RMM - Inequalities Marathon 601 - 700

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601. If $0 < a \leq b \leq \frac{c}{3c+1}$ then:

$$\sqrt[3]{3 + \frac{1}{c}} > \frac{1}{\sqrt[3]{c}} + \sqrt[3]{ab}$$

Proposed by Pavlos Trifon-Greece

Solution by Tran Hong-Dong Thap-Vietnam

Because: $0 < a \leq b \leq \frac{c}{3c+1} \to \sqrt[3]{ab} \leq \sqrt[3]{\left(\frac{c}{3c+1}\right)^2} \to \frac{1}{\sqrt[3]{c}} + \sqrt[3]{ab} \leq \sqrt[3]{\frac{1}{c}} + \sqrt[3]{\left(\frac{c}{3c+1}\right)^2}$

We need to prove:

$$\sqrt[3]{\frac{1}{c}} + \sqrt[3]{\left(\frac{c}{3c+1}\right)^2} < \sqrt[3]{3 + \frac{1}{c}} \overset{x=\sqrt[3]{\frac{1}{c}}>0}{\Leftrightarrow} x + \sqrt[3]{\left(\frac{1}{3 + x^3}\right)^2} < \sqrt[3]{3 + x^3} \leftrightarrow$$

$$x\sqrt[3]{(3 + x^3)^2} + 1 < 3 + x^3 \leftrightarrow x\sqrt[3]{(3 + x^3)^2} < 2 + x^3 \leftrightarrow$$

$$x^3(3 + x^3)^2 < (2 + x^3)^3 \leftrightarrow (2 + x^3)^3 - x^3(3 + x^3)^2 > 0$$

$$\leftrightarrow 3x^3 + 8 > 0; (\text{true by } x > 0)$$

602. If $a, b, c \in [0, 1)$ then:

$$\frac{1}{a^4 - 1} + \frac{1}{b^4 - 1} + \frac{1}{c^2 - 1} \leq \frac{2\sqrt{2}}{abc - 1}$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$\frac{1}{1 - a^4} + \frac{1}{1 - b^4} + \frac{1}{1 - c^2} \geq 2\sqrt{\frac{1}{(1 - a^4)(1 - b^4)}} + \frac{1}{1 - c^2} \geq$$

$$\geq 2\sqrt{2\sqrt{\frac{1}{(1 - a^4)(1 - b^4)}} \cdot \frac{1}{1 - c^2}} = 2\sqrt{2}\sqrt{\sqrt{\frac{1}{(1 - a^4)(1 - b^4)}} \cdot \frac{1}{1 - c^2}} \overset{(1)}{\geq} \frac{2\sqrt{2}}{1 - abc}$$

$$(1) \Leftrightarrow \frac{1}{(1 - a^4)(1 - b^4)(1 - c^2)^2} \geq \frac{1}{(1 - abc)^4} \Leftrightarrow$$

$$(1 - a^4)(1 - b^4)(1 - c^2)^2 \leq (1 - abc)^4$$

$$(1-a^4)(1-b^4)\le(1-a^2b^2)^2;(1)$$

$$\Leftrightarrow 1-a^4-b^4+a^4b^4\le 1-2a^2b^2+a^4b^4\Leftrightarrow a^4+b^4\ge 2a^2b^2\Rightarrow(1)-true.$$

Now, $(1-a^4)(1-b^4)(1-c^2)^2\le(1-a^2b^2)^2(1-c^2)^2\overset{(2)}{\le}(1-abc)^4$

$$(2)\Leftrightarrow(1-a^2b^2)(1-c^2)\le(1-abc)^2-true.$$

603. If $a,b,c>0$ then prove:

$$\frac{abc}{(a+b)(b+c)(c+a)}\le\frac{(a+b)(a+b+2c)}{(3a+3b+2c)^2}\le\frac{1}{8}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{abc}{(a+b)(b+c)(c+a)}\overset{(2)}{\le}\frac{(a+b)(a+b+2c)}{(3a+3b+2c)^2}\overset{(1)}{\le}\frac{1}{8}$$

$(1)\leftrightarrow[2(a+b)+(a+b+2c)]^2\ge 4[2(a+b)](a+b+2c)$, which is true because

$$(x+y)^2\ge 4xy.$$

Let $(p,q,r)=(\sum a,\sum ab,abc)$.

$$(2)\leftrightarrow r(3p-c)^2\le(p-c)(p+c)(pq-r)$$

$$\leftrightarrow 9p^2r-6prc+rc^2\le p^3q-p^2r-pqc^2+rc^2$$

$$\leftrightarrow p^2q-10pr+6rc-qc^2\ge 0$$

$$\leftrightarrow\left(\sum a^2+2\sum ab\right)\left(\sum ab\right)-10abc\sum a+6abc^2-\left(\sum ab\right)c^2\ge 0$$

$$\leftrightarrow\sum a^3b+2\sum a^2b^2-5abc\sum a+6abc^2-abc^2-bc^3-ac^3\ge 0$$

$$\leftrightarrow a^3b+a^3c+b^3a+b^3c+2\sum a^2b^2-5a^2bc-5ab^2c\ge 0$$

-which is true from AM-GM;

$$\begin{cases}b^3a+a^3c+a^2c^2+a^2c^2+a^2b^2\ge 5a^2bc\\ a^3b+b^3c+b^2c^2+b^2c^2+a^2b^2\ge 5ab^2c\end{cases}$$

604. If $a,b,c>0$ such that $a+b+c=6, ab+bc+ca=9, a<b<c$.

Prove that: $a\in(0,1), b\in(1,3), c\in(3,4)$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Morocco

Let $f(x) = (x-a)(x-b)(x-c) = x^3 - (\sum a)x^2 + (\sum ab)x - abc$

$f(x) = x^3 - 6x^2 + 9x - r; r = abc$. We have:

$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$

x	0	1	3	$+\infty$
$f(x)$	$\nearrow$		$\searrow$	$\nearrow$

$\rightarrow f(x) = 0$ admits 3 different solutions $a < b < c \rightarrow a \in (0,1), b \in (1,3), c \in (3,4)$

$$9 = ab + c(a+b) \overset{AM-GM}{\leq} \frac{(a+b)^2}{4} + c(6-c) = \frac{(6-c)^2}{4} + c(6-c) \rightarrow$$

$$0 \leq 3c(4-c) \rightarrow c \leq 4$$

$c = 4 \rightarrow a + b = 2, ab = 1 \rightarrow a = b = 1$ contradiction.

Therefore, $a \in (0,1), b \in (1,3), c \in (3,4)$

605. If $0 \leq \frac{a}{2} \leq b \leq a$ then:

$$2e^a + 3\sqrt[3]{e^a} + 1 \geq 4\sqrt{e^a} + e^b + e^{a-b}$$

Proposed by Pavlos Trifon-Greece

Solution by proposer

For $0 \leq \frac{a}{2} \leq b \leq a \Rightarrow (a,0,0) \succ (b, a-b, 0)$; (Karamata for $e^x \nearrow$) $\Rightarrow$

$e^a + 2 \geq e^b + e^{a-b} + 1 \Rightarrow e^a + 1 \geq e^b + e^{a-b}; (1)$. Equality for $a = b$.

Popoviu for $f(x) = e^x \Rightarrow$

$$f(a) + f(0) + f(0) + 3f\left(\frac{a+0+0}{3}\right) \geq 2\left(f\left(\frac{a+0}{2}\right) + f\left(\frac{0+0}{2}\right) + f\left(\frac{0+a}{2}\right)\right) \Rightarrow$$

$e^a + 2 + 3e^{\frac{a}{3}} \geq 4e^{\frac{a}{2}} + 2 \Rightarrow e^a + 3e^{\frac{a}{3}} \geq 4e^{\frac{a}{2}}; (2)$. Equality for $a = 0$.

From (1),(2) it follows that:

$$(e^a + 1) + \left(e^a + 3e^{\frac{a}{3}}\right) \geq \left(e^b + e^{a-b}\right) + 4e^{\frac{a}{2}} \Rightarrow$$

$$2e^a + 3e^{\frac{a}{3}} + 1 \geq e^b + e^{a-b} + 4e^{\frac{a}{2}}$$

Therefore, $2e^a + 3\sqrt[3]{e^a} + 1 \geq 4\sqrt{e^a} + e^b + e^{(a-b)}$

Equality holds if and only if $a = b = 0$.

606. If $x > 0, y \geq 0$ then:

$$\frac{(x+2y)^3-(x+y)^3}{2x+3y}+\frac{(x+y)^3-y^3}{2x+y}+\frac{y^3-(x+2y)^3}{2x+2y}\leq\frac{y^2}{2}$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\frac{(x+2y)^3-(x+y)^3}{2x+3y}+\frac{(x+y)^3-y^3}{2x+y}+\frac{y^3-(x+2y)^3}{2x+2y}\leq\frac{y^2}{2}\Leftrightarrow$$

$$(x+2y)^3\left(\frac{1}{2x+3y}-\frac{1}{2x+3y}\right)+(x+y)^3\left(\frac{1}{2x+y}-\frac{1}{2x+3y}\right)$$

$$+y^3\left(\frac{1}{2x+2y}-\frac{1}{2x+y}\right)\leq\frac{y^2}{2}\Leftrightarrow$$

$$-\frac{y(x+2y)^3}{(2x+3y)(2x+2y)}+\frac{2y(x+y)^3}{(2x+y)(2x+3y)}-\frac{y^4}{(2x+2y)(2x+y)}\leq\frac{y^2}{2}\Leftrightarrow$$

$$y\left[\frac{y}{2}+\frac{(x+2y)^2}{(2x+3y)(2x+2y)}+\frac{y^3}{(2x+2y)(2x+y)}-\frac{2(x+y)^3}{(2x+y)(2x+3y)}\right]\geq 0;(1)$$

Since $y \geq 0$, we need to prove:

$$\frac{y}{2}+\frac{(x+2y)^2}{(2x+3y)(2x+2y)}+\frac{y^3}{(2x+2y)(2x+y)}-\frac{2(x+y)^3}{(2x+y)(2x+3y)}\geq 0;\forall x>0;y\geq 0$$

$$y(2x+2y)(2x+3y)(2x+y)+(2x+y)(x+2y)^3+y^3(2x+3y)-2(x+y)^4\geq 0$$

$$\frac{y}{x}\left(2+\frac{2y}{x}\right)\left(2+\frac{3y}{x}\right)\left(2+\frac{y}{x}\right)+\left(2+\frac{y}{x}\right)\left(1+\frac{2y}{x}\right)^3+\left(\frac{y}{x}\right)^3\left(2+\frac{3y}{x}\right)-2\left(1+\frac{y}{x}\right)^4\geq 0$$

$$t(2+2t)(2+3t)(2+t)+(2+t)(1+2t)^3+t^3(2+3t)-2(1+t)^4\geq 0$$

$$\left(\because t=\frac{y}{x}\geq 0\right)$$

$15t^4+44t^3+42t^2+13t\geq 0\Leftrightarrow t(t+1)(15t^2+29t+13)\geq 0$, **which is true because:** $t\geq 0\Rightarrow t(t+1)\geq 0; 15t^2+29t+13>0\Rightarrow(1)$ **is true.**

607. If $0 < a \leq b \leq \frac{c}{3c+1}$ prove that:

$$\frac{a+1}{2a+1}+\frac{b+1}{2b+1}+\frac{c+1}{2c+1}<\frac{1}{a+1}+\frac{1}{b+1}+\frac{3c+1}{4c+1}$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$0 < a < \frac{c}{3c+1} < \frac{1}{3} \Rightarrow \frac{a+1}{2a+1} > \frac{4}{3(2a+1)} < \frac{4}{3(2a+2)} = \frac{2}{3(a+1)} < \frac{1}{a+1}; (1)$$

$$0 < b < \frac{c}{3c+1} < \frac{1}{3} \Rightarrow \frac{b+1}{2b+1} < \frac{4}{3(2b+1)} < \frac{4}{3(2b+2)} = \frac{2}{3(b+1)} < \frac{1}{b+1}; (2)$$

$\frac{c+1}{2c+1} \overset{(3)}{<} \frac{3c+1}{4c+1}$; $(3) \Leftrightarrow 4c^2 + 5c + 1 < 6c^2 + 5c + 1 \Leftrightarrow 2c^2 > 0$, which is true by:

$$\frac{c}{3c+1} > 0 \Leftrightarrow c(3c+1) > 0 \Leftrightarrow c \neq 0.$$

From (1),(2),(3) it follows that:

$$\frac{a+1}{2a+1} + \frac{b+1}{2b+1} + \frac{c+1}{2c+1} < \frac{1}{a+1} + \frac{1}{b+1} + \frac{3c+1}{4c+1}$$

Solution 2 by Michael Sterghiou-Greece

$$\frac{a+1}{2a+1} + \frac{b+1}{2b+1} + \frac{c+1}{2c+1} < \frac{1}{a+1} + \frac{1}{b+1} + \frac{3c+1}{4c+1}; (1)$$

We have: $\frac{a+1}{2a+1} - \frac{1}{a+1} = \frac{a^2}{(2a+1)(a+1)} < a^2 \Leftrightarrow -\frac{a^3(2a+3)}{(2a+1)(a+1)} < 0$ true.

Similarly: $\frac{b+1}{2b+1} - \frac{1}{b+1} < b^2$ and $\frac{c+1}{2c+1} - \frac{3c+1}{4c+1} = -\frac{2c^2}{(2c+1)(4c+1)}$

Now, $a^2, b^2 \leq \frac{c^2}{(3c+1)^2}$. Combining the above we need to show:

$\frac{2c^2}{(3c+1)^2} - \frac{2c^2}{(2c+1)(4c+1)} < 0$ or $-\frac{2c^4}{(2c+1)(3c+1)^2(4c+1)} < 0$ which is true.

Equality does not holds as $c = 0$.

608. If $a, b, c, d, e, f, g, h, i > 0$,

$a^2 + b^2 + c^2 = d^2 + e^2 + f^2 = g^2 + h^2 + i^2 = \sqrt[3]{2}$ then:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \cdot \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \leq 2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Abdul Hannan-Tezpur-India

Let: $A = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$. To prove: $det(A^T)det(A) \leq 2$.

Since: $det(A^T) = det(A)$, we must to prove $det^2A \leq 2$ or $|detA| \leq \sqrt{2}$.

By Hadamard's inequality, $|detA| \leq \|v_1\|\|v_2\|\|v_3\|$, where v_1, v_2, v_3 are the column vectors of A.

$$\Rightarrow |detA| \leq \sqrt{a^2+b^2+c^2}\cdot\sqrt{d^2+e^2+f^2}\cdot\sqrt{g^2+h^2+i^2} = \sqrt[6]{2}\cdot\sqrt[6]{2}\cdot\sqrt[6]{2} = \sqrt{2}$$

This proves the desired inequality.

Solution 2 by Ravi Prakash-New Delhi-India

Let: $\vec{x} = a\vec{m} + b\vec{n} + c\vec{p}; \vec{y} = d\vec{m} + e\vec{n} + f\vec{p}; \vec{z} = g\vec{m} + h\vec{n} + i\vec{p}$ then

$|\vec{x}| = |\vec{y}| = |\vec{z}| = \sqrt[3]{2}$. Also,

$\left|\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}\right| = \left|\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\right|$ = volume of parallelepiped with edges $\vec{x}, \vec{y}, \vec{z}$, hence

$$|[\vec{x}\ \vec{y}\ \vec{z}]| \leq |\vec{x}||\vec{y}||\vec{z}| = \sqrt[6]{8} = \sqrt{2}$$

Therefore, $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\cdot\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \leq 2$

Solution 3 by Geanina Tudose-Romania

$$a^2+b^2+c^2 = d^2+e^2+f^2 = g^2+h^2+i^2 = \sqrt[3]{2}$$

$$A = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\cdot\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} a^2+b^2+c^2 & \overbrace{ad+be+cf}^{\alpha} & \overbrace{ag+bh+ce}^{\beta} \\ ad+be+cf & d^2+e^2+f^2 & \overbrace{dg+eh+fi}^{\gamma} \\ ag+bh+ci & dg+eh+fi & g^2+h^2+i^2 \end{pmatrix} =$$

$$= \begin{pmatrix} \sqrt[3]{2} & \alpha & \beta \\ \alpha & \sqrt[3]{2} & \gamma \\ \beta & \gamma & \sqrt[3]{2} \end{pmatrix} \Rightarrow det(A) = 2 + 2\alpha\beta\gamma - \sqrt[3]{2}(\alpha^2+\beta^2+\gamma^2)$$

We want to show:

$$\sqrt[3]{2}(\alpha^2+\beta^2+\gamma^2) \geq 2\alpha\beta\gamma \Leftrightarrow \alpha^2+\beta^2+\gamma^2 \geq \sqrt[3]{4}\alpha\beta\gamma$$

Since: $\frac{\alpha^2+\beta^2+\gamma^2}{3} \geq \sqrt[3]{\alpha^2\beta^2\gamma^2} \Leftrightarrow \alpha^2+\beta^2+\gamma^2 \geq \sqrt[3]{3^3\alpha^2\beta^2\gamma^2}$

It suffices to show that:

$$\sqrt[3]{3^3\alpha^2\beta^2\gamma^2} \geq \sqrt[3]{4\alpha^3\beta^3\gamma^3} \Leftrightarrow \sqrt[3]{\alpha^2\beta^2\gamma^2}\left(\sqrt[3]{27} - \sqrt[3]{4\alpha\beta\gamma}\right) \geq 0; \ (1)$$

Since $(a-d)^2 + (b-e)^2 + (c-f)^2 \geq 0 \Rightarrow \sqrt[3]{2} + \sqrt[3]{2} - 2\alpha \geq 0$

$$\Rightarrow \alpha \le \sqrt[3]{2}, \beta \le \sqrt[3]{2}, \gamma \le \sqrt[3]{2} \Rightarrow \alpha\beta\gamma \le 2 < \frac{27}{4} \Rightarrow \sqrt[3]{\alpha\beta\gamma} < \sqrt[3]{\frac{27}{4}}$$

$$\Rightarrow \sqrt[3]{27} - \sqrt[3]{4\alpha\beta\gamma} > 0, \sqrt[3]{\alpha^2\beta^2\gamma^2} \ge 0 \Rightarrow (1) \text{ is true and } detA \le 2.$$

609. In ΔABC the following relationship holds:

$$\begin{vmatrix} 9R^2 & a^2 & b^2 & c^2 \\ a^2 & 9R^2 & c^2 & b^2 \\ b^2 & c^2 & 9R^2 & a^2 \\ c^2 & b^2 & a^2 & 9R^2 \end{vmatrix} > 0$$

Proposed by Daniel Sitaru-Romania

Solution by Abdul Hannan-Tezpur-India

We will this later:

$$det\begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 - y^2 = (x-y)(x+y); \quad (1)$$

For square matrices A, B of the same size, we have:

$$\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = \begin{pmatrix} A-B & 0 \\ B & A+B \end{pmatrix}$$

$$det\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} det\begin{pmatrix} A & B \\ B & A \end{pmatrix} det\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = det\begin{pmatrix} A-B & 0 \\ B & A+B \end{pmatrix} \Rightarrow$$

$$det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = det(A-B)det(A+B)$$

$$\text{Take } A = \begin{pmatrix} 9R^2 & a^2 \\ a^2 & 9R^2 \end{pmatrix}, B = \begin{pmatrix} b^2 & c^2 \\ c^2 & b^2 \end{pmatrix}$$

$$A+B = \begin{pmatrix} 9R^2+b^2 & a^2+c^2 \\ a^2+c^2 & 9R^2+b^2 \end{pmatrix} \text{ and } A-B = \begin{pmatrix} 9R^2-b^2 & a^2-c^2 \\ a^2-c^2 & 9R^2-b^2 \end{pmatrix}$$

$$\Rightarrow det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = det\begin{pmatrix} 9R^2+b^2 & a^2+c^2 \\ a^2+c^2 & 9R^2+b^2 \end{pmatrix} det\begin{pmatrix} 9R^2-b^2 & a^2-c^2 \\ a^2-c^2 & 9R^2-b^2 \end{pmatrix} \overset{(1)}{=}$$

$$= (9R^2-b^2+a^2-c^2)(9R^2-b^2-a^2+c^2)(9R^2+b^2+a^2+c^2)(9R^2+b^2-a^2-c^2)$$

Now,

$$9R^2 - b^2 + a^2 - c^2 \overset{Leibniz}{\ge} a^2+b^2+c^2-b^2+a^2-c^2 = 2a^2 > 0$$

$$9R^2 - b^2 - a^2 + c^2 \overset{Leibniz}{\ge} a^2+b^2+c^2-b^2-a^2+c^2 = 2c^2 > 0$$

$$9R^2 + b^2 + a^2 + c^2 > 0$$

$$9R^2 + b^2 - a^2 - c^2 \overset{Leibniz}{\ge} a^2+b^2+c^2+b^2-a^2-c^2 = 2b^2 > 0$$

$$\text{Therefore, } det\begin{pmatrix} A & B \\ B & A \end{pmatrix} > 0$$

10. Let $b_1, b_2, \dots, b_n \in \mathbb{R}$ such that for each $n \ge 1$, $b_{n+1}^2 \ge \frac{b_1^2}{1^3} + \frac{b_2^2}{2^3} + \dots + \frac{b_n^2}{n^3}$.

If N be the least positive integer satisfying the inequality:

$$\sum_{n=1}^{N} \frac{b_{n+1}}{b_1 + b_2 + \dots + b_n} \ge \frac{2021}{1015}$$

Find the value of N.

Proposed by Rajeev Rastogi-India

Solution by Adrian Popa-Romania

$$b_{n+1}^2 \ge \frac{b_1^2}{1^3} + \frac{b_2^2}{2^3} + \dots + \frac{b_n^2}{n^3} \overset{Bergstrom}{\ge} \frac{(b_1 + b_2 + \dots + b_n)^2}{1^3 + 2^3 + \dots + n^3}$$

$$\Rightarrow \left(\frac{b_{n+1}}{b_1 + b_2 + \dots + b_n}\right)^2 \ge \frac{1}{\left(\frac{n(n+1)}{2}\right)^2} \Rightarrow \frac{b_{n+1}}{b_1 + b_2 + \dots + b_n} \ge \frac{2}{n(n+1)}$$

$$\sum_{n=1}^{N} \frac{b_{n+1}}{b_1 + b_2 + \dots + b_n} \ge \sum_{n=1}^{N} \frac{2}{n(n+1)} = 2\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{N} - \frac{1}{N+1}\right)$$

$$\Rightarrow 9N \ge 2021 \Rightarrow N \ge \frac{2021}{9} = 224,(5), N \in \mathbb{N} \Rightarrow N = 225.$$

611. In ΔABC the following relationship holds:

$$\left(6\sqrt{3}+1\right)^2 r^2 \le \begin{vmatrix} \sqrt{a} & \sqrt{b} & \sqrt{c} & \sqrt{r} \\ \sqrt{c} & -\sqrt{r} & -\sqrt{a} & \sqrt{b} \\ \sqrt{b} & -\sqrt{a} & \sqrt{r} & -\sqrt{c} \\ \sqrt{r} & \sqrt{c} & -\sqrt{b} & -\sqrt{a} \end{vmatrix} \ge \frac{\left(6\sqrt{3}+1\right)^2 R^2}{4}$$

Proposed by Radu Diaconu-Romania

Solution by George Florin Şerban-Romania

$$\Delta = \begin{vmatrix} \sqrt{a} & \sqrt{b} & \sqrt{c} & \sqrt{r} \\ \sqrt{c} & -\sqrt{r} & -\sqrt{a} & \sqrt{b} \\ \sqrt{b} & -\sqrt{a} & \sqrt{r} & -\sqrt{c} \\ \sqrt{r} & \sqrt{c} & -\sqrt{b} & -\sqrt{a} \end{vmatrix} = (-1)^{1+1}\sqrt{a}\begin{vmatrix} -\sqrt{r} & -\sqrt{a} & \sqrt{b} \\ -\sqrt{a} & \sqrt{r} & -\sqrt{c} \\ \sqrt{c} & -\sqrt{b} & -\sqrt{a} \end{vmatrix} +$$

$$+(-1)^{1+2}\sqrt{b}\begin{vmatrix} \sqrt{c} & -\sqrt{a} & \sqrt{b} \\ \sqrt{b} & \sqrt{r} & -\sqrt{c} \\ \sqrt{r} & -\sqrt{b} & -\sqrt{a} \end{vmatrix} + (-1)^{1+3}\sqrt{c}\begin{vmatrix} \sqrt{c} & -\sqrt{r} & \sqrt{b} \\ \sqrt{b} & -\sqrt{a} & -\sqrt{c} \\ \sqrt{r} & \sqrt{c} & -\sqrt{a} \end{vmatrix} +$$

$$+(-1)^{1+4}\sqrt{r}\begin{vmatrix}\sqrt{c} & -\sqrt{r} & -\sqrt{a}\\ \sqrt{b} & -\sqrt{a} & \sqrt{r}\\ \sqrt{r} & \sqrt{c} & -\sqrt{b}\end{vmatrix}$$

$$\Delta=(a+b+c)^2+r^2+2r(a+b+c)=(a+b+c+r)^2=(2s+r)^2$$

$$\Delta\geq\left(2\cdot 3r\sqrt{3}+r\right)^2=\left(6\sqrt{3}r+r\right)^2=\left(6\sqrt{3}+1\right)^2r^2;\left(s\geq 3\sqrt{3}r-Mitrinovic\right)$$

$$\Delta\leq\left(2\cdot\frac{3\sqrt{3}R}{2}+r\right)^2=\left(3\sqrt{3}R+r\right)^2\overset{2r\leq R}{\leq}\left(3\sqrt{3}R+\frac{R}{2}\right)^2=\frac{\left(6\sqrt{3}+1\right)^2R^2}{4}$$

612.

$$x>0, A=\frac{\left(logx-1+\frac{1}{x}\right)\left(1-\frac{1}{x}\right)}{\sqrt{2+\frac{1}{x^2}-\frac{2}{x}}}+\frac{(x-1-logx)logx}{\sqrt{log^2x+1}}$$

$$B=\sqrt{x^2-2x+2}-\sqrt{2+\frac{1}{x^2}-\frac{2}{x}}; C=\frac{\left(logx-1+\frac{1}{x}\right)logx}{\sqrt{log^2x+1}}+\frac{(x-1-logx)(x-1)}{\sqrt{x^2-2x+2}}$$

Prove that: $A\leq B\leq C$

Proposed by Pavlos Trifon-Greece

Solution by proposer

If $x=1\Rightarrow A=B=C$.

For $x\in(0,1)\cup(1,\infty)$ we have:

Suppose $a<b<c$, let $f(x)=\sqrt{x^2+1}, f''(x)>0\Rightarrow\exists x_1\in(a,b), x_2\in(b,c)$ such that:

$$f'(x_1)=\frac{f(b)-f(a)}{b-a}, f'(x_2)=\frac{f(c)-f(b)}{c-b}$$

Now, $a<x_1<b, (f'\nearrow)\Rightarrow f'(a)<f'(x_1)<f'(b)\Rightarrow$

$$(b-a)f'(a)<f(b)-f(a)<(b-a)f'(b)$$

$$b<x_2<c, (f'\nearrow)\Rightarrow f'(b)<f'(x_2)<f'(c)\Rightarrow$$

$$(c-b)f'(b)<f(c)-f(b)<(c-b)f'(c)$$

Adding these up inequalities, we get:

$$(b-a)f'(a)+(c-b)f'(b)<f(c)-f(a)<(b-a)f'(b)+(c-b)f'(c);(1)$$

$$\forall x\in(0,1)\cup(1,\infty)\Rightarrow 1-\frac{1}{x}<logx<x-1$$

For $a = 1 - \frac{1}{x}, b = \log x. c = x - 1 \overset{(1)}{\Rightarrow} A < B < C.$

613. In ΔABC the following relationship holds:

$$\frac{8r}{R} \le \begin{vmatrix} -1 & \cos\frac{A-B}{2} & \cos\frac{C-A}{2} \\ \cos\frac{A-B}{2} & -1 & \cos\frac{B-C}{2} \\ \cos\frac{C-A}{2} & \cos\frac{B-C}{2} & -1 \end{vmatrix} \le 4$$

Proposed by Radu Diaconu-Romania

Solution by George Florin Şerban-Romania

$$\Delta = \begin{vmatrix} -1 & \cos\frac{A-B}{2} & \cos\frac{C-A}{2} \\ \cos\frac{A-B}{2} & -1 & \cos\frac{B-C}{2} \\ \cos\frac{C-A}{2} & \cos\frac{B-C}{2} & -1 \end{vmatrix} = -1 + 2\prod_{cyc}\cos\frac{A-B}{2} + \sum_{cyc}\cos^2\frac{A-B}{2}$$

$$\Delta = -1 + 2\frac{s^2 + r^2 + 2Rr}{8R^2} + \frac{s^2 + r^2 + 2Rr}{4R^2} + 1 =$$

$$= 2\frac{s^2 + r^2 + 2Rr}{4R^2} = \frac{s^2 + r^2 + 2Rr}{2R^2}$$

$$\Delta = \frac{s^2 + r^2 + 2Rr}{2R^2} \overset{Gerretsen}{\le} \frac{4R^2 + 4Rr + 3r^2 + r^2 + 2Rr}{2R^2} = \frac{4R^2 + 6Rr + 4r^2}{2R^2} =$$

$$= 2 + \frac{3r}{R} + \frac{3r^2}{r^2} \overset{(1)}{\le} 4$$

$$\frac{R}{r} = t \ge 2 \Rightarrow 2 + \frac{3}{t} + \frac{2}{t^2} \le 4 \Rightarrow 2t^2 + 3t + 2 \ge 0$$

$(t-2)(2t+1) \ge 0 \Rightarrow (1)$ **is true.**

$$\Delta = \frac{s^2 + r^2 + 2Rr}{2R^2} \ge \frac{8r}{R} \Rightarrow s^2 + r^2 + 2Rr \overset{(2)}{\ge} 16Rr \Leftrightarrow s^2 \ge 14Rr - r^2$$

But $s^2 \ge 16Rr - 5r^2 (Gerretsen)$

$$16Rr - 5r^2 \ge 14Rr - r^2 \Leftrightarrow 2Rr \ge 4r^2 \Leftrightarrow R \ge 2r (Euler)$$

614. If $x_1, x_2, \dots, x_n > 0, n \ge 2, \lambda > 0$ then:

$$\frac{nx_1 + 4(1+\lambda)}{nx_2^2 + n^2x_3^3 + \dots + n^{n-1}x_n^n} + \frac{n^2x_2^2 + 4(1+\lambda)}{x_1 + n^2x_3^3 + \dots + n^{n-1}x_n^n} + \dots +$$

$$+\frac{n^nx_n^n + 4(1+\lambda)}{x_1 + nx_2^2 + \dots + n^{n-2}x_{n-1}^{n-1}} \ge \frac{n^3x_1 + n^4x_2^2 + \dots + n^{n+2}x_n^n + 4n^2(1+\lambda)}{(n-1)(x_1 + nx_2^2 + \dots + n^{n-1}x_n^n)}$$

Proposed by Florică Anastase-Romania

Solution 1 by George Florin Şerban-Romania

$$S = x_1 + nx_2^2 + \dots + n^nx_n^n$$

$$\sum_{cyc}\frac{nx_1 + 4(1+\lambda)}{nx_2^2 + n^2x_3^3 + \dots + n^{n-1}x_n^n} = \sum_{cyc}\frac{nx_1 + 4(1+\lambda)}{S - x_1} =$$

$$= \frac{nx_1 + 4(1+\lambda)}{S - x_1} + \frac{n^2x_2^2 + 4(1+\lambda)}{S - nx_2^2} + \dots + \frac{n^nx_n^n + 4(1+\lambda)}{S - n^{n-1}x_n^n} \overset{(1)}{\ge}$$

$$\ge \frac{n^3x_1 + n^4x_2^2 + \dots + n^{n+2}x_n^n + 4n^2(1+\lambda)}{(n-1)(x_1 + nx_2^2 + \dots + n^{n-1}x_n^n)} = \frac{n^3S + 4n^2(1+\lambda)}{(n-1)S}$$

$$\sum_{cyc}\frac{nx_1 + 4(1+\lambda)}{nx_2^2 + n^2x_3^3 + \dots + n^{n-1}x_n^n} = \sum_{cyc}\frac{nx_1 + 4(1+\lambda)}{S - x_1} =$$

$$= \sum_{cyc}\frac{nx_1}{S - x_1} + 4(1+\lambda)\sum_{cyc}\frac{1}{S - x_1} = -\sum_{cyc}\frac{-nx_1 + nS - nS}{S - x_1} + 4(1+\lambda)\sum_{cyc}\frac{1}{S - x_1} =$$

$$= -\sum_{cyc}\frac{n(S - x_1)}{S - x_1} + nS\sum_{cyc}\frac{1}{S - x_1} + 4(1+\lambda)\sum_{cyc}\frac{1}{S - x_1} =$$

$$= -n^2 + nS\sum_{cyc}\frac{1}{S - x_1} + 4(1+\lambda)\sum_{cyc}\frac{1}{S - x_1} \overset{Bergstrom}{\ge}$$

$$\ge -n^2 + nS\frac{n^2}{\sum(S - x_1)} + 4(1+\lambda)\frac{n^2}{\sum(S - x_1)} =$$

$$= -n^2 + \frac{n^3S}{nS - S} + \frac{4n^2(1+\lambda)}{nS - S} = -n^2 + \frac{n^3S}{(n-1)S} + \frac{4n^2(1+\lambda)}{(n-1)S} =$$

$$= \frac{-n^3S + n^2S + n^3S + 4n^2S + 4n^2(1+\lambda)}{(n-1)S} = \frac{n^2S + 4n^2(1+\lambda)}{(n-1)S} =$$

$$=\frac{n^3x_1+n^4x_2^2+\dots+n^{n+2}x_n^n+4n^2(1+\lambda)}{(n-1)(x_1+nx_2^2+\dots+n^{n-1}x_n^n)}$$

Solution 2 by proposer

$$\frac{nx_1+4(1+\lambda)}{nx_2^2+n^2x_3^3+\dots+n^{n-1}x_n^n}+\frac{n^2x_2^2+4(1+\lambda)}{x_1+n^2x_3^3+\dots+n^{n-1}x_n^n}+\dots+$$

$$+\frac{n^nx_n^n+4(1+\lambda)}{x_1+nx_2^2+\dots+n^{n-2}x_{n-1}^{n-1}}\ge\frac{n^3x_1+n^4x_2^2+\dots+n^{n+2}x_n^n+4n^2(1+\lambda)}{(n-1)(x_1+nx_2^2+\dots+n^{n-1}x_n^n)}$$

