 + 2r^2}{2R^2} = 2 + \left(\frac{r}{R}\right)^2 \end{aligned}$$

Solution 2 by Aggeliki Papaspyropoulou-Greece

$$\begin{aligned} h_a + h_b + h_c &= 2F \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \\ (h_a + h_b + h_c)^2 &= \frac{4F^2(ab + bc + ca)^2}{a^2 b^2 c^2} = \frac{4a^2 b^2 c^2 (ab + bc + ca)^2}{16a^2 b^2 c^2 \cdot R^2} = \\ &= \frac{(ab + bc + ca)^2}{4R^2} \end{aligned}$$

So, it is enough to prove:

$$\begin{aligned} \frac{(ab + bc + ca)^2}{4R^2(a^2 + b^2 + c^2)} &\leq 2 + \frac{r^2}{R^2} \Leftrightarrow \\ (ab + bc + ca)^2 &\leq 8R^2(a^2 + b^2 + c^2) + 4r^2(a^2 + b^2 + c^2) \Leftrightarrow \\ (ab + bc + ca)^2 &\leq (a^2 + b^2 + c^2)(8R^2 + 4r^2) \Leftrightarrow \\ (ab + bc + ca)(ab + bc + ca) &\leq (a^2 + b^2 + c^2)(8R^2 + 4r^2); (1) \end{aligned}$$

$\because ab + bc + ca = a^2 + b^2 + c^2$, by (1) it is enough to prove:

$$\begin{aligned} ab + bc + ca &\leq 8R^2 + 4r^2 \Leftrightarrow s^2 + r^2 + 4Rr \leq 8R^2 + 4r^2 \Leftrightarrow \\ s^2 &\leq 8R^2 - 4Rr + 3r^2; (2) \end{aligned}$$

But: $s^2 \leq 4R^2 + 4Rr + 3r^2$; (Gerretsen); (3)

From (2)&(3) we must show:



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$4R^2 + 4Rr + 3r^2 \leq 8R^2 - 4Rr + 3r^2 \Leftrightarrow 4R(R - 2r) \geq 0$, which is true from

$$R \geq 2r \text{ (Euler).}$$

2790. If in } \Delta ABC, x, y, z > 0 \text{ then holds:}

$$\max \left\{ \sum_{cyc} yz \cos \frac{A}{2}, \sum_{cyc} yz \sin A \right\} \leq \frac{1}{2} (x + y + z) \sqrt{xy + yz + zx}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} \sum_{cyc} yz \sin A &= \\ &= \sum_{cyc} \sqrt{yz} \cdot \sqrt{yz} \sin A \stackrel{CBS}{\leq} \sqrt{\left(\sum_{cyc} yz \right) \left(\sum_{cyc} yz \sin^2 A \right)} \stackrel{?}{\leq} \frac{1}{2} \left(\sum_{cyc} x \right) \sqrt{\sum_{cyc} yz} \\ &\leftrightarrow \sum_{cyc} yza^2 \stackrel{(*)}{\leq} R^2 \left(\sum_{cyc} x \right)^2 \end{aligned}$$

$$\begin{aligned} \text{Let } O \text{ be the circumcenter of } \Delta ABC, \text{ we have: } & (x\overrightarrow{OA} + y\overrightarrow{OB} + z\overrightarrow{OC})^2 \geq 0 \\ \leftrightarrow & \sum_{cyc} x^2 OA^2 + \sum_{cyc} yz \cdot 2\overrightarrow{OB} \cdot \overrightarrow{OC} \geq 0 \\ \leftrightarrow & R^2 \sum_{cyc} x^2 + \sum_{cyc} yz(OB^2 + OC^2 - BC^2) \geq 0 \leftrightarrow R^2 \sum_{cyc} x^2 + 2R^2 \sum_{cyc} yz \\ & \geq \sum_{cyc} yza^2 \leftrightarrow R^2 \left(\sum_{cyc} x \right)^2 \geq \sum_{cyc} yza^2 \\ \rightarrow & (*) \text{ is true } \rightarrow \sum_{cyc} yz \sin A \leq \frac{1}{2} (x + y + z) \sqrt{xy + yz + zx} \quad (1) \end{aligned}$$

Using (1) for A triangle with angles $\frac{\pi - A}{2}$, $\frac{\pi - B}{2}$ and $\frac{\pi - C}{2}$

$$\rightarrow \sum_{cyc} yz \cos \frac{A}{2} \leq \frac{1}{2} \left(\sum_{cyc} x \right) \sqrt{xy + yz + zx} \quad (2)$$

$$(1), (2) \rightarrow \max \left\{ \sum_{cyc} yz \cos \frac{A}{2}, \sum_{cyc} yz \sin A \right\} \leq \frac{1}{2} (x + y + z) \sqrt{xy + yz + zx}.$$



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2791. In ΔABC the following relationship holds:

$$\sum \frac{a^2}{b+c} \leq \frac{1}{2} \sum a + \frac{R^2 - 4r^2}{r}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \sum \frac{a^2}{b+c} &= \sum \frac{(2s-(b+c))^2}{b+c} = 4s^2 \sum \frac{1}{b+c} - 3 \cdot 4s + \sum (b+c) \\
 &= 4s^2 \cdot \frac{5s^2 + r^2 + 4Rr}{2s(s^2 + r^2 + 2Rr)} - 8s = \\
 &= 2s \left(5 - \frac{4r^2 + 6Rr}{s^2 + r^2 + 2Rr} \right) - 8s \\
 &= 2s \left(1 - \frac{4r^2 + 6Rr}{s^2 + r^2 + 2Rr} \right) \stackrel{\text{Gerretsen}}{\lesssim} 2s \left(1 - \frac{4r^2 + 6Rr}{4R^2 + 4r^2 + 6Rr} \right) \\
 &\rightarrow \sum \frac{a^2}{b+c} - \frac{1}{2} \sum a \leq s \left(1 - \frac{4r^2 + 6Rr}{2R^2 + 2r^2 + 3Rr} \right) \\
 &= s \cdot \frac{2R^2 - 2r^2 - 3Rr}{2R^2 + 2r^2 + 3Rr} \stackrel{\text{Mitrinovic}}{\lesssim} \frac{3\sqrt{3}}{2} R \cdot \frac{(R-2r)(2R+r)}{2(R^2+r^2)+3Rr} \leq \\
 &\stackrel{\sqrt{3} \leq 2}{\underset{\text{AM-GM}}{\lesssim}} 3R \cdot \frac{(R-2r)(2R+r)}{2 \cdot 2Rr + 3Rr} = \frac{3(R-2r)(2R+r)}{7r} \stackrel{?}{\lesssim} \frac{R^2 - 4r^2}{r} \\
 &\leftrightarrow \frac{(R-2r)(R+11r)}{7r} \geq 0 \text{ which is true.}
 \end{aligned}$$

Therefore,
$$\sum \frac{a^2}{b+c} \leq \frac{1}{2} \sum a + \frac{R^2 - 4r^2}{r}.$$

2792. In acute ΔABC , P – point in plane of ΔABC , the following relationship holds:

$$AP \cdot \cos \frac{A}{2} + BP \cdot \cos \frac{B}{2} + CP \cdot \cos \frac{C}{2} \geq 2R + r$$

