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FEW OUTSTANDING LIMITS-(IV)

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Abstract: In this paper it was presented few outstanding limits using special sums.

Application 1. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \sum_{k=1}^n \sqrt{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt{k}} - \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i} - \sqrt{j})^2}{\sqrt{ij}} \right)$$

Daniel Sitaru

Solution.

$$\begin{aligned} \sum_{k=1}^n \sqrt{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt{k}} &= \sum_{k=1}^n \left(\sqrt{k} \cdot \frac{1}{\sqrt{k}} \right) + \frac{1}{\sqrt{1}} (\sqrt{2} + \dots + \sqrt{n}) + \frac{1}{\sqrt{2}} (\sqrt{3} + \dots + \sqrt{n}) + \dots \\ &\quad + \dots + \frac{1}{\sqrt{n-1}} \cdot \sqrt{n} + \sqrt{1} \left(\frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} \right) + \sqrt{2} \left(\frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \right) + \dots \\ &\quad + \dots + \sqrt{n-1} \cdot \frac{1}{\sqrt{n}} = n + \sum_{1 \leq i < j \leq n} \sqrt{\frac{i}{j}} + \sum_{1 \leq i < j \leq n} \sqrt{\frac{j}{i}} = \\ &= n + \sum_{1 \leq i < j \leq n} \left(\sqrt{\frac{i}{j}} + \sqrt{\frac{j}{i}} - 2 + 2 \right) = n + \sum_{1 \leq i < j \leq n} \left(\frac{i+j-2\sqrt{ij}}{\sqrt{ij}} + 2 \right) = \\ &= n + \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i} - \sqrt{j})^2}{\sqrt{ij}} + 2 \cdot \frac{n(n-1)}{2} = n^2 + \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i} - \sqrt{j})^2}{\sqrt{ij}} \end{aligned}$$

Hence,

$$\sum_{k=1}^n \sqrt{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt{k}} = n^2 + \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i} - \sqrt{j})^2}{\sqrt{ij}}$$

And then

$$\frac{1}{n^2} \sum_{k=1}^n \sqrt{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt{k}} - \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i} - \sqrt{j})^2}{\sqrt{ij}} = 1$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} \sum_{k=1}^n \sqrt{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt{k}} - \sum_{1 \leq i < j \leq n} \frac{(\sqrt{i} - \sqrt{j})^2}{\sqrt{ij}} \right) = 1$$



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Application 2. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left(\frac{\sin n}{n^3} \left(\sum_{k=1}^n \sqrt[3]{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt[3]{k}} - \sum_{1 \leq i < j \leq n} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} \right) \right)$$

Daniel Sitaru

Solution.

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} &= - \sum_{i=1}^n \sum_{j=1}^n (\sqrt[3]{i} - \sqrt[3]{j}) \left(\frac{1}{\sqrt[3]{i}} - \frac{1}{\sqrt[3]{j}} \right) = \\ &= - \sum_{i=1}^n \sum_{j=1}^n \left(1 + 1 - \sqrt[3]{\frac{i}{j}} - \sqrt[3]{\frac{j}{i}} \right) = - \sum_{i=1}^n \sum_{j=1}^n 2 + \sum_{i=1}^n \sum_{j=1}^n \sqrt[3]{\frac{j}{i}} = \\ &= -2n^2 + \sum_{i=1}^n \sum_{j=1}^n \sqrt[3]{\frac{i}{j}} + \sum_{i=1}^n \sum_{j=1}^n \sqrt[3]{\frac{j}{i}} = -2n^2 + 2 \left(\sum_{k=1}^n \sqrt[3]{k} \right) \left(\sum_{k=1}^n \frac{1}{\sqrt[3]{k}} \right) \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} &= -2n^2 + 2n^2 \left(\frac{1}{n^2} \sum_{k=1}^n \sqrt[3]{k} \right) \left(\sum_{k=1}^n \frac{1}{\sqrt[3]{k}} \right) \\ 2 \sum_{1 \leq i < j \leq n} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} &= -2n^2 + 2n^2 \left(\frac{1}{n^2} \sum_{k=1}^n \sqrt[3]{k} \right) \left(\sum_{k=1}^n \frac{1}{\sqrt[3]{k}} \right) \end{aligned}$$

So, it follows that

$$\frac{1}{n^2} \left(\sum_{k=1}^n \sqrt[3]{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt[3]{k}} - \sum_{1 \leq i < j \leq n} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} \right) = 1; \quad (1)$$