Hence, $\frac{nx_1+4(1+\lambda)}{n^4(x_2^2+nx_3^3+\dots+n^{n-2}x_n^n)}+\frac{n^2x_2^2+4(1+\lambda)}{n^3(x_1+n^2x_3^3+\dots+n^{n-1}x_n^n)}+\dots+$

$$+\frac{n^nx_n^n+4(1+\lambda)}{n^3(x_1+nx_2^2+\dots+n^{n-2}x_{n-1}^{n-1})}\ge\frac{nx_1+n^2x_2^2+\dots+n^nx_n^n+4(1+\lambda)}{n(n-1)(x_1+nx_2^2+\dots+n^{n-1}x_n^n)}$$

Hence, $\frac{nx_1+4(1+\lambda)}{n^2(n^2x_2^2+n^3x_3^3+\dots+n^nx_n^n)}+\frac{n^2x_2^2+4(1+\lambda)}{n^2(nx_1+n^3x_3^3+\dots+n^nx_n^n)}+\dots+\frac{n^nx_n^n+4(1+\lambda)}{n^2(nx_1+n^2x_2^2+\dots+n^{n-1}x_{n-1}^{n-1})}\ge$

$$\ge\frac{nx_1+n^2x_2^2+\dots+n^nx_n^n+4(1+\lambda)}{(n-1)(nx_1+n^2x_2^2+\dots+n^nx_n^n)}$$

Hence, $\frac{\frac{n^2x_1}{1+\lambda}+4}{\frac{n^2}{1+\lambda}(n^2x_2^2+n^3x_3^3+\dots+n^nx_n^n)}+\frac{\frac{nx_2^2}{1+\lambda}+4}{\frac{n^2}{1+\lambda}(nx_1+n^3x_3^3+\dots+n^nx_n^n)}+\dots+$

$$+\frac{\frac{nx_n}{1+\lambda}+4}{\frac{n^2}{1+\lambda}(nx_1+n^2x_2^2+\dots+n^{n-1}x_{n-1}^{n-1})}\ge\frac{\frac{x_1+x_2^2+\dots+x_n^n}{1+\lambda}+4}{\frac{n-1}{1+\lambda}(nx_1+n^2x_2^2+\dots+n^nx_n^n)}$$

$$a_1=\frac{nx_1}{1+\lambda},a_2=\frac{n^2x_2^2}{1+\lambda},\dots,a_n=\frac{n^nx_n^n}{1+\lambda}$$

Lemma: $a_1,a_2,\dots,a_n>0,n\ge 2$, **then**

$$\frac{na_1+4}{n^2(a_2+a_3+\dots+a_n)}+\frac{na_2+4}{n^2(a_1+a_3+\dots+a_n)}+\dots+\frac{na_n+4}{n^2(a_1+a_2+\dots+a_{n-1})}\ge$$

$$\ge\frac{a_1+a_2+\dots+a_n+4}{(n-1)(a_1+a_2+\dots+a_n)};\ (*)$$

Proof. Denote $S=a_1+a_2+\dots+a_n$, **we have:**

$$\frac{a_1}{S-a_1}+\frac{a_2}{S-a_2}+\dots+\frac{a_n}{S-a_n}=\frac{a_1^2}{a_1(S-a_1)}+\frac{a_2^2}{a_2(S-a_2)}+\dots+\frac{a_n^2}{a_n(S-a_n)}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(a_1+a_2+\dots+a_n)}{S^2-(a_1^2+a_2^2+\dots+a_n^2)}=\frac{S^2}{S^2-(a_1^2+a_2^2+\dots+a_n^2)}$$

$$\frac{S^2}{S^2-(a_1^2+a_2^2+\dots+a_n^2)}\ge\frac{n}{n-1}\Leftrightarrow(n-1)S^2\ge nS^2-n(a_1^2+a_2^2+\dots+a_n^2)\Leftrightarrow$$

$n(a_1^2+a_2^2+\dots+a_n^2)\ge S^2$ **true form CBS.**

Now,

$$\frac{na_1}{n^2(a_2+a_3+\cdots+a_n)}+\frac{na_2}{n^2(a_1+a_3+\cdots+a_n)}+\cdots+\frac{na_n}{n^2(a_1+a_2+\cdots+a_{n-1})}\geq\frac{1}{n-1};(1)$$

$$\frac{1}{S-a_1}+\frac{1}{S-a_2}+\cdots+\frac{1}{S-a_n}\overset{CBS}{\geq}\frac{n^2}{S-a_1+S-a_2+\cdots+S-a_n}=\frac{n^2}{(n-1)S}$$

$$\frac{4}{n^2(a_2+a_3+\cdots+a_n)}+\frac{4}{n^2(a_1+a_3+\cdots+a_n)}+\cdots+\frac{4}{n^2(a_1+a_2+\cdots+a_{n-1})}\geq$$
$$\geq\frac{4}{n^2}\cdot\frac{n^2}{(n-1)S}=\frac{4}{(n-1)S};\ (2)$$

From $(1)\&(2)$ we get:

$$\frac{na_1+4}{n^2(a_2+a_3+\cdots+a_n)}+\frac{na_2+4}{n^2(a_1+a_3+\cdots+a_n)}+\cdots+\frac{na_n+4}{n^2(a_1+a_2+\cdots+a_{n-1})}\geq$$
$$\geq\frac{1}{n-1}+\frac{4}{(n-1)S}=\frac{a_1+a_2+\cdots+a_n+4}{(n-1)(a_1+a_2+\cdots+a_n)}$$

615. If $0<a\leq b$ then:

$$\frac{\left(\frac{a+b}{2}+\sqrt{ab}-\frac{2ab}{a+b}\right)^{\frac{a+b}{2}+\sqrt{ab}-\frac{2ab}{a+b}}}{\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}}\geq\frac{\left(\sqrt{ab}\right)^{\sqrt{ab}}}{\left(\frac{2ab}{a+b}\right)^{2ab}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{\left(\frac{a+b}{2}+\sqrt{ab}-\frac{2ab}{a+b}\right)^{\frac{a+b}{2}+\sqrt{ab}-\frac{2ab}{a+b}}}{\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}}\geq\frac{\left(\sqrt{ab}\right)^{\sqrt{ab}}}{\left(\frac{2ab}{a+b}\right)^{2ab}}$$

$$\frac{\left(\frac{a+b}{2}+\sqrt{ab}-\frac{2ab}{a+b}\right)^{\frac{a+b}{2}}}{\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}}\cdot\frac{\left(\frac{a+b}{2}+\sqrt{ab}-\frac{2ab}{a+b}\right)^{\sqrt{ab}}}{\left(\sqrt{ab}\right)^{\sqrt{ab}}}\geq\frac{\left(\frac{a+b}{2}+\sqrt{ab}-\frac{2ab}{a+b}\right)^{\frac{2ab}{a+b}}}{\left(\frac{2ab}{a+b}\right)^{\frac{2ab}{a+b}}}$$

$$\left(1+\frac{2\sqrt{ab}}{a+b}-\frac{4ab}{(a+b)^2}\right)^{\frac{a+b}{2}}\left(\frac{a+b}{\sqrt{ab}}+1-\frac{2\sqrt{ab}}{a+b}\right)^{\sqrt{ab}}\geq\left(\frac{(a+b)^2}{4ab}+\frac{a+b}{2\sqrt{ab}}-1\right)^{\frac{2ab}{a+b}}$$

Let us denote: $a=x^2,b=y^2$, thus

$$\left(1+\frac{2xy}{x^2+y^2}-\left(\frac{2xy}{x^2+y^2}\right)^2\right)^{\frac{x^2+y^2}{2}}\left(\frac{x^2+y^2}{2xy}+1-\frac{2xy}{x^2+y^2}\right)^{xy}$$

$$\geq\left(\left(\frac{x^2+y^2}{2xy}\right)^2+\frac{x^2+y^2}{2xy}-1\right)^{\frac{2x^2y^2}{s^2+y^2}}$$

$$\left(1+\frac{1}{k}-\frac{1}{k^2}\right)^{\left(\frac{x^2+y^2}{2xy}\right)^2}\left(k+1-\frac{1}{k}\right)^{\frac{x^2+y^2}{2xy}}\geq(k^2+k-1)$$

$$\left(1+\frac{1}{k}-\frac{1}{k^2}\right)^{k^2}\left(k+1-\frac{1}{k}\right)^{k}\geq k^2+k-1$$

$$(1+k-1)(k^2+1-1)\geq k^2+k-1$$

$$k^3\geq k^2+k-1$$

$k^3-k^2-k+1\geq 0\Leftrightarrow(k-1)^2(k+1)\geq 0$ **true.**

Solution 2 by Khaled Abd Imouti-Damascus-Syria

Let $G=\sqrt{ab}, M=\frac{a+b}{2}$, then

$$\frac{\left(M\left(1+\frac{G}{M}-\frac{G}{M^2}^2\right)\right)^{M+G-\frac{G^2}{M}}}{M^M}\geq\frac{G^G}{\left(\frac{G^2}{M}\right)^{\frac{G^2}{M}}}\Leftrightarrow\frac{M^G\left(1+\frac{G}{M}-\frac{G^2}{M^2}\right)^{M+G-\frac{G^2}{M}}}{M^{\frac{G^2}{M}}}\geq\frac{G^G}{\left(\frac{G^2}{M}\right)^{\frac{G^2}{M}}}$$

$$M^G\left(1+\frac{G}{M}-\frac{G^2}{M^2}\right)^{M+G-\frac{G^2}{M}}\geq\frac{M^{\frac{G^2}{M}}G^G}{\left(\frac{G^2}{M}\right)^{\frac{G^2}{M}}}\Leftrightarrow M^G\left(1+\frac{G}{M}-\frac{G^2}{M^2}\right)^{M+G-\frac{G^2}{M}}\geq\left(\frac{M^2}{G^2}\right)^{\frac{G^2}{M}}\left(\frac{G}{M}\right)^G,$$

$$x=\frac{G}{M}\leq 1\Rightarrow(1+x-x^2)^{(1+x-x^2)M}\geq\left(\frac{1}{x^2}\right)^{\frac{G^2}{M}}x^G$$

$(1+x-x^2)^{1+x-x^2}\geq x^{-2x^2+x}$ **is true because,**

$1+x-x^2\geq x, \forall x\in[0,1]$ **and** $1+x-x^2\geq -2x^2+x\Leftrightarrow x^2+1\geq 0.$

616. $x,y,z\geq 0, z=max(x,y,z)$

$$A=9x^3y^3(x^3+y^3)+6x^2y^2z^2(x^6+y^6+z^6)+18z^{12}$$

$$B=(5x^6+5y^6+8z^6)(x^3y^3+x^2y^2z^2+z^6)$$

Prove that: $A\geq B$.

Proposed by Daniel Sitaru-Romania

Solution by George Florin Şerban-Romania

$$z=max(x,y,z)\to z\geq x,z\geq y,z\neq 0,a=\frac{x}{z}\in(0,1],b=\frac{y}{z}\in(0,1]$$

$$A\geq B\Leftrightarrow 9x^3y^3(x^3+y^3)+6x^2y^2z^2(x^6+y^6+z^6)+18z^{12}\geq$$

$$\geq(5x^6+5y^6+8z^6)(x^3y^3+x^2y^2z^2+z^6)\Leftrightarrow$$

$$9a^3b^3(a^3+b^3)+6a^2b^2(a^6+b^6+1)+18\geq(5a^6+5b^6+8)(a^3b^3+a^2b^2+1)\Leftrightarrow$$

$$(4a^3b^3+a^2b^2-5)(a^6+b^6-2)\overset{(*)}{\geq}0$$

$$(4a^3b^3-4a^2b^2+5a^2b^2-5)(a^6+b^6-2)\geq 0$$

$$[4a^2b^2(ab-1)+5(a^2b^2-1)](a^6+b^6-2)\geq 0$$

$$(ab-1)(4a^2b^2+5ab+5)(a^6+b^6-2)\geq 0\ true\ \forall a,b\in(0,1]\to(*)\ true.$$

Therefore, $A\geq B$.

617. If $0<x,y<\frac{\pi}{2}$ then:

$$\frac{sin^2x}{sin^6y}+\frac{sin^2y}{cos^6x}+\frac{cos^2x}{cos^6y}+\frac{cos^2y}{sin^6x}\geq 16$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{sin^2x}{sin^6y}+\frac{sin^2y}{cos^6x}+\frac{cos^2x}{cos^6y}+\frac{cos^2y}{sin^6x}\overset{AM-GM}{\geq}4\sqrt[4]{\frac{sin^2xsin^2ycos^2xcos^2y}{sin^6ycos^6xcos^6ysin^6x}}=$$

$$=\frac{4}{(sinxcosx)(sinycosy)}\overset{AM-GM}{\geq}\frac{4}{\frac{sin^2x+cos^2x}{2}\cdot\frac{sin^2y+cos^2y}{2}}$$

Therefore, $\frac{sin^2x}{sin^6y}+\frac{sin^2y}{cos^6x}+\frac{cos^2x}{cos^6y}+\frac{cos^2y}{sin^6x}\geq 16$

Solution 2 by Adrian Popa-Romania

$$\frac{\sin^2 x}{\sin^6 y}+\frac{\sin^2 y}{\cos^6 x}+\frac{\cos^2 x}{\cos^6 y}+\frac{\cos^2 y}{\sin^6 x}\overset{AM-GM}{\geq} 4\sqrt[4]{\frac{\sin^2 x\sin^2 y\cos^2 x\cos^2 y}{\sin^6 y\cos^6 x\cos^6 y\sin^6 x}}=$$

$$=4\sqrt[4]{\frac{1}{(\sin^4 x\cos^4 x)(\sin^4 y\cos^4 y)}}=4\sqrt[4]{\frac{1}{\frac{\sin^4(2x)}{16}\cdot\frac{\sin^4(2y)}{16}}}=$$

$$=\frac{16}{\sin(2x)\sin(2y)}\geq 16$$

$$\because \sin(2x),\sin(2y)\in(0,1]\Rightarrow \sin(2x)\sin(2y)\leq 1$$

Solution 3 by Samar Das-India

$$\frac{\left(\frac{\sin^2 x}{\sin^6 y}+\frac{\sin^2 y}{\cos^6 x}+\frac{\cos^2 x}{\cos^6 y}+\frac{\cos^2 y}{\sin^6 x}\right)}{4}\overset{AM-GM}{\geq}\sqrt[4]{\frac{\sin^2 x\sin^2 y\cos^2 x\cos^2 y}{\sin^6 y\cos^6 x\cos^6 y\sin^6 x}}$$

$$\frac{\sin^2 x}{\sin^6 y}+\frac{\sin^2 y}{\cos^6 x}+\frac{\cos^2 x}{\cos^6 y}+\frac{\cos^2 y}{\sin^6 x}\overset{AM-GM}{\geq} 4\sqrt[4]{\frac{\sin^2 x\sin^2 y\cos^2 x\cos^2 y}{\sin^6 y\cos^6 x\cos^6 y\sin^6 x}}=$$

$$=\frac{4}{\sin x\sin y\cos x\cos y};(1)$$

$$\frac{\sin x+\sin y+\cos x+\cos y}{4}\overset{AM-GM}{\geq}\sqrt[4]{\sin x\sin y\cos x\cos y}$$

$$\sin x\sin y\cos x\cos y\leq\left(\frac{\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)+\sqrt{2}\cos\left(y-\frac{\pi}{4}\right)}{4}\right)^4$$

$$\sin x\sin y\cos x\cos y\leq\left(\frac{\sqrt{2}+\sqrt{2}}{4}\right)^4=\frac{1}{2};(2)$$

$$\because \cos\left(x-\frac{\pi}{4}\right)\geq 1,\cos\left(y-\frac{\pi}{4}\right)\geq 1$$

From $(1),(2)$ we get:

$$\frac{\sin^2 x}{\sin^6 y}+\frac{\sin^2 y}{\cos^6 x}+\frac{\cos^2 x}{\cos^6 y}+\frac{\cos^2 y}{\sin^6 x}\geq\frac{4}{\frac{1}{4}}=16$$

Solution 4 by Tran Hong-Dong Thap-Vietnam

$$\frac{\sin^2 x}{\sin^6 y}+\frac{\sin^2 y}{\cos^6 x}+\frac{\cos^2 x}{\cos^6 y}+\frac{\cos^2 y}{\sin^6 x}=\frac{\left(\frac{\sin x}{\sin^2 y}\right)^2}{\sin^2 y}+\frac{\left(\frac{\sin y}{\cos^2 x}\right)^2}{\cos^2 x}+\frac{\left(\frac{\cos x}{\cos^2 y}\right)^2}{\cos^2 y}+\frac{\left(\frac{\cos y}{\sin^2 x}\right)^2}{\sin^2 x}\overset{C-S}{\geq}$$

$$\geq\frac{\left(\frac{\sin x}{\sin^2 y}+\frac{\sin y}{\cos^2 x}+\frac{\cos x}{\cos^2 y}+\frac{\cos y}{\sin^2 x}\right)^2}{\sin^2 x+\cos^2 x+\sin^2 y+\cos^2 y}=\frac{\Omega^2}{2}$$

$$\Omega=\frac{\sin x}{\sin^2 y}+\frac{\sin y}{\cos^2 x}+\frac{\cos x}{\cos^2 y}+\frac{\cos y}{\sin^2 x}=\frac{\left(\frac{1}{\sin y}\right)^2}{\frac{1}{\sin x}}+\frac{\left(\frac{1}{\cos x}\right)^2}{\frac{1}{\sin y}}+\frac{\left(\frac{1}{\cos y}\right)^2}{\frac{1}{\cos x}}+\frac{\left(\frac{1}{\sin x}\right)^2}{\frac{1}{\cos y}}\geq$$

$$\overset{C-S}{\geq}\frac{\left(\frac{1}{\sin y}+\frac{1}{\cos x}+\frac{1}{\cos y}+\frac{1}{\sin x}\right)^2}{\frac{1}{\sin x}+\frac{1}{\sin y}+\frac{1}{\cos x}+\frac{1}{\cos y}}=\frac{1}{\sin x}+\frac{1}{\sin y}+\frac{1}{\cos x}+\frac{1}{\cos y}\overset{C-S}{\geq}$$

$$\overset{C-S}{\geq}\frac{(1+1+1+1)^2}{\sin x+\cos x+\sin y+\cos y}=\frac{16}{\sin x+\cos x+\sin y+\cos y}\geq\frac{16}{\sqrt{2}+\sqrt{2}}=4\sqrt{2}$$

$$\left(\because 0<x,y<\frac{\pi}{2}\to 0<\sin x+\cos x\leq\sqrt{2};0<\sin y+\cos y\leq\sqrt{2}\right)$$

$$\Rightarrow P\geq\frac{\Omega^2}{2}\geq\frac{\left(4\sqrt{2}\right)^2}{2}=16$$

618. If $a,b,c,d>0$ then

$$4(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2)\geq(ab+cd)^2$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$2(1-a+a^2)^2\geq 1+a^4\Leftrightarrow(a-1)^4\geq 0 \text{ true.}$$

$$\Rightarrow\prod_{cyc}\left(2(1-a+a^2)^2\right)\geq\left((a^4+1)(b^4+1)\right)\left((1+c^4)(1+d^4)\right)\overset{BCS}{\geq}$$

$$\geq(a^2b^2+1)^2(1+c^2d^2)^2\overset{BCS}{\geq}(ab+cd)^2$$

Therefore,

$$4(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2)\geq(ab+cd)^2$$

619. Find all positive real numbers α, β such that:

$$(1+e^x)^\alpha \ge 1+\beta x, \forall x \ge 0$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Samar Das-West Bengal-India

$(1+e^x)^\alpha \ge 1+\beta x, \forall x \ge 0$, but $e^x > 1 \to$

$$1+e^x\alpha+\frac{\alpha(\alpha-1)}{2!}(e^x)^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}(e^x)^{3x}+\cdots \ge 1+\beta x$$

$$1+\alpha\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)+\frac{\alpha(\alpha-1)}{2!}\left(1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\cdots\right)$$
$$+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\left(1+\frac{3x}{1!}+\frac{(3x)^2}{2!}+\cdots\right) \ge 1+\beta x; (1)$$

Comparing both side:

$$1 \le 1+\frac{\alpha}{1!}+\frac{\alpha(\alpha-1)}{2!}+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}+\cdots$$

$1 \le (1+1)^\alpha \to 2^\alpha \ge 1$ its valid for $0 \le \alpha < \infty; (1)$

Again comparing the coefficient of x then:

$$\beta \le \frac{\alpha}{1!}+\frac{2}{1!}\cdot\frac{\alpha(\alpha-1)}{2!}+\frac{3}{1!}\cdot\frac{\alpha(\alpha-1)(\alpha-2)}{3!}+\cdots$$

$$\beta \le \alpha\left(1+\frac{\alpha}{1!}+\frac{\alpha(\alpha-1)}{2!}+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}+\cdots\right); (2)$$

$$\to \beta \le \alpha(1+1)^{\alpha-1} \to \beta \le \alpha 2^{\alpha-1}; (2)$$

From (1) we see that if we take $\alpha = 0$ then $eq^n(2)$ does not satisfies.

Then $\alpha > 0$ when $0 < \alpha \le 1$ then $0 < \beta \le 1$, then $\beta < \alpha \to 0 < \beta < \alpha \le 1$,

When $\beta < \alpha \to 0 < \beta < \alpha \le 1$.

When $\alpha > 1$ then $\frac{\beta}{\alpha} > 1 \to 1 < \alpha < \beta < \infty$.

620. If $a, b, c, d > 0$ then

$$4(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2) \ge (1+abcd)^2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$2(1-a+a^2)^2 \geq 1+a^4 \Leftrightarrow (a-1)^4 \geq 0 \text{ is true.}$$

$$\rightarrow 2(1-a+a^2)^2 \geq 1+a^4 (and\ analogs) \rightarrow$$

$$\prod_{cyc}(2(1-a+a^2)^2) \geq \prod_{cyc}(1+a^4) \overset{Holder}{\geq} (1+abcd)^4 \Leftrightarrow$$

$$4(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2) \geq (1+abcd)^2$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$4(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2) \geq (1+abcd)^2$$

$$\Leftrightarrow\ 4(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2) \geq (1+(ab)^2)(1+(cd)^2)$$

True because: $2(1-a+a^2)(1-b+b^2) \geq 1+(ab)^2$

$$2(1-b+b^2-a+ab-ab^2+a^2-a^2b+(ab)^2 \geq 1+(ab)^2$$

$$2+2b^2+2ab+2a^2+2ab \geq 1+(ab)^2+2a+2b+2a^2b+2ab^2$$

$$1+(ab)^2+2ab+2(a^2+b^2)+4ab \geq 2(a+b)+2a^2b+2ab^2+4ab$$

$$(1+ab)^2+(a+b)^2+(a+b)^2 \geq 2(a+b)+2a^2b+2ab^2+4ab$$

$2(1+ab)(a+b)+(a+b)^2 \geq 2(a+b)+2(a^2b+ab^2+2ab)$ true and

$$2(1-c+c^2)(1-d+d^2) \geq 1+(cd)^2$$

Therefore, $4(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2) \geq (1+abcd)^2$

621. If $0 \leq x < \frac{\pi}{16}$ then:

$$cos^{1216}x \geq cos(8x) \cdot cos^9(6x) \cdot cos^{34}(4x) \cdot cos^{71}(2x)$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$x \in \left[0, \frac{\pi}{16}\right] \Rightarrow cos(2x), cos(4x), cos(6x), cos(8x) > 0$$

$$cos^2x = \frac{1+cos(2x)}{2} \overset{AM-GM}{\geq} \sqrt{cos(2x)} \Rightarrow cos(2x) \overset{(1)}{\leq} cos^4x \Rightarrow$$

$$cos(4x) \overset{(1)}{\leq} cos^4(2x) \leq cos^{16}x; (2) \text{ and}$$

$$cos(8x) \overset{(1)}{\leq} cos^4(4x) \leq cos^{16}(2x) \leq cos^{16}x; (3)$$

$$\cos(3x)=\cos x(4\cos^2x-3)\overset{AM-GM}{\leq}\cos x(\cos^8x+1+1+1-3)=\cos^9x$$

$$\Rightarrow \cos(6x)\overset{(1)}{\leq}\cos^4(3x)\leq\cos^{36}x;(4)$$

From (1),...,(4) we get:

$$\cos^{1216}x\geq\cos(8x)\cdot\cos^9(6x)\cdot\cos^{34}(4x)\cdot\cos^{71}(2x)$$

622. If $0<x<\frac{\pi}{2}$ then:

$$\frac{8\sin^6x}{1+\cot x}+\frac{8\cos^6x}{1+\tan x}\geq\sin^3(2x)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$x\in\left(0,\frac{\pi}{2}\right)\Rightarrow\cos x,\sin x>0$$

$$\frac{8\sin^6x}{1+\cot x}+\frac{8\cos^6x}{1+\tan x}=\frac{8\sin^7x}{\sin x+\cos x}+\frac{8\cos^7x}{\cos x+\sin x}\overset{Holder}{\geq}\frac{8(\sin x+\cos x)^7}{2^6(\sin x+\cos x)}=$$

$$=\frac{1}{8}(\sin x+\cos x)^6\overset{AM-GM}{\geq}\frac{1}{8}\left(2\sqrt{\sin x\cos x}\right)^6=(2\sin x\cos x)^3=\sin^3(2x)$$

Solution 2 by Alex Szoros-Romania

$$\begin{cases}\sin x=a\in(0,1)\\ \cos x=b\in(0,1)\end{cases}\Rightarrow\begin{cases}a^2+b^2=1\\ \sin 2x=ab\end{cases}$$

$$\frac{\sin^6x}{1+\cot x}=\frac{a^6}{1+\frac{b}{a}}=\frac{a^7}{a+b};\frac{\cos^6x}{1+\tan x}=\frac{b^7}{a+b}$$

$$\frac{8a^7}{a+b}+\frac{8b^7}{a+b}\geq 8a^3b^3\Leftrightarrow\frac{a^7+b^7}{a+b}\geq a^3b^3\Leftrightarrow$$

$$(a^6+b^6)-ab(a^4+b^4)+a^2b^2(a^2+b^2)\geq 2a^3b^3\Leftrightarrow$$

$$(a^2+b^2)(a^4-a^2b^2+b^4)-ab(a^4+b^4)+a^2b^2\geq 2a^3b^3\Leftrightarrow$$

$$a^4+b^4-ab(a^4+b^4)\geq 2a^3b^3\Leftrightarrow$$

$$1-2a^2b^2-ab(1-2a^2b^2)\geq 2a^3b^3\Leftrightarrow$$

$$1-2a^2b^2-ab+2a^3b^3\geq 2a^3b^3\Leftrightarrow 2a^2b^2+ab\leq 1\Leftrightarrow$$

$4a^2b^2+2ab\leq 2\Leftrightarrow\sin^2(2x)+\sin(2x)\leq 2$, **which is true, because**

$$\sin(2x)\leq 1,\forall x\in\mathbb{R}$$

Solution 3 by Hafiz Iqbal-Indonesia

$$\frac{8sin^6x}{1+cotx}+\frac{8cos^6x}{1+tanx}=\frac{8sin^6x}{1+\frac{cosx}{sinx}}+\frac{8cos^6x}{1+\frac{sinx}{cosx}}=$$

$$=\frac{8sin^7x}{sinx+cosx}+\frac{8cos^7x}{sinx+cosx}\geq\frac{8(sinx+cosx)^7}{2^6(sinx+cosx)}=$$

$$=\frac{1}{8}(sinx+cosx)^6\overset{AM-GM}{\geq}\frac{1}{8}\left(\sqrt{4sinxcosx}\right)^6=(2sinxcosx)^3=sin^3(2x)$$

Solution 4 by Samar Das-India

$$\frac{8sin^6x}{1+cotx}+\frac{8cos^6x}{1+tanx}=8\left(\frac{sin^7x}{sinx+cosx}+\frac{cos^7x}{sinx+cosx}\right)=$$

$$=\frac{16}{sinx+cosx}\left(\frac{sin^7x+cos^7x}{2}\right)\geq\frac{16}{sinx+cosx}\left(\frac{sinx+cosx}{2}\right)^7=$$

$$=8\left(\frac{sinx+cosx}{2}\right)^6\overset{AM-GM}{\geq}8\left(\sqrt{sinxcosx}\right)^6=8sin^3xcos^3x=sin^3(2x)$$

Solution 5 by Fayssal Abdelli-Bejaia-Algerie

Let suppose that: $\frac{8sin^6x}{1+cotx}+\frac{8cos^6x}{1+tanx}<sin^3(2x);\ \ (A)$

$$\Leftrightarrow\frac{8sin^6x}{\frac{sinx+cosx}{sinx}}+\frac{8cos^6x}{\frac{cosx+sinx}{cosx}}<2^3sin^3x\cdot cos^3x\Leftrightarrow$$

$$\frac{sin^7x}{sinx+cosx}+\frac{cos^7x}{sinx+cosx}<sin^3x+cos^3x\Leftrightarrow$$

$$sin^7x+cos^7x<(sinx+cosx)sin^3x\cdot cos^3x\Leftrightarrow$$

$$sin^7x+cos^7x<sin^4xcos^3x+sin^3xcos^4x\Leftrightarrow$$

$$(sin^3x-cos^3x)(sin^4x-cos^4x)<0\Leftrightarrow$$

$$(sinx-cosx)(sin^2x+cos^2x+sinxcosx)(sin^2x-cos^2x)<0\Leftrightarrow$$

$(sinx-cosx)^2(sinx+cosx)(1+sinxcosx)<0$ contradiction, because

$$\begin{cases}(sinx-cosx)^2>0\\ sinx++cosx>0\\ 1+sinxcosx>0\end{cases};\forall x\in\left(0,\frac{\pi}{2}\right)$$

Solution 6 by Tran Hong-Dong Thap-Vietnam

Because: $0 < x < \frac{\pi}{2} \Rightarrow 0 < sinx; cosx < 1 \Rightarrow 0 < sin(2x) < 1$

$$\frac{8sin^6x}{1+cotx} + \frac{8cos^6x}{1+tanx} = \frac{8(sin^7x + cos^7x)}{sinx + cosx} =$$

$$= \frac{8\left(\frac{sin^8x}{sinx} + \frac{cos^8x}{cosx}\right)}{sinx + cosx} = \frac{8\left(\frac{(sin^4x)^2}{sinx} + \frac{(cos^4x)^2}{cosx}\right)}{sinx + cosx} \overset{BCS}{\geq}$$

$$\overset{BCS}{\geq} \frac{8\left(\frac{1}{2}(sin^2x + cos^2x)^2\right)^2}{(sinx + cosx)^2} = \frac{2}{1 + sin(2x)} \overset{(1)}{\geq} sin^3(2x)$$

$$(1) \Leftrightarrow \frac{2}{1+t} \geq t^3 (\because t = sin(2x) \Rightarrow t \in (0,1))$$

$$\Leftrightarrow t^4 + t^3 - 2 \leq 0 \Leftrightarrow (t-1)(t^3 + 2t^2 + 2t + 2) \leq 0, \text{ which is true, because:}$$

$$0 < t \leq 2 \Rightarrow t - 1 \leq 0, t^3 + 2t^2 + 2t + 2 > 0.$$

623. If $a, b > 1$ then:

$$\left(\frac{a-b}{a+b}\right)^2 < log\left(\frac{(a+b)^2}{4ab}\right)^{4(a+b)} < \frac{(a^2-b^2)^2}{ab}$$

Proposed by Nikos Ntorvas-Greece

Solution by Pavlos Trifon-Greece

$$logt < t - 1, t > 0; (1) \text{ and } logt > 1 - \frac{1}{t}, t > 0; (2)$$

$$log\frac{(a+b)^2}{4ab} \overset{(2)}{>} 1 - \frac{4ab}{(a+b)^2} \Rightarrow 4(a+b)log\frac{(a+b)^2}{4ab} > 4(a+b) - \frac{16ab}{a+b}$$

As long as we prove that:

$$4(a+b) - \frac{16ab}{a+b} > \left(\frac{a-b}{a+b}\right)^2 \Leftrightarrow 4(a+b)^3 - 16ab(a+b) > (a-b)^2$$

$$\Leftrightarrow (a+b)(4(a+b)^2 - 16ab) > (a-b)^2 \Leftrightarrow (a+b)(4a^2 + 4b^2 + 8ab - 16ab)$$

$$> (a-b)^2 \Leftrightarrow 4(a+b)(a-b)^2 > (a-b)^2; a \neq b \Leftrightarrow 4(a+b) > 1$$

Which is true because $a, b > 1 \Rightarrow a + b > 2 > \frac{1}{4} \Rightarrow 4(a+b) > 1.$

$$\log\frac{(a+b)^2}{4ab}\overset{(1)}{<}\frac{(a+b)^2}{4ab}-1\Rightarrow\log\left(\frac{(a+b)^2}{4ab}\right)^{4(a+b)}<\frac{(a+b)^3}{ab}-4(a+b)$$

As long as we prove that:

$$\frac{(a+b)^3}{ab}-4(a+b)<\frac{(a^2-b^2)^2}{ab}\Leftrightarrow(a+b)^3-4ab(a+b)<\frac{(a^2-b^2)^2}{ab}\Leftrightarrow$$

$$(a+b)^3-4ab(a+b)<(a-b)^2(a+b)^2\Leftrightarrow$$

$$(a+b)^2-4ab<(a-b)^2(a+b)\Leftrightarrow(a+b)^2<(a-b)^2(a+b)\Leftrightarrow$$

$(a\neq b); a+b>1$, **which is true because:** $(a,b>1\Rightarrow a+b>2>1)$.

