A SIMPLE PROOF FOR ABI-KHUZAM'S INEQUALITY

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Abstract. In this paper is presented an elementary, detailed proof for the famous Abi-Khuzam's inequality.

Lemma 1:
If $x, y, z, A, B, C \in \mathbb{R}; A + B + C = \pi$ then:

$$x^2 + y^2 + z^2 \geq 2(yz \cos A + zx \cos B + xy \cos C) \tag{1}$$

Proof.

$$\begin{gathered}
0 \leq (z - (x \cos B + y \cos A))^2 + (x \sin B - y \sin A)^2 = \\
= z^2 - 2z(x \cos B + y \cos A) + (x \cos B + y \cos A)^2 + \\
+ x^2 \sin^2 B + y^2 \sin^2 A - 2xy \sin A \sin B = \\
= z^2 - 2xz \cos B - 2zy \cos A + x^2(\sin^2 B + \cos^2 B) + \\
+ y^2(\cos^2 A + \sin^2 A) + 2xy(\cos A \cos B - \sin A \sin B) = \\
= x^2 + y^2 + z^2 - 2yz \cos A - 2zx \cos B + 2xy \cos(A + B) = \\
= x^2 + y^2 + z^2 - 2yz \cos A - 2zx \cos B + 2xy \cos(\pi - C) = \\
= x^2 + y^2 + z^2 - 2yz \cos A - 2zx \cos B - 2xy \cos C \\
0 \leq x^2 + y^2 + z^2 - 2yz \cos A - 2zx \cos B - 2xy \cos C \\
x^2 + y^2 + z^2 \geq 2(xy \cos C + yz \cos A + zx \cos B)
\end{gathered}$$

□

Lemma 2:
If $x, y, z, A, B, C \in \mathbb{R}; x, y, z > 0; A + B + C = \pi$ then:

$$x \cos A + y \cos B + z \cos C \leq \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right) \tag{2}$$

Proof.
Replace in (1):

$$x \to \sqrt{\frac{yz}{x}}; y \to \sqrt{\frac{zx}{y}}; z \to \sqrt{\frac{xy}{z}}$$

$$\left(\sqrt{\frac{yz}{x}}\right)^2 + \left(\sqrt{\frac{zx}{y}}\right)^2 + \left(\sqrt{\frac{xy}{z}}\right)^2 \geq 2\left(\sqrt{\frac{zx}{y}}\sqrt{\frac{xy}{z}} \cos A + \sqrt{\frac{xy}{z}} \cdot \sqrt{\frac{yz}{x}} \cos B + \sqrt{\frac{yz}{x}} \cdot \sqrt{\frac{zx}{y}} \cos C\right)$$

$$\frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right) \geq x \cos A + y \cos B + z \cos C$$

$$x \cos A + y \cos B + z \cos C \leq \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right)$$

□

Theorem (ABI-KHUZAM'S INEQUALITY)
If $x, y, z, t > 0; A, B, C, D \in \mathbb{R}; A + B + C + D = \pi$ then:

$$x \cos A + y \cos B + z \cos C + t \cos D \leq \sqrt{\frac{(xy + zt)(xz + yt)(xt + yz)}{xyzt}} \tag{3}$$

Proof.

$$\text{Denote: } p = \frac{1}{2}\Big(\frac{x}{y} + \frac{y}{x} + \frac{z}{t} + \frac{t}{z}\Big); q = \frac{xy + zt}{2}$$

By (2):

$$x \cos A + y \cos B + \sqrt{\frac{q}{p}} \cos(C + D) \leq \frac{1}{2}\left(\frac{xy}{\sqrt{\frac{q}{p}}} + \sqrt{\frac{q}{p}}\Big(\frac{x}{y} + \frac{y}{x}\Big)\right) \tag{4}$$

$$z \cos C + t \cos D + \sqrt{\frac{q}{p}} \cos(A + B) \leq \frac{1}{2}\left(\frac{zt}{\sqrt{\frac{q}{p}}} + \sqrt{\frac{q}{p}}\Big(\frac{z}{t} + \frac{t}{z}\Big)\right) \tag{5}$$

$\cos(A+B)+\cos(C+D) = \cos(A+B)+\cos(\pi-(A+B)) = \cos(A+B)-\cos(A+B) = 0$
By adding (4); (5):

$$x \cos A + y \cos B + z \cos C + t \cos D + \sqrt{\frac{q}{p}}(\cos(A + B) + \cos(C + D)) \leq$$

$$\leq \frac{1}{2}\left(\frac{xy + zt}{\sqrt{\frac{q}{p}}} + \sqrt{\frac{q}{p}}\Big(\frac{x}{y} + \frac{y}{x} + \frac{z}{t} + \frac{t}{z}\Big)\right)$$

$$x \cos A + y \cos B + z \cos C + t \cos D \leq$$

$$\leq \frac{xy + zt}{2} \cdot \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} \cdot \frac{1}{2}\Big(\frac{x}{y} + \frac{y}{x} + \frac{z}{t} + \frac{t}{z}\Big)$$

$$x \cos A + y \cos B + z \cos C + t \cos D \leq$$

$$\leq q\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} \cdot p = \sqrt{pq} + \sqrt{pq} = 2\sqrt{pq}$$

$$x \cos A + y \cos B + z \cos C + t \cos D \leq$$

$$\leq 2\sqrt{\frac{1}{2}\Big(\frac{x}{y} + \frac{y}{x} + \frac{z}{t} + \frac{t}{z}\Big)\frac{xy + zt}{2}} =$$

$$= \sqrt{4 \cdot \frac{x^2tz + y^2tz + z^2xy + t^2xy}{2xyzt} \cdot \frac{xy + zt}{2}} =$$

$$= \sqrt{\frac{xz(xt + yz) + yt(xt + yz)}{xyzt} \cdot (xy + zt)} =$$

$$= \sqrt{\frac{(xy + zt)(xz + yt)(xt + yz)}{xyzt}}$$

□

Corollary 1:
If $A, B, C, D \in \mathbb{R}; A + B + C + D = \pi$ then:

$$\cos A + \cos B + cosC + \cos D \leq 2\sqrt{2} \tag{6}$$

Proof.
We take in $(3): x = y = z = t \neq 0$. □

Corollary 2:
If $A, B, C \in \mathbb{R}; A + B + C = \frac{\pi}{2}$ then:

$$\cos A + \cos B + \cos C \leq 2\sqrt{2}$$

Proof.
We take in $(6): D = \frac{\pi}{2} \Rightarrow A + B + C = \pi - \frac{\pi}{2} \Rightarrow A + B + C = \frac{\pi}{2}; \cos D = 0$ □

Corollary 3:
If $x, y, z, t > 0$ then:

$$xyzt(x + y + z + t)^2 \leq 2(xy + zt)(xz + yt)(xt + yz)$$

Proof.
We take in $(3): A = B = C = D = \frac{\pi}{4} \Rightarrow$

$$\Rightarrow \cos A = \cos B = \cos C = \cos D = \frac{1}{\sqrt{2}}; A + B + C + D = \pi$$

$$\frac{1}{\sqrt{2}}(x + y + z + t) \leq \sqrt{\frac{(xy + zt)(xz + yt)(xt + yz)}{xyzt}}$$

By squaring:

$$\frac{(x + y + z + t)^2}{2} \leq \frac{(xy + zt)(xz + yt)(xt + yz)}{xyzt}$$

$$xyzt(x + y + z + t)^2 \leq 2(xy + zt)(xz + yt)(xt + yz)$$

Equality holds for $x = y = z = t$. □

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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