Therefore,

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left(\frac{\sin n}{n^3} \left(\sum_{k=1}^n \sqrt[3]{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt[3]{k}} - \sum_{1 \leq i < j \leq n} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} \right) \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{\sin n}{n} \left(\frac{1}{n^2} \sum_{k=1}^n \sqrt[3]{k} \cdot \sum_{k=1}^n \frac{1}{\sqrt[3]{k}} - \sum_{1 \leq i < j \leq n} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} \right) \right) \stackrel{(1)}{=} \lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0. \end{aligned}$$



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Application 3. Find:

$$\Omega = \lim_{n \rightarrow \infty} \frac{\log n}{n^2} \sum_{p=1}^n \left(\frac{1}{p^2} \sum_{k=1}^p \sqrt[3]{k} \cdot \sum_{k=1}^p \frac{1}{\sqrt[3]{k}} - \sum_{1 \leq i < j \leq p} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} \right)$$

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Solution.

Using (1) we have:

$$\frac{1}{p^2} \sum_{k=1}^p \sqrt[3]{k} \cdot \sum_{k=1}^p \frac{1}{\sqrt[3]{k}} - \sum_{1 \leq i < j \leq p} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} = 1$$

Hence,

$$\sum_{p=1}^n \left(\frac{1}{p^2} \sum_{k=1}^p \sqrt[3]{k} \cdot \sum_{k=1}^p \frac{1}{\sqrt[3]{k}} - \sum_{1 \leq i < j \leq p} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} \right) = n; \quad (2)$$

Therefore,

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \frac{\log n}{n^2} \sum_{p=1}^n \left(\frac{1}{p^2} \sum_{k=1}^p \sqrt[3]{k} \cdot \sum_{k=1}^p \frac{1}{\sqrt[3]{k}} - \sum_{1 \leq i < j \leq p} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} \right) \stackrel{(2)}{=} \\ &= \lim_{n \rightarrow \infty} \frac{\log n}{n^2} \cdot n = \lim_{n \rightarrow \infty} \frac{\log n}{n} \stackrel{\text{Stolz-C}}{=} 0. \end{aligned}$$

Application 4. Find:

$$\Omega = \lim_{m \rightarrow \infty} \frac{1}{m^2} \left(\lim_{n \rightarrow \infty} \sum_{n=1}^m \sum_{i=1}^n \sin \frac{(2i-1)a}{n^2} \right)$$

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Solution.

For $a > 0$. We prove that:

$$\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \sin \frac{(2i-1)a}{n^2} \right) = a$$

Using the well-known inequality: $x - \frac{x^3}{6} < \sin x < x, \forall x > 0 \Rightarrow$

$$\frac{a}{n^2} - \frac{1}{6} \cdot \frac{a^3}{n^6} < \sin \frac{a}{n^2} < \frac{a}{n^2}$$



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$$\frac{3a}{n^2} - \frac{1}{6} \cdot \frac{3^3 a^3}{n^6} < \sin \frac{3a}{n^2} < \frac{3a}{n^2}$$

$$\frac{(2n-1)a}{n^2} - \frac{1}{6} \cdot \frac{(2n-1)^3 a^3}{n^6} < \sin \frac{(2n-1)a}{n^2} < \frac{(2n-1)a}{n^2}$$

Summing, we get:

$$\frac{a}{n^2} \cdot \sum_{i=1}^n (2i-1) - \frac{a^3}{n^6} \cdot \sum_{i=1}^n (2i-1)^3 < \sum_{i=1}^n \sin \frac{(2i-1)a}{n^2} < \frac{a}{n^2} \cdot \sum_{i=1}^n (2i-1)$$

Let us denote:

$$a_n = \frac{a}{n^2} \cdot \sum_{i=1}^n (2i-1), b_n = x_n - \frac{a^3}{n^6} \cdot \sum_{i=1}^n (2i-1)^3$$

$$\Rightarrow a_n = a \Rightarrow b_n = a - \frac{a^3}{n^6} \cdot \sum_{i=1}^n (2i-1)^3 = a - \frac{a^3}{n^4(2n^2-1)}$$

So, $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = a$, then

$$\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \sin \frac{(2i-1)a}{n^2} \right) = a$$

For $a = j \in \{1, 2, \dots, m\}$ it follows that:

$$\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \sin \frac{(2i-1)j}{n^2} \right) = j$$

Therefore,

$$\Omega = \lim_{m \rightarrow \infty} \frac{1}{m^2} \left(\lim_{n \rightarrow \infty} \sum_{i=1}^m \sum_{i=1}^n \sin \frac{(2i-1)j}{n^2} \right) = \lim_{m \rightarrow \infty} \frac{1}{m^2} \sum_{j=1}^m j = \frac{1}{2}.$$

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Application 5. Find:

$$\Omega = \lim_{m \rightarrow \infty} \frac{1}{m^2} \left(\lim_{n \rightarrow \infty} \left(\sum_{k=1}^m \sum_{i=1}^n \sin \frac{(2i-1)\sqrt{k}}{n^2} \cdot \sum_{k=1}^m \sum_{i=1}^n \sin \frac{2i-1}{n^2\sqrt{k}} \right) - \sum_{1 \leq i < j \leq m} \frac{(\sqrt{i} - \sqrt{j})^2}{\sqrt{ij}} \right)$$

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Solution.

Using Application 3, we have:

$$\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \sin \frac{(2i-1)a}{n^2} \right) = a$$

For $a = \sqrt{k}$ and $a = \frac{1}{\sqrt{k}}$, we get:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{(2i-1)\sqrt{k}}{n^2} = \sqrt{k} \text{ and } \lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{(2i-1)}{n^2\sqrt{k}} = \frac{1}{\sqrt{k}}$$

Hence,

$$\begin{aligned} \Omega &= \lim_{m \rightarrow \infty} \frac{1}{m^2} \left(\lim_{n \rightarrow \infty} \left(\sum_{k=1}^m \sum_{i=1}^n \sin \frac{(2i-1)\sqrt{j}}{n^2} \cdot \sum_{k=1}^m \sum_{i=1}^n \sin \frac{2i-1}{n^2\sqrt{j}} \right) - \sum_{1 \leq i < j \leq m} \frac{(\sqrt{i} - \sqrt{j})^2}{\sqrt{ij}} \right) = \\ &= \lim_{m \rightarrow \infty} \frac{1}{m^2} \left(\sum_{k=1}^m \sqrt{k} \cdot \sum_{k=1}^m \frac{1}{\sqrt{k}} - \sum_{1 \leq i < j \leq m} \frac{(\sqrt{i} - \sqrt{j})^2}{\sqrt{ij}} \right) \end{aligned}$$

Using Application 1, it follows that:

$$\Omega = \lim_{m \rightarrow \infty} \frac{1}{m^2} \left(\sum_{k=1}^m \sqrt{k} \cdot \sum_{k=1}^m \frac{1}{\sqrt{k}} - \sum_{1 \leq i < j \leq m} \frac{(\sqrt{i} - \sqrt{j})^2}{\sqrt{ij}} \right) = 1.$$

Application 6. Find:

$$\Omega = \lim_{m \rightarrow \infty} \frac{1}{m^3} \left(\lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \sum_{i=1}^n \sin \frac{(2i-1)\sqrt[3]{k}}{n^2} \cdot \sum_{k=1}^n \sum_{i=1}^n \sin \frac{2i-1}{n^3\sqrt[3]{k}} \right) - \sum_{1 \leq i < j \leq m} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} \right)$$

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Solution.

Using Application 3, we have:

$$\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \sin \frac{(2i-1)a}{n^2} \right) = a$$

For $a = \sqrt[3]{k}$ and $a = \frac{1}{\sqrt[3]{k}}$ it follows that:



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$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{(2i-1)\sqrt[3]{k}}{n^2} = \sqrt[3]{k} \text{ and } \lim_{n \rightarrow \infty} \sum_{i=1}^n \sin \frac{(2i-1)}{n^2 \sqrt[3]{k}} = \frac{1}{\sqrt[3]{k}}$$

Hence,

$$\begin{aligned}\Omega &= \lim_{m \rightarrow \infty} \frac{1}{m^3} \left(\lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \sum_{i=1}^n \sin \frac{(2i-1)\sqrt[3]{k}}{n^2} \cdot \sum_{k=1}^n \sum_{i=1}^n \sin \frac{2i-1}{n^2 \sqrt[3]{k}} \right) - \sum_{1 \leq i < j \leq m} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} \right) \\ &= \lim_{m \rightarrow \infty} \frac{1}{m^3} \left(\sum_{k=1}^m \sqrt[3]{k} \cdot \sum_{k=1}^m \frac{1}{\sqrt[3]{k}} - \sum_{1 \leq i < j \leq m} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} \right)\end{aligned}$$

Now, using Application 2, it follows that:

$$\Omega = \lim_{m \rightarrow \infty} \frac{1}{m^3} \left(\lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \sum_{i=1}^n \sin \frac{(2i-1)\sqrt[3]{k}}{n^2} \cdot \sum_{k=1}^n \sum_{i=1}^n \sin \frac{2i-1}{n^2 \sqrt[3]{k}} \right) - \sum_{1 \leq i < j \leq m} \frac{(\sqrt[3]{i} - \sqrt[3]{j})^2}{\sqrt[3]{ij}} \right) = 0$$

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References:

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