624. Let $a,b,c>0$ such that $(a+b)(b+c)(c+a)=1$ and $\lambda\geq 2,\mu\geq 2$.

Find the minimum value of expression:

$$P=\frac{a}{b(b+\mu c)(a+\lambda c)^2}+\frac{b}{c(c+\mu a)(b+\lambda a)^2}+\frac{c}{a(a+\mu b)(c+\lambda b)^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$a,b,c>0\Rightarrow\frac{a}{a+\lambda c}+\frac{b}{b+\lambda a}+\frac{c}{c+\lambda b}=\frac{a^2}{a^2+\lambda ac}+\frac{b^2}{b^2+\lambda ba}+\frac{c^2}{c^2+\lambda bc}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(a+b+c)^2}{a^2+b^2+c^2+\lambda(ab+bc+ca)}\overset{(1)}{\geq}\frac{3}{1+\lambda}$$

$$(1)\Leftrightarrow(1+\lambda)(a+b+c)^2\geq 3\left(a^2+b^2+c^2+\lambda(ab+bc+ca)\right)$$

$\Leftrightarrow(\lambda-2)(a^2+b^2+c^2-ab-bc-ca)\geq 0$ **true because**

$\lambda\geq 2, a^2+b^2+c^2\geq ab+bc+ca\Rightarrow(1)$ **true. Now,**

$$P=\sum_{cyc}\frac{a}{b(b+\mu c)(a+\lambda c)^2}=\sum_{cyc}\frac{\left(\frac{a}{a+\lambda c}\right)^2}{ab(b+\mu c)}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc}\frac{a}{a+\lambda c}\right)^2}{ab^2+bc^2+ca^2+3\mu abc}\overset{(1)}{\geq}$$

$$\geq\frac{\left(\frac{3}{1+\lambda}\right)^2}{ab^2+bc^2+ca^2+3\mu abc}=Q$$

We have: $ab^2+bc^2+ca^2\overset{AM-GM}{\geq}3\sqrt[3]{(abc)^3}=abc;(2)$

$$9(a+b)(b+c)(c+a)\geq 8(a+b+c)(ab+bc+ca);(3)$$

$$\Leftrightarrow a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0 \text{ true for } a,b,c > 0$$

$$1 = (a+b)(b+c)(c+a) \geq 8abc \Rightarrow abc \leq \frac{1}{8}$$

$$\Rightarrow ab^2 + bc^2 + ca^2 + 3\mu abc = ab^2 + bc^2 + ca^2 + 3abc + 3(\mu-1)abc \overset{(2)}{\leq}$$

$$\leq (ab^2 + bc^2 + ca^2) + (ba^2 + cb^2 + ac^2) + 3(\mu-1)abc =$$

$$= (a+b+c)(ab+bc+ca) - 3abc + 3(\mu-1)abc =$$

$$= (a+b+c)(ab+bc+ca) + 3(\mu-2)abc \overset{(3),(4)}{\leq}$$

$$= (a+b+c)(ab+bc+ca) + 3(\mu-2)\cdot\frac{1}{8} = \frac{9+3(\mu-2)}{8} = \frac{3(\mu+1)}{8}$$

$$P \geq Q \geq \frac{\left(\frac{3}{1+\lambda}\right)^2}{\frac{3(\mu+1)}{8}} = \frac{24}{(\lambda+1)^2(\mu+1)}$$

$$\Rightarrow P_{min} = \frac{24}{(\lambda+1)^2(\mu+1)} \Leftrightarrow a = b = c = \frac{1}{2}$$

625. Let x, y and z be positive real numbers. Find the minimum value of:

$$f(x,y,z) = \sum_{cyc} \frac{x^3}{x^3 + y^3 + xyz}$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution 1 by Michael Sterghiou-Greece

$$f(x,y,z) = \sum_{cyc} \frac{x^3}{x^3 + y^3 + xyz}; \quad (1)$$

(1)-is homogeneous so WLOG $xyz = 1$. Let $x^3 = a$ and the analogs, so (1):

$$f(a,b,c) = \sum_{cyc} \frac{a}{a+b+1}; (2), abc = 1$$

Let $(p,q,r) = (\sum a, \sum ab, \prod a)$ with $r = 1$. We will show that:

$f \geq 1 \Leftrightarrow \frac{\sum a(b+c+1)(c+a+1)}{\prod(a+b+1)} \geq 1$ which reduces to:

$\frac{(\sum a^2 b) + p^2 + pq + p + q}{p^2 + pq + 2p + q} \geq 1$ which becomes $\sum a^2 b \geq a + b + c$ this is true because:

$$\begin{cases} a^2b + a^2b + c^2a \ge 3\sqrt[3]{a^3(abc)^2} = 3a \\ b^2c + b^2c + a^2b \ge 3b \\ c^2a + c^2a + b^2c \ge 2c \end{cases} \Rightarrow 3\sum a^2b \ge 3p.$$

Therefore,

$$f(x,y,z) = \sum_{cyc} \frac{x^3}{x^3 + y^3 + xyz} \ge 1$$

Equality holds of $x = y = z = 2$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y, z > 0$ we have:

$$f(x,y,z) = \sum_{cyc} \frac{x^3}{x^3 + y^3 + xyz} \ge 1$$

$$\sum x^3(z^3 + y^3 + xyz)(z^3 + x^3 + xyz) \ge \prod (x^3 + y^3 + xyz) \Leftrightarrow$$

$$3(xyz)^3 + 2(x^6y^3 + y^6z^3 + z^6x^3) + (x^6z^3 + z^6y^3 + y^6x^3)$$
$$+ (x^7yz + xy^7z + xyz^7) + (x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5) \ge$$
$$\ge 3(xyz)^3 + (x^6y^3 + y^6z^3 + z^6x^3) + (x^6z^3 + z^6y^3 + y^6x^3)$$
$$+ 3(x^4y^4z + x^4yz^4 + xy^4z^4) + (x^7yz + xy^7z + xyz^7)$$
$$+ 2(x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5)$$
$$\Leftrightarrow x^6y^3 + y^6z^3 + z^6x^3 \ge x^5y^2z^2 + x^2y^5z^2 + x^2y^2z^5\ (true).$$
$$x^6y^3 + x^6y^3 + x^6y^3 + x^6y^3 + z^6x^3 + z^6x^3 \ge 6x^5y^2z^2$$
$$y^6z^3 + y^6z^3 + y^6z^3 + y^6z^3 + x^6y^3 + x^6y^3 \ge 6x^2y^5z^2$$
$$z^6x^3 + z^6x^3 + z^6x^3 + z^6x^3 + y^6z^3 + x^6z^3 \ge 6x^2y^2z^5$$

Equality holds of $x = y = z = 2$.

Solution 3 by Abdul Aziz-Semarang-Indonesia

Let $x = \frac{b}{a}, y = \frac{c}{b}, = \frac{a}{c} \Rightarrow xyz = 1$.

$$\sum \frac{a^3}{a^3 + b^3 + abc} = \sum \frac{1}{1 + \left(\frac{b}{a}\right)^3 + \left(\frac{b}{c}\right)\left(\frac{c}{a}\right)} = \sum \frac{1}{1 + x^3 + \frac{x}{z}} =$$

$$=\sum\frac{z}{z+zx^3+x}=\sum\frac{zy}{zy+x^2+xy}=\sum\frac{(zy)^2}{(zy)(zy+x^2+xy)}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(xy+yz+zx)^2}{\sum zy(zy+x^2+xy)}\geq\frac{\sum zy(zy+x^2+xy)}{\sum zy(zy+x^2+xy)}=1$$

Equality holds of $x=y=z=2$.

626. If $a,b,c>0,\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1,\lambda\geq 1,$

$P=(a^2+\lambda bc)(b^2+\lambda ca)(c^2+\lambda ab)$, then find:

$$\Omega=max\{P\}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Michael Sterghiou-Greece

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1;(c);\quad \Omega=max\{P\};(1)$$

Let $(p,q,r)=(\sum a,\sum ab,abc)$. From $(c)\to$

$$1=\sum_{cyc}\frac{a^2}{a+ab}\overset{BCS}{\geq}\frac{p^2}{p+q}\to p+q\geq p^2\geq pq\to p\geq 2q;p^2\geq p+q.$$

Also, $p^2\leq p+q\leq p+\frac{p^2}{3}\to p\leq\frac{3}{2}$, equality holds when $a=b=c$.

Now, by AM-GM inequality, we get:

$$0\leq\left[\frac{\sum(a^2+\lambda bc)}{3}\right]^2=\left[\frac{p^2-2q+\lambda q}{3}\right]^3\leq\left[\frac{p+q-2q+\lambda q}{3}\right]^3=\left(\frac{p+(\lambda-1)q}{3}\right)^3\leq$$

$$\leq\left[\frac{p+(\lambda-1)\cdot\frac{p}{2}}{3}\right]^3=\left(\frac{p}{3}\right)^3\cdot\left(1+\frac{\lambda-1}{2}\right)^3\overset{p\leq\frac{3}{2}}{\leq}\left(\frac{\lambda+1}{4}\right)^3$$

-which is the required max achievement when $a=b=c=\frac{1}{2}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$1=\sum_{cyc}\frac{a}{1+b}\overset{BCS}{\geq}\frac{(\sum a)^2}{\sum a+\frac{1}{3}(\sum a)^2}\Leftrightarrow 1+\frac{1}{3}\sum_{cyc}a\geq\sum_{cyc}a\Leftrightarrow\sum_{cyc}a\leq\frac{3}{2};(1)$$

$$1 = \sum_{cyc} \frac{a}{1+b} = \sum_{cyc} \frac{a^2}{a+ab} \overset{BCS}{\geq} \frac{(\sum a)^2}{\sum a + \sum ab} \Leftrightarrow \sum_{cyc} a + \sum_{cyc} ab \geq \sum_{cyc} a^2 + 2\sum_{cyc} ab$$

$$\Leftrightarrow \sum_{cyc} a^2 + \sum_{cyc} ab \leq \sum_{cyc} a \overset{(1)}{\leq} \frac{3}{2}; (2)$$

$$P = \prod_{cyc}(a^2 + \lambda bc) \overset{AM-GM}{\leq} \left(\frac{\sum a^2 + \lambda \sum bc}{3}\right)^3 \overset{(2)}{\leq} \left(\frac{\frac{3}{2} + \frac{1}{3}(\lambda - 1)(\sum a)^2}{3}\right)^3 \overset{(1)}{\leq}$$

$$\leq \left(\frac{\frac{3}{2} + \frac{1}{3}(\lambda - 1)\left(\frac{3}{2}\right)^2}{3}\right)^3 = \left(\frac{\lambda + 1}{4}\right)^3 = max\{P\}.$$

Equality holds when $a = b = c = \frac{1}{2}$.

Solution 3 by Ruxandra Daniela Tonilă-Romania

$$(a^2 + \lambda bc)(b^2 + \lambda ca)(c^2 + \lambda ab) \leq \left(\frac{a^2 + b^2 + c^2 + \lambda(ab + bc + ca)}{3}\right)^3$$

$$P \leq \left(\frac{a^2 + b^2 + c^2}{3} + \frac{\lambda}{3}(ab + bc + ca)\right)^3$$

But, $ab + bc + ca \leq a^2 + b^2 + c^2 \rightarrow$

$$P \leq \left(\frac{a^2 + b^2 + c^2}{3} + (\lambda + 1)\right)^3$$

$$\begin{cases} P_{max} \Leftrightarrow \sum_{cyc} a^2 = \sum_{cyc} ab \Leftrightarrow a = b = c \\ \frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} = 1 \end{cases} \rightarrow \frac{3a}{1+a} = 1 \rightarrow a = b = c = \frac{1}{2}$$

$$P_{max} = \left(\frac{3a^2}{3}(\lambda + 1)\right) = \left(a^2(\lambda + 1)\right)^3 \rightarrow P_{max} = \frac{(\lambda + 1)^3}{64}$$

627. **For $x > 0$ and $m, n, p > 0$ –fixed, denote:**

$\Omega(m, n, p) = x_{min}, x_{min}$ –value of x such that

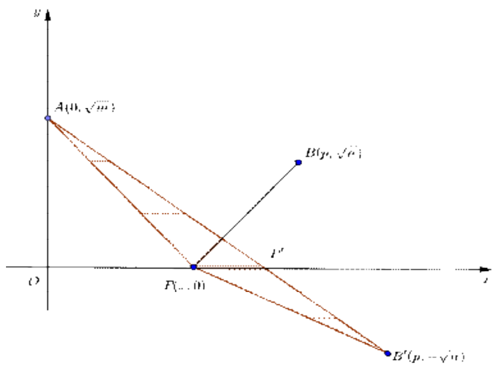
$\Omega(m, n, p)(x) = \left(\sqrt{n + (p - x)^2} + \sqrt{m + x^2}\right)$ has minimum value.

Prove that:

$$\sum_{cyc}\left(\sqrt{m}+\sqrt{n}\right)\Omega(m,n,p) \geq 3\sqrt{mnp}$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India



$$AP = \sqrt{m+x^2}, BP = \sqrt{(p-x)^2+n}$$

Reflect B **in the** x **−axis. Let image** $B'\left(p,-\sqrt{n}\right), PB = PB'$

$AP + PB$ **−is minimum when** P **is the point of intersection of** AB' **and the** x **−axis.**

Let P' divide the AB' **in the ratio** $k = 1$. $x_{p'} = \frac{kp}{k+1}, y_{p'} = \frac{\sqrt{m}-k\sqrt{n}}{k+1}$

As P' **lies on the** x **−axis,** $y_{p'} = 0 \Rightarrow k = \sqrt{\frac{m}{n}}, \because x_{p'} = \frac{p\sqrt{m}}{\sqrt{m}+\sqrt{n}}$

$$\Omega(m,n,p) = \frac{p\sqrt{m}}{\sqrt{m}+\sqrt{n}} \Rightarrow \left(\sqrt{m}+\sqrt{n}\right)\Omega(m,n,p) = p\sqrt{m}$$

$$\sum_{cyc}\left(\sqrt{m}+\sqrt{n}\right)\Omega(m,n,p) = \sum_{cyc} p\sqrt{m} = p\sqrt{m}+n\sqrt{p}+m\sqrt{n} \geq$$

$$\geq 3\sqrt[3]{p\sqrt{m}\cdot n\sqrt{p}\cdot m\sqrt{n}} = 3\sqrt{mnp}$$

628. If $A = \frac{\tan^3 x}{\sqrt{1+\cot^2 x}} + \frac{\cot^3 x}{\sqrt{1+\tan^2 x}}, x \in \left(0,\frac{\pi}{2}\right)$ **then find** $min\{A\}$.

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$A = \frac{\tan^3 x}{\sqrt{1+\cot^2 x}} + \frac{\cot^3 x}{\sqrt{1+\tan^2 x}}$, $f(x) = \frac{1}{\sqrt{x}}$ is decreasing and convexe function on $\left(0, \frac{\pi}{2}\right)$

$$\Rightarrow A = \tan^2 f\left(\cot^2 x(1+\cot^2 x)\right) + \cot^2 x f\left(\tan^2 x(1+\tan^2 x)\right) \overset{JENSEN}{\geq}$$

$$\geq (\tan^2 x + \cot^2 x) f\left(\frac{1+\cot^2 x + 1 + \tan^2 x}{\tan^2 x + \cot^2 x}\right)$$

$$\tan^2 x + \cot^2 x \overset{AM-GM}{\geq} 2 \Rightarrow 1 + \frac{2}{\tan^2 x + \cot^2 x} \leq 2 \Rightarrow A \geq 2f(2) = \sqrt{2} \Rightarrow$$

$min\{A\} = \sqrt{2}$. Equality holds when $x = \frac{\pi}{4}$.

629. If $A = \frac{\sec^3 x}{\sqrt{1+\csc^2 x}} + \frac{\csc^3 x}{\sqrt{1+\sec^2 x}}$, $x \in \left(0, \frac{\pi}{2}\right)$ then find $min\{A\}$.

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$A = \frac{\sec^3 x}{\sqrt{1+\csc^2 x}} + \frac{\csc^3 x}{\sqrt{1+\sec^2 x}}$, $f(x) = \frac{1}{\sqrt{x}}$ –decreasing and convexe on $\left(0, \frac{\pi}{2}\right)$

$$\Rightarrow A = \sec^2 x f\left(\cos^2 x(1+\csc^2 x)\right) + \csc^2 x f\left(\sin^2 x(1+\sec^2 x)\right) \overset{JENSEN}{\geq}$$

$$\geq (\sec^2 x + \csc^2 x) f\left(\frac{1+\csc^2 x + 1 + \sec^2 x}{\sec^2 x + \csc^2 x}\right) =$$

$$= (\sec^2 x + \csc^2 x) f\left(1 + \frac{2}{\sec^2 x + \csc^2 x}\right)$$

$\sec^2 x + \csc^2 x \geq \frac{4}{\cos^2 x + \sin^2 x} = 4$ and $1 + \frac{2}{\sec^2 x + \csc^2 x} \leq \frac{3}{2} \Rightarrow A \geq 4f\left(\frac{3}{2}\right) = \frac{4\sqrt{6}}{3} = min\{A\}$

Equality holds when $x = \frac{\pi}{4}$.

630. If $\frac{\pi}{4} \leq x \leq y \leq z < \frac{\pi}{2}$ then:

$$2\tan x + \left(\sum_{cyc} \tan x\right)\left(\sum_{cyc} \cot x\right) \leq 9 + 2\tan z$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

Let us denote: $a = \tan x; b = \tan y; c = \tan z;$ $\left(\frac{\pi}{4} \le x \le y \le z < \frac{\pi}{2}\right) \Rightarrow$

$1 \le a \le b \le c < +\infty$. We need to prove that:

$$2a\left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right) \le 9 + 2c \Leftrightarrow 2a + (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \le 9 + 2c \Leftrightarrow$$

$$2a + \frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b} + 3 \le 9 + 2c \Leftrightarrow$$

$$6 + 2c + \frac{ab(a+b) + bc(b+c) + ca(c+a)}{abc} - 2a \ge 0 \Leftrightarrow$$

$$\frac{6abc + 2abc^2 - (a^2+b^2)c - (a+b)c^2 - 2a^2bc - ab(a+b)}{abc} \ge 0 \Leftrightarrow$$

$$\big(2ab - (a+b)\big)c^2 + (6ab - 2a^2b - a^2 - b^2)c - ab(a+b) \ge 0; (1)$$

Let $f(c) = \big(2ab - (a+b)\big)c^2 + (6ab - 2a^2b - a^2 - b^2)c - ab(a+b)$

$$c \ge b \ge a \ge 1;$$

$$f'(c) = 2c\big(2ab - (a+b)\big) + 6ab - 2a^2b - a^2 - b^2 \overset{c \ge b}{\ge}$$

$$= 4ab^2 - 2ab - 2b^2 + 6ab - 2a^2b - a^2 - b^2 =$$

$$= (4a-3)b^2 + (4a - 2a^2)b - a^2 = f(b); (\because b \ge a \ge 1)$$

$$\Rightarrow f'(b) = 2(4a-3)b + 4a - 2a^2 \overset{b \ge a \ge 1}{\ge} 2(4a-3)a + 4a - 2a^2 =$$

$$= 2a(3a-1) \overset{a \ge 1}{\ge} 4 > 0 \Rightarrow f \nearrow [a, \infty) \Rightarrow$$

$$f(b) \ge f(a) = (4a-3)a^2 + (4a - 2a^2)a - a^2 = 2a^3 \ge 2 > 0 \Rightarrow f'(c) > 0, \forall c \ge b.$$

$$f(c) \ge f(b) = \big(2ab - (a+b)\big)b^2 + (6ab - 2a^2b - a^2 - b^2)b - ab(a+b) \overset{(2)}{\ge} 0$$

$$(2) \Leftrightarrow b(2ab - a - b) + (6ab - 2a^2b - a^2 - b^2) - a^2 - ab \ge 0$$

$$\Leftrightarrow 2ab^2 - ab - b^2 + 6ab - 2a^2b - a^2 - b^2 - a^2 - ab \ge 0$$

$$\Leftrightarrow 2ab^2 + 4ab - 2a^2b - 2a^2 - 2b^2 \ge 0$$

$$\Leftrightarrow ab^2 + 2ab - a^2b - a^2 - b^2 \ge 0$$

$$\Leftrightarrow ab(b-a) - (a-b)^2 \ge 0 \Leftrightarrow (b-a)\big(ab - (b-a)\big) \ge 0$$

$\Leftrightarrow (b-a)(b(a-1) + a) \ge 0$, which is true from $b \ge a \ge 1 \Rightarrow (2) \Rightarrow (1)$ is true.

Equality holds when $a = b = c = 1 \Leftrightarrow x = y = z = \frac{\pi}{4}$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let us denote: $a = \tan x; b = \tan y; c = \tan z;\ \left(\frac{\pi}{4} \le x \le y \le z < \frac{\pi}{2}\right) \Rightarrow$

$$1 \le a \le b \le c$$

$$\left(\sum_{cyc} \tan x\right)\left(\sum_{cyc} \cot x\right) = \left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right) = 3 + \sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right)$$

$$a \le b \le c \Rightarrow (c-b)(b-a) \ge 0 \Rightarrow bc + ab \ge b^2 + ac$$

$$\Rightarrow 1 + \frac{a}{c} \ge \frac{b}{c} + \frac{a}{b} \text{ and } \frac{c}{a} + 1 \ge \frac{b}{a} + \frac{c}{b}$$

$$\Rightarrow \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) \le 2 + \left(\frac{a}{c} + \frac{c}{a}\right)$$

$$\Rightarrow 2a + \left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right) \le 5 + 2a + 2\left(\frac{a}{c} + \frac{c}{a}\right) \overset{(*)}{\le} 9 + 2c$$

$$(*) \Leftrightarrow \frac{a}{c} + \frac{c}{a} - 2 \le c - a \Leftrightarrow (c-a)^2 \le ac(c-a) \Leftrightarrow (c-a)[c(a-1) + a] \ge 0$$

Which is true because $1 \le a \le c$/

Therefore,

$$2\tan x + \left(\sum_{cyc} \tan x\right)\left(\sum_{cyc} \cot x\right) \le 9 + 2\tan z$$

631. If $a, b > 0$, then prove:

$$2\sqrt{ab} \cdot \frac{\sin x}{x} + \frac{a+b}{2} \cdot \frac{\tan x}{x} \ge 3\sqrt{ab}, \forall x \in \left(0, \frac{\pi}{2}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$\frac{a+b}{2} \ge 2\sqrt{ab}, \forall a, b > 0$; (AM-GM)

So, it is suffices to prove that: $2 \cdot \frac{\sin x}{x} + \frac{\tan x}{x} > 3, \forall x \in \left(0, \frac{\pi}{2}\right)$

Let $f(x) = 2\sin x + \tan x - 3x, x \in \left[0, \frac{\pi}{2}\right]$. We have:

$$f'(x)=2\cos x+\frac{1}{\cos^2 x}-3=\frac{(1-\cos x)^2(1+2\cos x)}{\cos^2 x}\geq 0,\forall x\in\left[0,\frac{\pi}{2}\right]$$

So, f –is strictly increasing on $\left[0,\frac{\pi}{2}\right]$, thus

$$f(x)>f(0)=0,\forall x\in\left(0,\frac{\pi}{2}\right)\Rightarrow 2\cdot\frac{\sin x}{x}+\frac{\tan x}{x}>3,\forall x\in\left(0,\frac{\pi}{2}\right)$$

Therefore,

$$2\sqrt{ab}\cdot\frac{\sin x}{x}+\frac{a+b}{2}\cdot\frac{\tan x}{x}\geq 3\sqrt{ab},\forall x\in\left(0,\frac{\pi}{2}\right)$$

Solution 2 by Faysal Abdelli-Bejaia-Algerie

Let suppose:

$$2\sqrt{ab}\cdot\frac{\sin x}{x}+\frac{a+b}{2}\cdot\frac{\tan x}{x}\leq 3\sqrt{ab},\forall x\in\left(0,\frac{\pi}{2}\right);(1)$$

$$\left(\sqrt{a}-\sqrt{b}\right)^2\geq 0\Rightarrow a+b\geq\sqrt{ab}\Rightarrow\frac{a+b}{2}\geq\sqrt{ab}$$

$$(1)\Rightarrow\ 2\sqrt{ab}\cdot\frac{\sin x}{x}+\sqrt{ab}\cdot\frac{\tan x}{x}\leq 3\sqrt{ab},\forall x\in\left(0,\frac{\pi}{2}\right)$$

$$\Leftrightarrow 2\cdot\frac{\sin x}{x}+\frac{\tan x}{x}\leq 3\Leftrightarrow 2\sin x+\tan x\leq 3x.$$

Let $f(x)=2\sin x+\tan x-3x; f(x)<0;(2)$

$$f'(x)=2\cos x+\frac{1}{\cos^2 x}-3=\frac{2\cos^3 x-3\cos^2 x+1}{\cos^2 x}=$$

$$=\frac{2(\cos x-1)^2(2\cos x+1)}{2\cos^2 x}>0,\forall x\in\left(0,\frac{\pi}{2}\right)\Rightarrow f\text{ –increasing}$$

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}(2\sin x+\tan x-3x)=0\Rightarrow$$

$f(x)>0$ contradiction with (1)

Therefore,

$$2\sqrt{ab}\cdot\frac{\sin x}{x}+\frac{a+b}{2}\cdot\frac{\tan x}{x}\geq 3\sqrt{ab},\forall x\in\left(0,\frac{\pi}{2}\right)$$

632. If $a,b>0$ and $x\in\left(0,\frac{\pi}{2}\right)$ then prove:

$$(a+b)\cdot\frac{\sin x}{x}+\frac{2ab}{a+b}\cdot\frac{\tan x}{x}>\frac{6ab}{a+b}$$

$$a^2\cdot tan^k x+b^2\cdot sin^k x>2ab\cdot x^k, k>0$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Fayssal Abdelli-Bejaia-Algerie

$$(a+b)\cdot\frac{sinx}{x}+\frac{2ab}{a+b}\cdot\frac{tanx}{x}>\frac{6ab}{a+b}; (A)$$

Let suppose:

$$(a+b)\cdot\frac{sinx}{x}+\frac{2ab}{a+b}\cdot\frac{tanx}{x}\le\frac{6ab}{a+b}; (B)$$

$$(B)\to(a+b)^2\cdot\frac{sinx}{x}+2ab\cdot tanx\le 6ab$$

But: $a+b\ge 2\sqrt{ab}\ (AM-GM)\to(a+b)^2\ge 4ab$

$$4ab\cdot\frac{sinx}{x}+2ab\cdot\frac{tanx}{x}\le 6ab\to 2\cdot\frac{sinx}{x}+\frac{tanx}{x}\le 3\to 2sinx+tanx-3x\le 0$$

Let $f(x)=2sinx+tanx-3x, f(x)<0; (C)$

$$f'(x)=\frac{(cosx-1)^2(2cosx+1)}{cos^2x}>0,\forall x\in\left(0,\frac{\pi}{2}\right)\to$$

f –increasing on $\left(0,\frac{\pi}{2}\right)$

$$\lim_{x\to 0}f(x)=0;\ \lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}}f(x)=+\infty\to f(x)>0,\forall x\in\left(0,\frac{\pi}{2}\right)\text{ contradiction with }(C).$$

So, contradiction with (B), thus

$$(a+b)\cdot\frac{sinx}{x}+\frac{2ab}{a+b}\cdot\frac{tanx}{x}>\frac{6ab}{a+b}$$

$$a^2\cdot tan^k x+b^2\cdot sin^k x>2ab\cdot x^k, k>0, a,b>0; (D)$$

$$sinx, x,\forall x\in\left(0,\frac{\pi}{2}\right)\to sin^k x<x^k\to\frac{sin^k x}{cos^k x}<\frac{x^k}{cos^k x}\to a^2\cdot\frac{sin^k x}{cos^k x}<a^2\cdot\frac{x^k}{cos^k x}$$

Let suppose that (D) is true, then:

$$a^2\cdot\frac{x^k}{cos^k x}+b^2x^k>2abx^k\to\frac{a^2}{cos^k x}+b^2>2ab$$

We know that: $a^2+b^2>2ab$, but: $0<cos^k x<1,\forall x\in\left(0,\frac{\pi}{2}\right)\to$

$$\frac{1}{\cos^k x} > 1, \forall x \in \left(0, \frac{\pi}{2}\right) \to \frac{a^2}{\cos^k x} + b^2 > a^2 + b^2 \to \frac{a^2}{\cos^k x} + b^2 > 2ab$$

So, $a^2 \cdot \tan^k x + b^2 \cdot \sin^k x > 2ab \cdot x^k, k > 0, a, b > 0$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

1. By AM-GM, we have: $(a+b)^2 \geq 4ab \to a + b \geq \frac{4ab}{a+b}$.

So, it is suffices to prove that: $2 \cdot \frac{\sin x}{x} + \frac{\tan x}{x} > 3$.

Let $f(x) = 2\sin x + \tan x - 3x, \forall x \in \left[0, \frac{\pi}{2}\right]$; $f'(x) = \frac{(1-\cos x)^2(1+2\cos x)}{\cos^2 x} \geq 0$

So, f **is strictly increasing on** $\left[0, \frac{\pi}{2}\right]$, **then** $f(x) > f(0), \forall x \in \left(0, \frac{\pi}{2}\right) \to$

$$2 \cdot \frac{\sin x}{x} + \frac{\tan x}{x} > 3., \forall x \in \left(0, \frac{\pi}{2}\right)$$

Therefore,

$$(a+b) \cdot \frac{\sin x}{x} + \frac{2ab}{a+b} \cdot \frac{\tan x}{x} > \frac{6ab}{a+b}$$

2. By AM-GM, we have:

$$a^2 \cdot \tan^k x + b^2 \cdot \sin^k x \geq 2ab\sqrt{\tan x \cdot \sin x} \overset{(1)}{\geq} 2ab \cdot x^k \Leftrightarrow \tan x \cdot \sin x > x^2$$

Let $g(x) = \tan x \cdot \sin x - x^2, \forall x \in \left[0, \frac{\pi}{2}\right]$

$$g'(x) = \frac{\sin x}{\cos^2 x} + \sin x - 2x; \; g''(x) = \frac{1}{\cos x} + \frac{2\sin^2 x}{\cos^3 x} + \cos x - 2 =$$

$$= \frac{1}{\cos^3 x}(\cos^4 x - 2\cos^3 x - \cos^2 x + 2) =$$

$$= \frac{(1-\cos x)\left(1 + 2\cos x + \cos^2 x + (1 - \cos^3 x)\right)}{\cos^3 x}$$

$\to g'$ **–strictly increasing on** $\left[0, \frac{\pi}{2}\right] \to g'^{(x)} > g'(0) = 0, \forall x \in \left[0, \frac{\pi}{2}\right]$

$\to g$ **–strictly increasing on** $\left[0, \frac{\pi}{2}\right] \to g(x) > g(0) = 0, \forall x \in \left[0, \frac{\pi}{2}\right]$

$$\to \tan x \cdot \sin x > x^2, \forall x \in \left(0, \frac{\pi}{2}\right)$$

Therefore,

$$a^2 \cdot \tan^k x + b^2 \cdot \sin^k x > 2ab \cdot x^k, k > 0$$

633. If $x,y,z,t,u>0, n\in\mathbb{N}^*$ then:

$$x^n\cdot t^{n+1}+uyz\cdot\sqrt[n]{u}\geq tu(n+1)\cdot\sqrt[n+1]{\left(\frac{xyz}{n}\right)^n}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohammed Diai-Rabat-Morocco

Using Weighted AM-GM inequality, we get:

$$\frac{x^n\cdot t^{n+1}+n\left(\frac{uyz\cdot\sqrt[n]{u}}{n}\right)}{n+1}\geq\sqrt[n+1]{x^nt^{n+1}\left(\frac{xyz}{n}\right)^n}$$

$$x^n\cdot t^{n+1}+uyz\cdot\sqrt[n]{u}\geq(n+1)\cdot\sqrt[n+1]{(tu)^{n+1}\left(\frac{xyz}{n}\right)^n}$$

Therefore,

$$x^n\cdot t^{n+1}+uyz\cdot\sqrt[n]{u}\geq tu(n+1)\cdot\sqrt[n+1]{\left(\frac{xyz}{n}\right)^n}$$

634. If $x,y>0$ such that $\lambda\leq 16$ then find:

$$min\ P=\left(\lambda x^2+\frac{1}{y^2}\right)\left(\lambda y^2+\frac{1}{x^2}\right)$$

Proposed by Marin Chirciu-Romania

Solution by Florentin Vişescu-Romania

$$P=\left(\lambda x^2+\frac{1}{y^2}\right)\left(\lambda y^2+\frac{1}{x^2}\right)=\lambda^2x^2y^2+2\lambda+\frac{1}{x^2y^2}=\left(\lambda xy+\frac{1}{xy}\right)^2$$

Let $x,y>0$ with $x+y=2$, then x,y has roots of the equation:

$t^2-2t+q=0$, with $4-4q\geq 0$ and $q>0$, hence $q\in(0,1], xy=q$.

So, we find $min\ P=\left(\lambda q+\frac{1}{q}\right)^2,\ \ q\in(0,1]$.

Let $:(0,1]\to\mathbb{R},\ f(q)=\left(\lambda q+\frac{1}{q}\right)^2, f'(q)=2\left(\lambda q+\frac{1}{q}\right)\left(\lambda-\frac{1}{q^2}\right)$

$f'(q) = 0 \to \lambda q + \frac{1}{q} = 0 \to \lambda q^2 + 1 = 0$ no has solution.

If $\lambda > 0$ the equation no has solution.

If $\lambda < 0$ then $q = \frac{1}{i\sqrt{\lambda}}$.

If $\lambda \le 0$ no has solution.

If $\lambda > 0$ then $q = \frac{1}{\sqrt{\lambda}}$.

If $\lambda = 0 \to P = \left(\frac{1}{q}\right)^2$ and $min\ P = 1$.

If $\lambda \in (0,1) \to P(q)$ –decreasing and $minP = (\lambda + 1)^2$.

If $\lambda \in (-1,0) \to P(q)$ –decreasing and $min\ P = (\lambda + 1)^2$.

If $\lambda \ge 1 \to P(q)$ decreasing on $\left(0, \frac{1}{\sqrt{\lambda}}\right)$ and increasing on $\left(\frac{1}{\sqrt{\lambda}}, 1\right)$, then $min\ P = 4\lambda$.

If $\lambda \le -1 \to P(q)$ decreasing on $\left(0, \frac{1}{i\sqrt{\lambda}}\right)$ and increasing on $\left(\frac{1}{i\sqrt{\lambda}}, 1\right)$, then $min\ P = 0$.

