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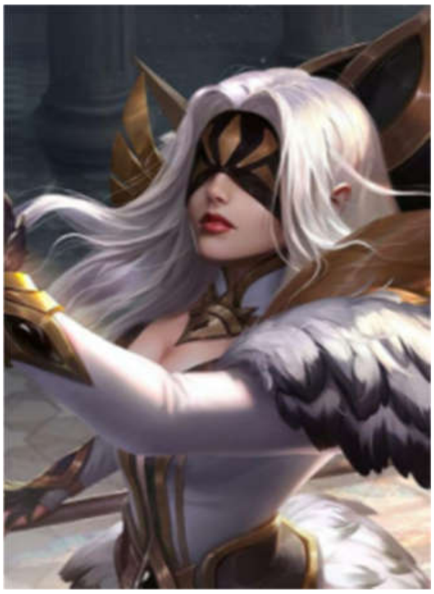

If we define the sequence $\{M_k(n)\}_{n\in N}$ with recurrence relation :

$$M_k(n+1) = k.M_k(n) + M_k(n-1), \forall n \in N^* \text{ with } k \neq 0,$$

$$M_k(0) = 0 \text{ and } M_k(1) = 1.$$

$$\text{Prove that : } \sum_{r=0}^{n} {M_k}^2(r) = \frac{M_k(n+1)\cdot M_k(n)}{k}$$

Proposed by Amrit Awasthi-India

Solution 1 by Kamel Gandouli Rezgui-Tunisia, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution 1 by Kamel Gandouli Rezgui-Tunisia

$$M_k(n+1) = kM_k(n) + M_k(n-1), M_k(0) = 0, M_k(1) = 1; n, k \in \mathbb{N}$$

$$\Rightarrow M_k(n+2) = k\cdot M_k(n+1) + M_k(n)$$

$$r^2 = kr + 1 \Rightarrow r^2 - kr - 1 = 0; \Delta = k^2 + 4 > 0 \text{ and let } \delta = \sqrt{k^2+4}$$

$$\alpha = \frac{k-\delta}{2}, \beta = \frac{k+\delta}{2} \Rightarrow \alpha\beta = -1 \Rightarrow \beta = -\frac{1}{\alpha}$$

$$\Rightarrow \alpha + \beta = k \Rightarrow \alpha - \frac{1}{\alpha} = k \Rightarrow \alpha^2 - 1 = \alpha k$$

$$\Rightarrow M_k(n) = a\alpha^n + b\beta^n, M_k(0) = 0 \Rightarrow a + b = 0 \Rightarrow a = -b$$

$$M_k(1)=1\Rightarrow a\alpha-a\beta=1\Rightarrow a=-\frac{1}{\delta}$$

$$\Rightarrow M_k(n)=-\frac{1}{\delta}\alpha^n+\frac{1}{\delta}\beta^n=\frac{1}{\delta}\left(\left(-\frac{1}{\alpha}\right)^n-\alpha^n\right)=\frac{1}{\delta}((-1)^n\alpha^{-n}-\alpha^n);\ (*)$$

$$M_k^2(n)=\frac{1}{\delta^2}\left(\left(\frac{1}{\alpha^2}\right)^n+\alpha^{2n}-2(-1)^n\right)$$

$$\Rightarrow\sum_{r=0}^{n}M_k^2(r)=\frac{1}{\delta^2}\left[\frac{1-\left(\frac{1}{\alpha^2}\right)^{n+1}}{1-\frac{1}{\alpha^2}}+\frac{1-\alpha^{2n+2}}{1-\alpha^2}-(1+(-1)^n)\right]=$$

$$=\frac{1}{\delta^2}\left[\frac{\alpha^2-\alpha^2\left(\frac{1}{\alpha^2}\right)^{n+1}}{\alpha^2-1}-\frac{1-\alpha^{2n+2}}{\alpha^2-1}-1+(-1)^{n+1}\right]\overset{\alpha^2-1=\alpha k}{=}$$

$$=\frac{1}{\delta^2}\left[\frac{\alpha^2-\alpha^{-2n}}{\alpha k}-\frac{1-\alpha^{2n+2}}{\alpha k}-\frac{\alpha^2-1}{\alpha k}+\frac{(-1)^{n+1}(\alpha^2-1)}{\alpha k}\right]=$$

$$=\frac{1}{\delta^2 k}(-\alpha^{-2n-1}+\alpha^{2n+1}+(-1)^{n+1}\alpha^{-1})$$

$$M_k(n)\cdot M_k(n+1)=\frac{1}{\delta^2}((-1)^n\alpha^{-n}-\alpha^n)((-1)^{n+1}\alpha^{-n-1}-\alpha^{n+1})=$$

$$=\frac{1}{\delta^2}(-\alpha^{-2n-1}+(-1)^{n+1}\alpha-(-1)^{n+1}\alpha^{-1}+\alpha^{2n+1})=$$

$$\frac{1}{k}\sum_{r=0}^{n}M_k^2(r)$$

Therefore,

$$\sum_{r=0}^{n}{M_k}^2(r)=\frac{M_k(n+1)\cdot M_k(n)}{k}.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$M_k(n+1)=k.M_k(n)+M_k(n-1)\ \leftrightarrow\ M_k(n)=\frac{1}{k}[M_k(n+1)-M_k(n-1)]$$

$$\rightarrow\ {M_k}^2(n)=\frac{1}{k}[M_k(n+1).M_k(n)-M_k(n).M_k(n-1)],\forall n\in N^*$$

$$\rightarrow \sum_{r=0}^{n} M_k{}^2(r) = \sum_{r=1}^{n} M_k{}^2(r) = \frac{1}{k}.\sum_{r=1}^{n}[M_k(r+1).M_k(r) - M_k(r).M_k(r-1)] =$$

$$= \frac{M_k(n+1).M_k(n) - M_k(1).M_k(0)}{k} = \frac{M_k(n+1).M_k(n)}{k}.$$

$$\textit{Therefore,} \quad \sum_{r=0}^{n} M_k{}^2(r) = \frac{M_k(n+1).M_k(n)}{k}.$$