R M M

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ABOUT A RMM INEQUALITY-XIX

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1) If $a,b,c>0$ such that $a+b+c=3$ then

$$\frac{a^3+b^3}{ab}+\frac{b^3+c^3}{bc}+\frac{c^3+a^3}{ca}\geq 6$$

Proposed by Carmen Chirfot-Romania

Solution. Lemma. **2) If $a,b>0$ then $\frac{a^3+b^3}{ab}\geq a+b$.**

Proof. We have: $\frac{a^3+b^3}{ab}\geq a+b \Leftrightarrow (a+b)(a-b)^2\geq 0$. Equality holds for $a=b$.

Solution 1. Let's get back to the main problem. Using Lemma, we get:

$$LHS=\sum_{cyc}\frac{b^3+c^3}{bc}\overset{Lemma}{\geq}\sum_{cyc}(b+c)=2\sum_{cyc}a=2\cdot 3=6.$$

Equality holds for $a=b=c=1$.

Solution 2. Using Holder Inequality, we have:

$$LHS=\sum_{cyc}\frac{b^3+c^3}{bc}=\sum_{cyc}\frac{b^2}{c}+\sum_{cyc}\frac{c^2}{b}\overset{Holder}{\geq}\frac{(\sum a)^2}{\sum a}+\frac{(\sum a)^2}{\sum a}=\sum_{cyc}a+\sum_{cyc}a=$$

$$=2\sum_{cyc}a=2\cdot 3=6$$

Equality holds for $a=b=c=1$.

Remark. The problem can be developed.

3) If $a,b,c>0$ such that $a+b+c=3$ and $\lambda\geq 0$ then

$$\frac{a^3+\lambda b^3}{ab}+\frac{b^3+\lambda c^3}{bc}+\frac{c^3+\lambda a^3}{ca}\geq 3(\lambda+1)$$

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Solution. Using Holder Inequality, we have:

$$LHS=\sum_{cyc}\frac{b^3+\lambda c^3}{bc}=\sum_{cyc}\frac{b^2}{c}+\lambda\sum_{cyc}\frac{c^2}{b}\overset{Holder}{\geq}\frac{(\sum a)^2}{\sum a}+\lambda\frac{(\sum a)^2}{\sum a}=\sum_{cyc}a+\lambda\sum_{cyc}a=$$

$$=(\lambda+1)\sum_{cyc}a=(\lambda+1)\cdot 3=3(\lambda+1)$$

Equality holds for $a=b=c=1$.

Note. For $\lambda=1$, we get Problem VIII.27, pag. 53, proposed by Carmen Chirfot, Romania in R.M.M. no.22.

Remark. The problem can be developed.

4) If $a,b,c>0$ such that $a+b+c=3$ and $\lambda\geq 0, n\in\mathbb{N}, n\geq 2$ then

$$\frac{a^n+\lambda b^n}{ab}+\frac{b^n+\lambda c^n}{bc}+\frac{c^n+\lambda a^n}{ca}\geq 3(\lambda+1)$$

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Solution. Using Holder Inequality, we get:

$$LHS=\sum_{cyc}\frac{b^n+\lambda c^n}{bc}=\sum_{cyc}\frac{b^{n-1}}{c}+\lambda\sum_{cyc}\frac{c^{n-1}}{b}\overset{Holder}{\geq}\frac{(\sum a)^{n-2}}{3^{n-3}\sum a}+\lambda\frac{(\sum a)^{n-2}}{3^{n-3}\sum a}=$$

$$=(1+\lambda)\frac{(\sum a)^{n-2}}{3^{n-3}\sum a}=\frac{(\lambda+1)}{3^{n-1}}\cdot 3^{n-2}=3(\lambda+1)$$

Equality holds for $a=b=c=1$.

Note. For $n=3$ and $\lambda=1$, we get Problem VIII.27, pag. 53, proposed by Carmen Chirfot, Romania in R.M.M. no.22.

REFERENCE:

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