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ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-XIV

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1) If $x, y, z \geq 1$ then

$$\sqrt{x^4-1}+\sqrt{y^4-1}+\sqrt{z^4-1}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 3.$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution. Lemma.

2) If $x \geq 1$ then $\sqrt{x^4-1}+\frac{1}{x}\geq x$.

Proof. Inequality can be written as:

$$\sqrt{x^4-1}+\frac{1}{x}\geq x \Leftrightarrow x\sqrt{x^4-1}+1\geq x^2 \Leftrightarrow x\sqrt{x^4-1}\geq x^2-1$$

$\Leftrightarrow x^2(x^4-1)\geq(x^2-1)^2 \Leftrightarrow (x^2-1)(x^4+1)\geq 0$, true from $x\geq 1$.

Equality holds for $x=1$. Let's get back to the main problem.

Using Lemma, summing and from $x, y, z\geq 1$, we get the conclusion.

Equality holds for $x=y=z=1$. The problem can be developed.

3) If $x, y, z \geq 1$ and $n\in\mathbb{N}, n\geq 2$ then

$$\sqrt{x^n-1}+\sqrt{y^n-1}+\sqrt{z^n-1}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 3.$$

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Solution. Lemma.

4) If $x\geq 1$ and $n\in\mathbb{N}, n\geq 2$ then $\sqrt{x^n-1}+\frac{1}{x}\geq x$.

Proof. Inequality is equivalent to:

$$\sqrt{x^n-1}+\frac{1}{x}\geq x \Leftrightarrow x\sqrt{x^n-1}+1\geq x^2 \Leftrightarrow x\sqrt{x^n-1}\geq x^2-1 \Leftrightarrow$$

$$x^2(x^n-1)\geq(x^2-1)^2 \Leftrightarrow$$

$x^2(x-1)(x^{n-1}+x^{n-2}+\cdots+x^2+x+1) \geq (x-1)^2(x+1)^2$, which follows from

$x-1 \geq 0$ and $x^2(x^{n-1}+x^{n-2}+\cdots+x^2+x+1) > (x-1)(x+1)^2 \Leftrightarrow$

$$x^{n+1}+x^n+x^{n-1}+\cdots+x^4+x^3+x^2 > x^3+x^2-x-1 \Leftrightarrow$$

$x^{n+1}+x^n+x^{n-1}+\cdots+x^4+x+1 > 0$, true from $x \geq 1$.

Let's get back to the main problem. Using Lemma, summing and from $x, y, z \geq 1$, we get the conclusion. Equality holds for $x=y=z=1$.

Note: For $n=4$ we get the proposed problem by George Apostolopoulos in RMM 9/2020.

REFERENCE:

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