Therefore,

$$min\ P = \begin{cases} (\lambda + 1)^2, & if\ \lambda \in (-1,1) \\ 4\lambda, & if\ \lambda \ge 1 \\ 0, & if\ \lambda \le -1 \end{cases}$$

635. Prove without any software:

$$e(e+2) < \frac{(e+1)(e+2)\log\left(1 + \frac{1}{e+1}\right)}{\log\left(1 + \frac{1}{e}\right)} < (e+1)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $f(x) = \log(1 + x); g(x) = \log x, x \in [e, e+1]$

By the Cauchy's mean value theorem there exists $\alpha \in (e, e+1)$ such that

$$\frac{f(e+2) - f(e+1)}{g(e+1) - g(e)} = \frac{f'(\alpha)}{g'(\alpha)} = \frac{\alpha}{\alpha + 1}$$

$$\Rightarrow \frac{\log\left(\frac{e+2}{e+1}\right)}{\log\left(\frac{e+1}{e}\right)} = 1 - \frac{1}{\alpha + 1}$$

Let be $e < \alpha < e+1, e+1 < \alpha+1 < e+2$, then

$$\frac{1}{e+2} < \frac{1}{\alpha+1} < \frac{1}{e+1} \Rightarrow -\frac{1}{e+1} < -\frac{1}{\alpha+1} < -\frac{1}{\alpha+2}$$

$$\frac{e}{e+1} < 1-\frac{1}{\alpha+1} < 1-\frac{1}{e+2}$$

$$\frac{e}{e+1} < \frac{log\left(\frac{e+2}{e+1}\right)}{log\left(\frac{e+1}{e}\right)} < \frac{e+1}{e+2} \Leftrightarrow e(e+2) < \frac{e(e+2)log\left(1+\frac{1}{e+1}\right)}{log\left(1+\frac{1}{e}\right)} < (e+1)^2$$

Solution 2 by Samar Das-India

If $b = a+h \Rightarrow \exists\theta \in (a,b)$ such that:

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}; (1)(0 < \theta < 1), (a \le a+\theta h \le a+h)$$

Let be the functions: $f(x) = log\, x; g(x) = log(x+1); (e \le e+\theta \le e+1); (2)$

$$f'(x) = \frac{1}{x}, g'(x) = \frac{1}{x+1}; (3)$$

From $(1),(2),(3)$ it follows that:

$$\frac{log(e+1)-loge}{log(e+2)-log(e+1)} = \frac{\frac{1}{e+\theta}}{\frac{1}{e+\theta+1}} = \frac{e+\theta+1}{e+\theta} = 1+\frac{1}{e+\theta}$$

$$\Rightarrow \frac{log\left(\frac{e+1}{e}\right)}{log\left(\frac{2+e}{e+1}\right)} = 1+\frac{1}{e+\theta} \Rightarrow \frac{log\left(\frac{e+1}{e}\right)-log\left(\frac{2+e}{e+1}\right)}{log\left(\frac{2+e}{e+1}\right)} = \frac{1}{e+\theta}$$

$$\Rightarrow \frac{log\left(\frac{2+e}{1+e}\right)}{log\left(\frac{(e+1)^2}{e(e+2)}\right)} - e = \theta$$

Since $0 < \theta < 1 \Rightarrow 0 < \frac{log\left(\frac{2+e}{1+e}\right)}{log\left(\frac{(e+1)^2}{e(e+2)}\right)} - e < 1$

$$\Rightarrow 1+e < 1+\frac{log\left(\frac{2+e}{1+e}\right)}{log\left(\frac{(e+1)^2}{e(e+2)}\right)} < 2+e \Rightarrow 1+e < \frac{log\left(\frac{e+1}{e}\right)}{log\left(\frac{(e+1)^2}{e(e+2)}\right)} < 2+e$$

$$\Rightarrow \frac{1}{1+e} > \frac{log\left(\frac{(e+1)^2}{e(e+2)}\right)}{log\left(\frac{e+1}{e}\right)} > \frac{1}{2+e} \Rightarrow 1-\frac{1}{1+e} < 1-\frac{log\left(\frac{(e+1)^2}{e(e+2)}\right)}{log\left(\frac{e+1}{e}\right)} < 1-\frac{1}{2+e}$$

$$\frac{e}{e+1} < \frac{log\left(\frac{e+2}{e+1}\right)}{log\left(1+\frac{1}{e}\right)} < \frac{e+1}{e+2} \Rightarrow \frac{e}{e+1} < \frac{log\left(1+\frac{1}{e+1}\right)}{log\left(1+\frac{1}{e}\right)} < \frac{e+1}{e+2}$$

Therefore,

$$e(e+2) < \frac{(e+1)(e+2)log\left(1+\frac{1}{e+1}\right)}{log\left(1+\frac{1}{e}\right)} < (e+1)^2$$

636. For $\frac{1}{e} < a < b$ prove that:

$$log\left(\frac{a}{b}\right)^2 \cdot log\left((ab)^{ab}\right) > log\left(\frac{\left(\frac{a^2}{e}\right)^{a^2}}{\left(\frac{b^2}{e}\right)^{b^2}}\right)$$

Proposed by Nikos Ntorvas-Greece

Solution by proposer

Let be $f(x) = x \cdot log^2 x, x \in (0,+\infty)$. f –is strictly convex

$$\because f''(x) = \frac{2(1+logx)}{x} > 0, x \in \left(\frac{1}{e},+\infty\right)$$

Therefore, for $a < b$ the area under the graph of $y = x \cdot log^2 x$ from $x = a$ to $x = b$ is less than area of the trapezoid with vertices $(a,0),(b,0),(a,f(a)),(b,f(b))$.

So, we have that:

$$(f(b) + \frac{f(a)(b-a)}{2} > \int_a^b f(x)dx$$

$$\Leftrightarrow \frac{a \cdot log^2 a + b \cdot log^2 b}{2}(b-a) > \left[\frac{x^2 \cdot log^2 x}{2} - \frac{x^2 \cdot logx}{2} + \frac{x^2}{4}\right]_a^b$$

$$\Leftrightarrow \frac{ab \cdot log^2 a - a^2 \cdot log^2 a + b^2 \cdot log^2 b - ab \cdot log^2 b}{2} >$$

$$>\frac{b^2\cdot log^2b}{2}-\frac{b^2\cdot logb}{2}+\frac{b^2}{4}-\frac{a^2\cdot log^2a}{2}+\frac{a^2\cdot loga}{2}-\frac{a^2}{4}$$

$$\Leftrightarrow 2ab(log^2a-log^2b)>2a^2\cdot loga-2b^2\cdot logb+b^2-a^2$$

$$\Leftrightarrow 2ab(loga-logb)(loga+logb)>a^2(2loga-1)-b^2(2logb-1)$$

$$\Leftrightarrow 2ab\cdot log\left(\frac{a}{b}\right)\cdot log(ab)>a^2\cdot log\left(\frac{a^2}{e}\right)-b^2\cdot log\left(\frac{b^2}{e}\right)$$

$$\Leftrightarrow log\left(\frac{a}{b}\right)^2\cdot log\left((ab)^{ab}\right)>log\left(\frac{\left(\frac{a^2}{e}\right)^{a^2}}{\left(\frac{b^2}{e}\right)^{b^2}}\right)$$

637. If $(L_n)_{n\geq 0}, L_0=2, L_1=1, L_{n+2}=L_{n+1}+L_n, \forall n\in\mathbb{N}$, is the Lucas's sequence, and $a,b,c\in\mathbb{R}_+^*$ such that $a+b+c\leq 24$, then:

$$\frac{L_n}{\sqrt{L_n^2+aL_{n+1}L_{n+2}}}+\frac{L_{n+1}}{\sqrt{L_{n+1}^2+bL_{n+2}L_n}}+\frac{L_{n+2}}{\sqrt{L_{n+2}^2+cL_nL_{n+1}}}\geq 1, \forall n\in\mathbb{N}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x=L_n, y=L_{n+1}, z=L_{n+2}, x,y,z>0$ and $f(x)=\frac{1}{\sqrt{x}}, x>0$

$$f-\text{convex}\overset{Using\ Jensen}{\rightarrow}\sum\frac{x}{\sqrt{x^2+ayz}}=\sum xf(x^2+ayz)\geq\left(\sum x\right)f\left(\frac{\sum x(x^2+ayz)}{\sum x}\right)$$

$$=\sqrt{\frac{(\sum x)^3}{\sum x^3+xyz\sum a}}\overset{\sum a\leq 24}{\geq}\sqrt{\frac{(\sum x)^3}{\sum x^3+24xyz}}\overset{?}{\geq}1\leftrightarrow\left(\sum x\right)^3\geq\sum x^3+24xyz$$

$$\leftrightarrow\sum z(x^2+y^2)\geq 6xyz\Leftrightarrow\sum z(x-y)^2\geq 0\text{, which is true, }(x,y,z>0)$$

Therefore, $\frac{L_n}{\sqrt{L_n^2+aL_{n+1}L_{n+2}}}+\frac{L_{n+1}}{\sqrt{L_{n+1}^2+aL_nL_{n+2}}}+\frac{L_{n+2}}{\sqrt{L_{n+2}^2+aL_{n+1}L_n}}\geq 1, \forall n\in IN$

638. If ABC is a right angled triangle with $\angle A=90^\circ$ and $(F_n)_{n\geq 0}, F_0=F_1=1, F_{n+2}=F_{n+1}+F_n$, for any positive integer n, i.e. the Fibonacci sequence, then prove that:

$$\frac{F_m^2}{\left(bF_n + cF_p\right)^2} + \frac{F_n^2}{\left(bF_p + cF_m\right)^2} + \frac{F_p^2}{(bF_m + cF_n)^2} \ge \frac{3}{2a^2}$$

for any positive integer m, n, p.

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x = F_m, y = F_n, z = F_p, x, y, z > 0$

$$\frac{F_m^2}{\left(bF_n + cF_p\right)^2} + \frac{F_n^2}{\left(bF_p + cF_m\right)^2} + \frac{F_p^2}{(bF_m + cF_n)^2} =$$

$$= \sum \frac{x^2}{(yb + zc)^2} \overset{CBS}{\ge} \sum \frac{x^2}{2(y^2b^2 + z^2c^2)} =$$

$$= \frac{1}{2}\sum \frac{x^4}{(xy)^2b^2 + (xz)^2c^2} \overset{CBS}{\ge} \frac{1}{2} \cdot \frac{(\sum x^2)^2}{\sum((xy)^2b^2 + (xz)^2c^2)} = \frac{(\sum x^2)^2}{2(b^2 + c^2)\sum(xy)^2} \ge$$

$$\overset{(\sum u)^2 \ge 3\sum uv}{\ge} \frac{3\sum(xy)^2}{2(b^2 + c^2)\sum(xy)^2} \overset{Pythagorean\ theorem}{=} \frac{3}{2a^2}$$

Therefore, $\frac{F_m^2}{\left(bF_n+cF_p\right)^2} + \frac{F_n^2}{\left(bF_p+cF_m\right)^2} + \frac{F_p^2}{(bF_m+cF_n)^2} \ge \frac{3}{2a^2}, \forall n, m, p \in \mathbb{N}^*$

639. For $a, b \in (0, \pi)$ prove that: $e^{e^{sinb-1}-1} \ge sinb(sina)^{\frac{1}{sina}-1}$

Proposed by Nikos Ntorvas-Greece

Solution by Mohammed Diai-Rabat-Morocco

Let be $f(a) = sinb(sina)^{\frac{1}{sina}-1} - e^{e^{sinb-1}-1}$

$$f'(a) = sinb\left(-\frac{cosa}{sin^2a} \cdot log(sina) + \left(\frac{1}{sina} - 1\right) \cdot \frac{cosa}{sina}\right)$$

$$= \frac{sinbcosa}{sin^2a}(1 - sina - log(sina))$$

We have: $sina \in (0, 1)$ so $-log(sina) > 0$ and $1 - sina > 0$

Therefore, f –is increasing on $\left[0, \frac{\pi}{2}\right]$ and decreasing on $\left[\frac{\pi}{2}, \pi\right]$

In both cases we get: $f(a) \le f\left(\frac{\pi}{2}\right) = sinb - e^{e^{sinb-1}-1}$

Let be $g(x) = x - e^{e^{x-1}-1}$, where $x = sinb \in (0,1)$

$$g'(x) = 1 - e^{x-1} \cdot e^{e^{x-1}-1} = 1 - e^{x+e^{x-1}-2} > 0$$

Because $x < 1 \to e^{x-1} < 1 \to x + e^{x-1} - 2 < 0$. Therefore, g –is increasing and we have:

$$g(x) < \lim_{x\to 1} g(x) = 0$$

So, we can assume that: $f(a) \leq sinb(sina)^{\frac{1}{sina}-1} - e^{e^{sinb-1}-1} < 0$

640. **Prove this cryptarithm:**

$$ACDEA \times BCDEB \leq ACDEB \times BCDEA$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Max Wong-Hong Kong

$$\overline{ACDEA} \times \overline{BCDEB} =$$

$$= (10001A + 10\overline{CDE})(10001B + 10\overline{CDE})$$

$$= 10001^2AB + 100010\overline{CDE}(A + B) + 100(\overline{CDE})^2$$

$$\overline{ACDEB} \times \overline{BCDEA}$$

$$= (10000A + B + 10\overline{CDE})(10000B + A + 10\overline{CDE})$$

$$= (10000A + B)(10000B + A) + 10\overline{CDE}(10001(B + A)) + 100(\overline{CDE})^2$$

$$\overline{ACDEA} \times \overline{BCDEB} \leq \overline{ACDEB} \times \overline{BCDEA}$$

$$\Leftrightarrow 10001^2AB \leq (10000A + B)(10000B + A)$$

$$\Leftrightarrow 10001^2AB \leq 10000^2AB + AB + 10000(A^2 + B^2)$$

$$\Leftrightarrow 20000AB \leq 10000(A^2 + B^2)$$

$$\Leftrightarrow 2AB \leq A^2 + B^2 \Leftrightarrow (A - B)^2 \geq 0 \Leftrightarrow \text{True}$$

$$\therefore \overline{ACDEA} \times \overline{BCDEB} \leq \overline{ACDEB} \times \overline{BCDEA}$$

641. **Solve for real numbers :**

$$e^{x^3+x^2+x} + tan^{-1}(x^3 + x) < 1$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ x^3+x^2+x=x\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right]\ and\ x^3+x=x(x^2+1),\forall x\in IR\to$$

$$If\ x<0\to\ x^3+x^2+x,x^3+x<0\to e^{x^3+x^2+x}<1\ and\ \tan^{-1}(x^3+x)<0$$

$$\to e^{x^3+x^2+x}+\tan^{-1}(x^3+x)<1\to(-\infty,0)\in S$$

$$If\ x\geq 0\to\ x^3+x^2+x,x^3+x\geq 0\to e^{x^3+x^2+x}\geq 1\ and\ \tan^{-1}(x^3+x)\geq 0$$

$$\to e^{x^3+x^2+x}+\tan^{-1}(x^3+x)\geq 1$$

$$Therefore, S=(-\infty,0)$$

642. **Let $A_1A_2\dots A_n, n\geq 3$ be a regular polygon, M a point on incircle and N a point on circumcircle of the polygon.**

Prove that:

$$\sum_{k=1}^{n}\frac{MA_k^{\ 4}}{NA_k^{\ 2}}\geq\frac{1}{4}\left(3+\cos\frac{2\pi}{n}\right)\sum_{k=1}^{n}MA_k^{\ 2}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let O, R and r be the center, circumradii and inradii, respectively, of the polygon.

$$Lemma\ 1:\ \forall P\ in\ the\ plane,\qquad\sum_{k=1}^{n}PA_k^{\ 2}=n(OP^2+R^2).$$

$$Proof: Consider\ the\ complex\ plane\ with\ the\ origin\ at\ point\ O.$$

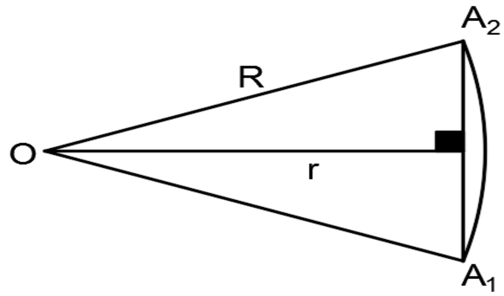
$$Let\ P(p), A_k(R\omega_k)\ where\ \omega_k\ are\ the\ nth-roots\ of\ unity, k=1,2,\dots,n.$$

$$\sum_{k=1}^{n}PA_k^{\ 2}=\sum_{k=1}^{n}(R\omega_k-p)(R\overline{\omega_k}-\overline{p})=R^2\sum_{k=1}^{n}|\omega_k|^2-R\overline{p}\sum_{k=1}^{n}\omega_k-Rp\sum_{k=1}^{n}\overline{\omega_k}+\sum_{k=1}^{n}|p|^2$$

$$=$$

$$\to\sum_{k=1}^{n}PA_k^{\ 2}=n(OP^2+R^2)\ \left(\therefore\ \sum_{k=1}^{n}\omega_k=\sum_{k=1}^{n}\overline{\omega_k}=0\ and\ |\omega_k|^2=1\ and\ |p|^2=OP^2\right)$$

$$Lemma\ 2: r=R\cos\frac{\pi}{n}$$



$$Proof: We\ have: \mu(A_1OA_2) = \frac{2\pi}{n}\ and\ \cos\left(\frac{1}{2}A_1OA_2\right) = \frac{r}{R} \to r = R\cos\frac{\pi}{n}$$

$$Using\ lemma\ 1, we\ have: \sum_{k=1}^{n} NA_k^{\ 2} = 2nR^2$$

$$and \sum_{k=1}^{n} MA_k^{\ 2} = n(r^2 + R^2) \overset{Lemma\ 2}{\cong} nR^2\left(\cos^2\left(\frac{\pi}{n}\right) + 1\right) = \frac{1}{2}nR^2\left(3 + \cos\frac{2\pi}{n}\right)$$

$$\to \sum_{k=1}^{n} \frac{MA_k^{\ 4}}{NA_k^{\ 2}} \overset{CBS}{\geq} \frac{\left(\sum_{k=1}^{n} MA_k^{\ 2}\right)^2}{\sum_{k=1}^{n} NA_k^{\ 2}} = \frac{\frac{1}{2}nR^2\left(3 + \cos\frac{2\pi}{n}\right)}{2nR^2}\sum_{k=1}^{n} MA_k^{\ 2}$$

$$= \frac{1}{4}\left(3 + \cos\frac{2\pi}{n}\right)\sum_{k=1}^{n} MA_k^{\ 2}$$

$$Therefore, \quad \sum_{k=1}^{n} \frac{MA_k^{\ 4}}{NA_k^{\ 2}} \geq \frac{1}{4}\left(3 + \cos\frac{2\pi}{n}\right)\sum_{k=1}^{n} MA_k^{\ 2}$$

643. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\frac{(\cos a - \cos b)\left(\sqrt{1 + b^2} + \sqrt{1 + a^2}\right)}{b + a} \geq b - a$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

If $a = b$, there is nothing to prove.

Suppose $a < b$, where $0 < a < b < \frac{\pi}{2}$ then

$$\frac{(\cos a - \cos b)\left(\sqrt{1 + b^2} + \sqrt{1 + a^2}\right)}{b + a} \geq b - a \Leftrightarrow$$

$$\frac{(\cos a-\cos b)(1+b^2-1-a^2)}{\sqrt{1+b^2}-\sqrt{1+a^2}}\geq b^2-a^2 \Leftrightarrow$$

$$\frac{\cos a-\cos b}{\sqrt{1+b^2}-\sqrt{1+a^2}}\geq 1$$

Let $f(x)=-\cos x$ and $g(x)=\sqrt{1+x^2}, a\leq x\leq b$. By the Cauchy's mean value theorem, we have:

$$\frac{-\cos b+\cos a}{\sqrt{1+b^2}-\sqrt{1+a^2}}=\frac{f'(c)}{g'(c)}=\frac{\sin c}{\frac{c}{\sqrt{1+c^2}}};(1)\ for\ same\ a<c<b.$$

Let $c=\tan\alpha$, where $0<\alpha<\tan\alpha<b<\frac{\pi}{2}$, that is $0<\alpha<\tan^{-1}\left(\frac{\pi}{2}\right)<\frac{\pi}{2}$

$$Now, \frac{\sin c}{\frac{c}{\sqrt{1+c^2}}}=\frac{\sin(\tan\alpha)}{\sin\alpha}>1;(2)$$

$$\because \alpha<\tan\alpha<\frac{\pi}{2}\Rightarrow \sin\alpha<\sin(\tan\alpha)$$

From $(1),(2)$ it follows that:

$$\frac{\cos a-\cos b}{\sqrt{1+b^2}-\sqrt{1+a^2}}\geq 1$$

644. For $0<a<b$, prove that: $(e^8b)^{\sqrt{ab}}a^{a(b+4)}<(e^8a)^a b^{\sqrt{ab}(a+4)}$

Proposed by Nikos Ntorvas-Greece

Solution by proposer

Let $f(x)=\frac{\ln x}{\sqrt{x}}, x>0$. f is concave $\forall x>0$

From Hermite-Hadamard Inequality we have that:

For $0<a<b$ it holds: $\big(f(a)+f(b)\big)(b-a)<2\int_a^b f(x)\,dx$ (1)

$$(1)\Leftrightarrow \left(\sqrt{b}\ln b+\sqrt{a}\ln a\right)(b-a)<2\left[2\sqrt{x}\ln x-4\sqrt{x}\right]_a^b$$

$$\Leftrightarrow \sqrt{b}(\ln b+\ln e^8)+\ln\frac{a^{4\sqrt{a}}}{b^{a\sqrt{b}}}<\sqrt{a}(\ln a+\ln e^8)+\ln\frac{b^{4\sqrt{b}}}{a^{b\sqrt{a}}}$$

$$\Leftrightarrow \ln\frac{(e^8b)^{\sqrt{b}}a^{4\sqrt{a}}}{b^{a\sqrt{b}}}<\ln\frac{(e^8a)^{\sqrt{a}}b^{4\sqrt{b}}}{a^{b\sqrt{a}}}$$

The function $g(x)=\ln x$ is strictly increasing on $(0,+\infty)$ so we have that:

$$(e^8b)^{\sqrt{b}}a^{\sqrt{a}(b+4)} < (e^8a)^{\sqrt{a}}b^{\sqrt{b}(a+4)}$$

The function $h(x) = x^{\sqrt{a}}$, strictly increasing on $(0, +\infty)$. Finally we have that:

$$(e^8b)^{\sqrt{ab}}a^{a(b+4)} < (e^8a)^{a}b^{\sqrt{ab}(a+4)}$$

645. If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\sin b - \sin a < \log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\sin b - \sin a = \sin x|_a^b = \int_a^b \cos x\,dx$$

$$\log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right) = \log\left(x+\sqrt{1+x^2}\right)\Big|_a^b = \int_a^b \frac{dx}{\sqrt{1+x^2}}$$

We must to prove that:

$$\int_a^b \cos x\,dx \leq \int_a^b \frac{dx}{\sqrt{1+x^2}} \leftrightarrow \cos x \leq \frac{1}{\sqrt{1+x^2}}, \forall x \in \left(0,\frac{\pi}{2}\right)$$

$$\leftrightarrow \cos x \cdot \sqrt{1+x^2} \leq 1, \forall x \in \left(0,\frac{\pi}{2}\right)$$

$$Let\ f(x) = \cos x \cdot \sqrt{1+x^2} \leq 1, f:\left(0,\frac{\pi}{2}\right) \to R,$$

$$f'(x) = -\sqrt{1+x^2}\cdot\sin x + \frac{x}{\sqrt{1+x^2}}\cdot\cos x$$

$$f'(x) \leq 0 \leftrightarrow x\cos x \leq (1+x^2)\sin x \leftrightarrow \frac{x}{1+x^2} \leq \tan x; (*)$$

$$But\ \frac{x}{1+x^2} \leq x \leq \tan x, \forall x \in \left(0,\frac{\pi}{2}\right) \to (*)\ is\ true \to f(x) \searrow and\ how\ f(0) = 1 \to$$

$$f(x) < 1 \to \sqrt{1+x^2}\cdot\cos x \leq 1\ is\ true.$$

Solution 2 by Ravi Prakash-New Delhi-India

Nothing to prove if $a = b$.

$$Let\ f(x) = \sin x, g(x) = \log\left(x+\sqrt{1+x^2}\right), 0 < a \leq b < \frac{\pi}{2}$$

By Cauchy M.V.T. $\exists c \in (a,b)$ *such that*:

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}\to\frac{\sin b-\sin a}{\log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)}=\frac{\cos c}{\frac{1}{\sqrt{1+c^2}}};(1)$$

$$Put\ c=\tan\alpha,0<a<\tan\alpha<b<\frac{\pi}{2}\to 0<\alpha<\tan^{-1}\left(\frac{\pi}{2}\right)<\frac{\pi}{2}$$

$$Now,\frac{\cos c}{\frac{1}{\sqrt{1+c^2}}}\le\frac{\cos(\tan\alpha)}{\cos\alpha};(2)$$

$$As\ 0<\alpha<\tan\alpha<\frac{\pi}{2}\to\cos(\tan\alpha)<\cos\alpha\to\frac{\cos(\tan\alpha)}{\cos\alpha}<1;(3)$$

From $(1),(2),(3)$ ***we get:***

$$\sin b-\sin a<\log\left(\frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}\right)$$

646.

$$\text{If } n\in N^*-\{1\},x_k\in R_+^*,\forall k=\overline{1,n},X_n(t)=\sum_{k=1}^{n}x_k^t,t\in R,$$

$$X_n(1)=\sum_{k=1}^{n}x_k=X_n,m\in[1,\infty)$$

Show that :

$$\sum_{k=1}^{n}x_k(X_n(-m)-x_k{}^{-m})\ge\frac{(n-1)n^m}{X_n{}^{m-1}}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$x_1\ge x_2\ge\cdots\ge x_n\to x_1{}^{-m}\le x_2{}^{-m}\le\cdots\le x_n{}^{-m}$$

$$\to X_n(-m)-x_1{}^{-m}\ge X_n(-m)-x_2{}^{-m}\ge\cdots\ge X_n(-m)-x_n{}^{-m}$$

$$\rightarrow \sum_{k=1}^{n} x_k(X_n(-m) - x_k^{-m}) \overset{Chebyshev}{\geq} \frac{1}{n}\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n}(X_n(-m) - x_k^{-m})\right)$$

$$= \frac{n-1}{n}.X_n.X_n(-m)$$

$$We\ have:\ X_n(-m).X_n^m = \left(\sum_{k=1}^{n}\frac{1}{x_k^m}\right)\left(\sum_{k=1}^{n} x_k\right)^m \overset{Hölder}{\geq} \left(\sum_{k=1}^{n} 1\right)^{m+1} = n^{m+1}$$

$$\rightarrow X_n.X_n(-m) \geq \frac{n^{m+1}}{X_n^{m-1}}$$

$$Therefore,\quad \sum_{k=1}^{n} x_k(X_n(-m) - x_k^{-m}) \geq \frac{(n-1)n^m}{X_n^{m-1}}$$

647. If $m \in N^*, a, c, s \in R_+, b, d, x_k, y_k \in R_+^*, k = \overline{1,m}, r \in [1,\infty)$ and

$$X_m = \sum_{k=1}^{m} x_k,\ Y_m = \sum_{k=1}^{m} y_k$$

Then show that :

$$\left(\sum_{k=1}^{m}(aX_m + bx_k)^r\right)\left(\sum_{k=1}^{m}\frac{1}{(cY_m + dy_k)^s}\right) \geq \frac{(am+b)^r}{(cm+d)^s}\cdot\frac{X_m^r}{Y_m^s}.m^{s-r+2}$$

Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \sum_{k=1}^{m}(aX_m + bx_k)^r \overset{Hölder}{\geq} \frac{1}{m^{r-1}}\left(\sum_{k=1}^{m}(aX_m + bx_k)\right)^r$$

$$= (am+b)^r.X_m^r.m^{-r+1}\ (1)$$

$$And:\ \left(\sum_{k=1}^{m}\frac{1}{(cY_m + dy_k)^s}\right)\left(\sum_{k=1}^{m}(cY_m + dy_k)\right)^s \overset{Hölder}{\geq} \left(\sum_{k=1}^{m} 1\right)^{s+1} = m^{s+1}$$

$$\rightarrow \sum_{k=1}^{m}\frac{1}{(cY_m + dy_k)^s} \geq \frac{m^{s+1}}{(cm+d)^s.Y_m^s}\quad (2)$$

$$(1),(2) \rightarrow \left(\sum_{k=1}^{m}(aX_m + bx_k)^r\right)\left(\sum_{k=1}^{m}\frac{1}{(cY_m + dy_k)^s}\right) \geq \frac{(am+b)^r}{(cm+d)^s}\cdot\frac{X_m^r}{Y_m^s}.m^{s-r+2}$$

648. Find:

$$\Omega = \underbrace{min}_{x\in\mathbb{R}}\left(\sqrt{x^2-8x+64}+\sqrt{x^2-6\sqrt{3}x+36}\right)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Asmat Qatea-Afghanistan

$$\text{Let } f(x) = \sqrt{x^2-8x+64}+\sqrt{x^2-6\sqrt{3}x+36} =$$
$$= \sqrt{(x-4)^2+48}+\sqrt{\left(x-3\sqrt{3}\right)^2+9}$$

$f(x)$ –is concave so it has minimum at point of x such that $f'(x) = 0$

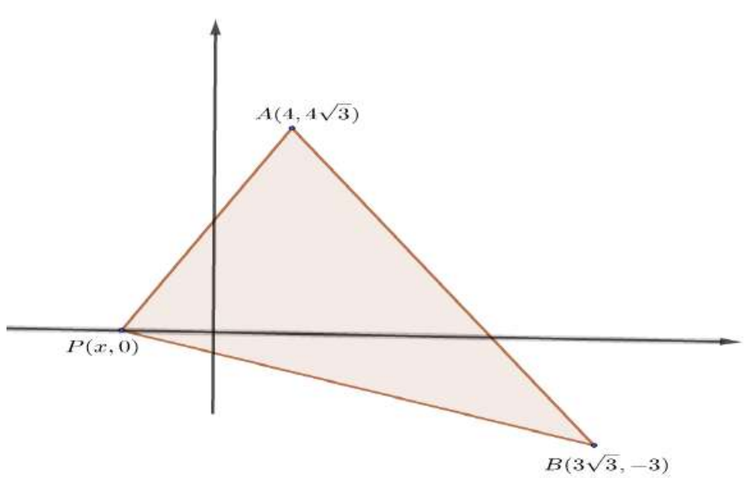
$$f'(x) = \frac{x-4}{\sqrt{(x-4)^2+48}}+\frac{x-3\sqrt{3}}{\sqrt{\left(x-3\sqrt{3}\right)^2+9}} = 0$$

$$(x-4)\sqrt{\left(x-3\sqrt{3}\right)^2+9}+\left(x-3\sqrt{3}\right)\sqrt{(x-4)^2+48} = 0$$
$$(x-4)^2\left(\left(x-3\sqrt{3}\right)^2+9\right) = \left(x-3\sqrt{3}\right)^2((x-4)^2+48)$$
$$9(x-4)^2 = 48\left(x-3\sqrt{3}\right)^2$$
$$x_1 = \frac{48}{4\sqrt{3}+3}, \qquad x_2 = \frac{24}{4\sqrt{3}-3}$$

$f(x_1) = 10$ and $f(x_2) = 10.37$. Hence:

$$\Omega = \underbrace{min}_{x\in\mathbb{R}}\left(\sqrt{x^2-8x+64}+\sqrt{x^2-6\sqrt{3}x+36}\right) = 10$$

Solution 2 by Ravi Prakash-New Delhi-India



Equality when $(x, 0)$ is the point of intersection of AB with the x –axis.

Equation of AB is: $\frac{y-4\sqrt{3}}{-3-4\sqrt{3}} = \frac{x-4}{3\sqrt{3}-4}$. For $y = 0 \to \frac{x-4}{3\sqrt{3}-4} = \frac{4\sqrt{3}}{3+4\sqrt{3}} \to$

$$x = \frac{\sqrt{3}(3\sqrt{3}-4)}{3+4\sqrt{3}} = \frac{1}{21}(48\sqrt{3}-75)$$

$$\sqrt{x^2-8x+64} + \sqrt{x^2-6\sqrt{3}x+36} =$$

$$= \sqrt{(x-4)^2+48} + \sqrt{(x-3\sqrt{3})^2+(-3)^2} = PB$$

$$\sqrt{x^2-6\sqrt{3}x+36} = \sqrt{(x-4)^2+(4\sqrt{3})^2} \to$$

$$\Omega = \sqrt{(x-4)^2+(4\sqrt{3})^2} + \sqrt{(x-3\sqrt{3})^2+(-3)^2} = PA + PB \geq AB =$$

$$= \sqrt{(4-3\sqrt{3})^2+(4\sqrt{3}+3)^2} = 10$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\sqrt{x^2-8x+64} = |(x-4)+4\sqrt{3}i|$$

$$\sqrt{x^2-6\sqrt{3}x+36} = |(3\sqrt{3}-x)+(-3)i|$$

$$\sqrt{x^2-8x+64} + \sqrt{x^2-6\sqrt{3}x+36} = |(x-4)+4\sqrt{3}i| + |(3\sqrt{3}-x)+(-3)i| \geq$$

$$\geq |(x-4)+(3\sqrt{3}-x)+(4\sqrt{3}-3)i| = \sqrt{(3\sqrt{3}-4)^2+(4\sqrt{3}-3)^2} = 10$$

649. If $a, b > 0, x \in \mathbb{R}$ then :

$$(1 + a\sin^2 x + b\cos^2 x)^{a\sin^2 x + b\cos^2 x} \leq (1+a)^{a\sin^2 x}.(1+b)^{b\cos^2 x}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohammed Amine Ben Ajiba-Tanger-Morocco

$$Let\ f(y) = y\log(1+y), y > 0. f'(y) = \log(1+y) + \frac{y}{1+y}\ and$$

$$f''(y) = \frac{1}{1+y} + \frac{1}{(1+y)^2} > 0$$

$$\overset{Using\ Jensen}{\Rightarrow} \quad \sin^2 x . f(a) + \cos^2 x . f(b) \geq$$

$$\geq f(a\sin^2 x + b\cos^2 x) \ \ (\because \sin^2 x + \cos^2 x = 1)$$

$$\leftrightarrow a\sin^2 x \log(1+a) + b\cos^2 x \log(1+b)$$

$$\geq (a\sin^2 x + b\cos^2 x)\log(1 + a\sin^2 x + b\cos^2 x)$$

$$\leftrightarrow \log\left((1+a)^{a\sin^2 x}.(1+b)^{b\cos^2 x}\right) \geq \log\left((1 + a\sin^2 x + b\cos^2 x)^{a\sin^2 x + b\cos^2 x}\right)$$

Therefore,

$$(1 + a\sin^2 x + b\cos^2 x)^{a\sin^2 x + b\cos^2 x} \leq (1+a)^{a\sin^2 x}.(1+b)^{b\cos^2 x}$$

650. If $a, m \in \mathbb{R}_+$ and $b, c, d, x, y, z \in \mathbb{R}_+^*, X = x + y + z, cX > d \cdot \max(x, y, z\}$, then:

$$\left(\frac{aX + bx}{cX - dx}\right)^{m+1} + \left(\frac{aX + by}{cX - dy}\right)^{m+1} + \left(\frac{aX + bz}{cX - dz}\right)^{m+1} \geq \frac{3(3a+b)^{m+1}}{(3c-d)^{m+1}}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by George Florin Șerban-Romania

$$\sum_{cyc}\left(\frac{aX + bx}{cX - dx}\right)^{m+1} \overset{Holder}{\geq} \frac{1}{3^m}\cdot\left(\sum_{cyc}\left(\frac{aX + bx}{cX - dx}\right)\right)^{m+1} \overset{(i)}{\geq} \frac{3(3a+b)^{m+1}}{(3c-d)^{m+1}}$$

$$\rightarrow \left(\sum_{cyc}\left(\frac{aX + bx}{cX - dx}\right)\right)^{m+1} \geq \frac{3^{m+1}(3a+b)^{m+1}}{(3c-d)^{m+1}}$$

$$cX > d \cdot \max(x, y, z\} \geq \{dx, dy, dz\} \rightarrow cX \geq dx, cX \geq dy, cX \geq dz$$

$$\rightarrow 3cX \geq d(x + y + z) \rightarrow 3cX \geq dX \rightarrow (3c - d)X \geq 0; (X > 0, 3c - d > 0)$$

$$\rightarrow \sum_{cyc}\left(\frac{aX + bx}{cX - dx}\right) \overset{(ii)}{\geq} \frac{3(3a+b)}{3c-d}$$

$$\sum_{cyc}\left(\frac{aX + bx}{cX - dx}\right) = \sum_{cyc}\frac{(aX + bx)^2}{(aX + bx)(cX - dx)} = \sum_{cyc}\frac{(aX + bx)^2}{acX^2 + (bc - ad)xX - bdx^2} \overset{Bergstrom}{\geq}$$

$$\geq \frac{\left(\sum(aX + bx)\right)^2}{\sum(acX^2 + (bc - ad)xX - bdx^2)} \overset{(x+y+z)^2 \leq 3(x^2+y^2+z^2)}{\geq} \frac{(3aX + bX)^2}{3acX^2 + (bc - ad)X^2 - \frac{bdx^2}{3}} =$$

$$=\frac{3(3a+b)^2}{X^2(9ac+3bc-3ad-bd)}=\frac{3(3a+b)}{3c-d}$$

Therefore,

$$\left(\frac{aX+bx}{cX-dx}\right)^{m+1}+\left(\frac{aX+by}{cX-dy}\right)^{m+1}+\left(\frac{aX+bz}{cX-dz}\right)^{m+1}\geq\frac{3(3a+b)^{m+1}}{(3c-d)^{m+1}}$$

