When is a Number
'A METALLIC NUMBER' !??

-Amrit Awasthi

amrit28awasthi@gmail.com

Abstract

American mathematician Ira Martin Gessel showed that a natural number 'n' is a Fibonacci number if and only $5n^2 \pm 4 = k^2$ for some $k \in N$. That is either $5n^2 + 4$ or $5n^2 - 4$ is a perfect square.

In this paper we'll proceed through similar steps and generalise this theorem for a number to be a metallic number. In order to get the desired result we'll first define these metallic sequence and also generalise and prove few properties exhibited by these metallic numbers analogues to Fibonacci numbers.

Introduction

It is known that the positive roots of the quadratic equation

$$x^2 - kx - 1 = 0.........(i)$$

is called metallic ratio , where $k \in N$.

lets solve our quadratic equation first

$$x^2 - kx - 1 = 0$$

Applying quadratic formula we get

$$x = \frac{k \pm \sqrt{k^2 + 4}}{2}$$

let's call the positive root as Δ_k

$$\Delta_k = \frac{k + \sqrt{k^2 + 4}}{2}$$

For k=1 we have the golden ratio

$$\Delta_1 = \frac{1 + \sqrt{5}}{4} = \phi$$

For k=2 we have the silver ratio

$$\Delta_2 = 1 + \sqrt{2}$$

It is also known the continued fraction of kth metallic ratio is of the form

$$\Delta_k = k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cdots \frac{\cdots + \cdots}{\cdots}}}}}}$$

Clearly for k=1 we get the continued fraction of golden ratio as

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots \frac{1}{1 + \cdots}}}}}}$$

Also follow that the ratio of two consecutive Fibonacci numbers that is the ratio of f_{n+1} to f_n converges to golden ratio as n approaches infinity.
Evidently ,

$$\frac{f_{n+1}}{f_n} = \frac{f_n + f_{n-1}}{f_n}$$

$$= 1 + \frac{f_{n-1}}{f_n} = 1 + \cfrac{1}{\frac{f_n}{f_{n-1}}}$$

$$= 1 + \cfrac{1}{1 + \frac{1}{\frac{f_{n-1}}{f_{n-2}}}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1 + \cdots}}}}$$

So we'll start by forming sequence $M_k(n)$ such that the ratios of two consecutive terms converge to kth metallic ratio.
Observe that this sequence should have recursive definition as

$$M_k(n) = k \cdot M_k(n-1) + M_k(n-2)$$

because then we'll have the ratio as

$$\frac{M_k(n+1)}{M_k(n)} = \frac{k \cdot M_k(n) + M_k(n-1)}{M_k(n)} = k + \frac{M_k(n-1)}{M_k(n)} = k + \cfrac{1}{\frac{M_k(n)}{M_k(n)}}$$

$$= k + \cfrac{1}{k + \frac{1}{\frac{M_k(n-1)}{M_k(n-2)}}} = k + \cfrac{1}{k + \frac{1}{k + \cdots}}$$

2

Clearly this is the continued fraction of our metallic ratio
In order to have a direct analogy with Fibonacci numbers we specify the start value as

$$M_k(0) = 0; M_k(1) = 1$$

<u>Note :</u> It is to be noted that $M_k(n)$ refers to n^{th} term of kth order Metallic sequence.By this we mean the nth term of the sequence defined by the recursive relation and start value as mentioned above. This notation will be used throughout this paper
Now ,We're ready to derive some generalisations now

<u>Important Results</u>

<u>Result I : Relation for higher powers</u>

If

$$x^2 = kx + 1$$

then $\forall n \in N$

$$x^n = M_k(n)x + M_k(n-1)$$

where $M_k(n)$ is the nth term of kth order Metallic sequence
<u>PROOF :</u>
We prove the above claim using principle of mathematical induction
<u>Step 1: Result holds for n=1</u>
for n=1 , we get

$$LHS = x^1 = x$$

$$RHS = M_k(1)x + M_k(0) = x$$

$$LHS = RHS$$

The result hold for n=1
<u>Step 2 : Assuming result is true for n</u>
let's assume that the result hold for some natural number n that is