Proposed by Bogdan Fuștei-Romania



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Solution 1 by Maohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 \left(\cos \frac{A}{2} + \cos \frac{B}{2}\right)^2 &= \frac{1 + \cos A}{2} + \frac{1 + \cos B}{2} + 2 \cos \frac{A}{2} \cdot \cos \frac{B}{2} \geq 1 + \frac{\cos A + \cos B}{2} \\
 &= 1 + \sin \frac{C}{2} \cdot \cos \left(\frac{A-B}{2}\right) \geq 1 \geq \cos^2 \frac{C}{2} \\
 \left(\because \frac{A-B}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \cos \left(\frac{A-B}{2}\right) \geq 0\right) &\rightarrow \cos \frac{A}{2} + \cos \frac{B}{2} \geq \cos \frac{C}{2} \quad (\text{And analogs}) \\
 \rightarrow a_1 = \cos \frac{A}{2}, b_1 = \cos \frac{B}{2}, c_1 &= \cos \frac{C}{2} \text{ can be the sides of triangle } \Delta_1 \text{ with area } F_1 \text{ such that :} \\
 16F_1^2 &= 2 \sum \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} - \sum \cos^4 \frac{A}{2} \\
 &= \frac{1}{2} \sum (1 + \cos A)(1 + \cos B) - \frac{1}{4} \sum (1 + \cos A)^2 = \\
 &= \frac{3}{4} + \frac{1}{2} \sum \cos A + \frac{1}{2} \sum \cos A \cos B - \frac{1}{4} \sum \cos^2 A \\
 &= \frac{3}{4} + \frac{1}{2} \left(1 + \frac{r}{R}\right) + \frac{1}{2} \left(\frac{s^2 + r^2}{4R^2} - 1\right) - \frac{1}{4} \left(3 - \frac{s^2 - r^2 - 4Rr}{2R^2}\right) \\
 \rightarrow 16F_1^2 &= \frac{s^2}{4R^2} \rightarrow F_1 = \frac{s}{8R} \quad (1)
 \end{aligned}$$

We know that for any triangles ΔABC and Δ_1 , and any point P in the plane of ΔABC ,

we have :

$$\begin{aligned}
 \sum a_1 \cdot AP &\geq \sqrt{\frac{1}{2} \sum a_1^2 (b^2 + c^2 - a^2) + 8FF_1} \quad (\text{Bottema}) \quad (2) \\
 \sum a_1^2 (b^2 + c^2 - a^2) &= \sum \cos^2 \frac{A}{2} (b^2 + c^2 - a^2) = 2s \sum (s - a) \cos A \\
 &= 2s^2 \sum \cos A - 2s \sum a \cos A = \\
 &= 2s^2 \left(1 + \frac{r}{R}\right) - 2s \cdot \frac{2sr}{R} = 2s^2 \left(1 - \frac{r}{R}\right) \rightarrow \sum a_1^2 (b^2 + c^2 - a^2) = 2s^2 \left(1 - \frac{r}{R}\right) \quad (3) \\
 (1), (2), (3) &\rightarrow \sum \cos \frac{A}{2} \cdot AP \geq \sqrt{\frac{1}{2} \cdot 2s^2 \left(1 - \frac{r}{R}\right) + 8sr \cdot \frac{s}{8R}} = s
 \end{aligned}$$



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$$ABC \text{ is an acute triangle} \rightarrow \prod \cos A \geq 0 \leftrightarrow \frac{s^2 - (2R + r)^2}{4R^2} \geq 0 \rightarrow s \geq 2R + r.$$

$$\text{Therefore, } AP \cdot \cos \frac{A}{2} + BP \cdot \cos \frac{B}{2} + CP \cdot \cos \frac{C}{2} \geq s \geq 2R + r.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 & \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} = \frac{s(s-a)}{bc} + \frac{s(s-b)}{ca} - \frac{s(s-c)}{ab} \\
 & = \frac{s}{abc}(a(s-a) + b(s-b) - c(s-c)) = \frac{s}{abc}(x(y+z) + y(z+x) - z(x+y)) \\
 & = \frac{s}{abc}(2xy) > 0 \\
 (\text{taking } x = s-a, y = s-b, z = s-c, \text{ which subsequently implies } a = y+z, b = z+x, c &= x+y \text{ and triangle inequality } \Rightarrow x, y, z > 0) \therefore \cos^2 \frac{C}{2} < \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} \\
 & < \left(\cos \frac{A}{2} + \cos \frac{B}{2} \right)^2 \Rightarrow \cos \frac{A}{2} + \cos \frac{B}{2} - \cos \frac{C}{2} > 0 \text{ and analogs} \\
 \Rightarrow \cos \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2} & \text{ form sides of a triangle} \rightarrow (1) \text{ with area } F_1 \\
 & = \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} - \sum_{\text{cyc}} \cos^4 \frac{A}{2}} \\
 & = \frac{1}{4} \sqrt{4 \sum_{\text{cyc}} \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} - \left(\sum_{\text{cyc}} \cos^2 \frac{A}{2} \right)^2} \\
 & = \frac{1}{4} \sqrt{4 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} \sum_{\text{cyc}} \sec^2 \frac{A}{2} - \frac{1}{4} \left(\sum_{\text{cyc}} (1 + \cos A) \right)^2} \\
 & = \frac{1}{4} \sqrt{\left(\frac{s^2}{4R^2} \right) \frac{(4R+r)^2 + s^2}{s^2} - \frac{(4R+r)^2}{4R^2}} = \frac{s}{8R} \stackrel{(i)}{=} \frac{s}{R}
 \end{aligned}$$

Via Bottema, for any arbitrary point P in the plane of ΔABC and for any arbitrary $\Delta A'B'C'$,
 $(AP \cdot a' + BP \cdot b' + CP \cdot c')^2$

$$\begin{aligned}
 & \geq \frac{1}{2} (a^2(b'^2 + c'^2 - a'^2) + b^2(c'^2 + a'^2 - b'^2) + c^2(a'^2 + b'^2 - c'^2)) \\
 & + 8[ABC][A'B'C'] \therefore \text{via (1) and choosing } a' = \cos \frac{A}{2}, b' = \cos \frac{B}{2}, c' = \cos \frac{C}{2}
 \end{aligned}$$



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$$\begin{aligned}
 & \left(AP \cdot \cos \frac{A}{2} + BP \cdot \cos \frac{B}{2} + CP \cdot \cos \frac{C}{2} \right)^2 \\
 & \geq \frac{1}{2} \sum_{\text{cyc}} a^2 \left(\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} - \cos^2 \frac{A}{2} \right) \\
 & + 8rsF_1 \stackrel{\text{via (i)}}{\cong} \frac{1}{2} \sum_{\text{cyc}} a^2 \left(\frac{s(s-b)}{ca} + \frac{s(s-c)}{ab} - \frac{s(s-a)}{bc} \right) + rs \left(\frac{s}{R} \right) \\
 & = \frac{s}{8Rrs} \left(2s \sum_{\text{cyc}} a^2(s-a) - \sum_{\text{cyc}} a^2(b^2+c^2-a^2) \right) + \frac{rs^2}{R} \\
 & = \frac{1}{8Rr} \left(4s^2(s^2-4Rr-r^2) - 4s^2(s^2-6Rr-3r^2) - 2 \sum_{\text{cyc}} a^2 b c \cos A \right) + \frac{rs^2}{R} \\
 & = \frac{1}{8Rr} \left(4s^2(2Rr+2r^2) - 8R^2rs \sum_{\text{cyc}} \sin 2A \right) + \frac{rs^2}{R} \\
 & = \frac{1}{8Rr} \left(4s^2(2Rr+2r^2) - 8R^2rs(2 \sin C \cos(A-B) - 2 \sin C \cos(A+B)) \right) + \frac{rs^2}{R} \\
 & = \frac{1}{8Rr} \left(4s^2(2Rr+2r^2) - 8R^2rs \cdot 4 \prod_{\text{cyc}} \sin A \right) + \frac{rs^2}{R} \\
 & = \frac{1}{8Rr} \left(4s^2(2Rr+2r^2) - 8R^2rs \cdot 4 \left(\frac{4Rrs}{8R^3} \right) \right) + \frac{rs^2}{R} \\
 & = \frac{8Rrs^2 + 8r^2s^2 - 16r^2s^2 + 8r^2s^2}{8Rr} = s^2 = s^2 - (2R+r)^2 + (2R+r)^2 \\
 & = 4R^2 \prod_{\text{cyc}} \cos A + (2R+r)^2 > (2R+r)^2 \quad \left(\because \Delta ABC \text{ is acute} \Rightarrow \prod_{\text{cyc}} \cos A > 0 \right) \\
 & \Rightarrow AP \cdot \cos \frac{A}{2} + BP \cdot \cos \frac{B}{2} + CP \cdot \cos \frac{C}{2} \geq 2R + r \quad (\text{QED})
 \end{aligned}$$