651. If $a\in\mathbb{R}_+, m\in[1,\infty)$ **and** $b,c,d,x,y,z\in\mathbb{R}_+^*, X=x+y+z,$

$cX>d\,max\{x,y,z\}$, **then**

$$\sum\frac{aX+bx}{(cX-dx)^m}\geq\frac{3^m(3a+b)}{(3c-d)^mX^{m-1}}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum\frac{aX+bx}{(cX-dx)^m}=aX\sum\frac{1}{(cX-dx)^m}+b\sum\frac{x}{(cX-dx)^m}$$

$$=aX\sum\frac{1}{(cX-dx)^m}+b\sum\frac{x^{m+1}}{(cXx-dx^2)^m}\geq$$

$$\overset{Hölder}{\geq}aX.\frac{(\sum 1)^{m+1}}{[\sum(cX-dx)]^m}+b.\frac{(\sum x)^{m+1}}{[\sum(cXx-dx^2)]^m}=\frac{3^{m+1}.aX}{[(3c-d)X]^m}+\frac{bX^{m+1}}{(cX^2-d\sum x^2)^m}\geq$$

$$\overset{3\sum x^2\geq X^2}{\geq}\frac{3^{m+1}.aX}{(3c-d)^mX^m}+\frac{bX^{m+1}}{\left(cX^2-d.\frac{1}{3}X^2\right)^m}=\frac{3^{m+1}.a}{(3c-d)^mX^{m-1}}+\frac{3^mbX^{m+1}}{(3c-d)^mX^{2m}}$$

$$Therefore,\quad\sum\frac{aX+bx}{(cX-dx)^m}\geq\frac{3^m(3a+b)}{(3c-d)^mX^{m-1}}$$

652. **1) Prove that :** $\sum_{k=1}^n\frac{1}{\left(ax_k+b\sqrt[n]{x_1x_2\dots x_n}\right)^m}\geq\frac{n^{m+1}}{(a+b)^m{X_n}^m}$

2) Prove that : $$\sum_{k=1}^n\frac{x_1x_2\dots x_n+c{x_k}^{2m+1}}{{x_k}^m(aX_n-bx_k)^m}\geq\frac{n^{2m+1}\cdot x_1x_2\dots x_n+c{X_n}^{2m+1}}{(an-b)^m{X_n}^{2m}}$$

$X_n=x_1+x_2+\dots+x_n,\quad a,b,x_1,x_2,\dots,x_n>0, m,n\in\mathbb{N}^*$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$1)\ We\ have:$$

$$\left(\sum_{k=1}^{n}\frac{1}{\left(ax_k+b\sqrt[n]{x_1x_2\dots x_n}\right)^m}\right)\left(\sum_{k=1}^{n}\left(ax_k+b\sqrt[n]{x_1x_2\dots x_n}\right)\right)^m\overset{Hölder}{\geq}\left(\sum_{k=1}^{n}1\right)^{m+1}=n^{m+1}$$

$$And\ \sum_{k=1}^{n}\left(ax_k+b\sqrt[n]{x_1x_2\dots x_n}\right)=aX_n+bn\sqrt[n]{x_1x_2\dots x_n}\overset{AM-GM}{\leq}aX_n+bX_n=(a+b)X_n$$

$$\to\sum_{k=1}^{n}\frac{1}{\left(ax_k+b\sqrt[n]{x_1x_2\dots x_n}\right)^m}\geq\frac{n^{m+1}}{\left(\sum_{k=1}^{n}\left(ax_k+b\sqrt[n]{x_1x_2\dots x_n}\right)\right)^m}\geq\frac{n^{m+1}}{\left((a+b)X_n\right)^m}=\frac{n^{m+1}}{(a+b)^mX_n{}^m}$$

$$Therefore,\quad\sum_{k=1}^{n}\frac{1}{\left(ax_k+b\sqrt[n]{x_1x_2\dots x_n}\right)^m}\geq\frac{n^{m+1}}{(a+b)^mX_n{}^m}$$

$$2)\ \sum_{k=1}^{n}\frac{x_1x_2\dots x_n+cx_k{}^{2m+1}}{x_k{}^m(aX_n-bx_k)^m}=x_1x_2\dots x_n\sum_{k=1}^{n}\frac{1}{x_k{}^m(aX_n-bx_k)^m}+c\sum_{k=1}^{n}\frac{x_k{}^{m+1}}{(aX_n-bx_k)^m}.$$

$$We\ have:\left(\sum_{k=1}^{n}\frac{1}{x_k{}^m(aX_n-bx_k)^m}\right)\left(\sum_{k=1}^{n}x_k\right)^m\left(\sum_{k=1}^{n}(aX_n-bx_k)\right)^m\overset{Hölder}{\geq}\left(\sum_{k=1}^{n}1\right)^{2m+1}=n^{2m+1}$$

$$\to\sum_{k=1}^{n}\frac{1}{x_k{}^m(aX_n-bx_k)^m}\geq\frac{n^{2m+1}}{X_n{}^m.\left[(an-b)X_n\right]^m}=\frac{n^{2m+1}}{(an-b)^mX_n{}^{2m}}$$

$$And:\left(\sum_{k=1}^{n}\frac{x_k{}^{m+1}}{(aX_n-bx_k)^m}\right)\left(\sum_{k=1}^{n}(aX_n-bx_k)\right)^m\overset{Hölder}{\geq}\left(\sum_{k=1}^{n}x_k\right)^{m+1}=X_n{}^{m+1}$$

$$\to\sum_{k=1}^{n}\frac{x_k{}^{m+1}}{(aX_n-bx_k)^m}\geq\frac{X_n{}^{m+1}}{\left[(an-b)X_n\right]^m}=\frac{X_n{}^{m+1}}{(an-b)^mX_n{}^m}=\frac{X_n{}^{2m+1}}{(an-b)^mX_n{}^{2m}}$$

$$\to\sum_{k=1}^{n}\frac{x_1x_2\dots x_n+cx_k{}^{2m+1}}{x_k{}^m(aX_n-bx_k)^m}\geq x_1x_2\dots x_n.\frac{n^{2m+1}}{(an-b)^mX_n{}^{2m}}+c.\frac{X_n{}^{2m+1}}{(an-b)^mX_n{}^{2m}}=\frac{n^{2m+1}.x_1x_2\dots x_n+cX_n{}^{2m+1}}{(an-b)^mX_n{}^{2m}}$$

653. If $a\in\mathbb{R}_+$ and $b,c,d,x,y,z\in\mathbb{R}_+^*$, $X=x+y+z$, $cX>d\max\{x,y,z\}$, then:

$$\sum_{cyc}\frac{aX+bx}{cX-dx}\ge\frac{3(3a+b)}{3c-d}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by George Florin Șerban-Romania

$$cX>d\max\{x,y,z\}\ge dx,dy,dz\to cX\ge dx,cX\ge dy,cX\ge dz$$

$$\to 3cX\ge d(x+y+z)\to 3cX\ge dX\to X(3c-d)\ge 0$$

$$X>0\to 3c-d>0\to\sum_{cyc}\frac{aX+bx}{cX-dx}\overset{?}{\ge}\frac{3(3a+b)}{3c-d}$$

$$\sum_{cyc}\frac{aX+bx}{cX-dx}=\sum_{cyc}\frac{(aX+bx)^2}{(aX+bx)(cX-dx)}=\sum_{cyc}\frac{(aX+bx)^2}{acX^2+(bc-ad)xX-bdx^2}\overset{Bergstrom}{\ge}$$

$$\ge\frac{\left(\sum(aX+bx)\right)^2}{\sum[(acX^2+(bc-ad)xX-bdx^2]}\ge\frac{(3aX+bX)^2}{3acX^2+(bc-ad)X^2-\frac{bdX^2}{3}}=$$

$$=\frac{3X^2(3a+b)^2}{X^2(9ac+3bc-3ad-bd)}=\frac{3(3a+b)^2}{3a(3c-d)+b(3c-d)}=$$

$$=\frac{3(3a+b)^2}{(3c-d)(3a+b)}=\frac{3(3a+b)}{3c-d}$$

654. Prove that :

$$\sum_{k=0}^{n}\frac{\binom{n}{k}}{2^{n+1}a-b\binom{n}{k}}\ge\frac{n+1}{2a(n+1)-b}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ (x_k)_{0\le k\le n}\ be\ a\ permutation\ of\ \binom{n}{k}\ such\ that: x_0\ge x_1\ge\cdots\ge x_n$$

$$\to\frac{1}{2^{n+1}a-bx_0}\ge\frac{1}{2^{n+1}a-bx_1}\ge\cdots\ge\frac{1}{2^{n+1}a-bx_n}.$$

$$\overset{Using\ Chebyshev}{\Rightarrow}\sum_{k=0}^{n}\frac{\binom{n}{k}}{2^{n+1}a-b\binom{n}{k}}=\sum_{k=0}^{n}\frac{x_k}{2^{n+1}a-bx_k}$$

$$\geq\frac{1}{n+1}\left(\sum_{k=0}^{n}x_k\right)\left(\sum_{k=0}^{n}\frac{1}{2^{n+1}a-bx_k}\right)$$

$$We\ know\ that:\sum_{k=0}^{n}x_k=2^n.$$

$$\to\sum_{k=0}^{n}\frac{\binom{n}{k}}{2^{n+1}a-b\binom{n}{k}}\geq\frac{2^n}{n+1}\left(\sum_{k=0}^{n}\frac{1}{2^{n+1}a-bx_k}\right)\overset{CBS}{\geq}\frac{2^n}{n+1}\cdot\frac{(n+1)^2}{\sum_{k=0}^{n}(2^{n+1}a-bx_k)}=$$

$$=\frac{2^n(n+1)}{2^{n+1}a(n+1)-b.2^n}=\frac{n+1}{2a(n+1)-b}.$$

$$Therefore,\quad\sum_{k=0}^{n}\frac{\binom{n}{k}}{2^{n+1}a-b\binom{n}{k}}\geq\frac{n+1}{2a(n+1)-b}$$

655. If $0<a\leq b$ then:

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\tan^{-1}b-\tan^{-1}\sqrt{ab}\right)\leq\sqrt{b}\left(\tan^{-1}b-\tan^{-1}a\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Samar Das-India

$$0<a\leq b\to 0<\sqrt{a}\leq\sqrt{b}\to 0<\sqrt{ab}<b;(1)$$

$$0<\sqrt{a}\leq\sqrt{b}\to 0<a\leq\sqrt{ab};(2)$$

$$\text{From }(1),(2)\to 0<a\leq\sqrt{ab}\leq b;(3)$$

Let: $f(x)=\tan^{-1}x$, $f'(x)=\frac{1}{1+x^2}$, when $\sqrt{ab}\leq c\leq b$, then applying Lagrange's Theorem

to $f(x)$ on $[\sqrt{ab},b]$, then $f'(c)=\frac{1}{1+c^2}$. Now, $f'(c)=\frac{f(b)-f(a)}{b-a}$; $(a\leq c\leq b)$

$$\to\frac{1}{1+c^2}=\frac{\tan^{-1}b-\tan^{-1}a}{b-\sqrt{ab}};\left(\sqrt{ab}\leq c\leq b\right)$$

$$\to\frac{\sqrt{b}}{1+c^2}=\frac{\tan^{-1}b-\tan^{-1}\sqrt{ab}}{\sqrt{b}-\sqrt{a}};(4),a\leq c=\sqrt{ab}\leq b$$

Then, applying Lagrange's Theorem to $f(x)$ on $[a,b]$ i.e. $a\leq\sqrt{ab}\leq b$.

$$f'(\sqrt{ab})=\frac{1}{1+ab}=\frac{f(b)-f(a)}{b-a}\to\frac{1}{1+ab}=\frac{\tan^{-1}b-\tan^{-1}a}{b-a};(5)$$

$$(5)\to\frac{\sqrt{b}+\sqrt{a}}{1+ab}=\frac{\tan^{-1}b-\tan^{-1}a}{\sqrt{b}-\sqrt{a}};(6)$$

From $(4),(6)$ it follows that:

$$\frac{\tan^{-1}b-\tan^{-1}a}{\sqrt{b}-\sqrt{a}}\cdot\frac{\sqrt{b}-\sqrt{a}}{\tan^{-1}b-\tan^{-1}a}=\frac{\sqrt{b}+\sqrt{a}}{1+ab}\cdot\frac{1+c^2}{\sqrt{b}}$$

$$\to\frac{\tan^{-1}b-\tan^{-1}a}{\tan^{-1}b-\tan^{-1}\sqrt{ab}}=\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}}\cdot\frac{1+c^2}{1+ab}$$

$$\to(\sqrt{b}+\sqrt{a})(\tan^{-1}b-\tan^{-1}\sqrt{ab})=\sqrt{b}(\tan^{-1}b-\tan^{-1}a)\cdot\frac{1+ab}{1+c^2};(7)$$

Now, $a\le\sqrt{ab}\le b$ and $\sqrt{ab}\le c\le b\to a\le\sqrt{ab}\le c\le b\to ab\le c^2\to$

$$1+ab\le 1+c^2\to\frac{1+ab}{1+c^2}\le 1;(8)$$

From $(7),(8)$ it follows that:

$$(\sqrt{a}+\sqrt{b})(\tan^{-1}b-\tan^{-1}\sqrt{ab})\le\sqrt{b}(\tan^{-1}b-\tan^{-1}a)$$

656. If $a,b,c>0,(a+b)^3+(b+c)^3+(c+a)^3=24$, then:

$$(a+b)(a^2+b^2)+(b+c)(b^2+c^2)+(c+a)(c^2+a^2)\ge 12$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\sum_{cyc}(a+b)(a^2+b^2)\overset{Power-Mean}{\geq}\sum_{cyc}(a+b)\cdot\frac{(a+b)^2}{2}=$$

$$=\frac{1}{2}\sum_{cyc}(a+b)^3=\frac{24}{2}=12$$

Solution 2 by Ravi Prakash-New Delhi-India

$$(a+b)(a^2+b^2)+(b+c)(b^2+c^2)+(c+a)(c^2+a^2)=$$

$$=\frac{1}{2}\{(a+b)[(a+b)^2+(a-b)^2]+(b+c)[(b+c)^2+(b-c)^2]$$

$$+(c+a)[(c+a)^2+(c-a)^2]\}=\frac{1}{2}[(a+b)^3+(b+c)^3+(c+a)^3]+$$

$$+\frac{1}{2}[(a+b)(a-b)^2+(b+c)(b-c)^2+(c+a)(c-a)^2]\ge\frac{24}{2}=12$$

Equality holds when $a-b=b-c=c-a \Leftrightarrow a=b=c=1$.

657. Let be positive numbers $x_1\le x_2\le\cdots\le x_n$. If $x_1x_2\dots x_n=n^n$ then:

$$\prod_{k=1}^{n}\left(1+e^{kx_k}\right)\ge\left(1+\sqrt{e^{n(n+1)}}\right)^n$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\prod_{k=1}^{n}\left(1+e^{kx_k}\right)\ge\left(1+\sqrt{e^{n(n+1)}}\right)^n;(*)$$

$$Let\ f(x)=\log(1+e^x),x>0.we\ have:\ f'(x)=\frac{e^x}{1+e^x}\ and\ f''(x)=\frac{e^x}{(1+e^x)^2}>0$$

$$\to f-convex\overset{Using\ Jensen}{\Rightarrow}\sum_{k=1}^{n}f(kx_k)\ge nf\left(\frac{1}{n}\sum_{k=1}^{n}kx_k\right)$$

$$\leftrightarrow\sum_{k=1}^{n}\log\left(1+e^{kx_k}\right)\ge n\log\left(1+e^{\frac{1}{n}\sum_{k=1}^{n}kx_k}\right)\leftrightarrow\prod_{k=1}^{n}\left(1+e^{kx_k}\right)\ge\left(1+e^{\frac{1}{n}\sum_{k=1}^{n}kx_k}\right)^n$$

$$So,we\ need\ to\ prove:\ \frac{1}{n}\sum_{k=1}^{n}kx_k\ge\frac{n(n+1)}{2}$$

$$We\ have:\ x_1\le x_2\le\cdots\le x_n\ and\ 1\le 2\le\cdots\le n$$

$$\overset{Using\ Chebyshev}{\Rightarrow}\sum_{k=1}^{n}kx_k\ge\frac{1}{n}\left(\sum_{k=1}^{n}k\right)\left(\sum_{k=1}^{n}x_k\right)\overset{AM-GM}{\ge}\frac{n+1}{2}.n\sqrt[n]{x_1x_2\dots x_n}=\frac{n^2(n+1)}{2}$$

$$\to\frac{1}{n}\sum_{k=1}^{n}kx_k\ge\frac{n(n+1)}{2}\ \to$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$(1+e^{x_1})(1+e^{2x_2})\dots(1+e^{2x_n})\ge\left(1+e^{\frac{(x_1+2x_2+\cdots+nx_n)}{n}}\right)^n\ge$$

$$\ge\left(1+e^{\left(\frac{(x_1+x_2+\cdots+x_n)(1+2+\cdots+n)}{n\cdot n}\right)}\right)^n\ge\left(1+e^{\frac{n^2}{n^2}\cdot\frac{n(n+1)}{2}}\right)^n=\left(1+\sqrt{e^{n(n+1)}}\right)^n,$$

Which is true, because $x_1x_2\dots x_n=n^n$ iff $\frac{x_1}{n}\cdot\frac{x_2}{n}\cdot\dots\cdot\frac{x_n}{n}=1$. Hence,

$$\frac{x_1+x_2+\dots+x_n}{n}\ge n\to x_1+x_2+\dots+x_n\ge n^2.$$

658.

If $a,b\in\mathbb{R}_+,c,u,v\in\mathbb{R}_+^*,n\in\mathbb{N},n\ge 2,x_ky_k\in\mathbb{R}_+^*,k=\overline{1,n},X_n=\sum_{k=1}^{n}x_k,Y_n=\sum_{k=1}^{n}y_k$

$bY_n+cX_n>v\cdot\max_{1\le k\le n}x_k$. Then prove that :

$$\sum_{k=1}^{n}\frac{aX_n+ux_k}{bY_n+cX_n-vx_k}\ge\frac{(an+u)nX_n}{bnY_n+(cn-v)X_n}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$If\ x_1\ge x_2\ge\dots\ge x_n\to aX_n+ux_1\ge aX_n+ux_2\ge\dots\ge aX_n+ux_n$$

$$and\ \frac{1}{bY_n+cX_n-vx_1}\ge\frac{1}{bY_n+cX_n-vx_2}\ge\dots\ge\frac{1}{bY_n+cX_n-vx_n}$$

$$\overset{Using\ Chebyshev}{\Rightarrow}\sum_{k=1}^{n}\frac{aX_n+ux_k}{bY_n+cX_n-vx_k}\ge\frac{1}{n}\left(\sum_{k=1}^{n}(aX_n+ux_k)\right)\left(\sum_{k=1}^{n}\frac{1}{bY_n+cX_n-vx_k}\right)\ge$$

$$\overset{CBS}{\ge}\frac{anX_n+uX_n}{n}\cdot\frac{n^2}{\sum_{k=1}^{n}(bY_n+cX_n-vx_k)}=\frac{(an+u)nX_n}{bnY_n+cnX_n-vX_n}$$

Therefore,

$$\sum_{k=1}^{n}\frac{aX_n+ux_k}{bY_n+cX_n-vx_k}\ge\frac{(an+u)nX_n}{bnY_n+(cn-v)X_n}.$$

659.

If $n\in\mathbb{N},n\ge 3,a,b,c,x_k>0,\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\prod_{k=1}^{n}x_k=a$, then:

$$\sum_{k=1}^{n}(bx_k^{\ n-1}+c).\frac{x_k^{\ 3}+x_{k+1}^{\ 3}}{x_k^{\ 2}+x_kx_{k+1}+x_{k+1}^{\ 2}}\ge\frac{2n}{3a}(ab+cn)\prod_{k=1}^{n}x_k$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: x_k^2 - x_kx_{k+1} + x_{k+1}^2 \geq \frac{1}{3}(x_k^2 + x_kx_{k+1} + x_{k+1}^2) \leftrightarrow (x_{k+1} - x_k)^2 \geq 0$$

$$\rightarrow \frac{x_k^3 + x_{k+1}^3}{x_k^2 + x_kx_{k+1} + x_{k+1}^2} = \frac{(x_k + x_{k+1})(x_k^2 - x_kx_{k+1} + x_{k+1}^2)}{x_k^2 + x_kx_{k+1} + x_{k+1}^2}$$

$$\geq \frac{x_k + x_{k+1}}{3}\ (And\ analogs)$$

$$\rightarrow \sum_{k=1}^{n}(bx_k^{n-1} + c).\frac{x_k^3 + x_{k+1}^3}{x_k^2 + x_kx_{k+1} + x_{k+1}^2} \geq \frac{1}{3}\sum_{k=1}^{n}(bx_k^{n-1} + c)(x_k + x_{k+1}) =$$

$$= \frac{b}{3}\sum_{k=1}^{n}x_k^n + \frac{b}{3}\sum_{k=1}^{n}x_k^{n-1}x_{k+1} + \frac{2c}{3}\sum_{k=1}^{n}x_k.$$

$$From\ AM - GM, we\ have: \sum_{k=1}^{n}x_k^n \geq n\prod_{k=1}^{n}x_k\ and\ \sum_{k=1}^{n}x_k^{n-1}x_{k+1}$$

$$\geq n\sqrt[n]{\prod_{k=1}^{n}(x_k^{n-1}x_{k+1})} = n\prod_{k=1}^{n}x_k$$

$$From\ CBS, we\ have: \sum_{k=1}^{n}x_k \geq \frac{n^2}{\sum_{k=1}^{n}\frac{1}{x_k}} = \frac{n^2}{a}\prod_{k=1}^{n}x_k$$

$$\rightarrow \sum_{k=1}^{n}(bx_k^{n-1} + c).\frac{x_k^3 + x_{k+1}^3}{x_k^2 + x_kx_{k+1} + x_{k+1}^2} \geq \frac{2b}{3}.n\prod_{k=1}^{n}x_k + \frac{2c}{3}.\frac{n^2}{a}\prod_{k=1}^{n}x_k$$

$$= \frac{2n}{3a}(ab + cn)\prod_{k=1}^{n}x_k.$$

660.

$$\text{If } n \in \mathbb{N}, n \geq, a, b, c, x_k > 0, \left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\prod_{k=1}^{n}x_k \leq d, \text{then:}$$

$$\sum_{k=1}^{n}(ax_k^{n-1} + bx_k^{n-1} + c).\frac{x_k^3 + x_{k+1}^3}{x_k^2 + x_kx_{k+1} + x_{k+1}^2} \geq \frac{2n}{3d}((a+b)d + cn)\prod_{k=1}^{n}x_k$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ x_k^2-x_kx_{k+1}+x_{k+1}^2\ge\frac{1}{3}(x_k^2+x_kx_{k+1}+x_{k+1}^2)\leftrightarrow(x_{k+1}-x_k)^2\ge 0$$

$$\to x_k^2-x_kx_{k+1}+x_{k+1}^2\ge\frac{1}{3}(x_k^2+x_kx_{k+1}+x_{k+1}^2),\forall k\in\{1,2,\dots,n\},x_{n+1}=x_1$$

$$\to\frac{x_k^3+x_{k+1}^3}{x_k^2+x_kx_{k+1}+x_{k+1}^2}=\frac{(x_k+x_{k+1})(x_k^2-x_kx_{k+1}+x_{k+1}^2)}{x_k^2+x_kx_{k+1}+x_{k+1}^2}\ge\frac{x_k+x_{k+1}}{3},\forall k\in\{1,2,\dots,n\},x_{n+1}=x_1$$

$$\to\sum_{k=1}^{n}(ax_k^{n-1}+bx_k^{n-1}+c).\frac{x_k^3+x_{k+1}^3}{x_k^2+x_kx_{k+1}+x_{k+1}^2}\ge\frac{1}{3}\sum_{k=1}^{n}\left((a+b)x_k^{n-1}+c\right)(x_k+x_{k+1})=$$

$$=\frac{a+b}{3}\sum_{k=1}^{n}x_k^n+\frac{a+b}{3}\sum_{k=1}^{n}x_k^{n-1}x_{k+1}+\frac{2c}{3}\sum_{k=1}^{n}x_k.$$

$$From\ AM-GM,we\ have:\ \sum_{k=1}^{n}x_k^n\ge n\prod_{k=1}^{n}x_k\ and\ \sum_{k=1}^{n}x_k^{n-1}x_{k+1}\ge n\sqrt[n]{\prod_{k=1}^{n}(x_k^{n-1}x_{k+1})}=n\prod_{k=1}^{n}x_k$$

$$From\ CBS,we\ have:\ \sum_{k=1}^{n}x_k\ge\frac{n^2}{\sum_{k=1}^{n}\frac{1}{x_k}}\ge\frac{n^2}{d}\prod_{k=1}^{n}x_k$$

$$\to\sum_{k=1}^{n}(ax_k^{n-1}+bx_k^{n-1}+c).\frac{x_k^3+x_{k+1}^3}{x_k^2+x_kx_{k+1}+x_{k+1}^2}\ge\frac{2(a+b)}{3}.n\prod_{k=1}^{n}x_k+\frac{2c}{3}.\frac{n^2}{d}\prod_{k=1}^{n}x_k=\frac{2n}{3d}((a+b)d+cn)\prod_{k=1}^{n}x_k.$$

661. If $x,y>0,x+y=\sqrt{\tan^{-1}\left(\frac{1}{5}\right)}$ **then:**

$$\frac{4x^2}{\pi}+\frac{y^2}{\tan^{-1}\left(\frac{1}{239}\right)}\ge\frac{1}{4}$$

Proposed by Daniel Sitaru-Romania

Solution by Asmat Qatea-Afghanistan

$$\frac{4x^2}{\pi}+\frac{y^2}{\tan^{-1}\left(\frac{1}{239}\right)}=\frac{x^2}{\frac{\pi}{4}}+\frac{y^2}{\tan^{-1}\left(\frac{1}{239}\right)}\overset{Bergstrom}{\geq}\frac{(x+y)^2}{\frac{\pi}{4}+\tan^{-1}\left(\frac{1}{239}\right)}$$

$$\frac{x^2}{\frac{\pi}{4}}+\frac{y^2}{\tan^{-1}\left(\frac{1}{239}\right)}\geq\frac{\tan^{-1}\left(\frac{1}{5}\right)}{\frac{\pi}{4}+\tan^{-1}\left(\frac{1}{239}\right)}=\frac{1}{4};(*)$$

Prove for $(*)$: $\tan x=\frac{1}{5}\to x=\tan^{-1}\left(\frac{1}{5}\right)$

$$\tan(2x)=\frac{2\tan x}{1-\tan^2 x}=\frac{2\cdot\frac{1}{5}}{1-\frac{1}{25}}=\frac{5}{12}$$

$$\tan(4x)=\frac{2\tan(2x)}{1-\tan^2(2x)}=\frac{2\cdot\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2}=\frac{120}{119}=1+\frac{1}{119}=\tan\frac{\pi}{4}+\frac{1}{119}$$

$$\tan\left(4x-\frac{\pi}{4}\right)=\frac{\tan(4x)-1}{1+\tan(4x)}=\frac{\frac{1}{119}}{\frac{239}{119}}=\frac{1}{239}$$

Therefore,

$$\frac{4x^2}{\pi}+\frac{y^2}{\tan^{-1}\left(\frac{1}{239}\right)}\geq\frac{1}{4}$$

662. Prove that if $0<a\leq b$ and $x_k\in\left(0,\frac{\pi}{2}\right),\forall k=\overline{1,n}$, then:

$$\sum_{k=1}^{n}(a\sin x_k+b\tan x_k)>(a+b)\sum_{k=1}^{n}x_k$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $b=a+c, c\geq 0$. For $0\leq x\leq\frac{\pi}{2}$, let $f(x)=a\sin x+b\tan x-(a+b)x$

$$f'(x)=a\cos x+b\sec^2 x-(a+b)=b\tan^2 x+b\sec^2 x-(a+b)=$$

$$=(a+c)\tan^2 x+a(\cos x-1)=a(\tan^2 x\cos x-1)+c\tan^2 x=$$

$= ag(x) + c\tan^2 x$, where $g(x) = \tan^2 x + \cos x - 1, 0 \le x \le \frac{\pi}{2}$

$$g'(x) = 2\tan x \sec^2 x - \sin x = \frac{\sin x}{\cos^3 x}(2 - \cos^3 x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$$

$$\to g(x) > g(0) = 0, \forall x \in \left(0, \frac{\pi}{2}\right)$$

$$f'(x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$$

Thus, $a\sin x + b\tan x > (a+b)x, \forall\, x \in \left(0, \frac{\pi}{2}\right)$

Therefore,

$$\sum_{k=1}^{n}(a\sin x_k + b\tan x_k) > (a+b)\sum_{k=1}^{n} x_k$$

Solution 2 by Mohamed Amine-Tanger-Morocco

$$We\ have: \forall k = \overline{1,n};\ \tan x_k \ge \sin x_k\ and\ b \ge a$$

$$\overset{Using\ Chebyshev}{\Rightarrow}\quad a\sin x_k + b\tan x_k \ge \frac{1}{2}(a+b)(\sin x_k + \tan x_k)$$

$$Now, we\ know\ that:\ 0 < x < \tan x < 1, \forall x \in \left(0, \frac{\pi}{4}\right)$$

$\to \sin x_k + \tan x_k$

$$= \frac{2\tan\frac{x_k}{2}}{1+\tan^2\frac{x_k}{2}} + \frac{2\tan\frac{x_k}{2}}{1-\tan^2\frac{x_k}{2}} \overset{CBS}{\ge} 2\tan\frac{x_k}{2}\cdot\frac{4}{\left(1+\tan^2\frac{x_k}{2}\right)+\left(1-\tan^2\frac{x_k}{2}\right)}$$

$$= 4\tan\frac{x_k}{2} > 4.\frac{x_k}{2} = 2x_k \to \sin x_k + \tan x_k > 2x_k, \forall k = \overline{1,n}$$

$$\to a\sin x_k + b\tan x_k > (a+b)x_k, \forall k = \overline{1,n}$$

$$Therefore,\quad \sum_{k=1}^{n}(a\sin x_k + b\tan x_k) > (a+b)\sum_{k=1}^{n} x_k$$

663. $a, b, c, d \ge 0, 2a + 3c \ge 10.$

Find $\Omega = minE.$

$$E = a^3 + b^3 + c^3 + d^3 + a^2 + 2b^2 + 3c^2 + 4d^2$$

Proposed by George-Florin Șerban-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$From\ AM-GM, we\ have:\ a^3 + 2^3 + 2^3 \ge 3.2.2.a = 12a \to a^3 \ge 12a - 16$$

$$And:\ a^2 + 2^2 \ge 2.2a = 4a \to a^2 \ge 4a - 4$$

$$Similarly, we\ have: \ c^3 \ge 12c - 16 \text{ and } c^2 \ge 4c - 4$$

$$\to E \overset{b,d\ge 0}{\gtrsim} (12a - 16) + (12c - 16) + (4a - 4) + 3(4c - 4) = 16a + 24c - 48 =$$

$$= 8(2a + 3c) - 48 \ge 8.10 - 48 = 32.$$

$$Equality\ holds\ when\ a = c = 2 \text{ and } b = d = 0.$$

$$Therefore, \quad \Omega = 32.$$

664. Prove that:

$$(n!)^{-\frac{1}{n}} \ge \frac{2}{n + 1 - 2\sqrt[n]{n!}\log\left(\frac{n+1}{2\sqrt[n]{n!}}\right)}, n \in \mathbb{N}, n \ge 1$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Ravi Prakash-New Delhi-India

Let $A = \frac{1}{n}(1 + 2 + 3 + \dots + n) = \frac{n+1}{2}$

$$G = (1 \cdot 2 \cdot 3 \cdot \ldots \cdot n)^{\frac{1}{n}} = (n!)^{\frac{1}{n}}$$

We can write the given inequality as: $\frac{1}{G} \ge \frac{2}{2A - 2G\log\left(\frac{A}{G}\right)} \Leftrightarrow A - G\log\left(\frac{A}{G}\right) \ge G$

$$A - G \ge G\log\left(\frac{A}{G}\right) \Leftrightarrow \frac{A}{G} - 1 \ge \log\left(\frac{A}{G}\right); (*)$$

As $A \ge G \to \frac{A}{G} \ge 1$ we get: $\log\left(\frac{A}{G}\right) = \log\left(\frac{A}{G} - 1 + 1\right) \le \frac{A}{G} - 1$

$\because \log(1 + x) \le x, \forall x \ge 0$.From $(*)$ it follows that

$$(n!)^{-\frac{1}{n}} \ge \frac{2}{n + 1 - 2\sqrt[n]{n!}\log\left(\frac{n+1}{2\sqrt[n]{n!}}\right)}$$

665. For $a, b \in \mathbb{R}, a \neq 0, b > 0$ prove that:

$$a^{2b(2a^2-1)} \cdot b^b > e^{a^2b-1}$$

Proposed by Nikos Ntorvas-Greece

Solution by Michael Sterghiou-Greece

$$a^{2b(2a^2-1)} \cdot b^b > e^{a^2b-1}; (1)$$

Let $a^2 = x$ then (1) written as: $x^{2x-1} \cdot b^b > e^{bx-1}$

Taking logarithms and rearranging, we get:

$$f(x)=\underbrace{[(2x-1)\log x-x]}_{g(x)}b+b\log b+1\geq 0$$

Now, $g'(x)=2\log x-\frac{1}{x}+1$ and $g''(x)=\frac{2x+1}{x^2}>0$ therefore, $g'(x)$ has one root in $(0,\infty)$ as strictly increasing. It is easy to see that this root is $x_0=1$ which is minimum for $g(x)\geq g(1)=-1$. As $b>0, f(x)\geq(-1)\cdot b+b\log b+1\overset{(*)}{\geq}-b+b\cdot\frac{b-1}{b}+1=0$.

We are done.$(*)$: $\log(1+b)\geq\frac{b}{b+1}$ for $b>-1$ so applying this for $b-1>-1$ we get

$$\log b\geq\frac{b-1}{b}$$

Equality holds for $b=1$ and $a^2=1$.

666. If $n\in\mathbb{N}$ and $0\leq x\leq 1$ then:

$$x^{2n}+(2n-1)x^n\leq(2n-1)x^{n+1}+x$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by George Florin Șerban-Romania

$$x^{2n}+(2n-1)x^n\leq(2n-1)x^{n+1}+x$$

$$\Leftrightarrow(2n-1)x^{n+1}+x-x^{2n}-(2n-1)x^n\overset{(*)}{\geq}0$$

$$\Leftrightarrow x(1-x^{2n-1})-(2n-1)x^n(1-x)\geq 0$$

$$\Leftrightarrow x(1-x)(1+x+\dots+x^{2n-1}-(2n-1)x^n]\geq 0$$

Now, $1-x\geq 0,\forall x\in[0,1]$ and

$$x+x^2+\dots+x^{2n-1}\overset{AM-GM}{\geq}(2n-1)\cdot\sqrt[2n-1]{x\cdot x^2\cdot\ldots\cdot x^{2n-1}}=$$

$$=(2n-1)\sqrt[2n-1]{x^{n(2n-1)}}=(2n-1)x^n$$

$$\Rightarrow x+x^2+\dots+x^{2n-1}-(2n-1)x^n\geq 0 \text{ true.}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$x^{2n}+(2n-1)x^n\leq(2n-1)x^{n+1}+x;(1)$$

If $x=0$ or $x=1$ is nothing to show.

Assume $0<x<1$, we can write (1) as

$$(2n-1)x^n(1-x)\le x(1-x^{2n-1})$$

$$\Leftrightarrow (2n-1)x^n(1-x)\le x(1-x)(1+x+\dots+x^{2n-2})$$

$$\Leftrightarrow (2n-1)x^{n-1}\le 1+x+\dots+x^{2n-1};\ (2)$$

$$RHS_{(2)}=1+x+\dots+x^{2n-2}\ge(2n-1)\sqrt[2n-1]{x^{1+2+\dots+(2n-2)}}$$

$$=(2n-1)\sqrt[2n-1]{x^{\frac{(2n-1)(2n-2)}{2}}}=(2n-1)x^{n-1}=LHS_{(2)}$$

Solution 3 by Michael Sterghiou-Greece

$$x^{2n}+(2n-1)x^n-(2n-1)x^{n+1}-x\le 0;\ (1)$$

Let $f(x)=x^{2n}+(2n-1)x^n-(2n-1)x^{n+1}-x$, then

$f'(x)=n(2n-1)x^{n+1}+2nx^{2n-1}-(n+1)(2n-1)x^n$ and

$$f''(x)=-n(2n-1)x^{n-2}\underbrace{[-2x^n+n(n-1)+x+1)}_{g(x)}$$

It is easy to see that the expression of $g(x)$ is zero if $n=1$ and $g(x)<0$ for $n>1$.

So, $f''(x)\le 0\Rightarrow f'(x)\searrow\Rightarrow f'(x)\le f'(0)=-1\Rightarrow f'(x)\le -1<0$ and $f(x)\searrow$ or

$f(x)\le f(0)$ for $x\ge 0$ or $f(x)\le 0$.

Equality holds for $x=0$ or $x=1$.

Solution 4 by Sire Ambrose-Albania

$$x^{2n}+(2n-1)x^n\le(2n-1)x^{n+1}+x\Rightarrow x^{2n}\le(2n-1)x^n(x-1)+x$$

Prove by induction. Test: $n=1: x^2\le x(x-1)+x=x^2$ true.

Assume that is true for all $n=k: x^{2k}\le(2k-1)x^k(x-1)+x;\ (1)$

We want to prove that: $x^{2k+2}\le(2k+1)x^{k+1}(x-1)+x$.

From (1) we have: $x^{2k+2}\le(2k-1)x^{k+2}(x-1)+x^3$

Because $x\in[0,1]\Rightarrow x^{k+2}\le x^{k+1}$

$$\Rightarrow x^{2k+2}\le(2k-1)x^{k+2}(x-1)+x^3\le(2k-1)x^{k+1}(x-1)+x^3$$

$$2k-1\le 2k+1$$

$$\Rightarrow x^{2k+2}\le(2k-1)x^{k+1}(x-1)+x^3\le(2k+1)x^{k+1}(x-1)+x^3\Leftrightarrow$$

$$x^{2k+2}\le(2k+1)x^{k+1}(x-1)+x^3\overset{x^3\le x}{\Leftrightarrow}$$

$$x^{2k+2}\le(2k+1)x^{k+1}(x-1)+x^3\le(2k+1)x^{k+1}(x-1)+x\Leftrightarrow$$

$$x^{2k+2} \leq (2k+1)x^{k+1}(-1) + x$$

667. If $x, y, z \in \mathbb{R}$ then:

$$\frac{|x-y|}{\sqrt{1+x^2+y^2+x^2y^2}} \leq \frac{|x-z|}{\sqrt{1+x^2+z^2+x^2z^2}} + \frac{|y-z|}{\sqrt{1+y^2+z^2+y^2z^2}}$$

Proposed by Jalil Hajimir-Canada

Solution by Ravi Prakash-New Delhi-India

Let $x = \tan\alpha, y = \tan\beta, z = \tan\gamma; -\frac{\pi}{2} < \alpha, \beta, \gamma < \frac{\pi}{2}$

Now,

$$\frac{|x-y|}{\sqrt{1+x^2+y^2+x^2y^2}} = \frac{|\tan\alpha - \tan\beta|}{\sqrt{(1+\tan^2\alpha)(1+\tan^2\beta)}} = \left|\frac{\tan\alpha - \tan\beta}{\sec\alpha \cdot \sec\beta}\right| = |\sin(\alpha-\beta)|$$

Similarly,

$$\frac{|x-z|}{\sqrt{1+x^2+z^2+x^2z^2}} = |\sin(\alpha-\gamma)|$$

$$\frac{|y-z|}{\sqrt{1+y^2+z^2+y^2z^2}} = |\sin(\beta-\gamma)|$$

Hence,

$$|\sin(\alpha-\beta)| = |\sin((\alpha-\gamma)-(\beta-\gamma))| =$$

$$= |\sin(\alpha-\gamma)\cos(\beta-\gamma) - \sin(\beta-\gamma)\cos(\alpha-\gamma)| \leq$$

$$\leq |\sin(\alpha-\gamma)| \cdot |\cos(\beta-\gamma)| - |\sin(\beta-\gamma)| \cdot |\cos(\alpha-\gamma)| \leq$$

$$\leq |\sin(\alpha-\gamma)| + |\sin(\beta-\gamma)|; |\cos\theta| \leq 1, \forall\theta \in \mathbb{R}$$

668. If $(F_n)_{n\geq 0}, F_0 = F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in \mathbb{N}$, then prove:

$$\frac{F_m^2}{(F_qF_n + F_{q+1}F_p)^2} + \frac{F_n^2}{(F_qF_p + F_{q+1}F_m)^2} + \frac{F_p^2}{(F_qF_m + F_{q+1}F_n)^2} \geq \frac{3}{F_{q+2}^2}, \forall m, n, p, q \in \mathbb{N}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Marian Ursărescu-Romania

$$\left(\frac{F_m}{F_qF_n + F_{q+1}F_p}\right)^2 + \left(\frac{F_n}{F_qF_p + F_{q+1}F_m}\right)^2 + \left(\frac{F_p}{F_qF_m + F_{q+1}F_n}\right)^2 \overset{CBS}{\geq}$$

$$\geq \frac{1}{3}\left(\frac{F_m}{F_qF_n + F_{q+1}F_p} + \frac{F_n}{F_qF_p + F_{q+1}F_m} + \frac{F_p}{F_qF_m + F_{q+1}F_n}\right)^2$$

We must show:

$$\left(\frac{F_m}{F_qF_n + F_{q+1}F_p} + \frac{F_n}{F_qF_p + F_{q+1}F_m} + \frac{F_p}{F_qF_m + F_{q+1}F_n}\right)^2 \geq \frac{9}{F_{q+2}^2} \Leftrightarrow$$

$$\frac{F_m}{F_qF_n + F_{q+1}F_p} + \frac{F_n}{F_qF_p + F_{q+1}F_m} + \frac{F_p}{F_qF_m + F_{q+1}F_n} \geq \frac{3}{F_{q+2}}; (1)$$

$$\frac{F_m}{F_qF_n + F_{q+1}F_p} + \frac{F_n}{F_qF_p + F_{q+1}F_m} + \frac{F_p}{F_qF_m + F_{q+1}F_n} =$$

$$= \frac{F_m^2}{F_qF_nF_m + F_{q+1}F_pF_m} + \frac{F_n^2}{F_qF_pF_n + F_{q+1}F_mF_n} + \frac{F_p^2}{F_qF_mF_p + F_{q+1}F_nF_p} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(F_m + F_n + F_p)^2}{(F_{q+1} + F_q)(F_mF_n + F_nF_p + F_pF_m)} = \frac{(F_m + F_n + F_p)^2}{F_{q+2}(F_mF_n + F_nF_p + F_pF_m)} \overset{CBS}{\geq}$$

$$\geq \frac{3(F_mF_n + F_nF_p + F_pF_m)}{F_{q+2}(F_mF_n + F_nF_p + F_pF_m)} = \frac{3}{F_{q+2}} \Rightarrow (1)\text{is true}.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{F_m^2}{(F_qF_n + F_{q+1}F_p)^2} + \frac{F_n^2}{(F_qF_p + F_{q+1}F_m)^2} + \frac{F_p^2}{(F_qF_m + F_{q+1}F_n)^2} \geq$$

$$\geq \frac{1}{3}\left(\frac{F_m^2}{F_qF_n + F_{q+1}F_p} + \frac{F_n^2}{F_qF_p + F_{q+1}F_m} + \frac{F_p^2}{F_qF_m + F_{q+1}F_n}\right)\left(\frac{1}{F_qF_n + F_{q+1}F_p}\right.$$

$$\left. + \frac{1}{F_qF_p + F_{q+1}F_m} + \frac{1}{F_qF_m + F_{q+1}F_n}\right) \geq$$

$$\geq \frac{3}{9} \cdot \frac{(F_m + F_n + F_p)^2}{F_qF_n + F_{q+1}F_p + F_qF_p + F_{q+1}F_m + F_qF_m + F_{q+1}F_n}$$

$$\cdot \frac{1}{F_qF_n + F_{q+1}F_p + F_qF_p + F_{q+1}F_m + F_qF_m + F_{q+1}F_n} =$$

$$= \frac{3(F_m + F_n + F_p)^2}{(F_q + F_{q+1})(F_m + F_n + F_p)(F_q + F_{q+1})(F_m + F_n + F_p)} = \frac{3}{(F_q + F_{q+1})^2} = \frac{3}{F_{q+2}^2}$$

669.

$$\text{Maximize } \prod_{i=1}^{n}(x_i^3 + x_i + 1) \text{ subject to}$$

$$\sum_{i=1}^{n} x_i^2 = n, x_i \geq 0$$

Proposed by Jalil Hajimir-Canada

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$3(x_i^3 + x_i + 1) \overset{?}{\leq} (x_i^2 + 2)^2 \leftrightarrow x_i^4 - 3x_i^3 + 4x_i^2 - 3x_i + 1 \geq 0$$

$$\leftrightarrow (x_i - 1)^2(x_i^2 - x_i + 1) \geq 0 \text{ which is true } \forall x_i \geq 0.$$

$$\rightarrow x_i^3 + x_i + 1 \leq \frac{1}{3}(x_i^2 + 2)^2, \forall i = \overline{1, n}.$$

$$\rightarrow \prod_{i=1}^{n}(x_i^3 + x_i + 1) \leq \prod_{i=1}^{n}\frac{1}{3}(x_i^2 + 2)^2 = \frac{1}{3^n}\left(\prod_{i=1}^{n}(x_i^2 + 2)\right)^2 \leq$$

$$\overset{AM-GM}{\leq} \frac{1}{3^n}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i^2 + 2)\right)^{2n} = \frac{1}{3^n}\left(\frac{1}{n}.3n\right)^{2n} = 3^n.$$

$$\rightarrow max\left\{\prod_{i=1}^{n}(x_i^3 + x_i + 1)\right\} = 3^n, \text{equality holds when } x_i = 1, \forall i = \overline{1, n}$$

670. If $x, y, z \in \mathbb{R}$ then prove:

$$\frac{25x}{21} + \frac{13y}{15} + \frac{33z}{35} \leq \sqrt{x^2 + y^2} + \sqrt{y^2 + z^2} + \sqrt{z^2 + x^2}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$9(x^2 + y^2) \geq (1^2 + 2^2)(x^2 + y^2) \overset{CBS}{\geq} (x + 2y)^2 \Rightarrow \sqrt{x^2 + y^2} \geq \frac{|x + 2y|}{3} \geq \frac{x + 2y}{3}; (1)$$

Analogous, $25(y^2 + z^2) \geq (1 + 4^2)(y^2 + z^2) \overset{CBS}{\geq} (y + 4z)^2 \Rightarrow \sqrt{y^2 + z^2} \geq \frac{y+4z}{5}; (2)$

$$49(z^2+x^2) \ge (1+6^2)(z^2+x^2) \overset{CBS}{\ge} (z+6x)^2 \Rightarrow \sqrt{x^2+y^2} \ge \frac{z+6x}{7};(3)$$

From (1),(2),(3) it follows that:

$$\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2} \ge \frac{x+2y}{3}+\frac{y+4z}{5}+\frac{z+6x}{7}=$$

$$=\frac{25x}{21}+\frac{13y}{15}+\frac{33z}{35}$$

Therefore,

$$\frac{25x}{21}+\frac{13y}{15}+\frac{33z}{35} \le \sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}$$

671. If $a,b,c>0$ then:

$$\frac{5}{a}+\frac{8}{b}+\frac{9}{c} \ge \frac{8}{a+b}+\frac{24}{b+c}+\frac{12}{c+a}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{8}{a+b}+\frac{24}{b+c}+\frac{12}{c+a} \le 8\left(\frac{1}{4a}+\frac{1}{4b}\right)+24\left(\frac{1}{4b}+\frac{1}{4c}\right)+12\left(\frac{1}{4c}+\frac{1}{4a}\right)=$$

$$=\frac{2}{a}+\frac{2}{b}+\frac{6}{b}+\frac{6}{c}+\frac{3}{c}+\frac{3}{a}=\frac{5}{a}+\frac{8}{b}+\frac{9}{c}$$

Solution 2 by Florică Anastase-Romania

$$\frac{5}{a}+\frac{8}{b}+\frac{9}{c}=5\cdot\frac{1}{a}+8\cdot\frac{1}{b}+9\cdot\frac{1}{c}=2\left(\frac{1}{a}+\frac{1}{b}\right)+6\left(\frac{1}{b}+\frac{1}{c}\right)+3\left(\frac{1}{c}+\frac{1}{a}\right)\overset{Bergstrom}{\ge}$$

$$\ge 2\cdot\frac{(1+1)^2}{a+b}+6\cdot\frac{(1+1)^2}{b+c}+3\cdot\frac{(1+1)^2}{c+a}=\frac{8}{a+b}+\frac{24}{b+c}+\frac{12}{c+a}$$

Solution 3 by Fayssal Abdelli-Bejaia-Algerie

$$\frac{5}{a}+\frac{8}{b}+\frac{9}{c}=\frac{2}{a}+\frac{2}{b}+\frac{3}{a}+\frac{3}{c}+\frac{6}{b}+\frac{6}{c} \ge \frac{\left(\sqrt{2}+\sqrt{2}\right)^2}{a+b}+\frac{\left(\sqrt{3}+\sqrt{3}\right)^2}{a+c}+\frac{\left(\sqrt{6}+\sqrt{6}\right)^2}{b+c}=$$

$$=\frac{8}{a+b}+\frac{24}{b+c}+\frac{12}{c+a}$$

672. $(x_n)_{n\in\mathbb{N}}>0, x_n(x_{n-1}+x_{n+1})<2x_{n-1}\cdot x_{n+1}, \forall n\ge 1$

Prove that: $x_0 \ge x_1$

Proposed by Dan Radu Seclăman-Romania

Solution by Kamel Gandouli Rezgui-Tunisia

$$x_n(x_{n-1}+x_{n+1})<2x_{n-1}\cdot x_{n+1} \Leftrightarrow x_nx_{n-1}+x_nx_{n+1}<2x_{n-1}x_{n+1} \Leftrightarrow$$

$$x_nx_{n-1} < x_{n+1}(2x_{n-1} - x_n) \Leftrightarrow 2x_{n-1} - x_n > 0 \Leftrightarrow x_{n-1} > \frac{x_n}{2}, \forall n \in \mathbb{N}$$

$$\Rightarrow \frac{x_{n+1}}{2}x_{n-1} < x_nx_{n-1} < x_{n+1}(2x_{n-1} - x_n) \Rightarrow \frac{x_{n-1}}{2} < 2x_{n-1} - x_n$$

$$\Rightarrow x_{n-1} > \frac{2}{3}x_n, \forall n \in \mathbb{N}$$

Suppose that: $\forall n \in \mathbb{N}: x_{n-1} > \frac{p}{p+1}x_n, p \in \mathbb{N}$ **and then** $x_{n-1} > \frac{p+1}{p+2}x_n, \forall n \in \mathbb{N}$.

$$\frac{p}{p+1}x_{n-1}x_{n+1} < x_nx_{n-1} < x_{n+1}(2x_{n-1} - x_n) \Leftrightarrow$$

$$\frac{p}{p+1}x_{n-1} < 2x_{n-1} - x_n \Leftrightarrow \left(2 - \frac{p}{p+1}\right)x_{n-1} > x_n \Leftrightarrow x_{n-1} > \frac{p+2}{p+1}x_n$$

So, $x_{n-1} > \frac{p+1}{p+2}x_n, \forall n \in \mathbb{N}, p \in \mathbb{N}$.

For $n = 1$, **it follows that** $x_0 > \frac{p}{p+1}x_1, p \to \infty$.

Therefore, $x_0 \geq x_1$

673. If $0 < x < 1$ then:

$$4\sin 2x \cdot \sin^2(1 - x) \leq 27x(1 - x)^2 \cdot \sin^3\left(\frac{2}{3}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*)\ 4\sin 2x . \sin^2(1 - x) \leq 27x(1 - x)^2 . \sin^3\left(\frac{2}{3}\right)$$

$$Since\ 0 < x < 1 < \frac{\pi}{2} \rightarrow \sin 2x, \sin(1 - x) > 0.$$

$$(*) \leftrightarrow 2\log 2 + \log(\sin 2x) + 2\log(\sin(1 - x))$$

$$\leq 3\log 3 + \log x + 2\log(1 - x) + 3\log\left(\sin\frac{2}{3}\right)$$

$$\leftrightarrow (\log(\sin 2x) - \log 2x) + 2(\log(\sin(1 - x)) - \log(1 - x)) \leq 3\left(\log\left(\sin\frac{2}{3}\right) - \log\frac{2}{3}\right)$$

$$Let\ f(x) = \log(\sin x) - \log x, x \in (0, \pi)$$

$$We\ have:\ f'(x) = \frac{\cos x}{\sin x} - \frac{1}{x}\ and\ f''(x) = -\frac{1}{\sin^2 x} + \frac{1}{x^2} = -\frac{(x - \sin x)(x + \sin x)}{x^2 . \sin^2 x}$$

$$Since\ x \geq \sin x, \forall x \in (0, \pi) \rightarrow f - concave\ on\ (0, \pi).$$

$$\overset{Using\ Jensen}{\Rightarrow} \quad f(2x)+f(1-x)+f(1-x)\le 3f\left(\frac{2x+(1-x)+(1-x)}{3}\right)=3f\left(\frac{2}{3}\right)$$

$$\leftrightarrow (\log(\sin 2x)-\log 2x)+2(\log(\sin(1-x))-\log(1-x))\le 3\left(\log\left(\sin\frac{2}{3}\right)-\log\frac{2}{3}\right)$$

$$Therefore,\quad 4\sin 2x.\sin^2(1-x)\le 27x(1-x)^2.\sin^3\left(\frac{2}{3}\right), \forall x\in(0,1).$$

674. If $a,b,c,d>0, abcd=1$ then:

$$\frac{a^2b^2}{a^3b^3+cd}+\frac{c^2d^2}{c^3d^3+ab}\ge\frac{8}{(a^2+b^2)^2+(c^2+d^2)^2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by George Florin Şerban-Romania

$$\sum_{cyc}\frac{a^2b^2}{a^3b^3+cd}=\sum_{cyc}\frac{(ab)^3}{(ab)^4+abcd}=\sum_{cyc}\frac{(ab)^3}{(ab)^4+1}\overset{\substack{ab=x\\cd=y}}{=}\frac{x^3}{x^4+1}+\frac{y^3}{y^4+1}\overset{xy=1}{=}$$

$$=\frac{x^3}{x^4+1}+\frac{\frac{1}{x^3}}{\frac{1}{x^4}+1}=\frac{x^3+x}{x^4+1}$$

$$\frac{8}{(a^2+b^2)^2+(c^2+d^2)^2}\le\frac{8}{(2ab)^2+(2cd)^2}=\frac{8}{4(x^2+y^2)}=\frac{2}{x^2+\frac{1}{x^2}}=$$

$$=\frac{2x^2}{1+x^4}\overset{(1)}{\le}\frac{x^3+x}{x^4+1}=\sum_{cyc}\frac{a^2b^2}{a^3b^3+cd}$$

$(1)\Leftrightarrow 2x^2\le x^3+x\Leftrightarrow x(x^2-2x+1)\ge 0\Leftrightarrow x(x-1)^2\ge 0$ **which is true** $\forall x>0$.

Therefore,

$$\frac{a^2b^2}{a^3b^3+cd}+\frac{c^2d^2}{c^3d^3+ab}\ge\frac{8}{(a^2+b^2)^2+(c^2+d^2)^2}$$

Solution 2 by Fayssal Abdelli-Bejaia-Algerie

$$A=\frac{a^2b^2}{a^3b^3+cd}+\frac{c^2d^2}{c^3d^3+ab}\overset{Bergstrom}{\ge}\frac{(ab+cd)^2}{a^3b^3+c^3d^3+ab+cd}\overset{AGM}{\ge}$$

$$\ge\frac{4}{a^3b^3+c^3d^3+ab+cd}\overset{(1)}{\ge}\frac{8}{(a^2+b^2)^2+(c^2+d^2)^2}$$

$$(1)\Leftrightarrow (a^2+b^2)^2+(c^2+d^2)^2\ge 2ab+2a^3b^3+2cd+2c^3d^3;(B)$$

$$a^2+b^2\ge 2ab\Rightarrow (a^2+b^2)^2\ge 4a^2b^2$$

$$c^2+d^2\ge 2cd\Rightarrow (c^2+d^2)^2\ge 4c^2d^2$$

$$4a^2b^2+4c^2d^2\ge 2ab+2a^3b^3+2cd+2c^2d^3\Leftrightarrow$$

$$4a^2b^2+4c^2d^2\ge 2ab(1+a^2b^2)+2cd(1+c^2d^2)$$

$$4a^2b^2\ge 2ab(1+a^2b^2)\Rightarrow 4a^2b^2-2ab-2a^3b^3\ge 0$$

$$2ab(-a^2b^2+2ab-1)\ge 0$$

$$-2ab(a^2b^2-2ab+1)\ge 0\Rightarrow ab=1 \text{ and similarly, } cd=1.$$

$$ab=1, cd=1\Rightarrow A\ge 1 \text{ and } \frac{8}{(a^2+b^2)^2+(c^2+d^2)^2}\le 1.$$

Therefore,

$$\frac{a^2b^2}{a^3b^3+cd}+\frac{c^2d^2}{c^3d^3+ab}\ge\frac{8}{(a^2+b^2)^2+(c^2+d^2)^2}$$

Solution 3 by Aggeliki Papaspyropoulou-Greece

$$a^2+b^2\ge 2ab\Rightarrow (a^2+b^2)^2\ge 4a^2b^2$$

$$c^2+d^2\ge 2cd\Rightarrow (c^2+d^2)^2\ge 4c^2d^2$$

$$\Rightarrow (a^2+b^2)^2+(c^2+d^2)^2\ge 4(a^2b^2+c^2d^2)\Rightarrow$$

$$\frac{8}{(a^2+b^2)^2+(c^2+d^2)^2}\le\frac{8}{4(a^2b^2+c^2d^2)}=\frac{2}{a^2b^2+c^2d^2}$$

So, it is enough to prove that:

$$\frac{a^2b^2}{a^3b^3+cd}+\frac{c^2d^2}{c^3d^3+ab}\ge\frac{2}{a^2b^2+c^2d^2};(*), abcd=1, a,b,c,d>0$$

$$x=ab, y=cd, xy=1 \text{ and } x+y\ge 2\sqrt{xy}\Rightarrow x+y\ge 2;(**)$$

$$\frac{a^2b^2}{a^3b^3+cd}+\frac{c^2d^2}{c^3d^3+ab}=\frac{x^2}{x^3+y}+\frac{y^2}{y^3+x}\overset{xy=1}{=}\frac{x^2}{x^3+y(xy)}+\frac{y^2}{y^3+x(xy)}=$$

$$=\frac{x^2}{x^3+xy^2}+\frac{y^2}{y^3+x^2y}=\frac{x}{x^2+y^2}+\frac{y}{x^2+y^2}=\frac{x+y}{x^2+y^2}\overset{(**)}{\ge}\frac{2}{x^2+y^2}$$

Therefore,

$$\frac{a^2b^2}{a^3b^3+cd}+\frac{c^2d^2}{c^3d^3+ab}\ge\frac{8}{(a^2+b^2)^2+(c^2+d^2)^2}$$

Solution 4 by Khaled Abd Imouti-Damascus-Syria

$$\text{Let: } ab = x, cd = y \Rightarrow \frac{a^2b^2}{a^3b^3+cd} + \frac{c^2d^2}{c^3d^3+ab} = \frac{x^3}{x^4+1} + \frac{y^3}{y^4+1} \overset{AGM}{\geq}$$

$$\geq 2 \cdot \sqrt{\frac{x^3y^3}{(x^4+1)(y^4+1)}} = \frac{2}{\sqrt{(x^4+1)(y^4+1)}} = \frac{8}{\sqrt{16(x^4+1)(y^4+1)}}$$

Let us prove that:

$$\sqrt{16(x^4+1)(y^4+1)} \leq (a^2+b^2)^2 + (c^2+d^2)^2 \Leftrightarrow$$

$$\sqrt{16(a^4b^4+1)(c^4d^4+1)} \leq (a^2+b^2)^2 + (c^2+d^2)^2$$

$$4\sqrt{a^4b^4+c^4d^4+1+1} \leq (a^2+b^2)^2 + (c^2+d^2)^2$$

$$4\sqrt{a^4b^4+c^4d^2+2a^2b^2c^2d^2} \leq (a^2+b^2)^2 + (c^2+d^2)^2$$

$4(a^2b^2+c^2d^2) \leq (a^2+b^2)^2 + (c^2+d^2)^2$, which is true from AM-GM.

Therefore,

$$\frac{a^2b^2}{a^3b^3+cd} + \frac{c^2d^2}{c^3d^3+ab} \geq \frac{8}{(a^2+b^2)^2+(c^2+d^2)^2}$$

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{a^2b^2}{a^3b^3+cd} + \frac{c^2d^2}{c^3d^3+ab} = \frac{1}{ab+(cd)^3} + \frac{1}{cd+(ab)^3} \geq \frac{4}{ab+cd+(ab)^3+(cd)^3} \geq$$

$$\geq \frac{8}{(a^2+b^2)^2+(c^2+d^2)^2} \Leftrightarrow$$

$a^4+b^4+c^4+d^4+2(ab)^2+2(vd)^2 \geq 2[(ab)^3+(cd)^3+ab+cd]$, true because

$$a^4+b^4+c^4+d^4 \geq 2(ab)^3+2(cd)^3$$

$$\frac{a^2}{(bcd)^2} + \frac{b^2}{(cda)^2} + \frac{c^2}{(dab)^2} + \frac{d^2}{(abc)^2} \overset{AGM}{\geq} 2\frac{ab}{(cd)^2} + 2\frac{cd}{(ab)^2}$$

$$2(ab)^2 + 2(cd)^2 \geq \frac{2(ab+cd)^2}{2} = (ab)^2 + (cd)^2 + 2 + 2 \geq 2ab + 2cd$$

Therefore,

$$\frac{a^2b^2}{a^3b^3+cd} + \frac{c^2d^2}{c^3d^3+ab} \geq \frac{8}{(a^2+b^2)^2+(c^2+d^2)^2}$$

675. Let $(F_n)_{n\geq 0}, F_0 = F_1 = 1, F_{n+2} = F_{n+1} + F_n$ be the Fibonacci's sequence, and T_k is the k^{th} triangular number defined by $T_k = \binom{k+1}{2}$ for all $k \geq 1$. Prove that:

$$\sum_{k=1}^{n} \frac{F_k^{2m+2}}{T_k^m} \geq \frac{3^m (F_n F_{n+1})^{m+1}}{n^m T_{n+1}^m}$$

for any positive integer $n \geq 1$ and for any positive real number m.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by George Florin Şerban-Romania

$$\sum_{k=1}^{n} \frac{F_k^{2m+2}}{T_k^m} = \sum_{k=1}^{n} \frac{\left(F_k^2\right)^{m+1}}{T_k^m} \overset{Radon}{\geq} \frac{\left(\sum_{k=1}^{n} F_k^2\right)^{m+1}}{\left(\sum_{k=1}^{n} T_k\right)^m} = \frac{(F_n F_{n+1})^{m+1}}{\left(\sum_{k=1}^{n} \frac{k(k+1)}{2}\right)^m} =$$

$$= \frac{2^m (F_n F_{n+1})^{m+1}}{\left(\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right)^m} = \frac{2^m (F_n F_{n+1})^{m+1}}{\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)^m} =$$

$$= \frac{2^m (F_n F_{n+1})^{m+1} \cdot 6^m}{[n(n+1)(2n+1) + 3n(n+1)]^m} = \frac{12^m (F_n F_{n+1})^{m+1}}{2^m n^m [(n+1)(n+2)]^m} =$$

$$= \frac{6^m (F_n F_{n+1})^{m+1}}{n^m \left[\frac{(n+1)(n+2)}{2}\right]^m \cdot 2^m} = \frac{3^m (F_n F_{n+1})^{m+1}}{n^m T_{n+1}^m}$$

Therefore,

$$\sum_{k=1}^{n} \frac{F_k^{2m+2}}{T_k^m} \geq \frac{3^m (F_n F_{n+1})^{m+1}}{n^m T_{n+1}^m}$$

676. If $n \in \mathbb{N}, n \geq 1, K(n) - K$ –function, then:

$$\left(\frac{1!}{n \cdot n!} + \frac{2!}{n \cdot n!} + \frac{3!}{n \cdot n!} + \cdots + \frac{n!}{n \cdot n!}\right)^n \cdot K(n+1) \geq n!$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

If $n \in \mathbb{N}, K(n+1) = 1 \cdot 2^2 \cdot 3^3 \cdot \ldots \cdot n^n$, also

$$\frac{1!+2!+\cdots+n!}{n}\geq(1!\cdot2!\cdot3!\cdot\ldots\cdot n!)^{\frac{1}{n}}\Rightarrow\left(\frac{1!+2!+\cdots+n!}{n}\right)^n\geq1!\cdot2!\cdot3!\cdot\ldots\cdot n!$$

$$\Rightarrow\left(\frac{1!}{n\cdot n!}+\frac{2!}{n\cdot n!}+\frac{3!}{n\cdot n!}+\cdots+\frac{n!}{n\cdot n!}\right)^n\cdot K(n+1)\geq$$

$$\geq\frac{1}{(n!)^n}(1!\cdot2!\cdot3!\cdot\ldots\cdot n!)(1\cdot2^2\cdot3^3\cdot\ldots\cdot n^n)=\frac{1^{n+1}\cdot2^{n+1}\cdot\ldots\cdot n^{n+1}}{(n!)^n}=$$

$$=\frac{(n!)^{n+1}}{(n!)^n}=n!$$

Solution 2 by Amrit Awasthi-India

It is known by AM-GM inequality:

$$\frac{a_1+a_2+\cdots+a_n}{n}\geq\sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}\Leftrightarrow\left(\frac{(a_1+a_2+\cdots+a_n)}{n}\right)^n\geq a_1\cdot a_2\cdot\ldots\cdot a_n$$

Substituting $a_i=i!,\forall i\in\mathbb{N}$ we have:

$$\left(\frac{1!+2!+\cdots+n!}{n}\right)^n\geq1!\cdot2!\cdot3!\cdot\ldots\cdot n!$$

$$\left(\frac{1!+2!+\cdots+n!}{n}\right)^n\cdot\frac{1}{1!\cdot2!\cdot3!\cdot\ldots\cdot(n-1)!}\geq n!;\ (*)$$

Now, we know that:

$$K(n)=\frac{(\Gamma(n))^{n-1}}{G(n)}$$

Here $K(n)$ –is K –function, $G(n)$ –is Barnes function and $\Gamma(n)$ –is Gamma fuction.

Substituting $n=n+1$, and rearranging we have

$$\frac{1}{G(n+1)}=\frac{K(n+1)}{(\Gamma(n+1))^n}=\frac{K(n+1)}{(n!)^n}\text{ and from }(*):$$

$$\left(\frac{1!+2!+\cdots+n!}{n}\right)^n\cdot\frac{K(n+1)}{(n!)^n}\geq n!$$

Therefore,

$$\left(\frac{1!}{n\cdot n!}+\frac{2!}{n\cdot n!}+\frac{3!}{n\cdot n!}+\cdots+\frac{n!}{n\cdot n!}\right)^n\cdot K(n+1)\geq n!$$

Solution 3 by Fayssal Abdelli-Bejaia-Algerie

$$A=\left(\frac{1!}{n\cdot n!}+\frac{2!}{n\cdot n!}+\frac{3!}{n\cdot n!}+\dots+\frac{n!}{n\cdot n!}\right)^n \text{ and } B=K(n+1)=1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n$$

$$A=\left(\frac{1!}{n\cdot n!}+\frac{2!}{n\cdot n!}+\frac{3!}{n\cdot n!}+\dots+\frac{n!}{n\cdot n!}\right)^n \overset{AGM}{\geq} \left(\frac{1}{n!}\right)^n\cdot 1!\cdot 2!\cdot\ldots\cdot n!$$

$$A\cdot B=\left(\frac{1!}{n\cdot n!}+\frac{2!}{n\cdot n!}+\frac{3!}{n\cdot n!}+\dots+\frac{n!}{n\cdot n!}\right)^n\cdot K(n+1)\geq$$

$$\geq\frac{1}{(n!)^n}(1!\cdot 2!\cdot 3!\cdot\ldots\cdot n!)(1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n);(C)$$

By induction we prove that: $P(n):\frac{1}{(n!)^n}(1!\cdot 2!\cdot 3!\cdot\ldots\cdot n!)(1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n)=n!$

For $P(1)$: $\frac{1}{1!}\cdot 1\cdot 1^1=1$ true.