$$x^n = M_k(n)x + M_k(n-1)$$

<u>Step 3 : To prove that result is true for n+1</u>

$$x^{n+1} = x \cdot x^n = x(M_k(n)x + M_k(n-1)) = M_k(n)x^2 + M_k(n-1)x = M_k(n)(kx+1) + M_k(n-1)x$$

$$= (k \cdot M_k(n) + M_k(n-1))x + M_k(n) = M_k(n+1)x + M_k$$

Hence the result holds $\forall n \in N$

Result II:(Binet like formula)

$$M_k(n) = \frac{(\Delta_k)^n - (k - \Delta_k)^n}{\sqrt{k^2 + 4}}$$

<u>PROOF</u> As the only two values that satisfy equation (i) are Δ_k and

$$\overline{\Delta_k} = k - \Delta_k$$

Using 'Result I' we get

$$(\Delta_k)^n = M_k(n)\Delta_k + M_k(n - 1)$$

$$(k - \Delta_k)^n = (k - \Delta_k)M_k(n) + M_k(n - 1)$$

$$\rightarrow (\Delta_k)^n - (k - \Delta_k)^n = (2\Delta_k - k)M_k(n) = (2 \cdot \frac{k + \sqrt{k^2 + 4}}{2} - k)M_k(n)$$

Rearranging we get

$$M_k(n) = \frac{(\Delta_k)^n - (k - \Delta_k)^n}{\sqrt{k^2 + 4}}$$

Example given for k=2 we get the silver sequence

$$0, 1, 2, 5, 12, 29, 70, 169, .., ..$$

Also the ratio of the consecutive terms converges to $\sqrt{2} + 1$ which is known as silver ratio.

Result III : Cassini like identity

$$\rightarrow M_k(n + 1) \cdot M_k(n - 1) - M_k^2(n) = (-1)^n$$

<u>PROOF</u> Note that Cassini identity for Fibonacci number states that $\forall n \in N$

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n$$

Now here we produce a similar result for metallic numbers
Note that

$$M_k(n + 1) = k \cdot M_k(n) + M_k(n - 1)$$

and

$$M_k(n - 1) = \frac{M_k(n) - M_k(n - 2)}{k}$$

Therefore we have

$$M_k(n + 1) \cdot M_k(n - 1) = (k \cdot M_k(n) + M_k(n - 1)) \left(\frac{M_k(n) - M_k(n - 2)}{k} \right)$$

$$= M_k^2(n) - M_k(n - 2) \cdot M_k(n) + \frac{M_k(n - 1) \cdot M_k(n)}{k} - \frac{M_k(n - 1) \cdot M_k(n - 2)}{k}$$

4

Transposing and rearranging we get

$$M_k(n+1) \cdot M_k(n-1) - M_k^2(n) = -M_k(n-2) \cdot M_k(n) + M_k(n-1) \left(\frac{M_k(n) - M_k(n-2)}{k} \right)$$

$$M_k(n+1) \cdot M_k(n-1) - M_k^2(n) = -(M_k(n-2) \cdot M_k(n) - M_k^2(n-1))$$

If we take

$$g_n = M_k(n+1) \cdot M_k(n-1) - M_k^2(n)$$

As proved above then we have

$$g_n = -g_{n-1} = (-1)^2 g_{n-2} = (-1)^3 g_{n-3} = .. = .. = (-1)^{n-1} g_1$$

But $g_1 = M_k(0)M_k(2) - M_k^2(1) = -1$ Hence we have

$$g_n = (-1)^n \cdot (-1) = (-1)^{n+1}$$

which implies that

$$\rightarrow M_k(n+1) \cdot M_k(n-1) - M_k^2(n) = (-1)^n$$

Result IV : Sum of first 'n' terms of kth order metallic numbers

$$\sum_{i=1}^{n} M_k(i) = \frac{M_k(n+2) - (k-1) \cdot M_k(n+1) - 1}{k}$$

<u>PROOF</u> We first use the recursive relation

$$M_k(n-1) = M_k(n+1) - k \cdot M_k(n)$$

plugging in n=2,3,4,..n+1

$$M_k(1) = M_k(3) - k \cdot M_k(2)$$
$$M_k(2) = M_k(4) - k \cdot M_k(3)$$
$$M_k(3) = M_k(5) - k \cdot M_k(4)$$
$$.......$$
$$........$$
$$.........$$
$$M_k(n) = M_k(n+2) - k \cdot M_k(n+1)$$

Adding vertically and manipulating we get

$$\sum_{i=1}^{n} M_k(i) = M_k(n+2) - (k-1) \cdot M_k(n+1) - 1 + (1-k) \sum_{i=1}^{n} M_k(i)$$

Finally transposing and dividing by k we get

$$\sum_{i=1}^{n} M_k(i) = \frac{M_k(n+2) - (k-1) \cdot M_k(n+1) - 1}{k}$$

Now as we're done with our important results we're ready to answer our main question.

When is a number Metallic Number !!?

Firstly we prove two lemmas

Lemma 1 :

The equation
$$y^2 - kxy - x^2 = \pm 1$$
is satisfied by $(x, y) = (M_k(n), M_k(n+1))$, $\forall n \geq 1$ k,x,y $\in$ N.

Proof using induction on n

Step 1 : Result holds for n=1

$$LHS = M_k^2(1) - k \cdot M_k(0) \cdot M_k(1) - M_k^2(0) = 1 - 0 - 0 = 1 = RHS$$

Result holds for n=1

Step 2 : Assuming result is true for n

let's assume that result holds true for some natural number 'n' that is

$$M_k^2(n) - k \cdot M_k(n) \cdot M_k(n-1) - M_k^2(n-1) = \pm 1$$

Step 3 : To prove the result holds for n+1

$$M_k^2(n+1) - k \cdot M_k(n+1) \cdot M_k(n) - M_k^2(n)$$

$$= (k \cdot M_k(n) + M_k(n-1))^2 - k(k \cdot M_k(n) + M_k(n-1)) \cdot M_k(n) - M_k^2(n)$$

$$= k^2 M_k^2(n) + M_k^2(n-1) + 2k M_k(n) \cdot M_k(n-1) - k^2 M_k^2(n) - k \cdot M_k(n) M_k(n-1) - M_k^2(n)$$

$$= M_k^2(n-1) + k \cdot M_k(n) M_k(n-1) - M_k^2(n)$$

$$= - \underbrace{(M_k^2(n) - k \cdot M_k(n) M_k(n-1) - M_k^2(n-1))}_{=\pm 1}$$

$$\rightarrow M_k^2(n+1) - k \cdot M_k(n+1) \cdot M_k(n) - M_k^2(n) = \mp 1$$

Hence the result holds for n+1.

Therefore by principle of mathematical induction the result is true $\forall n \in N$

Note :

We can prove it using Cassini like identity proved earlier with some little manipulations.

However for the sake of simplicity we've proved it using Principle of mathematical induction

Lemma 2 :

If (x, y) is a pair of positive integers satisfying the equation

$$y^2 - kxy - x^2 = \pm 1$$

then $(x, y) = (M_k(n), M_k(n+1))$ for some $n \geq 1$.

Proof : By induction on (x + y)

Firstly, we note that since x and y are positive, it must be that $x \leq y$, since if $x > y$ then the equation becomes less than -1

Secondly, if x = y then substituting y for x we obtain

$$y^2 - ky^2 - y^2 = -ky^2 = \pm 1$$

hence setting
$$(x, y) = (M_1(1), M_1(2) = (f_1, f_2)$$
, satisfies this equation.
Finally assuming that $1 \leq x < y$ and that

$$y^2 - kxy - x^2 = \pm 1$$

We have that the pair of positive integers

$$(a, b) = (y - kx, x)$$

also satisfies the equation $y^2 - kxy - x^2 = \pm 1$ since

$$b^2 - kab - b^2 = x^2 - k(y - kx)x - (y - kx)^2$$

$$= x^2 - kxy + k^2x^2 - y^2 - k^2x^2 + 2kxy$$

$$= x^2 + kxy - y^2 = -(y^2 - kxy - x^2) = \pm 1$$

Therefore, by induction

$$(a, b) = (M_k(n), M_k(n+1))$$

for some n, and hence
$$x = b = M_k(n+1)$$

and
$$y = a + kx = kM_k(n+1) + M_k(n)$$

meaning
$$(x, y) = (M_k(n+1), M_k(n+2))$$

Hence the result holds true $\forall n \in N$

7

Theorem 1

A positive integer 'n' is a kth order metallic number if and only if either

$$(k^2 + 4)n^2 + 4 = \lambda^2$$

or

$$(k^2 + 4)n^2 - 4 = \lambda^2$$

for some positive integer λ

Proof :

We first start with rewriting the equation

$$y^2 - kxy - x^2 \pm 1 = 0$$

Solving for y by applying quadratic formula we get

$$y = \frac{kx \pm \sqrt{k^2x^2 + 4(x^2 \pm 1)}}{2}$$

which on simplification gives

$$y = \frac{kx \pm \sqrt{(k^2 + 4)x^2 \pm 4}}{2}$$

Assuming y to be positive we can write

$$y = \frac{kx + \sqrt{(k^2 + 4)x^2 \pm 4}}{2}$$

Now
•Assuming x is a kth order metallic number $M_k(n)$ for some n, then by lemma 1, y must also be a Metallic number of kth order namely $M_k(n + 1)$. Thus for y to be a Metallic number (and hence an integer) $\sqrt{(k^2 + 4)x^2 \pm 4}$ must be an integer. Thus $\sqrt{(k^2 + 4)x^2 \pm 4}$ must be a perfect square.

•Conversely , Assuming $\sqrt{(k^2 + 4)x^2 \pm 4}$ be a perfect square and 'x' to be positive integer we have two cases

CASE 1: 'k' is even positive integer

then $k^2 + 4$ is also even hence two sub-cases arise.
SUB-CASE 1 → 'x' is even

$$\rightarrow s = kx + \sqrt{(k^2 + 4)x^2 \pm 4} = even + \sqrt{(even)(even) \pm (even)} = even$$

8

$\rightarrow y = \frac{1}{2} \cdot s$ is an integer

SUB-CASE 2 $\rightarrow$ 'x' is odd

$$\rightarrow s = kx + \sqrt{(k^2+4)x^2 \pm 4} = even + \sqrt{(even)(odd) \pm 4} = even$$

Hence again we get

$$y = \frac{1}{2} \cdot s \text{ is an integer}$$

CASE 2 : 'k' is odd positive integer

then $k^2 + 4$ is odd.Hence two sub-cases arise

SUB-CASE 1 $\rightarrow$ 'x' is odd

$$\rightarrow s = kx + \sqrt{(k^2+4)x^2 \pm 4} = odd + \sqrt{(odd)(odd) \pm 4} = odd + odd = even$$

Therefore we have

$$y = \frac{1}{2} \cdot s \text{ is a positive integer}$$

SUB-CASE 2$\rightarrow$ 'x' is even

$$\rightarrow s = kx + \sqrt{(k^2+4)x^2 \pm 4} = even + \sqrt{(odd)(even) \pm 4} = even + even = even$$

Hence again we get

$$y = \frac{1}{2} \cdot s \text{ is a positive integer}$$

Clearly if 'x' is a positive integer such that $\sqrt{(k^2+4)x^2 \pm 4}$ is a perfect square then we get 'y' to be a positive integer
Hence by lemma 2 both x and y must be metallic numbers

Therefore we conclude that a natural number 'n' will be a metallic number if and only if

$$(k^2+4)n^2 \pm 4 = \lambda^2$$

for some $\lambda \in N$

Corollary

for k=1 we get the gessel's theorem that is
' A natural number will be a Fibonacci number if and only if $5n^2 \pm 4$ is a perfect square

Also if we plug in k=2
then

$$(2^2 + 4)n^2 \pm 4 = \xi^2$$

$$8n^2 \pm 4 = 4(2n^2 \pm 1) = \xi^2$$

Replacing ξ with 2λ we get

$$4(2n^2 \pm 1) = 4\lambda^2$$

Cancelling out gives

$$2n^2 \pm 1 = \lambda^2$$

Hence a natural number 'n' is a 'silver number' if and only if $2n^2 \pm 1$ is a perfect square
Example given

$$n = 5 \to 2(5^2) - 1 = 7^2$$

$$n = 12 \to 2(12^2) + 1 = 289 = 17^2$$

Summary

To conclude, we have generalized gessel's theorem to metallic numbers while discussing and going through various properties exhibited by these numbers and sequences.

References

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AMRIT AWASTHI
Punjab , India
amrit28awasthi@gmail.com