2793. In ΔABC the following relationship holds:

$$\frac{\sqrt{6}}{2R} \leq \frac{1}{\sqrt{a(a+b)}} + \frac{1}{\sqrt{b(b+c)}} + \frac{1}{\sqrt{c(c+a)}} \leq \frac{\sqrt{6}}{4r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

$$\frac{1}{\sqrt{a(a+b)}} + \frac{1}{\sqrt{b(b+c)}} + \frac{1}{\sqrt{c(c+a)}} \geq 3 \sqrt[3]{\frac{1}{\sqrt{abc(a+b)(b+c)(c+a)}}}$$

We must to show:



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$$3\sqrt[6]{\frac{1}{abc(a+b)(b+c)(c+a)}} \geq \frac{\sqrt{6}}{2R} \Leftrightarrow \frac{3^6}{abc(a+b)(b+c)(c+a)} \geq \frac{3^3 \cdot 2^3}{2^6 R^6} \Leftrightarrow abc(a+b)(b+c)(c+a) \leq 3^3 \cdot 2^3 \cdot R^6; (1)$$

But $abc = 4Rrs$ and $(a+b)(b+c)(c+a) = 2s(s^2 + r^2 + 2Rr)$; (2)

From (1),(2) we must to show: $s^2r(s^2 + r^2 + 2Rr) \leq 27R^5$; (3)

From $R \geq 2r$ (Euler) and $s^2 \leq \frac{27R^2}{4}$ (Mitrinovic); (4)

From (3),(4) we must show: $s^2 + r^2 + 2Rr \leq 8R^2$; (5)

From $s^2 \leq 4R^2 + 4Rr + 3r^2$ (Gerretsen); (6)

From (5),(6) we must show:

$$s^2 + r^2 + 2Rr \leq 4R^2 + 6Rr + 4r^2 \leq 8R^2 \Leftrightarrow$$

$$6Rr + 4r^2 \leq 4R^2 \Leftrightarrow 3Rr + 2r^2 \leq 2R^2; (7)$$

$$3Rr + 2r^2 \leq \frac{2R^2}{2} + \frac{r^2}{2} = \frac{4R^2}{2} = 2R^2 \Rightarrow (7) \text{ it's true.}$$

$$\frac{1}{\sqrt{a(a+b)}} + \frac{1}{\sqrt{b(b+c)}} + \frac{1}{\sqrt{c(c+a)}} \leq \frac{\sqrt{6}}{4r} \Leftrightarrow$$

$$\frac{1}{\sqrt{2a(a+b)}} + \frac{1}{\sqrt{2b(b+c)}} + \frac{1}{\sqrt{2c(c+a)}} \leq \frac{\sqrt{3}}{4r}; (8)$$

$$\sqrt{2a(a+b)} \geq \frac{2}{\frac{1}{2a} + \frac{1}{a+b}} = \frac{4a(a+b)}{3a+b} \Rightarrow \frac{1}{\sqrt{2a(a+b)}} \leq \frac{3a+b}{4a(a+b)}; (9)$$

From (8),(9) we must show:

$$\frac{3a+b}{a(a+b)} + \frac{3b+c}{b(b+c)} + \frac{3c+a}{c(c+a)} \leq \frac{\sqrt{3}}{r} \Leftrightarrow$$

$$\frac{a+b+2a}{a(a+b)} + \frac{b+c+2b}{b(b+c)} + \frac{c+a+2c}{c(c+a)} \leq \frac{\sqrt{3}}{r} \Leftrightarrow$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 2 \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right) \leq \frac{\sqrt{3}}{r}; (10)$$

$$\text{But } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r} \text{ (Leuenberger 1960); (11)}$$

From (10),(11) we must show:



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$$2 \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right) \leq \frac{\sqrt{3}}{2r} \Leftrightarrow \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{\sqrt{3}}{2r}; \quad (12)$$

But: $\frac{1}{a+b} \leq \frac{1}{4} \left(\frac{1}{a} + \frac{1}{b} \right); \quad (13) \Leftrightarrow 4ab \leq (a+b)^2 \Leftrightarrow (a-b)^2 \geq 0$

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \leq \frac{1}{2} \cdot \frac{\sqrt{3}}{r} = \frac{\sqrt{3}}{2r} \Rightarrow (12) \text{ its true.}$$

2794. In ΔABC the following relationship holds:

$$\sum \frac{a^3}{b+c} \leq \frac{1}{2} \sum a^2 + \frac{R^3 - 8r^3}{r}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} \sum \frac{a^3}{b+c} &= \sum \frac{(2s - (b+c))^3}{b+c} \\ &= 8s^3 \sum \frac{1}{b+c} - 3 \cdot 3(2s)^2 + 3 \cdot 2s \sum (b+c) - \sum (b+c)^2 = \\ &= 8s^3 \cdot \frac{5s^2 + r^2 + 4Rr}{2s(s^2 + r^2 + 2Rr)} - 36s^2 + 24s^2 - 2(3s^2 - r^2 - 4Rr) \\ &= \frac{4s^2 \cdot (5s^2 + r^2 + 4Rr)}{s^2 + r^2 + 2Rr} - 18s^2 + 2r^2 + 8Rr \leq \\ &\stackrel{?}{\leq} \frac{1}{2} \sum a^2 + \frac{R^3 - 8r^3}{r} = s^2 - r^2 - 4Rr + \frac{R^3 - 8r^3}{r} \leftrightarrow \frac{-R^3 + 12Rr^2 + 11r^3}{r} \\ &\leq \frac{s^2 \cdot (-s^2 + 15r^2 + 22Rr)}{s^2 + r^2 + 2Rr} \\ &\leftrightarrow 2r(2R + r)(-R^3 + 12Rr^2 + 11r^3) \leq 2s^2(R^3 + 10Rr^2 + 4r^3 - rs^2) \quad (*) \end{aligned}$$

$$\begin{aligned} RHS_{(*)} &\stackrel{\substack{2s^2 \geq 27Rr \\ Gerretsen}}{\stackrel{?}{\geq}} 27Rr(R^3 - 4R^2r + 6Rr^2 + r^3) \stackrel{?}{\geq} LHS_{(*)} \\ &= r(-4R^4 - 2R^3r + 48R^2r^2 + 68Rr^3 + 22r^4) \\ &\leftrightarrow 31R^4 - 106R^3r + 114R^2r^2 - 41Rr^3 - 22r^4 \geq 0 \end{aligned}$$

$\leftrightarrow (R - 2r)(9R^3 + 22R^2(R - 2r) + 26Rr^2 + 11r^3) \geq 0$ which is true from Euler.

Therefore, $\sum \frac{a^3}{b+c} \leq \frac{1}{2} \sum a^2 + \frac{R^3 - 8r^3}{r}.$



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Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \sum \frac{a^3}{b+c} &= \sum \frac{a^3 + b^3 + c^3 - (b^3 + c^3)}{b+c} = \left(\sum a^3 \right) \sum \frac{1}{b+c} - \sum \frac{(b+c)(b^2 - bc + c^2)}{b+c} \\
 &= \frac{2s(s^2 - 6Rr - 3r^2)}{2s(s^2 + 2Rr + r^2)} \sum (c+a)(a+b) - 2 \sum a^2 + \sum ab \\
 &\Rightarrow \sum \frac{a^3}{b+c} - \frac{a^2 + b^2 + c^2}{2} \\
 &= \frac{s^2 - 6Rr - 3r^2}{s^2 + 2Rr + r^2} \left(\left(\sum a^2 + 2 \sum ab \right) + \sum ab \right) - \frac{5}{2} \sum a^2 + \sum ab \\
 &= \frac{(s^2 - 6Rr - 3r^2)(5s^2 + 4Rr + r^2)}{s^2 + 2Rr + r^2} - 5(s^2 - 4Rr - r^2) + s^2 + 4Rr + r^2 \\
 &= \frac{(s^2 - 6Rr - 3r^2)(5s^2 + 4Rr + r^2) - (s^2 + 2Rr + r^2)(4s^2 - 24Rr - 6r^2)}{s^2 + 2Rr + r^2} \\
 &= \frac{s^4 - (10Rr + 12r^2)s^2 + r^2(24R^2 + 18Rr + 3r^2)}{s^2 + 2Rr + r^2} \\
 &\Rightarrow \sum \frac{a^3}{b+c} - \frac{a^2 + b^2 + c^2}{2} - \frac{R^3 - 8r^3}{r} \\
 &= \frac{r(s^4 - (10Rr + 12r^2)s^2 + r^2(24R^2 + 18Rr + 3r^2)) - (R^3 - 8r^3)(s^2 + 2Rr + r^2)}{r(s^2 + 2Rr + r^2)} \leq 0 \\
 &\Leftrightarrow rs^4 - (R^3 - 8r^3 + r(10Rr + 12r^2))s^2 - (R^3 - 8r^3)(2Rr + r^2) \\
 &\quad + r^3(24R^2 + 18Rr + 3r^2) \stackrel{(*)}{\geq} 0
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, LHS of } (*) &\stackrel{\text{Gerretsen}}{\leq} \left(r(4R^2 + 4Rr + 3r^2) - (R^3 - 8r^3 + r(10Rr + 12r^2)) \right) s^2 \\
 &\quad - (R^3 - 8r^3)(2Rr + r^2) + r^3(24R^2 + 18Rr + 3r^2) \stackrel{?}{\geq} 0 \\
 &\Leftrightarrow (R^3 - 4R^2r + 6Rr^2 + r^3)s^2 + (R^3 - 8r^3)(2Rr + r^2) \stackrel{?}{\geq} r^3(24R^2 + 18Rr + 3r^2)
 \end{aligned}$$

Now, LHS of ()**

$$\begin{aligned}
 &= ((R - 2r)\{(R - r)^2 + r^2\} + 5r^3)s^2 \\
 &\quad + (R^3 - 8r^3)(2Rr + r^2) \stackrel{\text{Gerretsen}}{\geq} ((R - 2r)\{(R - r)^2 + r^2\} + 5r^3)(16Rr \\
 &\quad - 5r^2) + (R^3 - 8r^3)(2Rr + r^2)
 \end{aligned}$$



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$$\begin{aligned}
 & \left(\because (R - 2r)\{(R - r)^2 + r^2\} + 5r^3 \stackrel{\text{Euler}}{\geq} 5r^3 > 0 \right) \stackrel{?}{\geq} r^3(24R^2 + 18Rr + 3r^2) \\
 & \Leftrightarrow 9t^4 - 34t^3 + 46t^2 - 24t - 8 \stackrel{?}{\geq} 0 \quad \left(t = \frac{R}{r} \right) \\
 & \Leftrightarrow (t - 2)\{8t^2(t - 2) + t^3 + 14t + 4\} \stackrel{?}{\geq} 0 \\
 & \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \sum \frac{a^3}{b+c} \leq \frac{a^2 + b^2 + c^2}{2} + \frac{R^3 - 8r^3}{r} \quad (\text{QED})
 \end{aligned}$$

2795. In ΔABC the following relationship holds:

$$m_a^n + m_b^n + m_c^n \geq 3^{\frac{n}{4}+1} \cdot F^{\frac{n}{2}}, n \in \mathbb{N}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

From Holder Inequality, we have:

$$m_a^n + m_b^n + m_c^n \geq \frac{(m_a + m_b + m_c)^n}{3^{n-1}}$$

We must show:

$$(m_a + m_b + m_c)^n \geq 3^{n+\frac{n}{4}} \cdot F^{\frac{n}{2}} \Leftrightarrow m_a + m_b + m_c \geq 3^{\frac{5}{4}} \cdot \sqrt{F} = 3^{\frac{5}{4}} \sqrt{sr}; (1)$$

But: $m_a \geq \sqrt{s(s-a)}$; (2). From (1),(2) we must show:

$$\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c} \geq 3^{\frac{5}{4}} \sqrt{r}; (3)$$

$$\text{But: } \sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c} \geq 3 \sqrt[3]{\sqrt{(s-a)(s-b)(s-c)}} = 3 \sqrt[6]{sr^2}; (4)$$

From (3),(4) we must show:

$$3 \sqrt[6]{sr^2} \geq 3^{\frac{5}{4}} \sqrt{r} \Leftrightarrow \sqrt[6]{sr^2} \geq 3^{\frac{1}{4}} \sqrt{r} \Leftrightarrow sr^2 \geq 3^{\frac{3}{2}} r^3 \Leftrightarrow s \geq 3\sqrt{3}r \quad (\text{Mitrinovic})$$

2796. In ΔABC , prove that:

$$\sum \frac{a^3}{b^2 + c^2} \geq \frac{a+b+c}{2} + \frac{r(R-2r)}{R+3r}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam



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Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we may assume that : $a \geq b \geq c \rightarrow a^3 \geq b^3 \geq c^3$ and

$$\frac{1}{b^2 + c^2} \geq \frac{1}{c^2 + a^2} \geq \frac{1}{a^2 + b^2}$$

$$\begin{aligned}
& \text{Using Chebyshev} \quad \sum \frac{a^3}{b^2 + c^2} \geq \frac{1}{3} \left(\sum a^3 \right) \left(\sum \frac{1}{b^2 + c^2} \right) \stackrel{CBS}{\geq} \frac{1}{3} \left(\sum a^3 \right) \cdot \frac{9}{\sum (b^2 + c^2)} \\
& = \frac{3(a^3 + b^3 + c^3)}{2(a^2 + b^2 + c^2)} \\
& \rightarrow \sum \frac{a^3}{b^2 + c^2} - \frac{1}{2} \sum a \geq \frac{3(a^3 + b^3 + c^3)}{2(a^2 + b^2 + c^2)} - \frac{a + b + c}{2} \\
& = \frac{3(a^3 + b^3 + c^3) - (a + b + c)(a^2 + b^2 + c^2)}{2(a^2 + b^2 + c^2)} = \\
& = \frac{3.2s(s^2 - 3r^2 - 6Rr) - 2s.2(s^2 - r^2 - 4Rr)}{2.2(s^2 - r^2 - 4Rr)} \\
& = \frac{s(s^2 - 7r^2 - 10Rr)}{2(s^2 - r^2 - 4Rr)} \stackrel{\text{Mitrinovic}}{\geq} \frac{3\sqrt{3}r.(s^2 - 7r^2 - 10Rr)}{2(s^2 - r^2 - 4Rr)} \geq \\
& \stackrel{3\sqrt{3} \geq 5}{\geq} \frac{5r.(s^2 - 7r^2 - 10Rr)}{2(s^2 - r^2 - 4Rr)} \stackrel{?}{\geq} \frac{r(R - 2r)}{R + 3r} \\
& \leftrightarrow (3R + 19r)s^2 - r(42R^2 + 199Rr + 109r^2) \geq 0 \\
& \leftrightarrow (3R + 19r)[s^2 - (16Rr - 5r^2)] + r(R - 2r)(6R + 102r) \geq 0
\end{aligned}$$

Which is true from Euler and Gerretsen's inequality.

$$\text{Therefore, } \sum \frac{a^3}{b^2 + c^2} \geq \frac{a + b + c}{2} + \frac{r(R - 2r)}{R + 3r}.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
\sum \frac{a^3}{b^2 + c^2} &= \sum \frac{a^4}{ab^2 + ac^2} \stackrel{\text{Bergstrom}}{\geq} \frac{(a^2 + b^2 + c^2)^2}{ab(a + b) + bc(b + c) + ca(c + a)} \\
&= \frac{4(s^2 - 4Rr - r^2)^2}{2s(s^2 + 4Rr + r^2) - 12Rrs} = \frac{2(s^2 - 4Rr - r^2)^2}{s(s^2 - 2Rr + r^2)} \Rightarrow \sum \frac{a^3}{b^2 + c^2} - \frac{a + b + c}{2}
\end{aligned}$$



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$$\begin{aligned}
 & \geq \frac{2(s^2 - 4Rr - r^2)^2}{s(s^2 - 2Rr + r^2)} - s = \frac{2(s^2 - 4Rr - r^2)^2 - s^2(s^2 - 2Rr + r^2)}{s(s^2 - 2Rr + r^2)} \\
 & = \frac{s^4 - (14Rr + 5r^2)s^2 + 2r^2(4R + r)^2}{s(s^2 - 2Rr + r^2)} \stackrel{\text{Trucht}}{\geq} \frac{s^4 - (14Rr + 5r^2)s^2 + 6r^2s^2}{s(s^2 - 2Rr + r^2)} \\
 & = \frac{s(s^2 - 14Rr + r^2)}{s^2 - 2Rr + r^2}
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{\text{Gerretsen}}{\geq} \frac{s(2Rr - 4r^2)}{s^2 - 2Rr + r^2} \stackrel{?}{\geq} \frac{r(R - 2r)}{R + 3r} \stackrel{R - 2r \geq 0 \text{ via Euler}}{\Leftrightarrow} \frac{2s}{s^2 - 2Rr + r^2} \stackrel{?}{>} \frac{1}{R + 3r} \\
 & \Leftrightarrow 4s^2(R + 3r)^2 \stackrel{?}{\geq} (s^2 - 2Rr + r^2)^2 \stackrel{(i)}{\Leftrightarrow}
 \end{aligned}$$

$$\text{Now, } s^2 - (s^2 - 2Rr + r^2) = 2Rr - r^2 \stackrel{\text{Euler}}{\geq} 3r^2 > 0$$

$$\begin{aligned}
 & \Rightarrow s^2 \stackrel{(*)}{>} s^2 - 2Rr + r^2 \text{ and } s^2 - 2Rr + r^2 \stackrel{\text{Gerretsen}}{\leq} 4R^2 + 2Rr \\
 & \quad + 4r^2 \stackrel{?}{<} 4(R + 3r)^2 \Leftrightarrow 11Rr + 16r^2 \stackrel{?}{>} 0 \rightarrow \text{true} \\
 & \Rightarrow 4(R + 3r)^2 \stackrel{(**)}{\geq} s^2 - 2Rr + r^2 \therefore (*) . (**) \Rightarrow 4s^2(R + 3r)^2 > (s^2 - 2Rr + r^2)^2 \Rightarrow (i) \text{ is true} \\
 & \Rightarrow \sum \frac{a^3}{b^2 + c^2} - \frac{a + b + c}{2} \geq \frac{r(R - 2r)}{R + 3r} \Rightarrow \sum \frac{a^3}{b^2 + c^2} \geq \frac{a + b + c}{2} + \frac{r(R - 2r)}{R + 3r}
 \end{aligned}$$

2797. In ΔABC the following relationship holds:

$$\frac{8}{9} \left(\frac{R}{2r} \right)^{-2} \leq \frac{w_a}{w_b + w_c} \sin^2 A + \frac{w_b}{w_c + w_a} \sin^2 B + \frac{w_c}{w_a + w_b} \sin^2 C \leq \frac{9}{8} \cdot \frac{R}{2r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

$$\frac{a}{\sin A} = 2R \Rightarrow \sin^2 A = \frac{a^2}{4R^2}$$

For LHS, we must show:

$$\sum_{cyc} \frac{w_a}{w_b + w_c} \sin^2 A = \sum_{cyc} \frac{w_a}{w_b + w_c} \cdot \frac{a^2}{4R^2} \geq \frac{8}{9} \cdot \frac{4r^2}{R^2} \Leftrightarrow \sum_{cyc} \frac{a^2 w_a}{w_b + w_c} \geq 18r^2; (1)$$



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$$\sum_{cyc} \frac{a^2 w_a}{w_b + w_c} = \sum_{cyc} \frac{a^2 w_a^2}{w_a(w_b + w_c)} \stackrel{\text{Bergstrom}}{\geq} \frac{(aw_a + bw_b + cw_c)^2}{2(w_a w_b + w_b w_c + w_c w_a)}; (2)$$

But: $w_a \geq h_a$ (and analogs) $\Rightarrow (aw_a + bw_b + cw_c)^2 \geq (ah_a + bh_b + ch_c)^2 = 36F^2$; (3)

$$w_a w_b + w_b w_c + w_c w_a \leq w_a^2 + w_b^2 + w_c^2$$

But: $w_a \leq \sqrt{s(s-a)}$ then $w_a w_b + w_b w_c + w_c w_a \leq s(s-a+s-b+s-c) = s^2$; (4)

From (2),(3),(4) it follows that

$$\sum_{cyc} \frac{a^2 w_a}{w_b + w_c} \geq \frac{36F^2}{2s^2} = \frac{36s^2 r^2}{2s^2} = 18r^2 \Rightarrow (1) \text{ its true.}$$

For RHS, we must show:

$$\sum_{cyc} \frac{w_a}{w_b + w_c} \sin^2 A = \sum_{cyc} \frac{w_a}{w_b + w_c} \cdot \frac{a^2}{4R^2} \leq \frac{9}{8} \cdot \frac{R}{2r} \Leftrightarrow \sum_{cyc} \frac{w_a}{w_b + w_c} \leq \frac{9}{4} \cdot \frac{R^3}{r}; (5)$$

$$\text{Let } a \leq b \leq c \Rightarrow a^2 \leq b^2 \leq c^2 \text{ and } \frac{w_a}{w_b + w_c} \geq \frac{w_b}{w_c + w_a} \geq \frac{w_c}{w_a + w_b}$$

Because $w_a^2 + w_a w_c \geq w_b^2 + w_b w_c \Leftrightarrow (w_a - w_b)(w_a + w_b + w_c) \geq 0$ true.

From Chebyshev's Inequality, we have:

$$\sum_{cyc} \frac{a^2 w_a}{w_b + w_c} \leq \frac{1}{3} (a^2 + b^2 + c^2) \left(\frac{w_a}{w_b + w_c} + \frac{w_b}{w_c + w_a} + \frac{w_c}{w_a + w_b} \right); (6)$$

From (5),(6) we must show:

$$\frac{1}{3} (a^2 + b^2 + c^2) \left(\frac{w_a}{w_b + w_c} + \frac{w_b}{w_c + w_a} + \frac{w_c}{w_a + w_b} \right) \leq \frac{9}{4} \cdot \frac{R^3}{r}; (7)$$

But: $a^2 + b^2 + c^2 \leq 9R^2$ (Neuberg); (8). From (7),(8) we must show:

$$\frac{w_a}{w_b + w_c} + \frac{w_b}{w_c + w_a} + \frac{w_c}{w_a + w_b} \leq \frac{3}{4} \cdot \frac{R}{r}; (9)$$

But: $w_a \geq h_a \Rightarrow \frac{1}{w_a} \leq \frac{1}{h_a}$ (and analogs); (10). From (9),(10) we must show:

$$\sum_{cyc} \frac{w_a}{h_b + h_c} \leq \frac{3R}{4r} \Leftrightarrow \sum_{cyc} \frac{bc \cdot w_a}{2F\left(\frac{1}{b} + \frac{1}{c}\right)} \leq \frac{3R}{4r} \Leftrightarrow$$



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$$\sum_{cyc} \frac{bc \cdot w_a}{2F(b+c)} \leq \frac{3R}{4r} \Leftrightarrow \sum_{cyc} \frac{bc \cdot w_a}{4F\sqrt{bc}} \leq \frac{3}{4} \cdot \frac{R}{r} \Leftrightarrow \sum_{cyc} \frac{bc \cdot w_a}{\sqrt{bc}} \leq \frac{3RF}{r} \Leftrightarrow$$

$$\sum_{cyc} w_a \sqrt{bc} \leq 3Rs; (11)$$

From Cauchy inequality:

$$\left(\sum_{cyc} w_a \sqrt{bc} \right)^2 \leq \sum_{cyc} w_a^2 \cdot \sum_{cyc} bc \text{ but: } \sum_{cyc} bc \leq 9R^2 \text{ and } \sum_{cyc} w_a^2 \leq s^2 \Rightarrow$$

$$\left(\sum_{cyc} w_a \sqrt{bc} \right)^2 \leq 9R^2 s^2 \Leftrightarrow \sum_{cyc} w_a \sqrt{bc} \leq 3Rs \Rightarrow (11) \text{ its true.}$$

2798. In ΔABC the following relationship holds :

$$\sum_{cyc} \frac{a^2}{b+c-a} \geq \sum_{cyc} a + \frac{r^3(R-2r)}{5R^3+3r^3}$$

Proposed Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} \sum_{cyc} \frac{a^2}{b+c-a} &\stackrel{\text{Chebyshev}}{\geq} \frac{1}{3} \left(\sum_{cyc} a^2 \right) \left(\sum_{cyc} \frac{1}{b+c-a} \right) \stackrel{\text{CBS}}{\geq} \frac{1}{3} \left(\sum_{cyc} a^2 \right) \cdot \frac{9}{\sum (b+c-a)} \\ &= \frac{3(a^2 + b^2 + c^2)}{a+b+c} \\ &\rightarrow \sum_{cyc} \frac{a^2}{b+c-a} - \sum_{cyc} a \geq \frac{3(a^2 + b^2 + c^2)}{a+b+c} - \sum_{cyc} a \\ &= 2 \cdot \frac{(a^2 + b^2 + c^2) - (ab + bc + ca)}{a+b+c} = \frac{s^2 - 3r^2 - 12Rr}{s} \geq \\ &\stackrel{\text{Gerretsen}}{\geq} \frac{2 \cdot 4r(R-2r)}{3\sqrt{3}R} \stackrel{2 \geq \sqrt{3}}{\geq} \frac{4r(R-2r)}{3R} \stackrel{?}{\geq} \frac{r^3(R-2r)}{5R^3+3r^3} \Leftrightarrow 20R^3 + 12r^3 \geq 3Rr^2 \\ &\Leftrightarrow 19R^3 + (R-2r)(R^2 + 2Rr + r^2) + 14r^3 \\ &\geq 0 \text{ Which is true from Euler's inequality.} \end{aligned}$$



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$$\text{Therefore, } \sum_{\text{cyc}} \frac{a^2}{b+c-a} \geq \sum_{\text{cyc}} a + \frac{r^3(R-2r)}{5R^3+3r^3}.$$

2799. In ΔABC , prove that :

$$\sum \frac{a^2}{b^2+c^2} \geq \frac{3}{2} + \frac{r(R-2r)}{R^2+r^2}$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned} \sum \frac{a^2}{b^2+c^2} - \frac{3}{2} &= \sum \frac{a^4}{a^2b^2+c^2a^2} - \frac{3}{2} \stackrel{\text{Bergstrom}}{\geq} \frac{(a^2+b^2+c^2)^2}{2(a^2b^2+b^2c^2+c^2a^2)} - \frac{3}{2} = \\ &= -\frac{2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4)}{2(a^2b^2+b^2c^2+c^2a^2)} + \frac{1}{2} \stackrel{\Sigma x^2 \geq \Sigma xy}{\geq} -\frac{16F^2}{2abc(a+b+c)} + \frac{1}{2} \\ &= -\frac{16s^2r^2}{16s^2Rr} + \frac{1}{2} = \\ &= -\frac{r}{R} + \frac{1}{2} = \frac{r(R-2r)}{2Rr} \stackrel{\text{AM-GM}}{\geq} \frac{r(R-2r)}{R^2+r^2}. \\ \text{Therefore, } \sum \frac{a^2}{b^2+c^2} &\geq \frac{3}{2} + \frac{r(R-2r)}{R^2+r^2}. \end{aligned}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum \frac{a^2}{b^2+c^2} &= \sum \frac{a^4}{a^2b^2+a^2c^2} \stackrel{\text{Bergstrom}}{\geq} \frac{(a^2+b^2+c^2)^2}{2(a^2b^2+b^2c^2+c^2a^2)} = \frac{2(s^2-4Rr-r^2)^2}{(s^2+4Rr+r^2)^2-16Rrs^2} \\ &\Rightarrow \sum \frac{a^2}{b^2+c^2} - \frac{3}{2} - \frac{r(R-2r)}{R^2+r^2} \geq \\ &\frac{2(s^2-4Rr-r^2)^2}{(s^2+4Rr+r^2)^2-16Rrs^2} - \frac{3}{2} - \frac{r(R-2r)}{R^2+r^2} = \frac{2(s^2-4Rr-r^2)^2}{2\{(s^2+4Rr+r^2)^2-16Rrs^2\}} - \frac{r(R-2r)}{R^2+r^2} \\ &= \frac{s^4-(8Rr+14r^2)s^2+r^2(4R+r)^2}{2\{(s^2+4Rr+r^2)^2-16Rrs^2\}} - \frac{r(R-2r)}{R^2+r^2} \\ &= \frac{(R^2+r^2)\{s^4-(8Rr+14r^2)s^2+r^2(4R+r)^2\}-2r(R-2r)\{(s^2+4Rr+r^2)^2-16Rrs^2\}}{2(R^2+r^2)\{(s^2+4Rr+r^2)^2-16Rrs^2\}} \stackrel{?}{\geq} 0 \\ &\Leftrightarrow (R^2-2Rr+5r^2)s^4-rs^2(8R^3-2R^2r+44Rr^2+6r^3) \\ &\quad + r^2(16R^4-24R^3r+65R^2r^2+38Rr^3+5r^4) \stackrel{(i)}{\leq} 0 \end{aligned}$$



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$$\begin{aligned}
 & \text{Now, LHS of (i) } \stackrel{\text{Gerretsen}}{\geq} rs^2 \left((R^2 - 2Rr + 5r^2)(16R - 5r) - (8R^3 - 2R^2r + 44Rr^2 + 6r^3) \right) \\
 & \quad + r^2(16R^4 - 24R^3r + 65R^2r^2 + 38Rr^3 + 5r^4) \stackrel{?}{\geq} 0 \\
 & \Leftrightarrow (8R^3 - 35R^2r + 46Rr^2 - 31r^3)s^2 \\
 & \quad + r^2\{32(R - 2r)R^3 + 8R^3r + 65R^2r^2 + 38Rr^3 + 5r^4\} \stackrel{?}{\geq} 0 \\
 & \quad \quad \quad \text{(ii)}
 \end{aligned}$$

Case 1 $8R^3 - 35R^2r + 46Rr^2 - 31r^3 \geq 0$ and then, LHS of (ii)

$$\begin{aligned}
 & \geq r^2\{32(R - 2r)R^3 + 8R^3r + 65R^2r^2 + 38Rr^3 + 5r^4\} > 0 \because R \stackrel{\text{Euler}}{\geq} 2r \\
 & \Rightarrow \text{(ii) is true}
 \end{aligned}$$

Case 2 $8R^3 - 35R^2r + 46Rr^2 - 31r^3 < 0$ and then, LHS of (ii)

$$\begin{aligned}
 & = -(-(8R^3 - 35R^2r + 46Rr^2 - 31r^3)s^2) \\
 & + r^2\{32(R - 2r)R^3 + 8R^3r + 65R^2r^2 + 38Rr^3 + 5r^4\}
 \end{aligned}$$

Gerretsen

$$\begin{aligned}
 & \stackrel{?}{\geq} -(-(8R^3 - 35R^2r + 46Rr^2 - 31r^3)(4R^2 + 4Rr + 3r^2)) \\
 & + r^2\{32(R - 2r)R^3 + 8R^3r + 65R^2r^2 + 38Rr^3 + 5r^4\} \stackrel{?}{\geq} 0 \\
 & \Leftrightarrow 8t^5 - 23t^4 + 11t^3 + 5t^2 + 13t - 22 \stackrel{?}{\geq} 0 \quad \left(t = \frac{R}{r} \right) \\
 & \Leftrightarrow (t - 2)\left((t - 2)(8t^3 + 9t^2 + 15t + 29) + 69\right) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \\
 & \Rightarrow \text{(ii) is true} \because \text{combining cases (1), (2),}
 \end{aligned}$$

$$\begin{aligned}
 & \text{(ii) is true for all triangles} \Rightarrow \text{(i) is true for all triangles} \Rightarrow \sum \frac{a^2}{b^2 + c^2} - \frac{3}{2} - \frac{r(R - 2r)}{R^2 + r^2} \\
 & \geq 0 \Rightarrow \sum \frac{a^2}{b^2 + c^2} \geq \frac{3}{2} + \frac{r(R - 2r)}{R^2 + r^2} \quad (\text{QED})
 \end{aligned}$$

2800. In ΔABC , $A \geq B \geq C$, P – point in plane of ΔABC , the following

relationship holds

$$AP \cdot \cos \frac{A}{2} + BP \cdot \cos \frac{B}{2} + CP \cdot \cos \frac{C}{2} \geq \sqrt{3}(R + r)$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{aligned}
 & \left(\cos \frac{A}{2} + \cos \frac{B}{2}\right)^2 = \frac{1 + \cos A}{2} + \frac{1 + \cos B}{2} + 2 \cos \frac{A}{2} \cdot \cos \frac{B}{2} \geq 1 + \frac{\cos A + \cos B}{2} \\
 & = 1 + \sin \frac{C}{2} \cdot \cos \left(\frac{A-B}{2}\right) \geq 1 \geq \cos^2 \frac{C}{2} \\
 & \left(\because \frac{A-B}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \cos \left(\frac{A-B}{2}\right) \geq 0\right) \rightarrow \cos \frac{A}{2} + \cos \frac{B}{2} \geq \cos \frac{C}{2} \quad (\text{And analogs})
 \end{aligned}$$



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$$\rightarrow a_1 = \cos \frac{A}{2}, b_1 = \cos \frac{B}{2}, c_1$$

$= \cos \frac{C}{2}$ can be the sides of triangle Δ_1 with area F_1 such that :

$$\begin{aligned} 16F_1^2 &= 2 \sum \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} - \sum \cos^4 \frac{A}{2} \\ &= \frac{1}{2} \sum (1 + \cos A)(1 + \cos B) - \frac{1}{4} \sum (1 + \cos A)^2 = \\ &= \frac{3}{4} + \frac{1}{2} \sum \cos A + \frac{1}{2} \sum \cos A \cos B - \frac{1}{4} \sum \cos^2 A \\ &= \frac{3}{4} + \frac{1}{2} \left(1 + \frac{r}{R} \right) + \frac{1}{2} \left(\frac{s^2 + r^2}{4R^2} - 1 \right) - \frac{1}{4} \left(3 - \frac{s^2 - r^2 - 4Rr}{2R^2} \right) \\ &\rightarrow 16F_1^2 = \frac{s^2}{4R^2} \rightarrow F_1 = \frac{s}{8R} \quad (1) \end{aligned}$$

We know that for any triangles ΔABC and Δ_1 , and any point P in the plane of ΔABC , we have :

$$\sum a_1 \cdot AP \geq \sqrt{\frac{1}{2} \sum a_1^2 (b^2 + c^2 - a^2) + 8FF_1} \quad (\text{Bottema}) \quad (2)$$

$$\begin{aligned} \sum a_1^2 (b^2 + c^2 - a^2) &= \sum \cos^2 \frac{A}{2} (b^2 + c^2 - a^2) = 2s \sum (s - a) \cos A \\ &= 2s^2 \sum \cos A - 2s \sum a \cos A = \\ &= 2s^2 \left(1 + \frac{r}{R} \right) - 2s \cdot \frac{2sr}{R} = 2s^2 \left(1 - \frac{r}{R} \right) \rightarrow \sum a_1^2 (b^2 + c^2 - a^2) = 2s^2 \left(1 - \frac{r}{R} \right) \quad (3) \\ (1), (2), (3) \rightarrow \sum \cos \frac{A}{2} \cdot AP &\geq \sqrt{\frac{1}{2} \cdot 2s^2 \left(1 - \frac{r}{R} \right) + 8sr \cdot \frac{s}{8R}} = s \end{aligned}$$

So, it is suffices to prove : $s \geq \sqrt{3}(R + r) \leftrightarrow \sum \sin A \geq \sqrt{3} \sum \cos A$

$$\leftrightarrow \sum \sin \left(A - \frac{\pi}{3} \right) \geq 0 \leftrightarrow \sum \overset{(4)}{\sin x} \geq 0, \text{ where } x = A - \frac{\pi}{3}, y = B - \frac{\pi}{3}, z = C - \frac{\pi}{3}.$$

We have : $\pi > x \geq y \geq z \geq 0 \geq z$ and $x + y + z = 0 \rightarrow (4)$

$$\leftrightarrow \sin x + \sin y - \sin(x + y) \geq 0$$

$$\leftrightarrow \sin x + \sin y - (\sin x \cos y + \sin y \cos x) \geq 0$$

$$\leftrightarrow \sin x (1 - \cos y) + \sin y (1 - \cos x) \geq 0$$

Which is true because $\sin x, \sin y, 1 - \cos x, 1 - \cos y \geq 0$.

$$\text{Therefore, } AP \cdot \cos \frac{A}{2} + BP \cdot \cos \frac{B}{2} + CP \cdot \cos \frac{C}{2} \geq s \geq \sqrt{3}(R + r).$$



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Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 & \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} = \frac{s(s-a)}{bc} + \frac{s(s-b)}{ca} - \frac{s(s-c)}{ab} \\
 &= \frac{s}{abc}(a(s-a) + b(s-b) - c(s-c)) = \frac{s}{abc}(x(y+z) + y(z+x) - z(x+y)) \\
 &= \frac{s}{abc}(2xy) > 0
 \end{aligned}$$

(taking $x = s-a, y = s-b, z = s-c$, which subsequently implies $a = y+z, b = z+x, c = x+y$ and triangle inequality $\Rightarrow x, y, z > 0$) $\therefore \cos^2 \frac{C}{2} < \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2}$

$$\begin{aligned}
 &< \left(\cos \frac{A}{2} + \cos \frac{B}{2} \right)^2 \Rightarrow \cos \frac{A}{2} + \cos \frac{B}{2} - \cos \frac{C}{2} > 0 \text{ and analogs} \\
 &\Rightarrow \cos \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2} \text{ form sides of a triangle} \rightarrow (1) \text{ with area } F_1 \\
 &= \frac{1}{4} \sqrt{2 \sum_{\text{cyc}} \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} - \sum_{\text{cyc}} \cos^4 \frac{A}{2}} \\
 &= \frac{1}{4} \sqrt{4 \sum_{\text{cyc}} \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} - \left(\sum_{\text{cyc}} \cos^2 \frac{A}{2} \right)^2} \\
 &= \frac{1}{4} \sqrt{4 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} \sum_{\text{cyc}} \sec^2 \frac{A}{2} - \frac{1}{4} \left(\sum_{\text{cyc}} (1 + \cos A) \right)^2} \\
 &= \frac{1}{4} \sqrt{\left(\frac{s^2}{4R^2} \right) \frac{(4R+r)^2 + s^2}{s^2} - \frac{(4R+r)^2}{4R^2}} = \frac{s}{8R} \Rightarrow 8F_1 \stackrel{(i)}{\cong} \frac{s}{R}
 \end{aligned}$$

Via Bottema, for any arbitrary point P in the plane of ΔABC and for any arbitrary $\Delta A'B'C'$,

$$\begin{aligned}
 & (AP \cdot a' + BP \cdot b' + CP \cdot c')^2 \\
 & \geq \frac{1}{2} (a^2(b'^2 + c'^2 - a'^2) + b^2(c'^2 + a'^2 - b'^2) + c^2(a'^2 + b'^2 - c'^2)) \\
 & + 8[ABC][A'B'C'] \therefore \text{via (1) and choosing } a' = \cos \frac{A}{2}, b' = \cos \frac{B}{2}, c' = \cos \frac{C}{2}, \\
 & \quad \left(AP \cdot \cos \frac{A}{2} + BP \cdot \cos \frac{B}{2} + CP \cdot \cos \frac{C}{2} \right)^2 \\
 & \geq \frac{1}{2} \sum_{\text{cyc}} a^2 \left(\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} - \cos^2 \frac{A}{2} \right) \\
 & + 8rsF_1 \stackrel{\text{via (i)}}{\cong} \frac{1}{2} \sum_{\text{cyc}} a^2 \left(\frac{s(s-b)}{ca} + \frac{s(s-c)}{ab} - \frac{s(s-a)}{bc} \right) + rs \left(\frac{s}{R} \right)
 \end{aligned}$$



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$$\begin{aligned}
 &= \frac{s}{8Rrs} \left(2s \sum_{\text{cyc}} a^2(s-a) - \sum_{\text{cyc}} a^2(b^2 + c^2 - a^2) \right) + \frac{rs^2}{R} \\
 &= \frac{1}{8Rr} \left(4s^2(s^2 - 4Rr - r^2) - 4s^2(s^2 - 6Rr - 3r^2) - 2 \sum_{\text{cyc}} a^2 b c \cos A \right) + \frac{rs^2}{R} \\
 &= \frac{1}{8Rr} \left(4s^2(2Rr + 2r^2) - 8R^2rs \sum_{\text{cyc}} \sin 2A \right) + \frac{rs^2}{R} \\
 &= \frac{1}{8Rr} \left(4s^2(2Rr + 2r^2) - 8R^2rs(2 \sin C \cos(A-B) - 2 \sin C \cos(A+B)) \right) + \frac{rs^2}{R} \\
 &= \frac{1}{8Rr} \left(4s^2(2Rr + 2r^2) - 8R^2rs \cdot 4 \prod_{\text{cyc}} \sin A \right) + \frac{rs^2}{R} \\
 &= \frac{1}{8Rr} \left(4s^2(2Rr + 2r^2) - 8R^2rs \cdot 4 \left(\frac{4Rrs}{8R^3} \right) \right) + \frac{rs^2}{R} \\
 &= \frac{8Rrs^2 + 8r^2s^2 - 16r^2s^2 + 8r^2s^2}{8Rr} = s^2 \\
 &\Rightarrow \boxed{\text{AP.} \cos \frac{A}{2} + \text{BP.} \cos \frac{B}{2} + \text{CP.} \cos \frac{C}{2} \stackrel{(*)}{\geq} s}
 \end{aligned}$$

$$\text{Now, } s \geq \sqrt{3}(R+r) \Leftrightarrow \frac{s}{R} \geq \sqrt{3} \left(1 + \frac{r}{R} \right) \Leftrightarrow \sum \sin A \geq \sqrt{3} \sum \cos A \Leftrightarrow \sum \left(\frac{1}{2} \sin A - \frac{\sqrt{3}}{2} \cos A \right)$$

$$\geq 0$$

$$\Leftrightarrow \sum \left(\sin A \cdot \cos \frac{\pi}{3} - \cos A \cdot \sin \frac{\pi}{3} \right) \geq 0 \Leftrightarrow \sum \sin \left(A - \frac{\pi}{3} \right) \geq 0$$

$$\Leftrightarrow \sin \left(A - \frac{\pi}{3} \right) + \sin \left(B - \frac{\pi}{3} \right) + 2 \sin \left(\frac{C}{2} - \frac{\pi}{6} \right) \cos \left(\frac{C}{2} - \frac{\pi}{6} \right) \geq 0$$

$$\Leftrightarrow 2 \left(\sin \frac{A+B-\frac{2\pi}{3}}{2} \right) \left(\cos \frac{A-B}{2} \right) - 2 \left(\sin \left(\frac{\pi}{6} - \frac{C}{2} \right) \right) \left(\cos \left(\frac{\pi}{6} - \frac{C}{2} \right) \right) \geq 0$$

$$\Leftrightarrow 2 \left(\sin \left(\frac{\pi}{6} - \frac{C}{2} \right) \right) \left(\cos \frac{A-B}{2} - \cos \left(\frac{\pi}{6} - \frac{C}{2} \right) \right) \stackrel{(I)}{\geq} 0$$

$$\text{Now, } \cos \frac{A-B}{2} - \cos \left(\frac{\pi}{6} - \frac{C}{2} \right) = 2 \left(\sin \frac{\frac{A-B}{2} + \frac{\pi}{6} - \frac{C}{2}}{2} \right) \left(\sin \frac{\frac{\pi}{6} - \frac{C}{2} - \frac{A-B}{2}}{2} \right)$$

$$= 2 \sin \left(\frac{\frac{A-(\pi-A)}{2} + \frac{\pi}{6}}{2} \right) \sin \left(\frac{\frac{\pi}{6} - \frac{(\pi-B)-B}{2}}{2} \right)$$



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$$= 2 \left(\sin \left(\frac{A}{2} - \frac{\pi}{6} \right) \right) \left(\sin \left(\frac{B}{2} - \frac{\pi}{6} \right) \right)$$

$$\Rightarrow 2 \left(\sin \left(\frac{\pi}{6} - \frac{C}{2} \right) \right) \left(\cos \frac{A-B}{2} \right)$$

$$- \cos \left(\frac{\pi}{6} - \frac{C}{2} \right) \stackrel{(m)}{\cong} 4 \left(\sin \left(\frac{\pi}{6} - \frac{C}{2} \right) \right) \left(\sin \left(\frac{A}{2} - \frac{\pi}{6} \right) \right) \left(\sin \left(\frac{B}{2} - \frac{\pi}{6} \right) \right)$$

$$\text{Now, } \because 0 < \frac{C}{2} \leq \frac{\pi}{6} \therefore 0 \leq \frac{\pi}{6} - \frac{C}{2} < \frac{\pi}{6} \text{ and } \because \frac{\pi}{6} \leq \frac{A}{2}, \frac{B}{2} < \frac{\pi}{2} \therefore 0 \leq \frac{A}{2} - \frac{\pi}{6}, \frac{B}{2} - \frac{\pi}{6} < \frac{\pi}{3}$$

$$\therefore \boxed{\sin \left(\frac{\pi}{6} - \frac{C}{2} \right), \sin \left(\frac{A}{2} - \frac{\pi}{6} \right), \sin \left(\frac{B}{2} - \frac{\pi}{6} \right) \geq 0}$$

$$\Rightarrow 4 \left(\sin \left(\frac{\pi}{6} - \frac{C}{2} \right) \right) \left(\sin \left(\frac{A}{2} - \frac{\pi}{6} \right) \right) \left(\sin \left(\frac{B}{2} - \frac{\pi}{6} \right) \right)$$

$$\geq 0 \stackrel{\text{via (m)}}{\Rightarrow} 2 \left(\sin \left(\frac{\pi}{6} - \frac{C}{2} \right) \right) \left(\cos \frac{A-B}{2} - \cos \left(\frac{\pi}{6} - \frac{C}{2} \right) \right) \geq 0 \Rightarrow (l) \text{ is true}$$

$$\Rightarrow \boxed{s \stackrel{(**)}{\geq} \sqrt{3}(R+r)}$$

$$\therefore (*), (**) \Rightarrow AP \cdot \cos \frac{A}{2} + BP \cdot \cos \frac{B}{2} + CP \cdot \cos \frac{C}{2} \geq \sqrt{3}(R+r) \text{ (QED)}$$



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It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru