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ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-XIII

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1) If $a,b,c\in\mathbb{R}^*$ then

$$\frac{\sqrt{a^4+b^4}}{a^2+ab+b^2}+\frac{\sqrt{b^4+c^4}}{b^2+bc+c^2}+\frac{\sqrt{c^4+a^4}}{c^2+ca+a^2}\geq\sqrt{2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution. Inequality is equivalent to:

$$\sum_{cyc}\sqrt{\frac{a^4+b^4}{2(a^2+ab+b^2)}}\geq 1$$

***Lemma.*2) If $a,b\in\mathbb{R}$ then $9(a^4+b^4)\geq 2(a^2+ab+b^2)$**

Proof.

We have:

$$9(a^4+b^4)\geq 2(a^2+ab+b^2)\Leftrightarrow$$

$$7a^4-4a^3b-6a^2b^2-4ab^3+7b^4\geq 0\Leftrightarrow(a-b)^2(7a^2+10ab+7b^2)\geq 0$$

Equality holds for $a=b$.

Let's get back to the main problem. Using Lemma, we get:

$$\sum_{cyc}\sqrt{\frac{a^4+b^4}{2(a^2+ab+b^2)}}\overset{Lemma}{\geq}\sum_{cyc}\sqrt{\frac{1}{9}}=\sum_{cyc}\frac{1}{3}=1$$

Equality holds for $a=b=c$.

Remark: In same class of the problem.

3) If $a,b,c\in\mathbb{R}^*$ then:

$$\frac{\sqrt{a^4+b^4}}{a^2-ab+b^2}+\frac{\sqrt{b^4+c^4}}{b^2-bc+c^2}+\frac{\sqrt{c^4+a^4}}{c^2-ca+a^2}\leq 3\sqrt{2}$$

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Solution. Lemma. **4) If $a,b\in\mathbb{R}$ then $a^4+b^4\leq 2(a^2-ab+b^2)^2$.**

Proof.

We have $a^4 + b^4 \leq 2(a^2 - ab + b^2)^2 \Leftrightarrow a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \geq 0$

$\Leftrightarrow (a-b)^4 \geq 0$, true. Equality holds for $a = b$.

Let's get back to the main problem. Using Lemma, we get:

$$\sum_{cyc} \sqrt{\frac{a^4 + b^4}{2(a^2 - ab + b^2)}} \overset{Lemma}{\geq} \sum_{cyc} \sqrt{1} = 3$$

Equality holds for $a = b = c$.

REFERENCE:

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