Let suppose that: $P(n):\frac{1}{(n!)^n}(1!\cdot 2!\cdot 3!\cdot\ldots\cdot n!)(1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n)=n!$

$$P(n+1):\ \frac{1}{((n+1)!)^{n+1}}(1!\cdot 2!\cdot 3!\cdot\ldots\cdot n!\cdot(n+1)!)(1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n\cdot(n+1)^{n+1})$$
$$=(n+1)!$$

$$\frac{1}{\left((n+1)!\right)^{n+1}}(1!\cdot 2!\cdot 3!\cdot\ldots\cdot n!\cdot(n+1)!)(1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n\cdot(n+1)^{n+1})=$$

$$=P(n)\cdot\frac{(n+1)!}{n!}=\frac{n!}{n!}\cdot(n+1)!=(n+1)!$$

From $(A),(B),(C)$ it follows that:

$$\left(\frac{1!}{n\cdot n!}+\frac{2!}{n\cdot n!}+\frac{3!}{n\cdot n!}+\dots+\frac{n!}{n\cdot n!}\right)^n\cdot K(n+1)\geq n!$$

677. If $a,b,c\in(1,2)$, $f:(2,3)\to\mathbb{R}_+$ continuous with $f'(x)<0$ and $f''(x)<0,\forall x\in(2,3)$ then prove:

$$\sum_{cyc}f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right)\geq 2\cdot\sqrt[4]{\prod_{cyc}f\,(a+1)\cdot\sum_{cyc}f\,(a+1)}$$

Proposed by Florică Anastase-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Firstly, let's prove that : } (*) : \frac{(a+1)+(b+1)}{2} \geq \frac{(a+1)(b+1)}{1+\sqrt{ab}}, \forall a,b \in (1,2)$$

$$(*) \leftrightarrow (1+\sqrt{ab})(a+b+2) \geq 2(a+b)+2ab+2 \leftrightarrow (\sqrt{ab}-1)(a+b)$$
$$\geq 2\sqrt{ab}(\sqrt{ab}-1)$$
$$\leftrightarrow (\sqrt{ab}-1)(\sqrt{a}-\sqrt{b})^2 \geq 0, \text{which is true} \rightarrow 3 > \frac{(a+1)+(b+1)}{2}$$
$$\geq \frac{(a+1)(b+1)}{1+\sqrt{ab}} \ (\text{And analogs})$$

$$\text{Since } f \text{ is decreasing on } (2,3) \rightarrow f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right)$$
$$\geq f\left(\frac{(a+1)+(b+1)}{2}\right) \ (\text{And analogs})$$

$$\rightarrow \sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geq \sum_{cyc} f\left(\frac{(a+1)+(b+1)}{2}\right) \ \forall a,b,c \in (1,2) \ (1)$$

$$\text{Now, since } f \text{ is concave on } (2,3), \text{we have :}$$

$$\text{From Jensen : } 3f\left(\frac{1}{3}\sum_{cyc}(a+1)\right) \geq \sum_{cyc} f(a+1) \overset{AM-GM}{\geq} 3\sqrt[3]{\prod_{cyc} f(a+1)} \ (i)$$

$$\text{From Popoviciu : } 2\sum_{cyc} f\left(\frac{(a+1)+(b+1)}{2}\right) \geq 3f\left(\frac{1}{3}\sum_{cyc}(a+1)\right) + \sum_{cyc} f(a+1) \geq$$
$$\overset{(i)}{\geq} 3\sqrt[3]{\prod_{cyc} f(a+1)} + \sum_{cyc} f(a+1) \overset{AM-GM}{\geq} 4\sqrt[4]{\left(\sqrt[3]{\prod_{cyc} f(a+1)}\right)^3 \cdot \sum_{cyc} f(a+1)}$$
$$= 4\sqrt[4]{\prod_{cyc} f(a+1) \cdot \sum_{cyc} f(a+1)}$$

$$\rightarrow \sum_{cyc} f\left(\frac{(a+1)+(b+1)}{2}\right) \geq 2\sqrt[4]{\prod_{cyc} f(a+1) \cdot \sum_{cyc} f(a+1)} \ \forall a,b,c \in (1,2) \ (2)$$

$$\text{From } (1),(2) :$$

$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geq 2\sqrt[4]{\prod_{cyc} f(a+1) \cdot \sum_{cyc} f(a+1)}, \forall a,b,c \in (1,2)$$

Solution 2 by Remus Florin Stanca-Romania

Let's prove that:

$$f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geq \frac{f(a+1)}{2} + \frac{f(b+1)}{2}; f''x) < 0 \Rightarrow$$

f **–concave, from Jensen we get:** $\forall t_1, t_2 \in \mathbb{R}$ **with** $t_1 + t_2 = 1$ **and** $\forall x_1, x_2 \in I \Rightarrow$

$t_1 f(x_1) + t_2 f(x_2) \leq f(t_1 x_1 + t_2 x_2)$; **let** $t_1 = t_2 = \frac{1}{2}$ **and** $x_1 = a+1, x_2 = b+1$

$$\Rightarrow \frac{f(a+1)}{2} + \frac{f(b+1)}{2} \leq f\left(\frac{a+b+2}{2}\right); (1)$$

$f'(x) < 0 \Rightarrow f$ **–decreasing on** $(2,3) \Rightarrow 2 < \frac{a+b+2}{2} < 3, \forall a,b \in (1,2)$

We need to prove that:

$$\frac{a+b+2}{2} \geq \frac{(a+1)(b+1)}{1+\sqrt{ab}} \Leftrightarrow \frac{a+1+b+1}{(a+1)(b+1)} \geq \frac{2}{1+\sqrt{ab}} \Leftrightarrow \frac{1}{b+1} + \frac{1}{a+1} \geq \frac{2}{1+\sqrt{ab}}$$

Let $g: (0, \log 2) \to \mathbb{R};\ g(x) = \frac{1}{e^x+1}, g'(x) = -\frac{e^x}{(e^x+1)^2} = -\frac{1}{e^x+1} + \frac{1}{(e^x+1)^2}$

$g''(x) = \frac{e^x(e^x-1)}{(e^x+1)^3} > 0 \Rightarrow g$ **–convexe, from Jensen, we have:**

$$\frac{1}{2} \cdot \frac{1}{1+e^{\log b}} + \frac{1}{2} \cdot \frac{1}{1+e^{\log a}} \geq \frac{1}{1+e^{\frac{\log a+\log b}{2}}} = \frac{1}{1+\sqrt{ab}} \Leftrightarrow$$

$\frac{a+b+2}{2} \geq \frac{(a+1)(b+1)}{1+\sqrt{ab}}, f$ **–decreasing, then**

$$f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geq f\left(\frac{a+b+2}{2}\right) \overset{(1)}{\Rightarrow} \frac{f(a+1)}{2} + \frac{f(b+1)}{2} \leq f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \Rightarrow$$

$$\sum_{cyc} \frac{f(a+1)+f(b+1)}{2} \leq \sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \Leftrightarrow$$

$$\sum_{cyc} f(a+1) \leq \sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right); (2)$$

$$2\sqrt[4]{\prod_{cyc} f(a+1) \cdot \sum_{cyc} f(a+1)} \leq 2 \cdot \frac{2f(a+1)+2f(b+1)+2f(c+1)}{4} =$$

$$=\sum_{cyc} f(a+1)\overset{(2)}{\Rightarrow}$$

$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right)\ge 2\sqrt[4]{\prod_{cyc} f(a+1)\cdot\sum_{cyc} f(a+1)}\,,\forall a,b,c\in(1,2)$$

Solution 3 by proposer

$$a,b,c\in(1,2)\to(1+a),(1+b),(1+c)\in(2,3)$$

$$\frac{(a+1)(b+1)}{1+\sqrt{ab}}\le\frac{a+b+2}{2}\leftrightarrow\left(\sqrt{a}-\sqrt{b}\right)^2\left(\sqrt{ab}-1\right)\ge 0,\forall a,b$$

$$\in(1,2)\text{ and analogous}\ \dots\dots(1)$$

$$\frac{(a+1)(b+1)}{1+\sqrt{ab}}\ge\frac{(1+\sqrt{ab})^2}{1+\sqrt{ab}}=1+\sqrt{ab}>2$$

$$\frac{(a+1)(b+1)}{1+\sqrt{ab}}<3\leftrightarrow\left(\sqrt{ab}-1\right)\left(\sqrt{ab}-2\right)<0\text{ true.}$$

$$\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\ge 2\sqrt[4]{xyz(x+y+z)}\,,\forall x,y,z>0\ \dots\dots(2)$$

$$\text{Let } z=max\{x,y,z\}\text{ and } a=\frac{x}{z},b=\frac{y}{z}\,,a,b\in[0,1]\overset{(2)}{\Leftrightarrow}(a+b+ab)^2$$

$$\ge 4ab\sqrt{a^2+b^2+1}$$

But $(a+b+ab)^2=a^2b^2+(a+b)^2+2ab(a+b)\ge 2ab(a+b+2)$, then we have:

$$(a+b+2)^2\ge 4(a+b+1),\text{ true from}$$

$$(a+b+2)^2=(a+b)^2+4+4(a+b)\ge 4(a+b+1)\ge$$

$$\ge 4(a^2+b^2+1),\forall a,b\in[0,1]$$

From (1),(2) we have:

$$f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right)\ge f\left(\frac{(a+1)+(b+1)}{2}\right)\ge\frac{f(a+1)+f(b+1)}{2}$$

$$\ge\sqrt{f(a+1)\cdot f(b+1)}$$

$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right)\ge\sum_{cyc}\sqrt{f(a+1)\cdot f(b+1)}\ge$$

$$\geq 2\cdot\sqrt[4]{\left(\prod_{cyc}f(a+1)\right)\cdot\left(\sum_{cyc}f(a+1)\right)}$$

678. For all $x\in[-1,1]$, prove that:

$$\left|\frac{\left(x+\sqrt{x^2-1}\right)^{n+1}+\left(x-\sqrt{x^2-1}\right)^{n+1}+\left(x+\sqrt{x^2-1}\right)^{n-1}+\left(x-\sqrt{x^2-1}\right)^{n-1}}{\left(x+\sqrt{x^2-1}\right)^{n}+\left(x-\sqrt{x^2-1}\right)^{n}}\right|\leq 2$$

Proposed by Neculai Stanciu-Romania

Solution by Asmat Qatea-Afghanistan

$$\text{Let } x=\cos y\Rightarrow\begin{cases}x+\sqrt{x^2-1}=\cos y+i\sin y=e^{iy}\\x-\sqrt{x^2-1}=\cos y-i\sin y=e^{-iy}\end{cases}$$

$$\left|\frac{e^{iy(n+1)}+e^{-iy(n+1)}+e^{iy(n-1)}+e^{-ny(n-1)}}{e^{iy(n)}+e^{-iy(n)}}\right|\leq 2\Leftrightarrow$$

$$\left|\frac{\cos(n+1)y+\cos(n-1)y}{\cos(ny)}\right|\leq 2$$

$$\cos x=\frac{e^{ix}+e^{-nx}}{2}\Rightarrow$$

$$\left|\frac{2\cos\left(\frac{(n+1)y+(n-1)y}{2}\right)\cos\left(\frac{(n+1)y-(n-1)y}{2}\right)}{\cos ny}\right|\leq 2\Rightarrow|\cos y|\geq 1.$$

679. If $n\in\mathbb{N},n\geq 2$ then:

$$\sum_{k=2}^{n}\left(H_k+\sqrt[k]{k^{k-1}}\right)<\frac{(n-1)(n+4)}{2}$$

Proposed by Daniel Sitaru-Romania

Solution by Michael Sterghiou-Greece

$$\sum_{k=2}^{n}\left(H_k+\sqrt[k]{k^{k-1}}\right)<\frac{(n-1)(n+4)}{2};(1)$$

(1) holds for $n=2$ as $H_2+\sqrt{2}<3$ or $\frac{1}{2}+\sqrt{2}<3$ which is true.

Let (1) holds for $n = n$, that is

$$\sum_{k=2}^{n}\left(H_k + \sqrt[k]{k^{k-1}}\right) < \frac{(n-1)(n+4)}{2}$$

We will show that (1) holds for $n+1$ or

$$\sum_{k=2}^{n+1}\left(H_k + \sqrt[k]{k^{k-1}}\right) < \frac{n(n+5)}{2}; (3)$$

$$(3) \Rightarrow \left(\sum_{k=2}^{n} H_k\right) + \left(\sum_{k=2}^{n}\sqrt[k]{k^{k-1}}\right) + \frac{1}{n+1} + (n+1)^{\frac{n}{n+1}} < \frac{(n-1)(n+4)}{2} + n + 2$$

So, as (2) holds by the induction assumption, it is enough to prove that

$$\frac{1}{n+1} + (n+1)^{\frac{n}{n+1}} < (n+2); (4)$$

But $(n+1)^{\frac{n}{n+1}} < n+1$ so we get the stronger inequality $\frac{1}{n+1} + n + 1 < n + 2$ or $-\frac{n}{n+1} < 0$, which is true. Therefore, by induction (1) holds.

680. If $a, b, c \geq 0, a + b + c = 3$ then find:

$$\Omega = max\left\{2\sum a^3 + 15\sum ab + 6abc\right\}$$

Proposed by Dan Radu Seclăman-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$2\sum a^3 + 15\sum ab + 6abc = 2\sum a^3 + 5\left(\sum a\right)\left(\sum ab\right) + 6abc =$$

$$= 2\sum a^3 + 5\prod(a+b) + 11abc = 2\left(\sum a\right)^3 - \prod(a+b) + 11abc \leq$$

$$\overset{Cesaro}{\leq} 2.3^3 - 8abc + 11abc = 54 + 3abc \overset{AM-GM}{\leq} 54 + 3\left(\frac{1}{3}\sum a\right)^3 = 57.$$

$$With\ equality\ if\ a = b = c = 1.$$

$$Therefore, \quad \Omega = 57.$$

Solution 2 by George Florin Șerban-Romania

$$p = \sum a = 3, q = \sum ab, r = abc$$

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \Rightarrow abc \leq 1$$

$$2\sum a^3 + 15\sum ab + 6abc = 2(p^3 - 3pq + 3r) + 15q + 6r =$$

$$= 2p^3 - 6pq + 6r + 15q + 6r = 54 - 18q + 12r + 15q = 54 - 3q + 12r \overset{(*)}{\leq} 57$$

$$(*) \Leftrightarrow 3q \geq 12r - 3 \Leftrightarrow q \geq 4r - 1$$

$$pq \geq 9r \Rightarrow 3q \geq 9r \Rightarrow q \geq 3r \overset{(?)}{\geq} 4r - 1 \Rightarrow r \leq 1 \text{ true} \Rightarrow \Omega = 57.$$

Equality holds for $a = b = c = 1$.

681. If $x, y, z \in \mathbb{R}$ such that $x + y + z = 3, xy + yz + zx = 2$, then find:

$min(x)$ and $max(x)$

Proposed by Neculai Stanciu-Romania

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo

$$z = 3 - x - y, xy + yz + zx = 2 \Rightarrow$$

$$xy + y(3 - x - y) + x(3 - x - y) = 2$$

$$xy + 3y - xy - y^2 + 3x - x^2 - xy = 2$$

$$y^2 - 3y + xy + x^2 - 3x + 2 = 0$$

$$y^2 - (3 - x)y + x^2 - 3x + 2 = 0$$

$$\Delta = -3x^2 + 6x + 1 > 0$$

$$3x^2 - 6x - 1 = 0 \Rightarrow x_{1,2} = \frac{1 \pm 2\sqrt{3}}{3}$$

$$\min(x) = \frac{1 - 2\sqrt{3}}{3}, \max(x) = \frac{1 + 2\sqrt{3}}{3}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have : 2 = x(y+z) + yz \overset{AM-GM}{\leq} x(3-x) + \frac{1}{4}(y+z)^2 = 3x - x^2 + \frac{1}{4}(3-x)^2$$

$$\leftrightarrow -3x^2 + 6x + 1 \geq 0 \leftrightarrow \frac{3 - 2\sqrt{3}}{3} \leq x \leq \frac{3 - 2\sqrt{3}}{3}.$$

$$\rightarrow min(x) = \frac{3 - 2\sqrt{3}}{3}\ and\ max(x) = \frac{3 + 2\sqrt{3}}{3}.$$

$$x = min(x)\ for\ y = z = \frac{3+\sqrt{3}}{3}\ and\ x = max(x)\ for\ y = z = \frac{3-\sqrt{3}}{3}.$$

682. If $f : R \to R, f(x) = \sum_{k=0}^{n} a_k x^k, a_k \geq 0, \forall k = \overline{0,n}$ with $f(4) = 8$ and $f(9) = 18$, then determine $max(\ f(6)\ )$ and the function which realize this maximum.

Proposed by Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$f(4).f(9) = \left(\sum_{k=0}^{n} a_k 4^k\right)\left(\sum_{k=0}^{n} a_k 9^k\right) \overset{CBS}{\geq} \left(\sum_{k=0}^{n} \sqrt{a_k 4^k . a_k 9^k}\right)^2 = \left(\sum_{k=0}^{n} a_k 6^k\right)^2$$

$$= (\ f(6)\ )^2 \to max(\ f(6)\ ) = \sqrt{8.18} = 12.$$

$$With\ equality\ if\ \exists \lambda \in R / a_k 9^k = \lambda . a_k 4^k, \forall k = \overline{0,n}.$$

$$If\ \exists (i \neq j) \in \{0,1,..,n\}\ such\ that:\ a_i, a_j \neq 0 \to \frac{9^i}{4^i} = \frac{9^j}{4^j} = \lambda\ Absurd \to \exists!\, i \in \{0,1,..,n\}\ such\ that\ a_i \neq 0.$$

$$\to f(x) = a_i . x^i, \forall x \in R \to \frac{f(9)}{f(4)} = \frac{9}{4} = \frac{9^i}{4^i} \to i = 1\ and\ a_1 = \frac{f(4)}{4} = 2.$$

$$Therefore, f(x) = 2x, \forall x \in R.$$

683. If $a, b, c > 0$ such that: $a + 2b + 3c = 6$. Prove that :

$$S = 3\sqrt[3]{4}\sqrt[3]{\sqrt{a}(b^2+1)} + 5\sqrt[10]{128}\sqrt[5]{b(c^2+1)\sqrt{c^2+1}} + 4\sqrt[8]{128}\sqrt[4]{c\sqrt{c(a^2+1)}} \leq 24$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: 3\sqrt[3]{4}\sqrt[3]{\sqrt{a}(b^2+1)} = 2.3\sqrt[3]{\sqrt{a}.\sqrt{\frac{b^2+1}{2}}^{\,2}} \overset{AM-GM}{\leq} 2\left(\sqrt{a} + 2.\sqrt{\frac{b^2+1}{2}}\right)$$

$$5\sqrt[10]{128}\sqrt[5]{b(c^2+1)\sqrt{c^2+1}} = 2.5\sqrt[5]{\sqrt{b}^2\cdot\sqrt{\frac{c^2+1}{2}}^3} \overset{AM-GM}{\leq} 2\left(2\sqrt{b}+3\sqrt{\frac{c^2+1}{2}}\right)$$

$$4\sqrt[8]{128}\sqrt[4]{c\sqrt{c(a^2+1)}} = 2.4\sqrt[4]{\sqrt{c}^3\sqrt{\frac{a^2+1}{2}}} \overset{AM-GM}{\leq} 2\left(3\sqrt{c}+\sqrt{\frac{a^2+1}{2}}\right)$$

$$\to \frac{S}{2} \leq \left(\sqrt{a}+\sqrt{\frac{a^2+1}{2}}\right)+2\left(\sqrt{b}+\sqrt{\frac{b^2+1}{2}}\right)+3\left(\sqrt{c}+\sqrt{\frac{c^2+1}{2}}\right) \leq$$

$$\overset{CBS}{\leq} \sqrt{(1+1)\left(a+\frac{a^2+1}{2}\right)}+2\sqrt{(1+1)\left(b+\frac{b^2+1}{2}\right)}+3\sqrt{(1+1)\left(c+\frac{c^2+1}{2}\right)}$$

$$=(a+1)+2(b+1)+3(c+1)=(a+2b+3c)+6=6+6=12.$$

$$Therefore,\quad 3\sqrt[3]{4}\sqrt[3]{\sqrt{a}(b^2+1)}+5\sqrt[10]{128}\sqrt[5]{b(c^2+1)\sqrt{c^2+1}}$$

$$+4\sqrt[8]{128}\sqrt[4]{c\sqrt{c(a^2+1)}} \leq 24.$$

684. If $x,y,z,t \geq 0$ then:

$$x^2\cot\frac{\pi}{19}+y^2\cot\frac{2\pi}{19}+z^2\cot\frac{4\pi}{19}+t^2\tan\frac{8\pi}{19} \geq \left(x+y\sqrt{2}+2z+2t\sqrt{2}\right)^2\tan\frac{\pi}{19}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$From\ CBS, we\ have:$$

$$\left(x^2\cot\frac{\pi}{19}+y^2\cot\frac{2\pi}{19}+z^2\cot\frac{4\pi}{19}+t^2\tan\frac{8\pi}{19}\right)\left(\tan\frac{\pi}{19}+2\tan\frac{2\pi}{19}+4\tan\frac{4\pi}{19}\right.$$

$$\left.+8\cot\frac{8\pi}{19}\right) \geq \left(x+y\sqrt{2}+2z+2t\sqrt{2}\right)^2$$

$$We\ know\ that:\ \forall a \in \left(0,\frac{\pi}{2}\right);\ 2\cot 2a = \cot a - \tan a$$

$$\to 8\cot\frac{8\pi}{19} = 4\cot\frac{4\pi}{19}-4\tan\frac{4\pi}{19} = 2\cot\frac{2\pi}{19}-2\tan\frac{2\pi}{19}-4\tan\frac{4\pi}{19}$$

$$=\cot\frac{\pi}{19}-\tan\frac{\pi}{19}-2\tan\frac{2\pi}{19}-4\tan\frac{4\pi}{19}$$

$$\to \tan\frac{\pi}{19}+2\tan\frac{2\pi}{19}+4\tan\frac{4\pi}{19}+8\cot\frac{8\pi}{19}=\cot\frac{\pi}{19}$$

$$Therefore,$$

$$x^2\cot\frac{\pi}{19}+y^2\cot\frac{2\pi}{19}+z^2\cot\frac{4\pi}{19}+t^2\tan\frac{8\pi}{19}\ge\left(x+y\sqrt{2}+2z+2t\sqrt{2}\right)^2\tan\frac{\pi}{19}.$$

685. Let $a,b\ge 0$ such that: $2a^2+b^2+ab\ge 1$. Prove that:

$$(a+b)^2\ge a^2+b^2\ge\frac{6-2\sqrt{2}}{7}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$From\ AM-GM, We\ have:\ \frac{\sqrt{2}-1}{2}.a^2+\frac{\sqrt{2}+1}{2}.b^2\ge 2\sqrt{\frac{\sqrt{2}-1}{2}.\frac{\sqrt{2}+1}{2}}ab=ab$$

$$\to 1\le 2a^2+b^2+ab\le 2a^2+b^2+\frac{\sqrt{2}-1}{2}.a^2+\frac{\sqrt{2}+1}{2}.b^2=\frac{3+\sqrt{2}}{2}(a^2+b^2)$$

$$\to a^2+b^2\ge\frac{2}{3+\sqrt{2}}=\frac{6-2\sqrt{2}}{7}.$$

$$Therefore,\qquad (a+b)^2=(a^2+b^2)+2ab\ge a^2+b^2\ge\frac{6-2\sqrt{2}}{7}$$

686. If $x,y,z\in\mathbb{R}$ and $a,b,c,d>0, 2a\le c+d, b\le 2c, 3(a-c)=d-b$ then:

$$a\left(\sum x^2\right)^2+bxyz\sum x\ge c\left(\sum yz\right)^2+d\sum y^2z^2$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Șerban-Romania

First, we prove that $\sum x^4\ge xyz(x+y+z)$

We have: $x^4+\sum x^4\overset{AM-GM}{\ge}4\sqrt[4]{x^4(xyz)^4}=4x^2yz\Rightarrow 4\sum x^4\ge 4\sum x^2yz$

$$\Rightarrow\sum x^4\ge xyz(x+y+z).$$

Now, $\sum x^4\ge\sum x^2y^2\Leftrightarrow\sum(x^2-y^2)^2\ge 0$.

Hence, $a\sum x^4+2a\sum x^2y^2+bxyz\sum x\overset{(1)}{\ge}c\sum x^2y^2+cxyz\sum x+d\sum y^2z^2$

$$(1) \Leftrightarrow a\sum x^4 \geq (c+d-2a)\sum x^2y^2 + (2c-b)xyz\sum x; (2)$$

$$b \leq 2c \Rightarrow 2c-b \geq 0, 2a \leq c+d \Rightarrow c+d-2a \geq 0 \Rightarrow$$

$$(c+d-2a)\sum x^2y^2 + (2c-b)xyz\sum x \leq (c+d-2a)\sum x^4 + (2c-b)\sum x^4 =$$

$$= (c+d-2a+2c-b)\sum x^4 = (3c+d-2a-b)\sum x^4 \leq a\sum x^4 \Leftrightarrow$$

$$3c+d-2a-b \leq a \Leftrightarrow 3c-3a \leq b-d \Leftrightarrow 3(c-a) \leq b-d \Leftrightarrow b-d \leq b-d \text{ true.}$$

Hence,

$$a\left(\sum x^2\right)^2 + bxyz\sum x \geq c\left(\sum yz\right)^2 + d\sum y^2z^2$$

Therefore,

$$a\left(\sum x^2\right)^2 + bxyz\sum x \geq c\left(\sum yz\right)^2 + d\sum y^2z^2$$

687.

$$x_i, y_i > 0, i \in \overline{0,7}, \quad 512\sum_{i=0}^{7}(x_i+y_i) = 1225.$$

Prove that:

$$\sum_{i=0}^{7}\frac{\sin^6\left(\frac{i\pi}{8}\right)}{x_i} + \sum_{i=0}^{7}\frac{\cos^6\left(\frac{i\pi}{8}\right)}{y_i} > 1317$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

If $x_k, y_k > 0, k = \overline{1,n}, n \in \mathbb{N}^*, m \in \mathbb{N}; m \geq 3$ then:

$$\frac{y_1^m}{x_1} + \frac{y_2^m}{y_2} + \cdots + \frac{y_n^m}{x_n} \geq \frac{(y_1+y_2+\cdots+y_n)^m}{n^{m-2}(x_1+x_2+\cdots+x_n)}$$

Let $y_i = \sin^2\left(\frac{i\pi}{8}\right); i = \overline{0,7}, x_i = x_i, i = \overline{0,7}$

$$y_k = \cos^2\left(\frac{i\pi}{8}\right); k = \overline{8,15}, x_k = y_i, k = \overline{8,15}$$

Hence,

$$\sum_{i=0}^{7}\frac{\sin^6\left(\frac{i\pi}{8}\right)}{x_i} + \sum_{i=0}^{7}\frac{\cos^6\left(\frac{i\pi}{8}\right)}{y_i} = \sum_{i=0}^{7}\frac{\left(\sin^2\left(\frac{i\pi}{8}\right)\right)^3}{x_i} + \sum_{i=0}^{7}\frac{\left(\cos^2\left(\frac{i\pi}{8}\right)\right)^3}{y_i} \geq$$

$$\geq \frac{\left(\sum_{i=0}^{7}\left(\sin^2\left(\frac{i\pi}{8}\right)+\cos^2\left(\frac{i\pi}{n}\right)\right)\right)^3}{16^{3-2}\sum_{i=0}^{7}(x_i+y_i)} = \frac{8^3}{16\cdot\frac{1225}{512}} = \frac{32768}{2450} > 1317$$

688. If $a, b, c, d > 0, ad > bc$ then:

$$\frac{(5a+3b)(7a+5b)(9a+7b)}{(5c+3d)(7c+5d)(9c+7d)} > \left(\frac{a+b}{c+d}\right)^3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $k \geq 4$, $[(k+1)a+(k-1)b](c+d)-[(k+1)c+(k-1)d](a+b) =$

$$= (k+1)ac+(k-1)bc+(k+1)ad+(k-1)bd-$$

$$-[(k+1)ac+(k-1)da+(k+1)bc+(k-1)bd] = 2(ad-bc) > 0$$

Thus,

$$\frac{(k+1)a+(k-1)b}{(k+1)c+(k-1)d} > \frac{a+b}{c+d}$$

Taking $k = 4, 6, 8$, we get:

$$\frac{5a+3b}{5c+3d} > \frac{a+b}{c+d}, \frac{7a+5b}{7c+5d} > \frac{a+b}{c+d}, \frac{9a+7b}{9c+7d} > \frac{a+b}{c+d}$$

Multiplying these three inequalities, we get:

$$\frac{(5a+3b)(7a+5b)(9a+7b)}{(5c+3d)(7c+5d)(9c+7d)} > \left(\frac{a+b}{c+d}\right)^3$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{(5a+3b)(7a+5b)(9a+7b)}{(5c+3d)(7c+5d)(9c+7d)} \geq \left(\frac{3}{\frac{1}{\frac{5a+3b}{5c+3d}}+\frac{1}{\frac{7a+5b}{7c+5d}}+\frac{1}{\frac{9a+7b}{9c+7d}}}\right)^3 > \left(\frac{a+b}{c+d}\right)^3$$

$$3\left(\frac{c+d}{a+b}\right) > \frac{5c+3d}{5a+3b}+\frac{7c+5d}{7a+5b}+\frac{9c+7d}{9a+7b} \text{ true, because}$$

$\frac{c+d}{a+b} > \frac{5c+3d}{5a+3b} \Leftrightarrow (c+d)(5a+3b) > (5c+3d)(a+b) \Leftrightarrow ad > bc$ true. Similarly,

$$\frac{c+d}{a+b} > \frac{7c+5d}{7a+5b} \text{ and } \frac{c+d}{a+b} > \frac{9c+7d}{9a+7b}$$

689. If $x, y, z \in R^*, x > y > z$, then prove that:

$$\left(\frac{x}{z}\right)^y \cdot \left(\frac{y}{x}\right)^z \cdot \left(\frac{z}{y}\right)^x \leq 1$$

Proposed by Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\frac{x}{z}\right)^y \cdot \left(\frac{y}{x}\right)^z \cdot \left(\frac{z}{y}\right)^x \overset{(*)}{\leq} 1$$

$$(*) \leftrightarrow y\big(ln(x) - ln(z)\big) + z\big(ln(y) - ln(x)\big) + x\big(ln(z) - ln(y)\big) \leq 0$$

$$\leftrightarrow (y - z)ln(x) + (x - y)ln(z) \leq (x - z)ln(y)$$

$$Let\ f(x) = ln(x), x > 0.$$

$$We\ know\ that\ f\ is\ a\ concave\ function\ on\ (0, +\infty), y - z, x - y > 0$$

$$\rightarrow (y - z)ln(x) + (x - y)ln(z) \overset{Jensen}{\leq} [(y - z) + (x - y)].ln\left(\frac{(y - z)x + (x - y)z}{(y - z) + (x - y)}\right) =$$

$$= (x - z).ln\left(\frac{xy - yz}{x - z}\right) = (x - z)ln(y).$$

$$Therefore,\quad \left(\frac{x}{z}\right)^y \cdot \left(\frac{y}{x}\right)^z \cdot \left(\frac{z}{y}\right)^x \leq 1.$$

690. If $a_i, b_i > 0, i \in \overline{1,n}$, then:

$$\left(\sum_{i=1}^{n}(a_i + b_i)\right)^2 \geq 4\left(\sum_{i=1}^{n}\sqrt{a_i b_i}\right)\left(\sum_{i=1}^{n}\sqrt{\frac{a_i^2 + b_i^2}{2}}\right)$$

Proposed by Seyran Ibrahimov-Maasili-Azerbaijan

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ know\ that:\ 4ab \leq (a + b)^2, \forall a, b > 0\ (\because AM - GM)$$

$$\rightarrow 4\left(\sum_{i=1}^{n}\sqrt{a_i b_i}\right)\left(\sum_{i=1}^{n}\sqrt{\frac{a_i^2 + b_i^2}{2}}\right) \leq \left[\left(\sum_{i=1}^{n}\sqrt{a_i b_i}\right) + \left(\sum_{i=1}^{n}\sqrt{\frac{a_i^2 + b_i^2}{2}}\right)\right]^2$$

$$= \left[\sum_{i=1}^{n}\left(\sqrt{a_i b_i} + \sqrt{\frac{a_i^2 + b_i^2}{2}}\right)\right]^2 \leq$$

$$\overset{CBS}{\leq}\left(\sum_{i=1}^{n}\sqrt{(1+1)\left(a_ib_i+\frac{a_i^2+b_i^2}{2}\right)}\right)^2=\left(\sum_{i=1}^{n}(a_i+b_i)\right)^2.$$

Therefore,

$$\left(\sum_{i=1}^{n}(a_i+b_i)\right)^2\geq 4\left(\sum_{i=1}^{n}\sqrt{a_ib_i}\right)\left(\sum_{i=1}^{n}\sqrt{\frac{a_i^2+b_i^2}{2}}\right).$$

Solution 2 by Nguyen Van Canh-BenTre-Vietnam

Let $f(x)=\ln x$, $(\forall x>0)\to f''(x)=-\frac{1}{x^2}<0, (\forall x>0)$

$$\overset{Jensen}{\Rightarrow}\ 2f\left(\frac{x+y}{2}\right)\geq f(x)+f(y),\qquad \forall x,y>0$$

Now, choosing: $x=\sum_{i=1}^{n}\left(\sqrt{\frac{a_i^2+b_i^2}{2}}\right)$, $y=\sum_{i=1}^{n}\sqrt{a_ib_i}$

$$\to\ln\left(\sum_{i=1}^{n}\left(\sqrt{\frac{a_i^2+b_i^2}{2}}\right)\right)+\ln\left(\sum_{i=1}^{n}\sqrt{a_ib_i}\right)\leq 2\ln\left(\frac{\sum_{i=1}^{n}\left(\sqrt{\frac{a_i^2+b_i^2}{2}}\right)+\sum_{i=1}^{n}\sqrt{a_ib_i}}{2}\right)$$

$$=\ln\left(\frac{\sum_{i=1}^{n}\left(\sqrt{\frac{a_i^2+b_i^2}{2}}\right)+\sum_{i=1}^{n}\sqrt{a_ib_i}}{2}\right)^2\leftrightarrow$$

$$\ln\frac{\left(\sum_{i=1}^{n}\left(\sqrt{\frac{a_i^2+b_i^2}{2}}\right)+\sum_{i=1}^{n}\sqrt{a_ib_i}\right)^2}{4}\overset{(*)}{\leq}\ln\frac{\left(\sum_{i=1}^{n}(a_i+b_i)\right)^2}{4}$$

$$(*)\leftrightarrow\sum_{i=1}^{n}\left(\sqrt{\frac{a_i^2+b_i^2}{2}}\right)+\sum_{i=1}^{n}\sqrt{a_ib_i}\leq\sum_{i=1}^{n}(a_i+b_i)$$

$$\leftrightarrow\sum_{i=1}^{n}\left(\sqrt{\frac{a_i^2+b_i^2}{2}}\right)\leq\sum_{i=1}^{n}\left(a_i+b_i-\sqrt{a_ib_i}\right)$$

So, we need to prove:

$$\sqrt{\frac{{a_i}^2+{b_i}^2}{2}}\le a_i+b_i-\sqrt{a_ib_i},\qquad \forall i=1,2,\dots,n$$

$$\overset{a_i+b_i\ge 2\sqrt{a_ib_i}>\sqrt{a_ib_i}}{\Leftrightarrow}\ \frac{{a_i}^2+{b_i}^2}{2}\le\left(a_i+b_i-\sqrt{a_ib_i}\right)^2$$

$$\leftrightarrow {a_i}^2+{b_i}^2\le 2\left(a_i+b_i-\sqrt{a_ib_i}\right)^2,$$

$$\leftrightarrow {a_i}^2+{b_i}^2\le 2(a_i+b_i)^2-4(a_i+b_i)\sqrt{a_ib_i}+2a_ib_i,$$

$$\leftrightarrow {a_i}^2+{b_i}^2+6a_ib_i-4(a_i+b_i)\sqrt{a_ib_i}\ge 0,$$

$$\leftrightarrow (a_i+b_i)^2+4a_ib_i-4(a_i+b_i)\sqrt{a_ib_i}\ge 0,$$

$$\leftrightarrow \left(a_i+b_i-2\sqrt{a_ib_i}\right)^2\ge 0.$$

Which is true .

$$\to\left(\sum_{i=1}^n\sqrt{\frac{{a_i}^2+{b_i}^2}{2}}\right)\left(\sum_{i=1}^n\sqrt{a_ib_i}\right)\le\frac{\left(\sum_{i=1}^n(a_i+b_i)\right)^2}{4}$$

691. If $a_i,b_i>0, i\in\overline{1,n}$ then:

$$\left(\sum_{i=1}^n(a_i+b_i)\right)\left(\sum_{i=1}^n\frac{a_ib_i}{a_i+b_i}\right)\ge\left(\sum_{i=1}^n\sqrt{a_ib_i}\right)^2$$

Proposed by Seyran Ibrahimov-Maasili-Azerbaijan

Solution 1 by George Florin Şerban-Romania

$$\left(\sum_{i=1}^n\sqrt{a_ib_i}\right)^2=\left(\sum_{i=1}^n\sqrt{a_i+b_i}\cdot\frac{\sqrt{a_ib_i}}{\sqrt{a_i+b_i}}\right)^2\overset{CBS}{\le}\sum_{i=1}^n(a_i+b_i)\cdot\sum_{i=1}^n\frac{a_ib_i}{a_i+b_i}$$

Solution 2 by Fayssal Abdelli-Bejaia-Algerie

$$\left(\sum_{i=1}^n(a_i+b_i)\right)\left(\sum_{i=1}^n\frac{a_ib_i}{a_i+b_i}\right)\ge\left(\sum_{i=1}^n\sqrt{a_ib_i}\right)^2;(A)$$

$$\sum_{i=1}^n\frac{a_ib_i}{a_i+b_i}\overset{Brgstrom}{\ge}\frac{\left(\sum_{i=1}^n\sqrt{a_ib_i}\right)^2}{\sum_{i=1}^n(a_i+b_i)}$$

$$(A):\left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right)\ge\left(\sum_{i=1}^{n}(a_i+b_i)\right)\frac{\left(\sum_{i=1}^{n}\sqrt{a_ib_i}\right)^2}{\sum_{i=1}^{n}(a_i+b_i)}\ge\left(\sum_{i=1}^{n}\sqrt{a_ib_i}\right)^2$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right)=\sum_{i=1}^{n}a_ib_i+\sum_{1\le i<j\le n}\left(\frac{a_i+b_i}{a_j+b_j}\cdot a_jb_j+\frac{a_j+b_j}{a_i+b_i}\cdot a_ib_i\right)\ge$$

$$\ge\sum_{i=1}^{n}a_ib_i+2\sum_{1\le i<j\le n}\sqrt{a_ib_ia_jb_j}=\left(\sum_{i=1}^{n}\sqrt{a_ib_i}\right)^2$$

Solution 4 by proposer

We prove using mathematical induction:

$$n=1:(a_1+b_1)\cdot\frac{a_1b_1}{a_1+b_1}\ge\left(\sqrt{a_1b_1}\right)^2\Leftrightarrow a_1b_1=a_1b_1-true.$$

Assume for n:

$$\left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right)\ge\left(\sum_{i=1}^{n}\sqrt{a_ib_i}\right)^2$$

$$\text{Let: }\sum_{i=1}^{n}(a_i+b_i)=A;\ \sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}=B;C=\sum_{i=1}^{n}\sqrt{a_ib_i}$$

We assumed $AB\ge C^2$. We must show that:

$$\left(\sum_{i=1}^{n+1}(a_i+b_i)\right)\left(\sum_{i=1}^{n+1}\frac{a_ib_i}{a_i+b_i}\right)\ge\left(\sum_{i=1}^{n+1}\sqrt{a_ib_i}\right)^2\Leftrightarrow$$

$$\left(A+(a_{n+1}+b_{n+1})\right)\left(B+\frac{a_{n+1}b_{n+1}}{a_{n+1}+b_{n+1}}\right)\ge\left(C+\sqrt{a_{n+1}b_{n+1}}\right)^2$$

$$AB+A\cdot\frac{a_{n+1}b_{n+1}}{a_{n+1}+b_{n+1}}+B(a_{n+1}+b_{n+1})+a_{n+1}b_{n+1}\ge C^2+2C\sqrt{a_{n+1}b_{n+1}}+a_{n+1}b_{n+1}$$

$$\Leftrightarrow AB+A\cdot\frac{a_{n+1}b_{n+1}}{a_{n+1}+b_{n+1}}+B(a_{n+1}+b_{n+1})\ge C^2+2C\sqrt{a_{n+1}b_{n+1}};(but\ AB\ge C^2)$$

$$A\cdot\frac{a_{n+1}b_{n+1}}{a_{n+1}+b_{n+1}}+B(a_{n+1}+b_{n+1})\overset{AM-GM}{\ge}2\sqrt{AB\cdot a_{n+1}b_{n+1}}\ge2\sqrt{C^2a_{n+1}b_{n+1}}=$$

$$=2C\sqrt{a_{n+1}b_{n+1}}.\text{ Proved.}$$

692. If $0 < b \le 1 \le a$ then prove:

$$\left(\frac{1}{\frac{a^3}{b^3}+a^3+\frac{3}{4}}+7\right).\sqrt[3]{\left(\frac{a}{b}+a\right)^2} > \frac{9.\sqrt[9]{128}}{2}$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger Morocco

$$We\ have:$$

$$\left(\frac{1}{\frac{a^3}{b^3}+a^3+\frac{3}{4}}+7\right).\sqrt[3]{\left(\frac{a}{b}+a\right)^2} > 7.\sqrt[3]{(1+1)^2} = 7.\sqrt[3]{2^2} \overset{?}{\gtrsim} \frac{9.\sqrt[9]{128}}{2} \leftrightarrow 7.2^{\frac{8}{9}} > 9$$

$$Which\ is\ true\ because\ 7.2^{\frac{8}{9}} > 7.2^{\frac{1}{2}} \overset{?}{\gtrsim} 9 \leftrightarrow 49.2 > 81.$$

$$Therefore,\quad \left(\frac{1}{\frac{a^3}{b^3}+a^3+\frac{3}{4}}+7\right).\sqrt[3]{\left(\frac{a}{b}+a\right)^2} > \frac{9.\sqrt[9]{128}}{2}.$$

Solution 2 by Nguyen Van Canh-BenTre-Vietnam

For $0 < b \le 1 \le a$ we have:

$$\sqrt[3]{\left(\frac{a}{b}+a\right)^2} \overset{AM-GM}{\ge} \sqrt[3]{\left(\frac{2a^2}{b}\right)^2} = \sqrt[3]{4}\cdot\sqrt[3]{\frac{a^2\cdot a^2}{b^2}} \overset{a\ge 1}{\ge} \sqrt[3]{4}\cdot\sqrt[3]{\left(\frac{a}{b}\right)^2};(1)$$

Again, for $0 < b \le 1 \le a$, we have:

$$\frac{1}{\frac{a^3}{b^3}+a^3+\frac{3}{4}}+7 > 0+7 = 7;(2)$$

From (1),(2) we have:

$$\left(\frac{1}{\frac{a^3}{b^3}+a^3+\frac{3}{4}}+7\right).\sqrt[3]{\left(\frac{a}{b}+a\right)^2} > 7\cdot\sqrt[3]{4}\cdot\sqrt[3]{\left(\frac{a}{b}\right)^2} \overset{(3)}{\ge} \frac{9\cdot\sqrt[9]{128}}{2}$$

$$(3) \Leftrightarrow 7\cdot\sqrt[3]{\left(\frac{a}{b}\right)^2} > \frac{9\sqrt[3]{2}}{2} \Leftrightarrow 9t^2 > \frac{9\sqrt[9]{2}}{2};\left(\because t = \sqrt[3]{\left(\frac{a}{b}\right)} \ge 1\right)$$

But: $\frac{9\sqrt[9]{2}}{2} < \frac{9\sqrt[3]{2^3}}{2} = \frac{9\sqrt[3]{2}}{2} \overset{(4)}{<} 7 \overset{1\le t^2}{\le} 7t^2$

$$(4) \Leftrightarrow 14 > 9\sqrt[3]{2} \Leftrightarrow 14^3 > 9^3 \cdot 2 \Leftrightarrow 2744 > 1458 \ (true).$$

Solution 3 by proposer

$$0 < b \le 1 \le a \Rightarrow \left(\frac{a}{b} + a, \frac{1}{2}, \frac{1}{2}\right) \sim \left(\frac{a}{b}, a, 1\right); (Karamata\ f(x) = x^3) \Rightarrow$$

$$\left(\frac{a}{b} + a\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 \ge \left(\frac{a}{b}\right)^3 + a^3 + 1^3 \Rightarrow$$

$$\left(\frac{a}{b} + a\right)^3 \ge \frac{a^3}{b^3} + a^3 + \frac{3}{4} \Rightarrow \frac{2}{\left(\frac{a}{b} + a\right)^3} \le \frac{2}{\frac{a^3}{b^3} + a^3 + \frac{3}{4}}; (1)$$

Equality holds for $\left(\frac{a}{b} + a = \frac{a}{b}\right), \left(a = \frac{1}{2}\right), \left(\frac{1}{2} = 1\right)$ impossible.

$$\sqrt[9]{\left(\left(\frac{a}{b} + a\right)^3\right)^2 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^4} \overset{AM-HM}{\ge} \frac{9}{\frac{2}{\left(\frac{a}{b} + a\right)^3} + \frac{3}{\frac{1}{2}} + \frac{4}{\frac{1}{2}}} \overset{(1)}{>} \frac{9}{\frac{2}{\frac{a^3}{b^3} + a^3 + \frac{3}{4}} + 14}$$

$$\Rightarrow \sqrt[9]{\left(\frac{a}{b} + a\right)^6 \cdot \frac{1}{2^7}} > \frac{9}{\frac{2}{\frac{a^3}{b^3} + a^3 + \frac{3}{4}} + 14} \Rightarrow$$

$$\left(\frac{1}{\frac{a^3}{b^3} + a^3 + \frac{3}{4}} + 7\right) \cdot \sqrt[3]{\left(\frac{a}{b} + a\right)^2} > \frac{9 \cdot \sqrt[9]{128}}{2}$$

693. **If $a, b, x_k \in \left(0, \frac{\pi}{2}\right), k = \overline{1, n}$ such that $a + \sqrt{a^2 + 4ab} > 2ab$ then:**

$$\sum_{k=1}^{n} \frac{a + b\cos x_k}{(3 - x_k^2)\cos^2 x_k} \ge (a + b)\sum_{k=1}^{n} x_k^2$$

Proposed by Florică Anastase-Romania

Solution by Daniela Ruxandra Tonilă-Romania

If $x \in \left(0, \frac{\pi}{2}\right)$ such that $a + \sqrt{a^2 + 4ab} > 2ab$ then: $(a + b\cos x)\tan x \ge (a + b)x; (1)$

Proof. If $f(x) = (a + b\cos x)\tan x - (a + b)x$ then,

$$f'(x)=\frac{b(1-\cos x)\left(\frac{a+\sqrt{a^2+4ab}}{2a}-\cos x\right)\left(\frac{\sqrt{a^2+4ab}-a}{2b}+\cos x\right)}{\cos^2 x}\geq 0$$

$$\Rightarrow f(x)\geq f(0)$$

If $x\in\left(0,\frac{\pi}{2}\right)$ then $\frac{1}{x(3-x^2)\sin 2x}\geq\frac{1}{2}$.

Proof. $\frac{2}{\sin 2x}\geq 2\geq x(3-x^2)$. But $2\geq x(3-x^2)\Leftrightarrow(x-1)^2(x+2)\geq 0$, true. Thus,

$$\frac{1}{x(3-x^2)\sin 2x}\geq\frac{1}{2};(2)$$

From (1),(2) it follows that:

$$\frac{1}{(3-x^2)\sin 2x}\geq\frac{x}{2}\Rightarrow\frac{a+b\cos x}{(3-x^2)\cdot\cos^2 x}\geq(a+b)x^2$$

Therefore,

$$\sum_{k=1}^{n}\frac{a+b\cos x_k}{(3-x_k^2)\cos^2 x_k}\geq(a+b)\sum_{k=1}^{n}x_k^2$$

694. If $a,b>0$ then prove:

$$\log^2\left(\frac{a}{b}\right)+\frac{9ab}{a^2+ab+b^2}\geq 3$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$\log^2\left(\frac{a}{b}\right)+\frac{9ab}{a^2+ab+b^2}\geq 3$$

$$\log^2\left(\frac{a}{b}\right)+\frac{9\left(\frac{a}{b}\right)}{\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)+1}\geq 3$$

Let $f(x)=\log^2 x+\frac{9x}{x^2+x+1}, x\in(0,\infty)$

$$f'(x)=\frac{2\log x}{x}+\frac{9(1-x^2)}{(x^2+x+1)^2}\Rightarrow f\searrow(0,1)\ and\ f\nearrow(1,\infty)$$

$$f'(x)=0\Leftrightarrow x_0=1\Rightarrow\min f(x)=3$$

Therefore,

$$f(x) = \log^2 x + \frac{9x}{x^2 + x + 1} \geq 3, x \in (0, \infty)$$

Equality holds for $a = b$.

Solution 2 by Ravi Prakash-New Delhi-India

If $a = b$, there is nothing to prove. Assume $a < b$.

Let $a = r\cos\theta, b = r\sin\theta, \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. The inequality becomes as:

$$\log^2(\cot\theta) + \frac{9\sin\theta\cos\theta}{1 + \sin\theta\cos\theta} \geq 3; (1)$$

$$\log^2(\cot\theta) + \frac{9\cot\theta}{1 + \cot\theta + \cot^2\theta} \geq 3$$

Let $f(\theta) = \log^2(\cot\theta) + \frac{9\cot\theta}{1+\cot\theta+\cot^2\theta}, \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ then

$$f'(\theta) = 2\log(\cot\theta) \cdot (-\csc^2\theta) - \frac{9\csc\theta\,(\cot^2\theta + \cot\theta)}{(1 + \cot\theta + \cot^2\theta)^2} > 0, \forall\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$\Rightarrow f(\theta) \geq f\left(\frac{\pi}{4}\right), \forall\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \Rightarrow (1) \text{ is true.}$$

Similarly for $a > b$.

Therefore,

$$\log^2\left(\frac{a}{b}\right) + \frac{9ab}{a^2 + ab + b^2} \geq 3$$

Equality holds for $a = b$.

Solution 3 by Michael Sterghiou-Greece

$$\log^2\left(\frac{a}{b}\right) + \frac{9ab}{a^2 + ab + b^2} \geq 3; (1)$$

Due to symmetry WLOG let $a \leq b$, $(1) \Rightarrow \log^2 x + \frac{9x}{x^2+x+1} - 3 \geq 0; (2)$, where $x = \frac{a}{b} \leq 1$.

It is well-known that: $\frac{x}{x+1} < \log(1 + x) < x, \forall x > -1$ or

$$\frac{x-1}{x} \leq \log x \leq x - 1, \forall x \geq 1 \ (equality\ for\ x = 1)$$

So (2) becomes the stronger: $\left(\frac{x-1}{x}\right)^2 + \frac{9x}{x^2+x+1} - 3 \geq 0 \Leftrightarrow \frac{(1-x)^3(2x+1)}{x^2(x^2+x+1)} \geq 0, \forall x \leq 1$.

Therefore,

$$\log^2\left(\frac{a}{b}\right) + \frac{9ab}{a^2 + ab + b^2} \geq 3$$

Equality holds for $a = b$.

695. If $a, b > 0$ then

$$8(a^5 + b^5) \geq 5(a^2 - b^2)(a^3 - b^3) + \sqrt{8}\left(\sqrt{a^2 + b^2}\right)^5$$

Proposed by Asmat Qatea-Afghanistan

Solution by Ravi Prakash-New Delhi-India

Put $a = r \cdot \cos\theta, b = r \cdot \sin\theta; r \geq 0, \theta \in \left(0, \frac{\pi}{2}\right)$. The inequality becomes:

$$8(\cos^5 a + \sin^5\theta) \geq 5(\cos^2\theta - \sin^2\theta)(\cos^3\theta + \sin^3\theta) + \sqrt{8}$$

$$3(\cos^5\theta + \sin^5\theta) + 5\sin^2\theta\cos^2\theta(\sin\theta + \cos\theta) \geq \sqrt{8}$$

$$8(\sin\theta + \cos\theta)[3\cos^4\theta - 3\cos^3\theta\sin\theta + 8\cos^2\theta\sin^2\theta - 3\cos\theta\sin^3\theta + 3\sin^4\theta] \geq \sqrt{8}$$

$$(\cos\theta + \sin\theta)[3 + 2\cos^2\theta\sin^2\theta - 3\cos\theta\sin\theta] \geq \sqrt{8}$$

Let $f(\theta) = (\cos\theta + \sin\theta)\left\{3 - \frac{3}{2}\sin 2\theta + \frac{1}{2}\sin^2 2\theta\right\}; \theta \in \left(0, \frac{\pi}{2}\right)$ then

$$f'(\theta) = (\cos\theta - \sin\theta)\left[3 - \frac{3}{2}\sin 2\theta + \frac{1}{2}\sin^2 2\theta - 3(\cos\theta + \sin\theta)^2\right.$$

$$\left. + 2\sin 2\theta(\cos\theta + \sin\theta)^2\right] =$$

$$= (\cos\theta + \sin\theta)\left[\frac{5}{2}\sin^2 2\theta - \frac{5}{2}\sin 2\theta\right] = \frac{5}{2}\sin 2\theta(\sin\theta - \cos\theta)(1 - \sin 2\theta)$$

For $\theta \in (0, \pi), \sin 2\theta > 0, 1 - \sin 2\theta \geq 0$

$$f'(\theta) = 0 \Rightarrow \theta = \frac{\pi}{4}$$

$$\text{Also, } f'(\theta) = \begin{cases} < 0; if\ 0 < \theta < \frac{\pi}{4} \\ > 0; if\ \frac{\pi}{4} < \theta < \frac{\pi}{2} \end{cases}$$

Thus, $f'(\theta)$ is minimum when $\theta = \frac{\pi}{4}$. Hence,

$$f(\theta) \geq f\left(\frac{\pi}{4}\right), \theta \in \left(0, \frac{\pi}{2}\right)$$

$$f(\theta) \geq \sqrt{2}\left\{3 - \frac{3}{2} + \frac{1}{2}\right\} = \sqrt{8}$$

696. If $x_k \in (0,1), k = \overline{1,n}$ then prove:

$$\sum_{k=1}^{n} \frac{\tan x_k}{(3-x_k^2)\sin^2 x_k \sin^{-1} x_k} \geq n - \sum_{k=1}^{n} x_k^2$$

Proposed by Florică Anastase-Romania

Solution 1 by Kamel Gandouli Rezgui-Tunisia

$$(3-x^2)(1-x^2)\sin^{-1} x \sin 2x \leq 2; (?)$$

$$\sin^{-1} x \sin 2x \leq \frac{2}{(1-x^2)(3-x^2)}; (1)$$

$$(3-x^2)\sin 2x \leq 3 \Rightarrow \sin 2x \leq \frac{3}{3-x^2}$$

$$h(x) = \sin^{-1} x \leq \frac{2}{3-3x^2}; (?)$$

If $k(x) = \sin^{-1} x - x^2 - \frac{\pi}{5}$

$$K(x) = \frac{1}{\sqrt{1-x^2}} - 2x = \frac{1-2\sqrt{1-x^2}}{\sqrt{1-x^2}} = \frac{\left(x-\frac{\sqrt{3}}{2}\right)\left(x+\frac{\sqrt{3}}{2}\right)}{\left(1+\sqrt{1-x^2}\right)\sqrt{1-x^2}}$$

$$\min_{x\in(0,1)} K(x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{3}{4} - \frac{\pi}{5} = \frac{8\pi-15}{60} < 0$$

$$\lim_{x\to 0}\left(\sin^{-1} x - x^2 - \frac{\pi}{5}\right) < 0 \text{ and } \lim_{x\to 1}\left(\sin^{-1} x - x^2 - \frac{\pi}{5}\right) = \frac{\pi}{2} - 1 - \frac{\pi}{5} < 0$$

$$\Rightarrow \sin^{-1} x - x^2 - \frac{\pi}{5} < 0$$

$$f(x) = \frac{2}{3-3x^2} - x^2 - \frac{\pi}{5}; \min_{x\in(0,1)} f(x) = 0 \Rightarrow f(x) \geq 0 \Rightarrow$$

$$\sin^{-1} x < x^2 + \frac{\pi}{5} < \frac{2}{3-3x^2} \Rightarrow (3-x^2)(1-x^2)\sin^{-1} x \sin 2x \leq 2$$

$$\sin^{-1} x \sin 2x \leq \frac{2}{(3-x^2)(1-x^2)}$$

$$\frac{\tan x}{\sin^{-1} x \sin^2 x\,(3-x^2)} \geq 1 - x^2; \forall x \in (0,1)$$

Hence, $\forall k \in \overline{1,n}, x_k \in (0,1) \Rightarrow \frac{\tan x_k}{\sin^{-1} x_k \sin^2 x_k (3-x_k^2)} \geq 1 - x_k^2$

Therefore,

$$\sum_{k=1}^{n}\frac{\tan x_k}{(3-x_k^2)\sin^2 x_k\ \sin^{-1}x_k}\geq n-\sum_{k=1}^{n}x_k^2$$

Solution 2 by proposer

Lemma.

If $x\in(0,1)$ then $\frac{1}{x(3-x^2)\sin 2x}\geq\frac{1}{2}$.

Proof. $\frac{2}{\sin 2x}\geq 2\geq x(3-x^2)$. But $2\geq x(3-x^2)\Leftrightarrow(x-1)^2(x+2)\geq 0$, true. Thus,

$$\frac{1}{(3-x^2)\sin 2x}\geq\frac{x}{2};(1)$$

Lemma.

If $x\in(0,1)$, then $x\geq(1-x^2)\sin^{-1}x;(2)$

Proof.

If $f(x)=\frac{x}{1-x^2}-\sin^{-1}x$, then $f'(x)=\frac{1+x^2-(1-x^2)\sqrt{1-x^2}}{(1-x^2)^2}\overset{t=\sqrt{1-x^2}}{=}\frac{(1-t)(t^2+2t+2)}{t^4}\geq 0$,

therefore $f(x)\geq f(0)=0$. From (1),(2) it follows that:

$$\frac{1}{(3-x^2)\sin 2x}\geq\frac{x}{2}\geq\frac{1-x^2}{2}\sin^{-1}x$$

$$\frac{1}{(3-x^2)\sin 2x\sin^{-1}x}\geq\frac{1}{2}(1-x^2)$$

$$\frac{\tan x}{(3-x^2)\sin^2 x\ \sin^{-1}x}\geq 1-x^2$$

Therefore,

$$\sum_{k=1}^{n}\frac{\tan x_k}{(3-x_k^2)\sin^2 x_k\ \sin^{-1}x_k}\geq n-\sum_{k=1}^{n}x_k^2$$

697. For $x,y,z>2$ prove:

$$\log\left(\frac{(3+x)^2(3+y)^2(3+z)^2}{8}\right)^3\leq\sum_{cyc}(x^2+8)\sin\frac{\pi}{x}$$

Proposed by Florică Anastase-Romania

Solution 1 by Ruxandra Daniela Tonilă-Romania

$$\log\left(\frac{(3+x)^2(3+y)^2(3+z)^2}{8}\right)^3 = 3\log\left(\frac{(3+x)^2(3+y)^2(3+z)^2}{8}\right) =$$

$$= 3\left(\log\frac{(3+x)^2}{2} + \log\frac{(3+y)^2}{2} + \log\frac{(3+z)^2}{2}\right) \overset{Bergstrom}{\leq}$$

$$\leq 3(\log(9+x^2) + \log(9+y^2) + \log(9+z^2))$$

Let be the function $f: (2,\infty) \to \mathbb{R}, f(x) = \frac{x}{3} + 2 - \log(9+x^2)$

$$f'(x) = \frac{1}{3} - \frac{2x}{9+x^2} = \frac{(x-3)^2}{3(9+x^2)} > 0; \forall x > 2 \Rightarrow f \nearrow (x) \geq f(2) = \frac{2}{3} + 2 - \log 13 > 0$$

$$\Leftrightarrow \log(9+x^2) \leq \frac{x}{3} + 2 \Leftrightarrow 3\sum_{cyc}\log(9+x^2) \leq \sum_{cyc}(x+6)$$

$$x,y,z > 2; (Jordan) \Rightarrow \sum_{cyc}(x^2+8)\cdot\sin\left(\frac{\pi}{x}\right) \geq \sum_{cyc}(x^2+8)\cdot\frac{2}{\pi}\cdot\frac{\pi}{x}$$

$$\Leftrightarrow \sum_{cyc}(x^2+8)\cdot\sin\left(\frac{\pi}{x}\right) \geq \sum_{cyc}\frac{2x^2+16}{x}$$

We have to prove that:

$$\sum_{cyc}\frac{2x^2+16}{x} \geq \sum_{cyc}(x+6) \Leftrightarrow \frac{2x^2+16}{x} \geq x+6 \Leftrightarrow 2x^2+16-x^2-6x \geq 0$$

$\Leftrightarrow x^2-6x+16 \geq 0 \Leftrightarrow (x-3)^2+7 \geq 0$, which is true for all $x > 2$.

Therefore,

$$\sum_{cyc}(x^2+8)\cdot\sin\left(\frac{\pi}{x}\right) \geq \sum_{cyc}\frac{2x^2+16}{x} \geq \sum_{cyc}(x+6) \geq 3\sum_{cyc}\log(9+x^2) \geq$$

$$\geq \log\left(\frac{(3+x)^2(3+y)^2(3+z)^2}{8}\right)^3$$

Solution 2 by proposer

If $x > 2$, then $\frac{\pi}{x} < \frac{\pi}{2}, \tan\frac{\pi}{x} > \frac{\pi}{x} > \frac{3}{x}$,

$$\cos^2\frac{\pi}{x} = \frac{1}{1+\tan^2\frac{\pi}{x}} < \frac{1}{1+\left(\frac{\pi}{x}\right)^2} < \frac{1}{1+\left(\frac{3}{x}\right)^2} = \frac{x^2}{x^2+9}$$

$$\Leftrightarrow \sin\frac{\pi}{x} > \frac{3}{\sqrt{x^2+9}}; (1)$$

$$\log(1+t) \leq \frac{t}{\sqrt{1+t}}, \forall t \geq 0; (2)$$

$$t = x^2 + 8; (2) \Rightarrow \log(9 + x^2) \le \frac{8 + x^2}{\sqrt{9 + x^2}} \Leftrightarrow \frac{\log(9 + x^2)}{8 + x^2} \le \frac{1}{\sqrt{9 + x^2}}; (3)$$

From $(1), (2), (3)$ it follows that:

$$\frac{\log(9 + x^2)}{8 + x^2} \le \frac{1}{\sqrt{9 + x^2}} \le \frac{1}{3}\sin\frac{\pi}{x} \Leftrightarrow \log(9 + x^2) \le \frac{1}{3}(x^2 + 8)\sin\frac{\pi}{x}$$

$$\log\left(\frac{(3 + x)^2}{2}\right) \le \frac{1}{3}(x^2 + 8)\sin\frac{\pi}{x}$$

Therefore,

$$\log\left(\frac{(3 + x)^2(3 + y)^2(3 + z)^2}{8}\right)^3 \le \sum_{cyc}(x^2 + 8)\sin\frac{\pi}{x}$$

698. **Prove without any software:**

$$\cos^{-1}\left(\frac{1}{\pi^2}\right) - \cos^{-1}\left(\frac{1}{e^2}\right) > \frac{\pi}{2k}\left(e^{2k} - \pi^{2k}\right), k \in \mathbb{N} - \{0\}$$

Proposed by Rovsen Pirguliyev, Alisa Pirguliyeva-Sumgait-Azerbaijan

Solution by Adrian Popa-Romania

$$f(x) = \cos^{-1}x \Rightarrow f \searrow \text{ and } e < \pi \Rightarrow \frac{1}{e^2} > \frac{1}{\pi^2} \Rightarrow \cos^{-1}\left(\frac{1}{e^2}\right) < \cos^{-1}\left(\frac{1}{\pi^2}\right)$$

$$\cos^{-1}\left(\frac{1}{\pi^2}\right) - \cos^{-1}\left(\frac{1}{e^2}\right) > 0; (1)$$

$$e^{2k} < \pi^{2k}; \forall k \in \mathbb{N} - \{0\} \Rightarrow e^{2k} - \pi^{2k} < 0 \Rightarrow \frac{\pi}{2k}\left(e^{2k} - \pi^{2k}\right) < 0; (2)$$

From (1),(2) it follows that:

$$\cos^{-1}\left(\frac{1}{\pi^2}\right) - \cos^{-1}\left(\frac{1}{e^2}\right) > \frac{\pi}{2k}\left(e^{2k} - \pi^{2k}\right), k \in \mathbb{N} - \{0\}$$

699. **Let** $A = x^2 - xy + y^2, B = 53x^3 + 17x^2y - 56xy^2 + 53y^3,$

$$C = 101x^4 + 146x^3y - 111x^2y^2 - 73xy^3 + 101y^4,$$

$$D = 31x^5 + 48x^4y + 33x^3y^2 - 88x^2y^3 - xy^4 + 31y^5.$$

Prove that : $Ax + By^2 + Cx^2 + Dy^3 \ge 0, \forall x, y \ge 0.$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: A = x^2 + y^2 - xy \overset{AM-GM}{\geq} 2xy - xy = xy \geq 0.$$

$$x^3 + y^3 + y^3 \overset{AM-GM}{\geq} 3xy^2 \rightarrow 19(x^3 + 2y^3) \geq 57xy^2 \geq 56xy^2 \rightarrow B \geq 0.$$

$$Also, 100x^4 + 36y^4 \overset{AM-GM}{\geq} 2\sqrt{100x^4.36y^4} = 120x^2y^2 \geq 111x^2y^2$$

$$And\ x^3y + y^4 + y^4 \geq 3\sqrt[3]{x^3y.y^4.y^4} = 3xy^3 \rightarrow 25(x^3y + 2y^4) \geq 75xy^3 \geq 73xy^3$$

$$\rightarrow (100x^4 + 36y^4) + 25(x^3y + 2y^4) = 100x^4 + 25x^3y + 86y^4 \geq 111x^2y^2 + 73xy^3$$

$$\rightarrow C \geq 0.$$

$$Now, we\ have: x^5 + x^3y^2 \overset{AM-GM}{\geq} 2\sqrt{x^5.x^3y^2} = 2x^4y \rightarrow 31(x^5 + x^3y^2) \geq 62x^4y\ (1)$$

$$And: 100x^4y + 25y^5 \overset{AM-GM}{\geq} 2\sqrt{100x^4y.25y^5}$$

$$= 100x^2y^3\ (2)\ and\ x^2y^3 + y^5 \overset{AM-GM}{\geq} 2xy^4\ (3)$$

$$\rightarrow D \geq 31(x^5 + x^3y^2) + 48x^4y - 88x^2y^3 - xy^4 + 31y^5 \overset{(1)}{\geq} (100x^4y + 25y^5)$$

$$-88x^2y^3 - xy^4 + 6y^5 \geq$$

$$\overset{(2)}{\geq} x^2y^3 + y^5 - xy^4 \overset{(3)}{\geq} xy^4 \geq 0 \rightarrow D \geq 0.$$

$$A, B, C, D \geq 0 \rightarrow Ax + By^2 + Cx^2 + Dy^3 \geq 0, \forall x, y \geq 0.$$

700. For $\forall m, n \in \mathbb{N}$ prove that: $\frac{H_n}{n} + \frac{H_m}{m} \leq \frac{m+2}{m+1} + \frac{\psi}{n\cdot m}$, where H_n – is n^{th} harmonic number and

$$\psi = \sum_{k=1}^{m} \frac{\zeta_n(k+1)}{k+1}; \zeta_n(s) = \sum_{k=1}^{n} \frac{1}{k^s}$$

Proposed by Amrit Awasthi-India

Solution by proposer

We start with a little rearrangement

$$\frac{1}{k} = \sqrt[r]{1\cdot 1\cdot 1\ldots\cdot 1\cdot\frac{1}{k}} \overset{AM-GM}{\leq} \frac{1+1+1+\cdots+1+\frac{1}{k^r}}{r} = \frac{r-1}{r} + \frac{1}{r\cdot k^r}$$

Here, there are $(r-1)$ 1's in the root and $r \in \mathbb{N}$ and $r \geq 2$.

$$\sum_{k=1}^{n}\frac{1}{k}\leq\sum_{k=1}^{n}\frac{r-1}{r}+\sum_{k=1}^{n}\frac{1}{r\cdot k^{r}}$$

$$H_n\leq n\left(1-\frac{1}{r}\right)+\frac{\zeta_n(r)}{r}$$

$$\sum_{r=2}^{m+1}H_n\leq\sum_{r=2}^{m+1}n\left(1-\frac{1}{r}\right)+\sum_{r=2}^{m+1}\frac{\zeta_n(r)}{r}$$

$$m\cdot H_n+n\cdot H_m\leq n\cdot\frac{(m+1)^2-1}{m+1}+\psi=n\cdot m\left(\frac{m+2}{m+1}\right)+\psi$$

Therefore,

$$\frac{H_n}{n}+\frac{H_m}{m}\leq\frac{m+2}{m+1}+\frac{\psi}{n\cdot m}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru