RMM - Cyclic Inequalities Marathon 801 - 900

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DANIEL SITARU

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Proposed by

Daniel Sitaru – Romania,Marin Chirciu-Romania,

Mehmet Şahin-Ankara-Turkiye,Lucian Tuţescu-Romania,

Rajeev Rastogi-India, D.M.Bătinețu-Giurgiu-Romania,

Flaviu Cristian Verde-Romania,Neculai Stanciu-Romania, George Apostolopoulos-Messolonghi-Greece,Olimjon Jalilov-Uzbekistan,

Marian Ursărescu-Romania,Seyran Ibrahimov-Maasilli-Azerbaijan,

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Lazaros Zachariadis-Greece,Serlea Kabay-Liberia

Dan Radu Seclăman-Romania

Nikos Ntorvas-Greece, Amrit Awasthi-India

Bogdan Fuştei-Romania

Nguyen Van Canh-Ben Tre-Vietnam

Solutions by

Daniel Sitaru – Romania, Marian Ursărescu-Romania, Ruxandra Daniela Tonilă-Romania, Mohamed Amine Ben Ajiba-Tanger-Morocco, George Florin Şerban-Romania, Samar Das-India, Nikos Ntorvas-Greece,Asmat Qatea-Afghanistan, Sanong Huayrerai-Thailand,Lazaros Zachariadis-Greece,Ravi Prakash-India,Adrian Popa-Romania,Tran Hong-DongThap-Vietnam, Abdul Aziz-Semarang-Indonesia, Samir Cabiyev-Azerbaijan, Aggeliki Papaspyropoulou-Greece, Fayssal Abdelli-Bejaia-Algerie, Remus Florin Stanca-Romania, Daoudi Abdessattar-Tunisia,Florică Anastase-Romania, Michael Sterghiou-Greece, Sire Ambrose-Albania,Florentin Vişescu-Romania, Khaled Abd Imouti-Damascus-Syria, Amrit Awasthi-India, Avishek Mitra-West Bengal-India, Abdallah El Farissi-Bechar-Algerie,Marian Dincă-Romania,Ihsan Yucel-Turkiye,Kamel Gandouli Rezgui-Tunisia,Soumava Chakraborty-Kolkata-India,Nguyen Van Canh-BenTre-Vietnam,Michael Sterghiou-Greece

801. If $x,y,z>0$ such that $xy+yz+zx=1$ then prove:

$$\frac{1}{x^2+1}+\frac{1}{y^2+1}+\frac{1}{z^2+1}\le\frac{9}{4}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

Because $x,y,z>0$ such that $xy+yz+zx=1\to\exists\Delta ABC$ such that

$$x=\tan\frac{A}{2},y=\tan\frac{B}{2},z=\tan\frac{C}{2}$$

Inequality becomes as:

$$\sum_{cyc}\frac{1}{1+\tan^2\frac{A}{2}}\le\frac{9}{4}\Leftrightarrow\sum_{cyc}\cos^2\frac{A}{2}\le\frac{9}{4};\ \ (1)$$

But $\sum_{cyc}\cos^2\frac{A}{2}=\frac{4R+r}{2R};\ \ (2)$

From $(1),(2)$ we must show: $\frac{4R+r}{2R}\le\frac{9}{4}\Leftrightarrow 8R+2r\le 9R\Leftrightarrow 2r\le R(Euler)$.

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$\frac{1}{x^2+1}+\frac{1}{y^2+1}+\frac{1}{z^2+1}=1-\frac{x^2}{1+x^2}+1-\frac{y^2}{1+y^2}+1-\frac{z^2}{1+z^2}=3-\sum_{cyc}\frac{x^2}{x^2+1}$$

We have to prove that:

$$3-\sum_{cyc}\frac{x^2}{x^2+1}\le\frac{9}{4}\Leftrightarrow\sum_{cyc}\frac{x^2}{x^2+1}\ge\frac{3}{4}$$

$$\because\sum_{cyc}x^2\ge\sum_{cyc}xy=1\to\sum_{cyc}x^2\ge 1;\ \ (1)$$

$$\sum_{cyc}\frac{x^2}{x^2+1}\overset{Bergstrom}{\ge}\frac{(\sum x)^2}{\sum(x^2+1)}\ge\frac{3}{4}\Leftrightarrow$$

$$4\left(\sum_{cyc}x\right)^2\ge 3\sum_{cyc}(x^2+1)$$

$$\Leftrightarrow 4(x^2+y^2+z^2+2xy+2yz+2zx)\ge 3x^2+3y^2+3z^2+9$$

$$\Leftrightarrow \sum_{cyc} x^2 + 8\sum_{cyc} xy \ge 9, \because \sum_{cyc} xy = 1 \to \sum_{cyc} x^2 \ge 1(true).$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{1}{x^2+1} = \sum_{cyc}\frac{1}{x^2+\sum xy} = \sum_{cyc}\frac{1}{(x+y)(x+z)} = \frac{\sum y+z}{\prod(x+y)} =$$

$$= \frac{2\sum x}{(\sum x)(\sum xy) - xyz} = \frac{2\sum x}{\sum x - xyz} \overset{(*)}{\leq} \frac{9}{4}$$

$(*) \Leftrightarrow 9xyz \le \sum x \Leftrightarrow 9 \le \sum\left(\frac{1}{xy}\right)$, which is true from CBS:

$$\sum_{cyc}\frac{1}{xy} \ge \frac{9}{\sum xy} = 9$$

Therefore,

$$\sum_{cyc}\frac{1}{x^2+1} \le \frac{9}{4}$$

802. If $x, y, z > 0$ then:

$$\frac{9\sqrt{2}}{2}\cdot\frac{\sqrt[3]{xyz}}{x^2+y^2+z^2} \le \sum_{cyc}\frac{\sqrt{x(y+z)}}{y^2+z^2} \le \frac{x^2+y^2+z^2}{\sqrt{2}\cdot xyz}$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{\sqrt{x(y+z)}}{y^2+z^2} \overset{AM-GM}{\geq} \sum_{cyc}\frac{\sqrt{2x\sqrt{yz}}}{y^2+z^2} = \sqrt{2}\cdot\sqrt[4]{xyz}\sum_{cyc}\frac{\sqrt[4]{x}}{y^2+z^2} \overset{CBS}{\geq}$$

$$\ge \sqrt{2}\cdot\sqrt[4]{xyz}\cdot\frac{(\sum\sqrt[8]{x})}{\sum(y^2+z^2)} \overset{AM-GM}{\geq} \sqrt{2}\cdot\sqrt[4]{xyz}\cdot\frac{(3\cdot\sqrt[24]{xyz})}{2\sum x^2} = \frac{9\sqrt{2}}{2}\cdot\frac{\sqrt[3]{xyz}}{x^2+y^2+z^2}$$

$$\sum_{cyc}\frac{\sqrt{x(y+z)}}{y^2+z^2} \overset{AM-GM}{\leq} \sum_{cyc}\frac{\sqrt{x(y+z)}}{2yz} = \frac{1}{2xyz}\sum_{cyc} x\sqrt{x(y+z)} =$$

$$= \frac{1}{2\sqrt{2}xyz}\sum_{cyc} x\sqrt{2x(y+z)} \overset{AM-GM}{\leq} \frac{1}{2\sqrt{2}xyz}\sum_{cyc} x\cdot\frac{(2x)+(y+z)}{2} =$$

$$=\frac{1}{4\sqrt{2}xyz}\left(2\sum_{cyc}x^2+2\sum_{cyc}xy\right)\overset{\sum xy\le\sum x^2}{\leq}\frac{x^2+y^2+z^2}{\sqrt{2}xyz}$$

Therefore,

$$\frac{9\sqrt{2}}{2}\cdot\frac{\sqrt[3]{xyz}}{x^2+y^2+z^2}\leq\sum_{cyc}\frac{\sqrt{x(y+z)}}{y^2+z^2}\leq\frac{x^2+y^2+z^2}{\sqrt{2}\cdot xyz}$$

803. If $a,b,c>0, 2021(ab+bc+ca)>a+b+c$ then:

$$2021(a+b+c)>3$$

Proposed by Lucian Tuţescu-Romania

Solution 1 by George Florin Şerban-Romania

$$(a+b+c)^2\geq 3(ab+bc+ca)>\frac{3}{2021}(a+b+c)\rightarrow$$

$$a+b+c>\frac{3}{2021}\rightarrow 2021(a+b+c)>3$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(a+b+c)^2\geq 3(ab+bc+ca)$$

$$2021(a+b+c)^2\geq 3\cdot 2021(ab+bc+ca)>3(a+b+c)$$

$$2021(a+b+c)>3$$

804. If $a,b,c>0$ then:

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=1\Rightarrow a+b+c\geq 6$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Samar Das-India

$$\sum_{cyc}\frac{1}{a+1}\geq\frac{(1+1+1)^2}{\sum(a+1)}=\frac{9}{\sum a+3}$$

$$\rightarrow 1\geq\frac{9}{\sum a+3}\rightarrow\sum a+3\geq 9\rightarrow\sum a\geq 6$$

Solution 2 by Nikos Ntorvas-Greece

$$\text{Let } f(t)=\frac{1}{t+1}, t>0$$

$$f'(t)=-\frac{1}{(t+1)^2}<0, f\text{ —strictly decreasing on }(0,\infty)$$

$$f''(x)=\frac{2}{(t+1)^3}>0, f\text{ —strictly convex on }(0,\infty)$$

From Jensen Inequality we have for $a,b,c>0$:

$$\frac{f(a)+f(b)+f(c)}{3}\ge f\left(\frac{a+b+c}{3}\right)\Leftrightarrow$$

$$\frac{\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}}{3}\ge f\left(\frac{a+b+c}{3}\right)\Leftrightarrow\frac{1}{3}\ge f\left(\frac{a+b+c}{3}\right);(1)$$

We have that $f(2)=\frac{1}{3}$ and $f-\searrow(0,\infty)$: $(1)\Leftrightarrow$

$$f(2)\ge f\left(\frac{a+b+c}{3}\right)\Leftrightarrow 2\le\frac{a+b+c}{3}\Leftrightarrow a+b+c\ge 6$$

805. Let $a,b,c>0$ such that $a+b+c=6$ then:

$$\sum_{cyc}\frac{(a+2)(b+2)^4}{\sqrt[3]{(2ac)^2}+\frac{4}{3}}\ge 576$$

Proposed by Rajeev Rastogi-India

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{(a+2)(b+2)^4}{\sqrt[3]{(2ac)^2}+\frac{4}{3}}\overset{AM-GM}{\ge}\sum_{cyc}\frac{(a+2)(b+2)^4}{\frac{ac+ac+4}{3}+\frac{4}{3}}=\frac{3}{2}\sum_{cyc}\frac{(a+2)(b+2)^4}{ac+4}=$$

$$=\frac{3}{2}\left(\prod_{cyc}(a+2)\right)\sum_{cyc}\frac{(a+2)(b+2)^4}{(ac+4)(b+2)(c+2)}=$$

$$=\frac{3}{2}\left(\prod_{cyc}(a+2)\right)\sum_{cyc}\frac{(b+2)^4}{(abc+2ac+4b+8)(c+2)}\overset{CBS}{\ge}$$

$$\ge\frac{3}{2}\left(\prod_{cyc}(a+2)\right)\cdot\frac{\left(\sum(b+2)\right)^4}{3\left(\sum(abc+2ac+4b+8)\right)\left(\sum(c+2)\right)}=$$

$$=\frac{1}{2}\left(\prod_{cyc}(a+2)\right)\cdot\frac{(6+\sum a)^3}{3abc+48+2\sum ab}=864\cdot\frac{\prod(a+2)}{3abc+48+2\sum ab}\overset{(*)}{\geq}576$$

$$(*)\Leftrightarrow 3\prod_{cyc}(a+2)\geq 2\left(3abc+48+2\sum_{cyc}ab\right)$$

$$\Leftrightarrow 3\left(abc+32+2\sum_{cyc}ab\right)\geq 2\left(3abc+48+2\sum_{cyc}ab\right)$$

$$\Leftrightarrow 2\sum_{cyc}ab\geq 3abc\Leftrightarrow\sum_{cyc}\frac{1}{a}\geq\frac{3}{2}$$

Which is true, from CBS inequality, we have:

$$\sum_{cyc}\frac{1}{a}\geq\frac{9}{\sum a}=\frac{3}{2}$$

Therefore,

$$\sum_{cyc}\frac{(a+2)(b+2)^4}{\sqrt[3]{(2ac)^2}+\frac{4}{3}}\geq 576$$

806. If $a,b,c,d>0$ such that $\sum a=4$ then:

$$\sum_{cyc}\frac{a^2}{a+2b^3}\geq\frac{4}{3}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{a^2}{a+2b^3}=\sum_{cyc}\left(a-\frac{2ab^2}{a+2b^3}\right)=\sum_{cyc}a-\sum_{cyc}\frac{2ab^3}{a+b^3+b^3}\overset{AM-GM}{\geq}$$

$$\geq 4-\frac{2}{3}\sum_{cyc}\sqrt[3]{a^2b^3}\overset{AM-GM}{\geq}4-\frac{2}{3}\sum_{cyc}\frac{b+ab+ab}{3}=$$

$$=4-\frac{2}{9}\sum_{cyc}a-\frac{4}{9}(ab+bc+cd+da)=$$

$$=\frac{28}{9}-\frac{4}{9}(a+c)(b+d)\overset{AM-GM}{\overset{\smile}{\geq}}\frac{28}{9}-\frac{4}{9}\left(\frac{(a+c)+(b+d)}{2}\right)^2=\frac{28}{9}-\frac{16}{9}=\frac{4}{3}$$

Therefore,

$$\sum_{cyc}\frac{a^2}{a+2b^3}\geq\frac{4}{3}$$

807. **If $x,y,z>0$ that $xy+yz+zx=1$ then:**

$$\frac{1}{3x^2+2}+\frac{1}{3y^2+2}+\frac{1}{3z^2+2}\leq 1$$

Proposed by Marin Chirciu – Romania

Solution by Asmat Qatea-Afghanistan

$$xy+yz+zx=1\Rightarrow\frac{1}{3x^2+2}+\frac{1}{3y^2+2}+\frac{1}{3z^2+2}\leq 1$$

$$\frac{1}{3x^2+2}+\frac{1}{3y^2+2}+\frac{1}{3z^2+2}=\frac{3}{2}-\frac{3}{2}\left(\frac{x^2}{3x^2+2}+\frac{y^2}{3y^2+2}+\frac{z^2}{3z^2+2}\right)$$

$$\frac{x^2}{3x^2+2}+\frac{y^2}{3y^2+2}+\frac{z^2}{3z^2+2}\overset{Bergstrom}{\geq}\frac{(x+y+z)^2}{3(x^2+y^2+z^2)+6}$$

$$\frac{3}{2}-\frac{3}{2}\left(\frac{x^2}{3x^2+2}+\frac{y^2}{3y^2+2}+\frac{z^2}{3z^2+2}\right)\leq\frac{3}{2}-\frac{3(x+y+z)^2}{6(x^2+y^2+z^2)+12}\overset{?}{\leq}1$$

$$6(x+y+z)^2\geq 6(x^2+y^2+z^2)+12$$

$$(x+y+z)^2\geq x^2+y^2+z^2+2\Rightarrow 2\geq 2\ \text{(True)}$$

808. **If $x,y,z\geq 1$ then:**

$$\frac{x\sqrt[3]{y}+y\sqrt[3]{x}}{x\sqrt{y}+y\sqrt{x}}+\frac{y\sqrt[3]{z}+z\sqrt[3]{y}}{y\sqrt{z}+z\sqrt{y}}+\frac{z\sqrt[3]{x}+x\sqrt[3]{z}}{z\sqrt{x}+x\sqrt{z}}\leq 3$$

Proposed by Mehmet Şahin-Ankara-Turkiye

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{x\sqrt[3]{y}+y\sqrt[3]{x}}{x\sqrt{y}+y\sqrt{x}}\overset{?}{\leq}1\Leftrightarrow x\sqrt[3]{y}+y\sqrt[3]{x}\leq x\sqrt{y}+y\sqrt{x}\Leftrightarrow$$

$x\sqrt[3]{y}(\sqrt[6]{y}-1)+y\sqrt[3]{x}-1)\geq 0$, which is true because $x,y\geq 1\rightarrow \frac{x\sqrt[3]{y}+y\sqrt[3]{x}}{x\sqrt{y}+y\sqrt{x}}\leq 1$

Therefore,

$$\frac{x\sqrt[3]{y}+y\sqrt[3]{x}}{x\sqrt{y}+y\sqrt{x}}+\frac{y\sqrt[3]{z}+z\sqrt[3]{y}}{y\sqrt{z}+z\sqrt{y}}+\frac{z\sqrt[3]{x}+x\sqrt[3]{z}}{z\sqrt{x}+x\sqrt{z}}\leq 3$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x,y,z\geq 1$, we give $x=a^6,y=b^6,z=c^6$, hence

$$\frac{x\sqrt[3]{y}+y\sqrt[3]{x}}{x\sqrt{y}+y\sqrt{x}}+\frac{y\sqrt[3]{z}+z\sqrt[3]{y}}{y\sqrt{z}+z\sqrt{y}}+\frac{z\sqrt[3]{x}+x\sqrt[3]{z}}{z\sqrt{x}+x\sqrt{z}}=\frac{a^6b^2+a^2b^6}{a^6b^3+a^3b^6}+\frac{b^6c^2+b^2c^6}{b^6c^3+b^3c^6}+\frac{c^6a^2+c^2a^6}{c^6a^3+c^3a^6}$$

$$=\frac{a^4+b^4}{a^4b+ab^4}+\frac{b^4+c^4}{b^4c+bc^4}+\frac{c^4+a^4}{c^4a+ca^4}\leq\frac{a^4+b^4}{a^4+b^4}+\frac{b^4+c^4}{b^4+c^4}+\frac{c^4+a^4}{c^4+a^4}=3$$

Therefore,

$$\frac{x\sqrt[3]{y}+y\sqrt[3]{x}}{x\sqrt{y}+y\sqrt{x}}+\frac{y\sqrt[3]{z}+z\sqrt[3]{y}}{y\sqrt{z}+z\sqrt{y}}+\frac{z\sqrt[3]{x}+x\sqrt[3]{z}}{z\sqrt{x}+x\sqrt{z}}\leq 3$$

Solution 3 by Lazaros Zachariadis-Thessaloniki-Greece

$$a^4(b-1)+b^4(a-1)\geq 0;\forall a,b\in[1,\infty)\Leftrightarrow$$

$$a^4b+ab^4-a^4-b^4\geq 0\Leftrightarrow ab(a^3+b^3)\geq a^4+b^4$$

$$\Leftrightarrow\frac{1}{ab(a^3+b^3)}\leq\frac{1}{a^4+b^4}\Leftrightarrow\frac{a^4+b^4}{ab(a^3+b^3)}\leq 1$$

$$\Leftrightarrow\frac{(ab)^2(a^4+b^4)}{(ab)^3(a^3+b^3)}\leq 1\Leftrightarrow\frac{a^6b^2+a^2b^6}{a^6b^3+a^3b^6}\leq 1$$

For $x,y,z\geq 1$, we give $x=a^6,y=b^6,z=c^6$, hence

$$\frac{x\sqrt[3]{y}+y\sqrt[3]{x}}{x\sqrt{y}+y\sqrt{x}}\leq 1$$

Therefore,

$$\frac{x\sqrt[3]{y}+y\sqrt[3]{x}}{x\sqrt{y}+y\sqrt{x}}+\frac{y\sqrt[3]{z}+z\sqrt[3]{y}}{y\sqrt{z}+z\sqrt{y}}+\frac{z\sqrt[3]{x}+x\sqrt[3]{z}}{z\sqrt{x}+x\sqrt{z}}\leq 3$$

809. If $x,y,z>0$ such that $xy+yz+zx=1$ and $0\le\lambda\le\frac{3}{2}$ then

$$\frac{1}{\lambda x^2+1}+\frac{1}{\lambda y^2+1}+\frac{1}{\lambda z^2+1}\le\frac{9}{\lambda+3}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{1}{\lambda x^2+1}+\frac{1}{\lambda y^2+1}+\frac{1}{\lambda z^2+1}\le\frac{9}{\lambda+3};(*)$$

$$(*)\Leftrightarrow\sum_{cyc}\left(1-\frac{\lambda x^2}{\lambda x^2+1}\right)\le\frac{9}{\lambda+3}\Leftrightarrow 3-\sum_{cyc}\frac{\lambda x^2}{\lambda x^2+1}\le\frac{9}{\lambda+3}\Leftrightarrow\sum_{cyc}\frac{\lambda x^2}{\lambda x^2+1}\overset{\lambda\ge0}{\overbrace{\le}}\frac{3}{\lambda+3}$$

By CBS we have:

$$\sum_{cyc}\frac{\lambda x^2}{\lambda x^2+1}\ge\frac{(\sum x)^2}{\sum(\lambda x^2+1)}\overset{?}{\overbrace{\ge}}\frac{3}{\lambda+3}$$

$$\Leftrightarrow(\lambda+3)\sum_{cyc}x^2+2(\lambda+3)\ge3\lambda\sum_{cyc}x^2+9$$

$(3-2\lambda)(\sum x^2-1)\ge0$, which is true because $\lambda\le\frac{3}{2}$ and $\sum x^2\ge\sum xy=1$

Therefore,

$$\frac{1}{\lambda x^2+1}+\frac{1}{\lambda y^2+1}+\frac{1}{\lambda z^2+1}\le\frac{9}{\lambda+3}$$

810. If $a,b>0$ then

$$\frac{2(a^2+b^2)+3(a+b)^2+20ab}{4}+\frac{28a^2b^2}{(a+b)^2}\le\left(\sqrt{\frac{a^2+b^2}{2}}+\frac{(\sqrt{a}+\sqrt{b})^2}{2}+\frac{2ab}{a+b}\right)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*)\frac{2(a^2+b^2)+3(a+b)^2+20ab}{4}+\frac{28a^2b^2}{(a+b)^2}\le\left(\sqrt{\frac{a^2+b^2}{2}}+\frac{(\sqrt{a}+\sqrt{b})^2}{2}+\frac{2ab}{a+b}\right)^2$$

We have:

$$\sqrt{\frac{a^2+b^2}{2}} \ge \frac{a+b}{2} (QM \ge AM); (1)$$

$$\sqrt{ab} \ge \frac{1}{\frac{1}{a}+\frac{1}{b}} \ge \frac{2ab}{a+b} (GM \ge HM); (2)$$

$$RHS_{(*)} = \left(\sqrt{\frac{a^2+b^2}{2}} + \frac{a+b}{2} + \sqrt{ab} + \frac{2ab}{a+b}\right)^2 \overset{(2)}{\ge} \left(\sqrt{\frac{a^2+b^2}{2}} + \frac{a+b}{2} + \frac{4ab}{a+b}\right)^2 =$$

$$= \frac{a^2+b^2}{2} + \frac{(a+b)^2}{2} + \frac{16a^2b^2}{(a+b)^2} + 2\sqrt{\frac{a^2+b^2}{2}}\left(\frac{a+b}{2} + \frac{4ab}{a+b}\right) + \frac{2(a+b)}{2} \cdot \frac{4ab}{a+b} \overset{(1)}{\ge}$$

$$\ge \frac{2(a^2+b^2)+(a+b)^2}{4} + \frac{16a^2b^2}{(a+b)^2} + \frac{(a+b)^2}{2} + 4ab + 4ab =$$

$$= \frac{2(a^2+b^2)+3(a+b)^2+20ab}{4} + \frac{16a^2b^2}{(a+b)^2} + 3ab \overset{(2)}{\ge}$$

$$\ge \frac{2(a^2+b^2)+3(a+b)^2+20ab}{4} + \frac{16a^2b^2}{(a+b)^2} + 3\left(\frac{2ab}{a+b}\right)^2 = LHS_{(*)}$$

Therefore,

$$\frac{2(a^2+b^2)+3(a+b)^2+20ab}{4} + \frac{28a^2b^2}{(a+b)^2} \le \left(\sqrt{\frac{a^2+b^2}{2}} + \frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{2} + \frac{2ab}{a+b}\right)^2$$

Solution 2 by Ravi Prakash-New Delhi-India

Let $Q = \sqrt{\frac{a^2+b^2}{2}}, A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$, then $H \le G \le A \le Q$.

$$\left(\sqrt{\frac{a^2+b^2}{2}} + \frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{2} + \frac{2ab}{a+b}\right)^2 - \frac{1}{4}[2(a^2+b^2)+3(a+b)^2+20ab] - \frac{28a^2b^2}{(a+b)^2}$$

$$= (Q+A+G+H)^2 - Q^2 - 3A^2 - 5G^2 - 7H^2 =$$

$$= 2QA + 2QG + 2QH + 2AG + 2AH + 2GH - 2A^2 - 4G^2 - 6H^2 =$$

$$= 2A(Q-A) + 2G(Q-G) + 2H(Q-H) + 2G(A-G) + 2H(A-H) + 2H(G-H) \ge 0$$

The inequality follows.

Equality when $A=G=H=Q\Leftrightarrow a=b$.

811. If $x,y,z>0$ such that $xy+yz+zx=3$ and $-3\le\lambda\le1$ then

$$x^2+y^2+z^2+\lambda xyz(x+y+z)\ge3(\lambda+1)$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$x^2+y^2+z^2+\lambda xyz(x+y+z)\ge3(\lambda+1);(*)$$

$$(*)\Leftrightarrow\frac{1}{3}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)+\lambda xyz\sum_{cyc}x\le\frac{1}{3}(\lambda+1)\left(\sum_{cyc}xy\right)^2$$

$$\Leftrightarrow\sum_{cyc}xy(x^2+y^2)+(3\lambda+1)xyz\sum_{cyc}x\ge(\lambda+1)\sum_{cyc}(xy)^2+2(\lambda+1)xyz\sum_{cyc}x$$

$$\Leftrightarrow\sum_{cyc}xy(x^2+y^2)\ge(\lambda+1)\sum_{cyc}(xy)^2+(1-\lambda)xyz\sum_{cyc}x$$

We have:

$$\sum_{cyc}xy(x^2+y^2)\overset{AM-GM}{\ge}2\sum_{cyc}(xy)^2\overset{?}{\ge}(\lambda+1)\sum_{cyc}(xy)^2+(1-\lambda)xyz\sum_{cyc}x$$

$$\Leftrightarrow(1-\lambda)\sum_{cyc}(xy)^2\ge(1-\lambda)xyz\sum_{cyc}x$$

Which is true because $\lambda\le1$ and $\sum(xy)^2\overset{\sum a^2\ge\sum ab}{\ge}\sum(xy)(xz)=xyz\sum x$

Therefore,

$$x^2+y^2+z^2+\lambda xyz(x+y+z)\ge3(\lambda+1)$$

812. If $a_1,a_2,\dots,a_n>0$ such that $a_1^2+a_2^2+\dots+a_{n+1}^2=a_1a_2\dots a_{n+1}$ then

$$(a_1-n)(a_2-n)\dots(a_{n+1}-n)\ge1$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$a_1^2+a_2^2+\dots+a_{n+1}^2=a_1a_2\dots a_{n+1};(*)$$

We have:

$$\prod_{i=1}^{n+1} a_i - n\prod_{i=1}^{n+1} a_i = (a_j - n)\prod_{\substack{i=1\\ i\neq j}}^{n+1} a_i^n - n\prod_{\substack{i=1\\ i\neq j}}^{n+1} a_i =$$

$$= \left(\sum_{\substack{i=1\\ i\neq j}}^{n+1} a_i^n - n\prod_{\substack{i=1\\ i\neq j}}^{n+1} a_i\right) + a_j^n \overset{AM-GM}{\gtrsim} 0 + a_j^n = a_j^n \to \forall i\{1,2,\dots,n+1\},$$

$$a_j - n \geq a_j^n\left(\prod_{\substack{i=1\\ i\neq j}}^{n+1} a_i\right)^{-1}$$

Therefore,

$$\prod_{j=1}^{n+1}(a_j - n) \geq \prod_{j=1}^{n+1}\left[a_j^n\left(\prod_{\substack{i=1\\ i\neq j}}^{n+1} a_i\right)^{-1}\right] = 1$$

813. If $a,b,c,d > 0$ such that $a^3+b^3+c^3+d^3 = abcd$ then

$$(a-3)(b-3)(c-3)(d-3) \geq 1$$

Proposed by Marin Chirciu-Romania

Solution by Ravi Prakash-New Delhi-India

$$(a-3)bcd = abcd - 3bcd = a^3 + (b^3+c^3+d^3-3bcd) \geq a^3 \to a-3 \geq \frac{a^3}{bcd}$$

Analogously,

$$b-3 \geq \frac{b^2}{acd}, c-3 \geq \frac{c^3}{abc}, d-3 \geq \frac{cd^3}{abc}$$

Multiplying these four inequalities, we get:

$$(a-3)(b-3)(c-3)(d-3) \geq \frac{a^3b^3c^3d^3}{a^3b^3c^3d^3} = 1$$

814. If $m,n,p,x,y,z > 0$ then:

$$\sum_{cyc}\frac{ny+pz}{mx+\sqrt{(mx+2ny)(mx+2pz)}} \geq \frac{3}{2}$$

Proposed by D.M. Bătinețu-Giurgiu, Flaviu Cristian Verde – Romania

Solution 1 by Marian Ursărescu – Romania

$$\sqrt{(mx+2ny)(mx+2pz)} \le \frac{2mx+2ny+2pz}{2} = mx+ny+pz \Rightarrow \text{we must show:}$$

$$\sum \frac{ny+pz}{2mx+ny+pz} \ge \frac{3}{2} \quad (1)$$

Let $mx = a, ny = b, pz = c$ (2)

From (1)+(2) we must show:

$$\sum \frac{b+c}{2a+b+c} \ge \frac{3}{2} \Leftrightarrow \sum \frac{(b+c)^2}{2a(b+c)+(b+c)^2} \ge \frac{3}{2} \quad (3)$$

From Bergstom: $\sum \frac{(b+c)^2}{2a(b+c)+(b+c)^2} \ge \frac{4(a+b+c)^2}{2\sum a(b+c)+\sum(b+c)^2} \Leftrightarrow$

$$\sum \frac{(b+c)^2}{2a(b+c)+(b+c)^2} \ge \frac{2(a+b+c)^2}{3(ab+ac+bc)+a^2+b^2+c^2} \quad (4)$$

From (3)+(4) we must show: $\frac{2(a+b+c)^2}{3(ab+bc+ac)+a^2+b^2+c^2} \ge \frac{3}{2} \Leftrightarrow$

$$\Leftrightarrow 4(a+b+c)^2 \ge 9(ab+ac+bc)+3(a^2+b^2+c^2) \Leftrightarrow$$

$$\Leftrightarrow a^2+b^2+c^2 \ge ab+ac+bc \text{ (true)}$$

Solution 2 by Adrian Popa – Romania

$$\sqrt{(mx+2ny)(mx+2pz)} \overset{MG\le MA}{\le} \frac{mx+2ny+mx+2pz}{2} = mx+ny+pz \Rightarrow$$

$$\Rightarrow S = \sum_{cyc} \frac{ny+pz}{mx+\sqrt{(mx+2ny)(mx+2pz)}} \ge \sum_{cyc} \frac{ny+pz}{mx+(mx+ny+pz)} \quad (1)$$

We denote: $\left.\begin{matrix} mx+ny=a \\ ny+pz=b \\ pz+mx=c \end{matrix}\right\} \Rightarrow mx+ny+pz = \frac{a+b+c}{2}$

$$mx = \frac{a+b+c}{2} - b = \frac{a-b+c}{2}; ny = \frac{a+b-c}{2}; pz = \frac{b+c-a}{2}$$

Replacing in (1) $\Rightarrow S \ge \frac{b}{\frac{a-b+c}{2}+\frac{a+b+c}{2}} + \frac{c}{\frac{a-c+b}{2}+\frac{a+b+c}{2}} + \frac{a}{\frac{b-a+c}{2}+\frac{a+b+c}{2}} = \frac{b}{a+c} + \frac{c}{a+b} + \frac{a}{b+c} \ge \frac{3}{2}$ (True)

(Nesbit inequality)

815. If $a, b, c, d > 1$ then

$$\sum_{cyc} log_a \left(\frac{b^4+c^4+d^4}{b^3+c^3+d^3}\right) \ge 4$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

$$\text{From Chebyshev's inequality: } b^4+c^4+d^4 \ge \frac{1}{3}(b^3+c^3+d^3)(b+c+d)$$

$$\to \frac{b^4+c^4+d^4}{b^3+c^3+d^3} \ge \frac{1}{3}(b+c+d) \overset{AM-GM}{\ge} \sqrt[3]{bcd} \text{ and } a>1$$

$$\to \log_a\left(\frac{b^4+c^4+d^4}{b^3+c^3+d^3}\right) \ge \log_a\left(\sqrt[3]{bcd}\right) = \frac{1}{3}\log_a(bcd)(\text{and analogs})$$

$$\sum_{cyc}\log_a\left(\frac{b^4+c^4+d^4}{b^3+c^3+d^3}\right) \ge \frac{1}{3}\sum_{cyc}\log_a(bcd) =$$

$$= \frac{1}{3}[(\log_a b+\log_b a)+(\log_a c+\log_c a)+(\log_a d+\log_d a)+(\log_b c+\log_c b)$$

$$+(\log_b d+\log_d b)+(\log_c d+\log_d c)] \overset{AM-GM}{\ge} \frac{12}{3} = 4$$

Solution 2 by Ravi-Prakash-New Delhi-India

$$\text{For } x,y,z>0:\ 3(x^4+y^4+z^4)-(x^3+y^3+z^3)(x+y+z) =$$

$$= 2(x^4+y^4+z^4)-x(y^3+z^3)-y(x^3+z^3)-z(x^3+y^3) =$$

$$= (x^3-y^3)(x-y)+(y^3-z^3)(y-z)+(z^3-x^3)(z-x) \ge 0$$

$$\to \frac{x^4+y^4+z^4}{x^3+y^3+z^3} \ge \frac{x+y+z}{3} \ge \sqrt[3]{xyz}$$

$$\log_a\left(\frac{b^4+c^4+d^4}{b^3+c^3+d^3}\right) \ge \log_a\left(\sqrt[3]{bcd}\right) = \frac{1}{3}\log_a(bcd)(\text{and analogs})$$

$$\sum_{cyc}\log_a\left(\frac{b^4+c^4+d^4}{b^3+c^3+d^3}\right) \ge \frac{1}{3}\sum_{cyc}\log_a(bcd) = \frac{1}{3}\left(\sum_{cyc}\log_a b+\log_b a\right) \overset{AM-GM}{\ge} \frac{12}{3} = 4$$

816. Given $a,b,c>0$ such that : $\sum a^4 = \frac{1}{2}$, then prove that :

$$\sum \frac{1+3b^4+c^4}{b^2(2a^2+3c^2)} \ge 6$$

Proposed by Rajeev Rastogi-India

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum \frac{1}{b^2(2a^2+3c^2)} \overset{CBS}{\geq} \frac{9}{\sum b^2(2a^2+3c^2)} = \frac{9}{5\sum a^2b^2} \overset{\sum xy \leq \sum x^2}{\geq} \frac{9}{5\sum a^4} = \frac{18}{5} \quad (1)$$

$$\sum \frac{b^4}{b^2(2a^2+3c^2)} \overset{CBS}{\geq} \frac{(\sum b^2)^2}{\sum b^2(2a^2+3c^2)} = \frac{(\sum b^2)^2}{5\sum a^2b^2} \overset{3\sum xy \leq (\sum x)^2}{\geq} \frac{3}{5} \quad (2)$$

$$\sum \frac{c^4}{b^2(2a^2+3c^2)} \overset{CBS}{\geq} \frac{(\sum c^2)^2}{\sum b^2(2a^2+3c^2)} = \frac{(\sum c^2)^2}{5\sum a^2b^2} \overset{3\sum xy \leq (\sum x)^2}{\geq} \frac{3}{5} \quad (3)$$

$$(1)+3\times(2)+(3) \to \sum \frac{1+3b^4+c^4}{b^2(2a^2+3c^2)} \geq \frac{18}{5} + 3.\frac{3}{5} + \frac{3}{5} = 6$$

817. If $a,b,c>0$ then:

$$\sum \frac{a}{\sqrt{a^2+(n^2-1)bc}} \geq \frac{3}{n}, n \geq 3$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

From Hölder, we have :

$$\left(\sum \frac{a}{\sqrt{a^2+(n^2-1)bc}}\right)^2 \left(\sum a(a^2+(n^2-1)bc)\right) \geq \left(\sum a\right)^3$$

$$\to \left(\sum \frac{a}{\sqrt{a^2+(n^2-1)bc}}\right)^2 \geq \frac{(\sum a)^3}{\sum a^3 + 3(n^2-1)abc} \overset{?}{\geq} \left(\frac{3}{n}\right)^2 \leftrightarrow n^2\left(\sum a\right)^3$$

$$\geq 9\sum a^3 + 27(n^2-1)abc$$

$$\leftrightarrow (n^2-9)\sum a^3 + 3n^2\prod(a+b) \geq 27(n^2-1)abc$$

$$\textit{We have}: \sum a^3 \overset{AM-GM}{\geq} 3abc \textit{ and } \prod(a+b) \overset{Cesaro}{\geq} 8abc \textit{ and } n^2-9 \geq 0$$

$$\to (n^2-9)\sum a^3 + 3n^2\prod(a+b) \geq 3(n^2-9)abc + 24n^2abc = 27(n^2-1)abc$$

Therefore,

$$\sum \frac{a}{\sqrt{a^2+(n^2-1)bc}} \geq \frac{3}{n}, n \geq 3$$

818. If $x,y,z>0$, prove that :

$$\sum\frac{x^2}{z^3(zx+y^2)}\ge\frac{3}{2xyz}$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\sum_{cyc}\frac{x^2}{z^3(zx+y^2)}=\sum_{cyc}\frac{(x^2y)^2}{x^2y^2z^3(xz+y^2)}=\frac{1}{x^2y^2z^2}\sum_{cyc}\frac{(x^2y)^2}{xz^2+y^2z}\overset{CBS}{\ge}$$

$$\ge\frac{1}{x^2y^2z^2}\cdot\frac{(x^2y+y^2z+z^2x)^2}{xy(x+y)+yz(y+z)+zx(z+x)}\overset{(1)}{\ge}\frac{3}{2xyz}$$

$$(1)\Leftrightarrow 2(x^2y+y^2z+z^2x)^2\ge 3xyz\sum_{cyc}(xy(x+y))$$

$$\Leftrightarrow[(x^2y+y^2z+z^2x)^2-3xyz(x^2y+y^2z+z^2x)]$$
$$+[(x^2y+y^2z+z^2x)^2-3xyz(xy^2+yz^2+zx^2)]\ge 0$$

$$\Leftrightarrow[(x^2y+y^2z+z^2x)(x^2y+y^2z+z^2x-3xyz)]+[(x^2y+y^2z+z^2x)^2$$
$$-3xyz(xy^2+yz^2+zx^2)]\ge 0;(2), \textit{which is true, because}:$$

$$x^2y+y^2z+z^2x\overset{AM-GM}{\ge}3\sqrt[3]{(xyz)^3}=3xyz$$

$$\rightarrow(x^2y+y^2z+z^2x)(x^2y+y^2z+z^2x-3xyz)\ge 0;(3)$$

$$\because(a+b+c)^2\ge 3(ab+bc+ca)\ \textit{for}\ a=x^2,b=y^2,c=z^2\ \textit{it follows that}:$$

$$(x^2y+y^2z+z^2x)^2\ge 3xyz(xy^2+yz^2+zx^2)$$

$$\rightarrow(x^2y+y^2z+z^2x)^2-3xyz(xy^2+yz^2+zx^2)\ge 0;(4)$$

From $(3),(4)$ it follows that: (2) is true→ (1) is true. Proved!

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum\frac{x^2}{z^3(zx+y^2)}\ge\frac{3}{2xyz}(*)\leftrightarrow\sum\frac{\left(\frac{x}{z}\right)^2}{\frac{z}{y}+\frac{y}{x}}\ge\frac{3}{2}$$

$$\textit{We have}:\ \sum\frac{\left(\frac{x}{z}\right)^2}{\frac{z}{y}+\frac{y}{x}}\overset{CBS}{\ge}\frac{\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)^2}{2\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)}=\frac{1}{2}\left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)\overset{AM-GM}{\ge}\frac{3}{2}$$

$$Therefore,\quad \sum\frac{x^2}{z^3(zx+y^2)}\ge\frac{3}{2xyz}$$

819. If $a,b,c>0$ such that $a+b+c=1$ and $\lambda\ge 0$ then:

$$(\lambda a+b)(\lambda b+c)(\lambda c+a)+(1+a+\lambda b)(1+b+\lambda c)(1+c+\lambda a)\le$$
$$\le\left(\frac{\lambda+1}{3}\right)^3+\left(1+\frac{\lambda+1}{3}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$(\lambda a+b)(\lambda b+c)(\lambda c+a)\le\left(\frac{(\lambda+1)(a+b+c)}{3}\right)^3\overset{\sum a=1}{=}\left(\frac{\lambda+1}{3}\right)^3;(1)$$

$$(1+a+\lambda b)(1+b+\lambda c)(1+c+\lambda a)\le\left(\frac{3+(a+b+c)(\lambda+1)}{3}\right)^3\overset{\sum a=1}{=}$$

$$=\left(\frac{3+\lambda+1}{3}\right)^3=\left(1+\frac{\lambda+1}{3}\right)^3;(2)$$

From $(1),(2)$ *it follows that:*

$$(\lambda a+b)(\lambda b+c)(\lambda c+a)+(1+a+\lambda b)(1+b+\lambda c)(1+c+\lambda a)\le$$
$$\le\left(\frac{\lambda+1}{3}\right)^3+\left(1+\frac{\lambda+1}{3}\right)^3$$

820. If $a,b,c>0, a+b+c=3$ then:

$$\frac{5a-17}{a^2-3a-1}+\frac{5b-17}{b^2-3b-1}+\frac{5c-17}{c^2-3c-1}\le 3\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Abdul Aziz-Semarang-Indonesia

Since $0<a<3$ then $(a-1)^2(a-3)\le 0\leftrightarrow\frac{(a-4)^2}{1+3a-a^2}\le\frac{3}{a}$. Similarly,

$$(b-1)^2(b-3)\le 0\leftrightarrow\frac{(b-4)^2}{1+3b-b^2}\le\frac{3}{b}$$

$$(c-1)^2(c-3)\le 0\leftrightarrow\frac{(c-3)^2}{1+3c-c^2}\le\frac{3}{c}$$

Hence,

$$\sum_{cyc}\frac{(a-4)^2}{1+3a-a^2}\le 3\sum_{cyc}\frac{1}{a}\leftrightarrow\sum_{cyc}\left(\frac{17-5a}{1+3a-a^2}-1\right)\le 3\sum_{cyc}\frac{1}{a}$$

$$\leftrightarrow\sum_{cyc}\frac{5a-17}{a^2-3a-1}\le 3\left(1+\sum_{cyc}\frac{1}{a}\right)$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{5a-17}{a^2-3a-1}+\frac{5b-17}{b^2-3b-1}+\frac{5c-17}{c^2-3c-1}\le 3\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\frac{17-5a}{1+3a-a^2}+\frac{17-5b}{1+3b-b^2}+\frac{17-5c}{1+3c-c^2}\le 3+\frac{3}{a}+\frac{3}{b}+\frac{3}{c}$$

$$\text{Iff }\frac{17-5a}{1+3a-a^2}\le 1+\frac{3}{a}=\frac{a+3}{a}\leftrightarrow 17a-5a^2\le a+3a^2-a^3+3+9a-3a$$

$$\leftrightarrow a^3-5a^2+7a\le 3\leftrightarrow a(a^2-5a+7)\le 3\leftrightarrow a\left[\left(a-\frac{5}{2}\right)^2+\frac{3}{4}\right]\le 3\ true\ \forall a<3$$

and $a+b+c=3$. Similarly,

$$\frac{17-5b}{1+3b-b^2}\le 1+\frac{3}{b},\frac{17-5c}{1+3c-c^2}\le 1+\frac{3}{c}$$

821. If $0<a,b,c\le 1<\lambda$ then:

$$\sum_{cyc}\frac{(2\lambda-1)a-5\lambda-2}{a^2-3a-1}\le 3+\lambda\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Șerban-Romania

$$\frac{(2\lambda-1)a-5\lambda-2}{a^2-3a-1}\overset{?}{\le}1+\frac{\lambda}{a}=\frac{\lambda+a}{a},0<a,b,c\le 1<\lambda$$

Let $f:(0,1)\to\mathbb{R}, f(x)=x^2-3x-1, f'(x)=2x-3<2-3<0$, then $f\searrow(0,1)$

$$\lim_{x\to 0}f(x)=-1,\lim_{x\to 1}f(x)=-3\to f(x)<0,\forall x\in(0,1)$$

$$(2\lambda-1)a-5\lambda-2\le(2\lambda-1)\cdot 1-5\lambda-2=-3\lambda-3<0$$

$$2\lambda-1>2\cdot 1-1=1>0\to\frac{(2\lambda-1)a-5\lambda-2}{-a^2+3a+1}\le\frac{\lambda+a}{a}$$

$$\to(1-2\lambda)a^2+(5\lambda+2)a\le -a^2\lambda+3a\lambda+\lambda-a^3+3a^2+a$$

$\rightarrow (a-1)^2(a-\lambda) \le 0$, which is true $\forall a \in \mathbb{R}, a < \lambda$.

Therefore,

$$\sum_{cyc} \frac{(2\lambda-1)a-5\lambda-2}{a^2-3a-1} \le 3+\lambda\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

822. If $x, y, z > 0$ and $\lambda \ge 0$ then:

$$\sum_{cyc} \frac{x^2}{z^3(zx+\lambda y^2)} \ge \frac{3}{(\lambda+1)xyz}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

Let $x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$, then inequality becomes as:

$$\sum_{cyc} \frac{b^2c^4}{ab^2+\lambda a^2c} \ge \frac{3abc}{\lambda+1}; (1)$$

$$\sum_{cyc} \frac{(bc^2)^2}{ab^2+\lambda a^2c} \overset{Bergstrom}{\ge} \frac{(bc^2+ca^2+ab^2)^2}{(\lambda+1)(ab^2+bc^2+ca^2)} = \frac{bc^2+ca^2+ab^2}{\lambda+1}; (2)$$

From $(1),(2)$ we must show:

$$\frac{bc^2+ca^2+ab^2}{\lambda+1} \ge \frac{3abc}{\lambda+1} \leftrightarrow bc^2+ca^2+ab^2 \ge 3abc, \text{true},$$

Because: $bc^2+ca^2+ab^2 \ge 3\sqrt[3]{a^3b^3c^3} = 3abc$.

Solution 2 by Samir Cabiyev-Azerbaijan

$$\sum_{cyc} \frac{x^2}{z^3(zx+\lambda y^2)} = \sum_{cyc} \frac{\left(\frac{x}{y}\right)^2}{z^3\left(\frac{xz}{y^2}+\lambda\right)} \overset{Bergstrom}{\ge} \frac{\left(\sum \frac{x}{y}\right)^2}{\sum \frac{xz^4}{y^2}+\lambda\sum x^3} = \frac{\left(\sum \frac{x}{y}\right)^2}{\sum \frac{z^2}{y^2}\cdot xz^2+\lambda\sum x^3}; (1)$$

$$\because \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^2 \ge \left(3\cdot\sqrt[3]{\frac{xyz}{yzx}}\right)^2 = 9; x^3+y^3+z^3 \ge 3xyz$$

$$\sum_{cyc} \frac{z^2}{y^2}\cdot xz^2 \ge 3\sqrt[3]{\prod_{cyc}\left(\frac{z^2}{y^2}\cdot xz^2\right)} \ge 3\sqrt[3]{(xyz)^3} = 3xyz, \text{else } (1) \rightarrow$$

$$\frac{9}{\lambda\cdot 3xyz+3xyz}\ge\frac{9}{3xyz\cdot(\lambda+1)}=\frac{3}{(\lambda+1)xyz}$$

823. If $a,b,c>0$ such that $a^2+b^2+c^2=12$ and $n\in\mathbb{N},n\ge 2$ then:

$$\sum_{cyc}\frac{a^{2n}}{\sqrt{a^3+1}}\ge 4^n$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Şerban-Romania

$$\sum_{cyc}\frac{a^{2n}}{\sqrt{a^3+1}}=\sum_{cyc}\frac{a^{2n}}{\sqrt{(a+1)(a^2-a+1)}}\overset{AM-GM}{\ge}\sum_{cyc}\frac{(a^2)^n}{\frac{(a+1)+(a^2-a+1)}{2}}=$$

$$=2\sum_{cyc}\frac{(a^2)^n}{a^2+2}\overset{Holder}{\ge}2\cdot\frac{(a^2+b^2+c^2)^n}{3^{n-2}\sum(a^2+2)}=\frac{2\cdot 12^n}{3^{n-2}\cdot 18}=\frac{12^n}{3^{n-2}\cdot 9}=4^n$$

824. If $a,b,c>0,abc=1,0\le n\le 2$ then:

$$\sum a^3+n\sum\frac{ab}{a^2+b^2}\ge\frac{3}{2}(n+2)$$

Proposed by Marin Chirciu-Romania

Solution 1 by Asmat Qatea-Afghanistan

$$\sum_{cyc}a^3\overset{AM-GM}{\ge}3;(*)$$

$$\left(\sum_{cyc}\frac{ab}{a^2+b^2}\right)\left(\sum_{cyc}(a^2+b^2)\right)\overset{Holder}{\ge}\left(\sum_{cyc}\sqrt{ab}\right)^2\overset{AM-GM}{\ge}9$$

$$\sum_{cyc}\frac{ab}{a^2+b^2}\ge\frac{9}{\sum(a^2+b^2)}=\frac{3}{2};(**)$$

From $(*),(**)$ we have:

$$\sum a^3+n\sum\frac{ab}{a^2+b^2}\ge\frac{3}{2}(n+2)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ \sum\frac{ab}{a^2+b^2}\overset{AM-GM}{\leq}\sum\frac{ab}{2ab}=\frac{3}{2}\ (1)$$

$$and:\ \sum a^2b\overset{AM-GM}{\leq}\sum\frac{1}{3}(a^3+a^3+b^3)=\sum a^3\ \ (2)$$

$$\to\sum a^3+n\sum\frac{ab}{a^2+b^2}=\sum a^3+2\sum\frac{ab}{a^2+b^2}-(2-n)\sum\frac{ab}{a^2+b^2}\geq$$

$$\overset{(1)}{\geq}\sum a^3+2\sum\frac{1}{c(a^2+b^2)}-\frac{3}{2}(2-n)\overset{CBS}{\geq}\sum a^3+2.\frac{9}{\sum c(a^2+b^2)}-\frac{3}{2}(2-n)\geq$$

$$\overset{(2)}{\geq}\sum a^3+2.\frac{9}{2\sum a^3}-\frac{3}{2}(2-n)\overset{AM-GM}{\geq}2\sqrt{\sum a^3.\frac{9}{\sum a^3}}-\frac{3}{2}(2-n)=\frac{3}{2}(n+2).$$

$$Therefore,\quad \sum a^3+n\sum\frac{ab}{a^2+b^2}\geq\frac{3}{2}(n+2)$$

Solution 3 by Aggeliki Papaspyropoulou-Greece

For $n=0$: $a^3+b^3+c^3\geq 3\sqrt[3]{(abc)^3}=3$ true.

For $0<n\leq 2$:

$$a^3+b^3+c^3+n\left[\frac{abc}{c(a^2+b^2)}+\frac{abc}{a(b^2+c^2)}+\frac{abc}{b(a^2+c^2)}\right]\geq\frac{3}{2}(n+2)\Leftrightarrow$$

$$a^3+b^3+c^3+n\left[\frac{1^2}{c(a^2+b^2)}+\frac{1^2}{a(b^2+c^2)}+\frac{1^2}{b(a^2+c^2)}\right]\geq\frac{3}{2}(n+2)$$

$$a^3+b^3+c^3+n\left[\sum_{cyc}\frac{1^2}{c(a^2+b^2)}\right]\geq a^3+b^3+c^3+n\cdot\frac{9}{\sum ab(a+b)};(1)$$

$$a^3+b^3+c^3+3abc\geq\sum ab(a+b)\Leftrightarrow$$

$$a^3+b^3+c^3+3\geq\sum ab(a+b);(Schur's)$$

$$\frac{9n}{\sum ab(a+b)}\geq\frac{9n}{a^3+b^3+c^3+3};(2)$$

It is enough to prove that:

$$a^3+b^3+c^3+\frac{9n}{3+a^3+b^3+c^3}\geq\frac{3}{2}(n+2);(3)$$

$$(3)\Leftrightarrow 2(a^3+b^3+c^3)^2+6(a^3+b^3+c^3)+18n\geq(3n+6)(3+\sum a^3);(4)$$

$a^3+b^3+c^3=x\geq 3$. So, $(4)\Leftrightarrow$

$$2x^2+6x+18n\geq 9n-18+3nx+6x\Leftrightarrow$$

$$2x^2-3nx+9n-18\geq 0;\ (x\geq 3, 0<n\leq 2)$$

$$\Leftrightarrow (x-3)(2x+6-3n)\geq 0;\ (*), x\geq 3$$

$x-3\geq 0, 3n\leq 6$ and $2x+6\geq 12\rightarrow 12-3n>6\rightarrow (*)$ is true.

Equality holds if $a=b=c=1$.

Solution 4 by Fayssal Abdelli-Bejaia-Algerie

For $n=0$: $a^3+b^3+c^3\geq 3\sqrt[3]{(abc)^3}=3$ true.

$$(A): \sum a^3+n\sum\frac{ab}{a^2+b^2}\geq\frac{3}{2}(n+2)\Leftrightarrow$$

$$3+n\left(\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}\right)\geq\frac{3}{2}(n+2)\Leftrightarrow$$

$$3+n\left(\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}\right)\geq\frac{3}{2}n+3\Leftrightarrow$$

$$n\left(\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}\right)\geq\frac{3}{2}n$$

For $n\neq 0$ we have:

$$\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}\geq\frac{3}{2};\ (B)$$

We have: $(a-b)^2\geq 0\rightarrow a^2+b^2\geq 2ab\rightarrow\frac{ab}{a^2+b^2}\leq\frac{1}{2};\ (and\ analogs)$

$$\rightarrow\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}\leq\frac{3}{2};\ (C)$$

From $(B),(C)$ it follows that:

$$\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2}=\frac{3}{2}$$

Therefore,

$$\sum a^3+n\sum\frac{ab}{a^2+b^2}\geq 3+n\sum\frac{ab}{a^2+b^2}=3+n\cdot\frac{3}{2}=\frac{3}{2}(n+2)$$

825. If $a,b,c>0$ then:

$$\left(\frac{a}{a+b+c}\right)^{\frac{a}{b+c}}\cdot\left(\frac{b}{a+b+c}\right)^{\frac{b}{c+a}}\cdot\left(\frac{c}{a+b+c}\right)^{\frac{c}{a+b}}\ge\sqrt{\frac{abc}{(a+b+c)^3}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Remus Florin Stanca-Romania

Let $f(x)x\cdot\log\left(\frac{x}{x+1}\right),\frac{\partial f}{\partial x}=(x\log x-x\log(x+1))'=\log x+1-\log(x+1)-\frac{x}{x+1}$

$$\to\frac{\partial^2 f}{\partial x^2}=\frac{1}{x}-\frac{1}{x+1}-\left(1-\frac{1}{x+1}\right)'=\frac{1}{x}-\frac{1}{x+1}-\frac{1}{(x+1)^2}=\frac{1}{x(x+1)^2}$$

$\to f$ is convexe, now from Jensen's Inequality: $\forall x_1,x_2x_3\in R$ and $t_1,t_2,t_3\in(0,1)$ with $t_1+t_2+t_3=1$, we have: $t_1f(x_1)+t_2f(x_2)+t_3f(x_3)\ge f(t_1x_1+t_2x_2+t_3x_3)$

Let $x_1=\frac{a}{b+c},x_2=\frac{b}{c+a},x_3=\frac{c}{a+b}$ and $t_1=t_2=t_3=\frac{1}{3}$

$$\frac{1}{3}\sum_{cyc}\frac{a}{b+c}\log\left(\frac{\frac{a}{b+c}}{\frac{a}{b+c}+1}\right)\ge\left(\frac{1}{3}\sum_{cyc}\frac{a}{b+c}\right)\log\left(\frac{\frac{1}{3}\sum\frac{a}{b+c}}{\frac{1}{3}\sum\frac{a}{b+c}+1}\right)\ge$$

$$\ge\frac{3}{2}\cdot\frac{1}{3}\cdot\log\left(\frac{\frac{1}{3}\sum\frac{a}{b+c}}{\frac{1}{3}\sum\frac{a}{b+c}+1}\right)=\frac{1}{2}\cdot\log\left(\frac{\frac{1}{3}\sum\frac{a}{b+c}}{\frac{1}{3}\sum\frac{a}{b+c}+1}\right);(1)$$

Let $g(x)=\frac{x}{x+1}\to\frac{\partial g}{\partial x}=\frac{1}{(x+1)^2}>0\to g$ is increasing. So,

$$\frac{1}{3}\sum_{cyc}\frac{a}{b+c}\ge\frac{1}{3}\cdot\frac{3}{2}=\frac{1}{2}\to g\left(\frac{1}{3}\sum_{cyc}\frac{a}{b+c}\right)\ge g\left(\frac{1}{2}\right)$$

$$\to\frac{\frac{1}{3}\sum\frac{a}{b+c}}{\frac{1}{3}\sum\frac{a}{b+c}+1}\ge\frac{\frac{1}{2}}{\frac{1}{2}+1}=\frac{3}{2}\to\frac{1}{2}\cdot\log\left(\frac{\frac{1}{3}\sum\frac{a}{b+c}}{\frac{1}{3}\sum\frac{a}{b+c}+1}\right)\ge\frac{1}{2}\log\left(\frac{1}{3}\right)\to$$

$$\overset{(1)}{\Rightarrow}\frac{1}{3}\sum_{cyc}\left(\frac{a}{b+c}\log\left(\frac{a}{a+b+c}\right)\right)\ge\frac{1}{2}\cdot\log\left(\frac{1}{3}\right);(2)$$

$$\sqrt[3]{abc}\le\frac{a+b+c}{3}\to\frac{\sqrt[3]{abc}}{a+b+c}\le\frac{1}{3}\to\log\left(\frac{1}{3}\right)\ge\log\left(\frac{\sqrt[3]{abc}}{a+b+c}\right)\overset{(2)}{\Rightarrow}$$

$$\frac{1}{3}\sum_{cyc}\left(\frac{a}{b+c}\log\left(\frac{a}{a+b+c}\right)\right)\geq\frac{1}{2}\log\left(\frac{\sqrt[3]{abc}}{a+b+c}\right)$$

$$\rightarrow\sum_{cyc}\log\left(\frac{a}{a+b+c}\right)^{\frac{a}{b+c}}\geq\log\left(\sqrt{\frac{abc}{(a+b+c)^3}}\right)$$

Therefore,

$$\left(\frac{a}{a+b+c}\right)^{\frac{a}{b+c}}\cdot\left(\frac{b}{a+b+c}\right)^{\frac{b}{c+a}}\cdot\left(\frac{c}{a+b+c}\right)^{\frac{c}{a+b}}\geq\sqrt{\frac{abc}{(a+b+c)^3}}$$

Solution 2 by Daoudi Abdessattar-Tunisia

$x,y,z\in(0,1)$ such that $x+y+z=1$. We want to prove:

$$x^{\frac{x}{1-x}}\cdot y^{\frac{y}{1-y}}\cdot z^{\frac{z}{1-z}}\geq(xyz)^{\frac{1}{2}}\rightarrow\sum_{cyc}\frac{\log x}{1-x}\geq\frac{3}{2}\sum_{cyc}\log x$$

Let $f(t)=\frac{\log t}{1-t}-\frac{3}{2}\log t\,,0<t<1\rightarrow f'(t)=\frac{1}{1-t}-\frac{1}{2t}+\frac{\log t}{(1-t)^2}$

$f''(t)=\frac{1}{2t^2}+\frac{g(t)}{t(1-t)^3}>0$, because $g(t)>0$

Using Jensen's Inequality, we get:

$\sum f(t)\geq 3f\left(\frac{1}{3}\right)$ and inequality is proved.

Now, we take: $x=\frac{a}{a+b+c},y=\frac{b}{a+b+c},z=\frac{c}{a+b+c};a,b,c>0$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\left(\frac{a}{a+b+c}\right)^{\frac{a}{b+c}}\cdot\left(\frac{b}{a+b+c}\right)^{\frac{b}{c+a}}\cdot\left(\frac{c}{a+b+c}\right)^{\frac{c}{a+b}}\geq\sqrt{\frac{abc}{(a+b+c)^3}}$$

$$\rightarrow\left(\frac{a}{a+b+c}\right)^{\frac{a}{b+c}-\frac{1}{2}}\cdot\left(\frac{b}{a+b+c}\right)^{\frac{b}{c+a}-\frac{1}{2}}\cdot\left(\frac{c}{a+b+c}\right)^{\frac{c}{a+b}-\frac{1}{2}}\geq 1$$

$$\rightarrow\left(\frac{\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\right)}{\frac{\frac{2a-(b+c)}{2(b+c)}}{\frac{a}{a+b+c}}+\frac{\frac{2b-(c+a)}{2(c+a)}}{\frac{b}{a+b+c}}+\frac{\frac{2c-(a+b)}{2(a+b)}}{\frac{c}{a+b+c}}}\right)^{\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}}\geq 1$$

$$\rightarrow \sum_{cyc} \frac{a+b+c}{a(b+c)}\left(2a-(b+c)\right) \le 2\sum_{cyc} \frac{a}{b+c} - \frac{3}{2}$$

$$\rightarrow \sum_{cyc} \left(\frac{1}{b+c} + \frac{1}{a}\right)\left(2a-(b+c)\right) \le 2\sum_{cyc} \frac{a}{b+c} - \frac{3}{2}$$

$$\rightarrow \sum_{cyc} \left(\frac{a}{b} + \frac{b}{a}\right) \ge 6 \ (\mathrm{AM-GM})$$

826. If $x, y, z > 0, \lambda \ge \frac{7}{8}$ then:

$$\lambda \sum \frac{x}{y} + \frac{8xyz}{\prod(x+y)} \ge 3\lambda + 1$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\lambda \sum \frac{x}{y} + \frac{8xyz}{\prod(x+y)} \ge 3\lambda + 1; (*)$$

$$(*) \leftrightarrow \lambda\left(\sum \frac{x}{y} - 3\right) + \frac{8xyz}{\prod(x+y)} \ge 1$$

$$Since \sum \frac{x}{y} - 3 \overset{AM-GM}{\ge} 0 \rightarrow It\ is\ suffices\ to\ prove: \frac{7}{8}\left(\sum \frac{x}{y} - 3\right) + \frac{8xyz}{\prod(x+y)} \ge 1$$

$$\leftrightarrow 7\sum \frac{x}{y} + \frac{64}{\prod\left(\frac{x}{y}+1\right)} \ge 29 \ (**)$$

$$Let\ a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}, p = \sum a \ and\ q = \sum ab, abc = 1$$

$$\rightarrow (**) \leftrightarrow 7\sum a + \frac{64}{\prod(a+1)} \ge 29 \leftrightarrow 7p + \frac{64}{p+q+2} \ge 29$$

$$We\ know\ that: p^2 \ge 3q \rightarrow It\ is\ suffices\ to\ prove: 7p + \frac{64}{p + \frac{p^2}{3} + 2} \ge 29$$

$$\leftrightarrow 7p^3 - 8p^2 - 45p + 18 \ge 0 \leftrightarrow (p-3)\left(7p^2 + 11p + 2(p-3)\right) \ge 0$$

$$Which\ is\ true\ because\ p \overset{AM-GM}{\ge} 3. (\because abc = 1)$$

$$Therefore,\quad \lambda\sum\frac{x}{y}+\frac{8xyz}{\prod(x+y)}\ge 3\lambda+1$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{5x-(2y+3z)}{y}+\frac{5y-(2z+3x)}{z}+\frac{5z-(2x+3y)}{x}\ge$$

$$\ge\frac{1}{3}\big(5x-(2y+3z)+5y-(2z+3x)+5z-(2x+3y)\big)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Hence,

$$5\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\ge 3\left(\frac{x}{z}+\frac{z}{y}+\frac{y}{x}\right)+6$$

$$5(x^2z+z^2y+y^2x)\ge 3(x^2y+y^2z+z^2x)+6xyz$$

$$8(x^2z+z^2y+y^2x)\ge 3(x+y)(y+z)(z+x)$$

$$\frac{x^2z+z^2y+y^2x}{xyz}\ge\frac{3}{8}\cdot\frac{(x+y)(y+z)(z+x)}{xyz}$$

$$\frac{\lambda(x^2z+z^2y+y^2x)}{xyz}\ge\frac{3\lambda}{8}\cdot\frac{(x+y)(y+z)(z+x)}{xyz}$$

$$\lambda\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)+\frac{8xyz}{(x+y)(y+z)(z+x)}\ge 3\lambda+1$$

$$\frac{3\lambda}{8}\cdot\frac{(x+y)(y+z)(z+x)}{xyz}+\frac{8xyz}{(x+y)(y+z)(z+x)}\ge 3\lambda+1$$

$$a\coloneqq\frac{(x+y)(y+z)(z+x)}{8xyz}$$

$$3\lambda a+\frac{1}{a}\ge 3\lambda+1, a\ge 1\to 3\lambda a^2+1\ge 3\lambda a+a$$

$$3\lambda a(a-1)\ge(a-1),\ \text{true because } a\ge 1.$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

Using Walker's inequality:

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\ge\frac{1}{3}(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=$$

$$=\frac{1}{3}\cdot\frac{(x+y+z)(xy+yz+zx)}{xyz}=\frac{1}{3}\cdot\frac{(x+y+z)(xy+yz+zx)-xyz+xyz}{xyz}=$$

$$=\frac{(x+y)(y+z)(z+x)}{3xyz}+\frac{1}{3}$$

$$Let\ t=\frac{(x+y)(y+z)(z+x)}{xyz}\overset{AM-GM}{\geq}\frac{2\sqrt{xy}\cdot 2\sqrt{yz}\cdot 2\sqrt{yz}}{xyz}=8\rightarrow t\geq 8$$

Since $\lambda\geq\frac{7}{8}$, we need to prove that:

$$\lambda\left(\frac{t}{3}+\frac{1}{3}\right)+\frac{8}{t}\geq 3\lambda+1\Leftrightarrow\lambda\left(\frac{t}{3}-\frac{8}{3}\right)+\frac{8}{t}-1\geq 0$$

$$\frac{\lambda}{3}(t-8)+\frac{8-t}{t}\geq 0\Leftrightarrow(t-8)\left(\frac{\lambda}{3}-\frac{1}{t}\right)\geq 0$$

$$(t-8)\cdot\frac{\lambda t-3}{3t}\geq 0$$

Which is true because $t\geq 8,\lambda\geq\frac{7}{8}\rightarrow t-8\geq 0$ and $\lambda t-3\geq\frac{7}{8}\cdot 8-3=4>0.$

827. Let $a,b,c>0$. Prove that :

$$\sum\frac{ab}{\sqrt{a+b}}>2\sqrt{abc}$$

Proposed by Olimjon Jalilov-Uzbekistan

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$From\ H\ddot{o}lder, we\ have:\left(\sum\frac{ab}{\sqrt{a+b}}\right)^2\left(\sum ab(a+b)\right)\geq\left(\sum ab\right)^3$$

$$\rightarrow\left(\sum\frac{ab}{\sqrt{a+b}}\right)^2\geq\frac{(\sum ab)^3}{\sum ab(a+b)}\overset{?}{>}4abc$$

$$\leftrightarrow\left(\sum ab\right)^3>4abc\sum ab(a+b)=4\sum(ab)^2(ac)+4\sum(ab)^2(bc)$$

$$\leftrightarrow\left(\sum x\right)^3>4\sum xy(x+y)\ ,where\ x=ab,y=bc,z=ca.$$

$$\leftrightarrow\sum x^3+6xyz>\sum xy(x+y)$$

$$Which\ is\ true\ from\ Schur's\ inequality\left(\sum x^3+3xyz\geq\sum xy(x+y)\right).$$

$$Therefore,\quad\sum\frac{ab}{\sqrt{a+b}}>2\sqrt{abc}$$

Solution 2 by Daoudi Abdessattar-Tunisia

$$\frac{\sqrt{x}}{\sqrt{y+z}}+\frac{\sqrt{y}}{\sqrt{z+x}}+\frac{\sqrt{z}}{\sqrt{x+y}}=\sum_{cyc}\frac{x}{\sqrt{x(y+z)}}\overset{Am-GM}{\geq}\sum_{cyc}\frac{2x}{x+y+z}=2;(1)$$

The inequality is strict since $xyz\neq 0$.

Let $x=ab, y=bc, z=ca, (a,b,c>0)$

$$(1)\rightarrow\sum_{cyc}\frac{\sqrt{ab}}{\sqrt{ac+bc}}>2\rightarrow\sum_{cyc}\frac{ab}{\sqrt{a+b}}>2\sqrt{abc}$$

828. If $a,b,c>0$ then:

$$\frac{a}{3b+\sqrt[7]{ab^6}}+\frac{b}{3c+\sqrt[7]{bc^6}}+\frac{c}{3a+\sqrt[7]{ca^6}}\geq\frac{3}{8}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum\frac{a}{3b+\sqrt[7]{ab^6}}\overset{AM-GM}{\geq}\sum\frac{a}{3b+\frac{a+6b}{7}}=7\sum\frac{a}{27b+a}=7\sum\frac{a^2}{27ab+a^2}\geq$$

$$\overset{CBS}{\geq}7.\frac{(\sum a)^2}{\sum(27ab+a^2)}=\frac{7\sum a^2+14\sum ab}{27\sum ab+\sum a^2}\overset{?}{\geq}\frac{3}{4}$$

$$\leftrightarrow 28\sum a^2+56\sum ab\geq 81\sum ab+3\sum a^2 \;\leftrightarrow\; \sum a^2\geq\sum ab$$

Which is true.

Therefore,

$$\sum\frac{a}{3b+\sqrt[7]{ab^6}}\geq\frac{3}{4}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$ we give $a=x^7, b=y^7, c=z^7$, hence

$$\frac{a}{3b+\sqrt[7]{ab^6}}+\frac{b}{3c+\sqrt[7]{bc^6}}+\frac{c}{3a+\sqrt[7]{ca^6}}=\frac{x^7}{3y^7+xy^6}+\frac{y^7}{3z^7+yz^6}+\frac{z^7}{3x^6+zx^6}=$$

$$=\frac{x^7}{y^6(3y+x)}+\frac{y^7}{z^6(3z+y)}+\frac{z^7}{x^6(3x+z)}=$$

$$=\frac{\left(\frac{x}{y}\right)^6}{3\left(\frac{y}{x}\right)+1}+\frac{\left(\frac{y}{z}\right)^6}{3\left(\frac{z}{y}\right)+1}+\frac{\left(\frac{z}{x}\right)^6}{3\left(\frac{x}{z}\right)+1}\geq\frac{\left(\left(\frac{x}{y}\right)^3+\left(\frac{y}{z}\right)^3+\left(\frac{z}{x}\right)^3\right)^2}{3\left(\frac{y}{x}+\frac{x}{z}+\frac{z}{y}\right)+3}=$$

$$=\frac{\left(\frac{x}{y}\right)^6+\left(\frac{y}{z}\right)^6+\left(\frac{z}{x}\right)^6+2\left(\left(\frac{x}{z}\right)^3+\left(\frac{z}{y}\right)^3+\left(\frac{y}{x}\right)^3\right)}{3\left(\frac{y}{x}+\frac{x}{z}+\frac{z}{y}\right)+3}\ge$$

$$\ge\frac{\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)+2\left(\frac{x}{z}+\frac{z}{y}+\frac{y}{x}\right)}{3\left(\frac{y}{x}+\frac{x}{z}+\frac{z}{y}\right)+3}>\frac{3}{8}$$

Solution 3 by Daoudi Abdessattar-Tunisia

$$\sum_{cyc}\frac{a}{3b+\sqrt[7]{ab^6}}\overset{AM-GM}{\ge}\sum_{cyc}\frac{a}{3b+\frac{a+6b}{7}}=7\sum_{cyc}\frac{a}{27b+a}=7\sum_{cyc}\frac{a^2}{27ab+a^2}\overset{CBS}{\ge}$$

$$\ge 7\cdot\frac{(\sum a)^2}{\sum(27ab+a^2)}=\frac{7\sum a^2+14\sum ab}{27\sum ab+\sum a^2}\overset{(*)}{\ge}\frac{3}{4}$$

$$(*)\Leftrightarrow 28\sum a^2+56\sum ab\ge 81\sum ab+3\sum a^2\Leftrightarrow \sum a^2\ge\sum ab\ (true\ by\ AM-GM)$$

Therefore,

$$\frac{a}{3b+\sqrt[7]{ab^6}}+\frac{b}{3c+\sqrt[7]{bc^6}}+\frac{c}{3a+\sqrt[7]{ca^6}}\ge\frac{3}{8}$$

829. If $m<1, n\in IN_{>1}, a_k, b_k>0$ and $\sum_{k=1}^n a_k=a, a_{n+1}=a_1$, then :

$$\sum_{k=1}^n\left(\frac{a_k}{a_{k+1}}+\frac{1}{a_k}\right)^{\frac{1}{m}}\ge\left(\frac{2}{\sqrt{a}}\right)^{\frac{1}{m}}.n^{1+\frac{1}{2m}}$$

Proposed by D.M.Bătinetu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\sum_{k=1}^n\left(\frac{a_k}{a_{k+1}}+\frac{1}{a_k}\right)^{\frac{1}{m}}\overset{AM-GM}{\ge}\sum_{k=1}^n\left(2\sqrt{\frac{a_k}{a_{k+1}}.\frac{1}{a_k}}\right)^{\frac{1}{m}}=2^{\frac{1}{m}}\sum_{k=1}^n\frac{1}{a_{k+1}^{\frac{1}{2m}}}$$

$$And:\left(\sum_{k=1}^n\frac{1}{a_{k+1}^{\frac{1}{2m}}}\right)\left(\sum_{k=1}^n a_{k+1}\right)^{\frac{1}{2m}}\overset{Hölder}{\ge}\left(\sum_{k=1}^n 1\right)^{1+\frac{1}{2m}}=n^{1+\frac{1}{2m}}$$

$$\to\sum_{k=1}^n\frac{1}{a_{k+1}^{\frac{1}{2m}}}\ge\left(\frac{1}{\sqrt{a}}\right)^{\frac{1}{m}}.n^{1+\frac{1}{2m}}$$

Therefore,

$$\sum_{k=1}^{n}\left(\frac{a_k}{a_{k+1}}+\frac{1}{a_k}\right)^{\frac{1}{m}} \geq \left(\frac{2}{\sqrt{a}}\right)^{\frac{1}{m}}.n^{1+\frac{1}{2m}}$$

830. If $a,b,c,d \in (0,1)$ *or* $a,b,c,d \in (1,\infty)$ then :

$$\log_{bc^2d^3}(a^3b^2c)+\log_{cd^2a^3}(b^3c^2d)+\log_{da^2b^3}(c^3d^2a)+\log_{ab^2c^3}(d^3a^2b) \geq 4$$

Proposed by Marian Ursărescu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*) : \log_{bc^2d^3}(a^3b^2c)+\log_{cd^2a^3}(b^3c^2d)+\log_{da^2b^3}(c^3d^2a)+\log_{ab^2c^3}(d^3a^2b) \geq 4$$

Let $a,b,c,d \in (1,\infty), x=\log a, y=\log b, z=\log c, t=\log d, x,y,z,t>0.$

$$(*) \leftrightarrow \sum\frac{3\log a+2\log b+\log c}{\log b+2\log c+3\log d} \geq 4 \leftrightarrow \sum\frac{3x+2y+z}{y+2z+3t} \geq 4$$

$$\leftrightarrow \sum\left(\frac{3x+2y+z}{y+2z+3t}+1\right) \geq 8 \leftrightarrow \sum\frac{3(x+y+z+t)}{y+2z+3t} \geq 8$$

$$\leftrightarrow 3\left(\sum x\right)\left(\sum\frac{1}{y+2z+3t}\right) \geq 8 \leftrightarrow \frac{1}{2}\left[\sum(y+2z+3t)\right].\left(\sum\frac{1}{y+2z+3t}\right) \geq 8$$

Which is true from CBS, because :

$$\left[\sum(y+2z+3t)\right].\left(\sum\frac{1}{y+2z+3t}\right) \geq \left(\sum 1\right)^2 = 16$$

If $a,b,c,d \in (0,1),$

let $x=-\log a, y=-\log b, z=-\log c, t=-\log d, x,y,z,t>0$

Similarly to first case, we have : $\sum\log_{bc^2d^3}(a^3b^2c) \geq 4.$

Therefore,

$$\log_{bc^2d^3}(a^3b^2c)+\log_{cd^2a^3}(b^3c^2d)+\log_{da^2b^3}(c^3d^2a)+\log_{ab^2c^3}(d^3a^2b) \geq 4$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\log_{bc^2d^3}(a^3b^2c)+\log_{cd^2a^3}(b^3c^2d)+\log_{da^2b^3}(c^3d^2a)+\log_{ab^2c^3}(d^3a^2b) =$$

$$=\frac{\log(a^3b^2c)}{\log(bc^2d^3)}+\frac{\log(b^3c^2d)}{\log(cd^2a^3)}+\frac{\log(c^3d^2a)}{\log(da^2b^3)}+\frac{\log(d^3a^2b)}{\log(ab^2c^3)} \geq$$

$$\geq 4\cdot\sqrt[4]{\frac{\log(a^3b^2c)}{\log(bc^2d^3)}\cdot\frac{\log(b^3c^2d)}{\log(cd^2a^3)}\cdot\frac{\log(c^3d^2a)}{\log(da^2b^3)}\cdot\frac{\log(d^3a^2b)}{\log(ab^2c^3)}} = 4$$

Because $\frac{log(a^3b^2c)}{log(bc^2d^3)}, \frac{log(b^3c^2d)}{log(cd^2a^3)}, \frac{log(c^3d^2a)}{log(da^2b^3)}, \frac{log(d^3a^2b)}{log(ab^2c^3)} > 0, \forall a,b,c,d \in (\mathbf{0},\mathbf{1})$ or $(\mathbf{1},\infty)$

831. If $a,b,c \geq 0$, then:

$$\sum \frac{(a^6+b^6)(a^8+b^8)}{(a^5+b^5)(a^{11}+b^{11})} \leq \sum \frac{1}{a^2-ab+b^2}$$

Proposed Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{(a^6+b^6)(a^8+b^8)}{(a^5+b^5)(a^{11}+b^{11})} \overset{?}{\leq} \frac{1}{a^2-ab+b^2} \leftrightarrow$$

$$(a^3+b^3)(a^6+b^6)(a^8+b^8) \leq (a+b)(a^5+b^5)(a^{11}+b^{11})$$

$$From\ CBS, we\ have:\ (a+b)(a^5+b^5) \geq (a^3+b^3)^2$$

$$\rightarrow It\ is\ suffices\ to\ prove:$$

$$(a^6+b^6)(a^8+b^8) \leq (a^3+b^3)(a^{11}+b^{11})$$

$$\leftrightarrow a^8b^6+a^6b^8 \leq a^{11}b^3+a^3b^{11} \leftrightarrow (ab)^3(a^5-b^5)(a^3-b^3) \geq 0$$

$$Which\ is\ true, because\ a^5-b^5\ and\ a^3-b^3\ have\ the\ same\ sign.$$

$$\rightarrow \frac{(a^6+b^6)(a^8+b^8)}{(a^5+b^5)(a^{11}+b^{11})} \leq \frac{1}{a^2-ab+b^2}\ (and\ analogs)$$

$$Therefore,$$

$$\sum \frac{(a^6+b^6)(a^8+b^8)}{(a^5+b^5)(a^{11}+b^{11})} \leq \sum \frac{1}{a^2-ab+b^2}$$

832. If $a_i > 0, i \in \overline{1,n}$ then:

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \sqrt[n]{\prod_{i=1}^{n} a_i} + \log\left(\frac{\frac{1}{n}\sum_{i=1}^{n} a_i}{\sqrt[n]{\prod_{i=1}^{n} a_i}}\right)^{\sqrt[n]{\prod_{i=1}^{n} a_i}}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Ravi Prakash-New Delhi-India

$$\text{Let } A = \frac{1}{n}\sum_{i=1}^{n} a_i > 0 \text{ and } G = \sqrt[n]{\prod_{i=1}^{n} a_i} > 0$$

We have to show that: $A \geq G + G \cdot \log\left(\frac{A}{G}\right)$

If $A = G$, there is nothing to show.

Assume $A > G$. Let $f(x) = x, x \in [G, A]$ and $g(x) = \log x, x \in [G, A]$
By the Cauchy's mean value theorem, $\exists c \in (G, A)$ such that

$$\frac{f(A) - f(G)}{g(A) - g(G)} = \frac{f'(c)}{g'(c)} = c > G$$

$$A - G > G \cdot \log\left(\frac{A}{G}\right) \to A > G + G \cdot \log\left(\frac{A}{G}\right)$$

833. If $a, b, c, d > 1$ then

$$\sum_{cyc} \log_a\left(\frac{b^{n+1} + c^{n+1} + d^{n+1}}{b^n + c^n + d^n}\right) \geq 4, n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$From\ Chebyshev's:\ (b^n + c^n + d^n)(b + c + d) \leq 3(b^{n+1} + c^{n+1} + d^{n+1})$$

$$\frac{b^{n+1} + c^{n+1} + d^{n+1}}{b^n + c^n + d^n} \geq \frac{b + c + d}{3} \geq \sqrt[3]{bcd}; (and\ analogs)$$

$$We\ must\ show: \sum_{cyc} \log_a\left(\sqrt[3]{bcd}\right) \geq 4 \Leftrightarrow \frac{1}{3}\sum_{cyc} \log_a(bcd) \geq 4$$

$$\Leftrightarrow \frac{1}{3}\sum_{cyc}(\log_a b + \log_b a) \geq 4, true\ because \log_a b + \log_b a \geq 2, \forall a, b > 1.$$

834. Let $a, b, c \in (0, 1)$ such that $a + b + c = 1$, then prove:

$$\frac{a}{\sqrt{4b^3 + c + 6}} + \frac{b}{\sqrt{4c^3 + a + 6}} + \frac{c}{\sqrt{4a^3 + b + 6}} < \frac{\sqrt{3}}{3}$$

Proposed by Florică Anastase-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$4b^3 + c + 6 > 3, \forall b, c \in (0, 1) \to \frac{a}{\sqrt{4b^3 + c + 6}} < \frac{a}{\sqrt{3}}\ (and\ analogs)$$

$$\to \sum \frac{a}{\sqrt{4b^3 + c + 6}} < \sum \frac{a}{\sqrt{3}} = \frac{\sqrt{3}}{3}\sum a = \frac{\sqrt{3}}{3}$$

$$Therefore,$$

$$\sum \frac{a}{\sqrt{4b^3+c+6}} < \frac{\sqrt{3}}{3}$$

Solution 2 by proposer

$$\frac{a}{\sqrt{4b^3+c+6}} + \frac{b}{\sqrt{4c^3+a+6}} + \frac{c}{\sqrt{4a^3+b+6}} =$$

$$= \sqrt{a} \cdot \sqrt{\frac{a}{4b^3+c+6}} + \sqrt{b} \cdot \sqrt{\frac{b}{4c^3+a+6}} + \sqrt{c} \cdot \sqrt{\frac{c}{4a^3+b+6}} \overset{CBS}{\leq}$$

$$\leq \sqrt{(a+b+c) \cdot \left(\frac{a}{4b^3+c+6} + \frac{b}{4c^3+a+6} + \frac{c}{4a^3+c+6}\right)} =$$

$$= \sqrt{1 \cdot \left(\frac{a}{4b^3+c+6} + \frac{b}{4c^3+a+6} + \frac{c}{4a^3+c+6}\right)}; (1)$$

Now,

$$4x^3 - 3x + 1 = 4x^3 + 4x^2 - 4x^2 - 4x + x + 1 = (4x^2 - 4x + 1)(x+1) =$$

$$= (2x-1)^2(x+1) \geq 0, \forall x \geq -1, \text{then } 4x^3 \geq 3x - 1, \forall x \geq -1.$$

Hence,

$$4a^3 + b + 6 \geq 3a - 1 + b + 6 = 3a + b + 2 + 3 > 3a + b + 2b + 3c$$
$$= 3(a+b+c)$$

So, we have:

$$\frac{c}{4a^3+b+6} < \frac{c}{3(a+b+c)}$$

Similarly, we get:

$$\frac{a}{4b^3+c+6} < \frac{a}{3(a+b+c)} \text{ and } \frac{b}{4c^3+a+6} < \frac{b}{3(a+b+c)}$$

Thus,

$$\frac{a}{4b^3+c+6} + \frac{b}{4c^3+a+6} + \frac{c}{4a^3+c+6} < \frac{1}{3}; (2)$$

From $(1), (2)$ it follows that:

$$\frac{a}{\sqrt{4b^3+c+6}} + \frac{b}{\sqrt{4c^3+a+6}} + \frac{c}{\sqrt{4a^3+b+6}} < \frac{\sqrt{3}}{3}$$

835. If $a, b, c > 1$ then prove that:

$$\log_{ab^2c^2} a + \log_{a^2bc^2} b + \log_{a^2b^2c} c \geq \frac{3}{5}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Marian Ursărescu-Romania

$$\log_{ab^2c^2} a + \log_{a^2bc^2} b + \log_{a^2b^2c} c \geq \frac{3}{5}$$

$$\Leftrightarrow \sum_{cyc} \frac{\log a}{\log ab^2c^2} \geq \frac{3}{5} \Leftrightarrow \sum_{cyc} \frac{\log a}{\log a + 2\log b + 2\log c} \geq \frac{3}{5}; (1)$$

Let us denote: $\log a = x, \log b = y, \log c = z; x, y, z > 0; (2)$

From $(1), (2)$ we must show that:

$$\sum_{cyc} \frac{x}{x + 2y + 2z} \geq \frac{3}{5}; (3)$$

$$\sum_{cyc} \frac{x}{x + 2y + 2z} = \sum_{cyc} \frac{x^2}{x^2 + 2xy + 2xz} \overset{Bergstrom}{\geq} \frac{(x+y+z)^2}{x^2 + y^2 + z^2 + 4xy + 4yz + 4zx}; (4)$$

From $(3), (4)$ we must show that:

$$\frac{(x+y+z)^2}{x^2 + y^2 + z^2 + 4xy + 4yz + 4zx} \geq \frac{3}{5} \Leftrightarrow$$

$$5(x+y+z)^2 \geq 3(x^2+y^2+z^2) + 12(xy+yz+zx) \Leftrightarrow$$

$$5(x^2+y^2+z^2) + 10(xy+yz+zx) \geq 3(x^2+y^2+z^2) + 12(xy+yz+zx) \Leftrightarrow$$

$$x^2 + y^2 + z^2 \geq xy + yz + zx \text{ } true.$$

836. If $x, y, z \in (0, 1)$, then prove:

$$\sum \frac{z^2}{(1-x^2)(y^2+yz+z^2)} \geq \frac{3\sqrt{3}}{2} \cdot \frac{xy+yz+zx}{x+y+z}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$From\ AM-GM, we\ have:\ x^2 + \frac{\sqrt{3}}{9x} + \frac{\sqrt{3}}{9x} \geq 3\sqrt[3]{x^2 \cdot \frac{\sqrt{3}}{9x} \cdot \frac{\sqrt{3}}{9x}} = 1$$

$$\rightarrow 1 - x^2 \leq \frac{2\sqrt{3}}{9x} \ (And\ analogs) \rightarrow$$

$$\sum \frac{z^2}{(1-x^2)(y^2+yz+z^2)} \ge \frac{9}{2\sqrt{3}}\sum \frac{xz^2}{y^2+yz+z^2} = \frac{3\sqrt{3}}{2}\sum \frac{(zx)^2}{xy^2+xyz+xz^2} \ge$$

$$\overset{CBS}{\ge} \frac{3\sqrt{3}}{2}\cdot\frac{(\sum zx)^2}{\sum(xy^2+xyz+xz^2)} = \frac{3\sqrt{3}}{2}\cdot\frac{(xy+yz+zx)^2}{(xy+yz+zx)(x+y+z)}$$

$$Therefore, \quad \sum \frac{z^2}{(1-x^2)(y^2+yz+z^2)} \ge \frac{3\sqrt{3}}{2}\cdot\frac{xy+yz+zx}{x+y+z}$$

837. Prove that:

$$\frac{a}{3a^2+2b^2+c^2} \le \frac{1}{18}\left(\frac{2}{b}+\frac{1}{c}\right), \forall a,b,c>0$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \frac{3a^2+2b^2+c^2}{a} = \frac{3a^2+(2b^2+c^2)}{a} \overset{AM-GM}{\ge} \frac{2\sqrt{3a^2.(2b^2+c^2)}}{a}$$

$$= 2\sqrt{3(b^2+b^2+c^2)} \ge$$

$$\overset{CBS}{\ge} 2(b+b+c) \overset{CBS}{\ge} 2.\frac{9}{\frac{1}{b}+\frac{1}{b}+\frac{1}{c}} = \frac{18}{\frac{2}{b}+\frac{1}{c}}$$

$$Therefore, \quad \frac{a}{3a^2+2b^2+c^2} \le \frac{1}{18}\left(\frac{2}{b}+\frac{1}{c}\right), \forall a,b,c>0$$

Solution 2 by Michael Sterghiou-Greece

$$\frac{a}{3a^2+2b^2+c^2} \le \frac{1}{18}\left(\frac{2}{b}+\frac{1}{c}\right), \forall a,b,c>0; (1)$$

$$\to (3a^2+2b^2+c^2)(2c+b) \ge 18abc; (2)$$

But $(3a^2+2b^2+c^2)(2c+b) \overset{AM-GM}{\ge} \left(6\cdot\sqrt[6]{a^6b^4c^2}\right)\cdot\left(3\cdot\sqrt[3]{bc^2}\right) = 18abc$

Equality for $a=b=c=1$.

838. If $x,y,z \in (0,1)$, then:

$$\sum \frac{z^2}{(1-x^2)(y^2+yz+z^2)} \ge \frac{3\sqrt{3}}{2}\cdot\frac{xy+yz+zx}{x+y+z}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$From\ AM-GM, we\ have:\ x^2+\frac{\sqrt{3}}{9x}+\frac{\sqrt{3}}{9x}\geq 3\sqrt[3]{x^2\cdot\frac{\sqrt{3}}{9x}\cdot\frac{\sqrt{3}}{9x}}=1$$

$$\to 1-x^2\leq\frac{2\sqrt{3}}{9x}\ \ (And\ analogs)\to$$

$$\sum\frac{z^2}{(1-x^2)(y^2+yz+z^2)}\geq\frac{9}{2\sqrt{3}}\sum\frac{xz^2}{y^2+yz+z^2}=\frac{3\sqrt{3}}{2}\sum\frac{(zx)^2}{xy^2+xyz+xz^2}\geq$$

$$\overset{CBS}{\geq}\frac{3\sqrt{3}}{2}\cdot\frac{(\sum zx)^2}{\sum(xy^2+xyz+xz^2)}=\frac{3\sqrt{3}}{2}\cdot\frac{(xy+yz+zx)^2}{(xy+yz+zx)(x+y+z)}$$

$$Therefore,$$

$$\sum\frac{z^2}{(1-x^2)(y^2+yz+z^2)}\geq\frac{3\sqrt{3}}{2}\cdot\frac{xy+yz+zx}{x+y+z}$$

839. If $a,b,c>0$ then :

$$\sum\frac{(a+b)^4}{13abc+3c^3}\geq 3\sqrt[3]{abc}$$

Proposed by Lazaros Zachariadis-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum\frac{(a+b)^4}{13abc+3c^3}=\sum\frac{(a+b)^4}{c(13ab+3c^2)}\overset{Hölder}{\geq}\frac{[\sum(a+b)]^4}{3(\sum c)[\sum(13ab+3c^2)]}=\frac{16(\sum a)^3}{3[7\sum ab+3(\sum a)^2]}\geq$$

$$\overset{3(\sum ab)\leq(\sum a)^2}{\geq}\frac{16(\sum a)^3}{7(\sum a)^2+9(\sum a)^2}=\sum a\overset{AM-GM}{\geq}3\sqrt[3]{abc}.$$

$$Therefore,\quad \sum\frac{(a+b)^4}{13abc+3c^3}\geq 3\sqrt[3]{abc}$$

Solution 2 by Michael Sterghiou-Greece

$$\sum\frac{(a+b)^4}{13abc+3c^3}\geq 3\sqrt[3]{abc}\ ;(1)$$

$$abc=1, p=\sum a, q=\sum ab$$

$$LHS_{(1)}\overset{CBS}{\geq}\frac{[\sum(a+b)^2]^2}{(\sum a^3)+13}\overset{(?)}{\geq}9\Leftrightarrow\frac{4(p^2-q)^2}{p^3-3pq+16}\overset{(?)}{\geq}9$$

Let $f(p) = 4p^4 - 9p^3 - 8p^2q + 27pq + 4q^2 - 144 \geq 0$

$$f'(p) = (16p - 27)(p^2 - q) > 0, abc = 1 \to p \geq 3, q \geq 3.$$

$$\to f(p) \geq f(3) = 4q^2 + 9q - 63 \geq 4 \cdot 3^2 + 9 \cdot 3 - 63 = 0$$

Equality holds for $a = b = c = 1$.

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum \frac{(a+b)^4}{13abc + 3c^3} \geq 3\sqrt[3]{abc}, a = x^3, b = y^3, c = z^3 \to$$

$$\sum \frac{(x^3+y^3)^4}{13(x+y)^3 + 3z^9} \geq 3xyz$$

$$\frac{1}{3}\left(\sum (x^3+y^3)^4\right)\left(\sum \frac{1}{13(x+y)^3 + 3z^9}\right) \geq 3xyz$$

$$\frac{1}{3}\left(\sum (x^3+y^3)^4\right)\left(\frac{9}{\sum(13(x+y)^3 + 3z^9)}\right) \geq 3xyz$$

$$\sum (x^3+y^3)^4 \geq xyz\left(39(xyz)^3 + 3(x^9+y^9+z^9)\right)$$

$$x^{12} + 4x^9y^3 + 6x^6y^6 + 4x^3y^9 + y^{12} + y^9z^3 + 6y^6z^6 + 4y^3z^9 + z^{12} + 4z^9x^3 + 6z^6x^6 + z^9x^3 + 4z^3x^9 + x^{12} \geq 39(xyz)^4 + 3(x^{10}yz + y^{10}zx + z^{10}xy)$$

Which is true because : $x^{12} + x^9y^3 + x^9z^3 \geq 3x^{10}yz$; $(and\ analogs)$ and

$$x^{12} + y^{12} + z^{12} + 3x^9y^3 + 6x^6y^6 + 3y^9x^3 + 3y^9z^3 + 6x^6z^6 + 3z^9y^3 + 3z^9x^3 + 6z^6x^6 + 3z^3x^9 \geq 39(xyz)^4$$

840. If $0 \leq x_i \leq 1, i \in [1, 2022]$ and $y_n = \prod_{k \neq n} x_k$. Prove that:

$$\sum_{n=1}^{2022} \frac{x_n}{1 + y_n} \leq 2021$$

Proposed by Serlea Kabay-Liberia

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ us\ prove:\ P(m)\ :\ \forall (a_i)_{1 \leq i \leq m} \in [0,1], (m-1) + \prod_{i=1}^{m} a_i \geq \sum_{i=1}^{m} a_i, \forall m \in N_{\geq 2}.$$

$$For\ m = 2:\ a_1, a_2 \in [0,1] \to (1-a_1)(1-a_2) \geq 0 \leftrightarrow 1 + a_1a_2 \geq a_1 + a_2\ (1)$$

$$Now, we\ suppose\ that\ P(m)\ is\ true\ for\ a\ m \geq 2.$$

$$Let\ a_{m+1}\in[0,1]\to(m-1)a_{m+1}+\prod_{i=1}^{m+1}a_i\geq\sum_{i=1}^{m}a_{m+1}a_i$$

$$\leftrightarrow m+(m-1)a_{m+1}+\prod_{i=1}^{m+1}a_i\geq\sum_{i=1}^{m}(1+a_{m+1}a_i)\overset{(1)}{\gtrsim}\sum_{i=1}^{m}(a_{m+1}+a_i)$$

$$=(m-1)a_{m+1}+\sum_{i=1}^{m+1}a_i$$

$$\to P(m+1)\ is\ true\ \to\forall(a_i)_{1\leq i\leq m}\in[0,1],(m-1)+\prod_{i=1}^{m}a_i\geq\sum_{i=1}^{m}a_i,\forall m\in N_{\geq 2}\ \ (*)$$

$$\sum_{i=1}^{2022}x_i=x_n+\sum_{i=1,i\neq n}^{2022}x_i\overset{(*)}{\lesssim}x_n+2020+y_n\overset{x_n\leq 1}{\lesssim}2021+y_n\overset{y_n\geq 0}{\lesssim}2021(1+y_n)$$

$$\to\frac{1}{1+y_n}\leq\frac{2021}{\sum_{i=1}^{2022}x_i}\to\frac{x_n}{1+y_n}\leq\frac{2021x_n}{\sum_{i=1}^{2022}x_i},\forall n\in\overline{1,2022}$$

$$Therefore,\sum_{n=1}^{2022}\frac{x_n}{1+y_n}\leq 2021\sum_{n=1}^{2022}\frac{x_n}{\sum_{i=1}^{2022}x_i}=2021$$

Solution 2 by Sire Ambrose-Albania

$$y_n=\prod_{k=1,k\neq n}^{2022}x_k\geq\prod_{k=1}^{2022}x_k,\frac{1}{1+y_n}\leq\frac{1}{1+\prod_{k=1}^{n}x_k}$$

$$\sum_{n=1}^{2022}\frac{x_n}{1+y_n}\leq\sum_{n=1}^{2022}\frac{x_n}{1+\prod_{k=1}^{2022}x_k}=\frac{1}{1+\prod_{k=1}^{2022}x_k}\sum_{k=1}^{2022}x_n$$

At maximum of the sum

$\frac{1}{1+\prod_{k=1}^{2022}x_k}\sum_{k=1}^{2022}x_n,\prod_{k=1}^{2022}x_k=0$ →exist at least one $x_i\in\{x_n\},x_i=0$

$$\sum_{n=1}^{2022}\frac{x_n}{1+y_n}\leq x_1+x_2+\cdots+x_i+\cdots+x_{2022},x_i=0\to$$

$$\sum_{n=1}^{2022}\frac{x_n}{1+y_n}\leq x_1+x_2+\cdots+x_{2022}\leq 1+1\ldots+1=2021$$

841. If $a,b,c>0$ then

$$\sum_{cyc}a^{11}\cdot\left(\sum_{cyc}a\right)^2\ge\sum_{cyc}a^7\cdot\sum_{cyc}a^4\cdot\sum_{cyc}a^2$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{From CBS, we have}:\left(\sum_{cyc}a^{11}\right)\left(\sum_{cyc}a^3\right)\ge\left(\sum_{cyc}a^7\right)^2\ (1)$$

$$\left(\sum_{cyc}a^7\right)\left(\sum_{cyc}a\right)\ge\left(\sum_{cyc}a^4\right)^2\ (2)\quad\left(\sum_{cyc}a^4\right)\left(\sum_{cyc}a^2\right)\ge\left(\sum_{cyc}a^3\right)^2\ (3)$$

$$\left(\sum_{cyc}a^3\right)\left(\sum_{cyc}a\right)\ge\left(\sum_{cyc}a^2\right)^2\ (4)$$

$$(1)\times(2)\times(3)\times(4)\to\sum_{cyc}a^{11}\cdot\left(\sum_{cyc}a\right)^2\ge\sum_{cyc}a^7\cdot\sum_{cyc}a^4\cdot\sum_{cyc}a^2$$

$$\text{Therefore,}\quad\sum_{cyc}a^{11}\cdot\left(\sum_{cyc}a\right)^2\ge\sum_{cyc}a^7\cdot\sum_{cyc}a^4\cdot\sum_{cyc}a^2$$

842. If $a,b,c>0$ such that $a^2+b^2+c^2=3$ and $\lambda\ge0$ then:

$$\frac{a}{a+\lambda bc}+\frac{b}{b+\lambda ca}+\frac{c}{c+\lambda ab}\le\frac{3}{1+\lambda abc}$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Șerban-Romania

$$\sum_{cyc}\frac{a}{a+\lambda bc}=\sum_{cyc}\left(1-\frac{\lambda bc}{a+\lambda bc}\right)=3-\lambda\sum_{cyc}\frac{bc}{a+\lambda bc}\overset{(*)}{\le}\frac{3}{1+\lambda abc}$$

$$(*) \Leftrightarrow \lambda \sum_{cyc} \frac{bc}{a+\lambda bc} \geq 3 - \frac{3}{1+\lambda abc} = \frac{3\lambda abc}{1+\lambda abc}$$

If $\lambda = 0 \Rightarrow \sum \frac{a}{a} = \sum 1 = 3 \geq \frac{3}{1}$ true.

If $\lambda > 0$ then

$$\frac{1}{abc} \sum_{cyc} \frac{bc}{a+\lambda bc} \overset{(?)}{\geq} \frac{3}{1+\lambda abc} \Rightarrow \sum_{cyc} \frac{1}{a(a+\lambda bc)} \geq \frac{1}{1+\lambda abc}$$

$$\Rightarrow \sum_{cyc} \frac{1}{a^2+\lambda abc} \overset{Bergstrom}{\geq} \frac{9}{\sum a^2 + \lambda \sum abc} = \frac{9}{3+3\lambda abc} = \frac{1}{1+\lambda abc}$$

Therefore,

$$\frac{a}{a+\lambda bc} + \frac{b}{b+\lambda ca} + \frac{c}{c+\lambda ab} \leq \frac{3}{1+\lambda abc}$$

843. If $x, y, z > 0$ and $\lambda \geq 1$ then

$$\frac{x}{\lambda x+y+z} + \frac{y}{x+\lambda y+z} + \frac{z}{x+y+\lambda z} \leq \frac{3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Șerban-Romania

If $\lambda = 1 \Rightarrow \sum \frac{x}{x+y+z} = 1 \leq \frac{3}{3} = 1$ true.

If $\lambda > 1$ let $\begin{cases} \lambda x+y+z = a \\ x+\lambda y+z = b \\ x+y+\lambda z = c \end{cases} \Rightarrow x+y+z = \frac{a+b+c}{\lambda+2}$

$$\lambda x + \frac{a+b+c}{\lambda+2} - x = a \Rightarrow (\lambda-1)x = a - \frac{a+b+c}{\lambda+2} = \frac{a(\lambda+1)-b-c}{\lambda+2}$$

$$\Rightarrow x = \frac{a(\lambda+1)-b-c}{(\lambda-1)(\lambda+2)}; y = \frac{-a+b(\lambda+1)-c}{(\lambda-1)(\lambda+2)}; z = \frac{-a-b+c(\lambda+1)}{(\lambda-1)(\lambda+2)}$$

$$\sum_{cyc} \frac{x}{\lambda x+y+z} = \sum_{cyc} \frac{a(\lambda+1)-b-c}{(\lambda-1)(\lambda+2)a} = \sum_{cyc} \frac{a(\lambda+1)}{(\lambda-1)(\lambda+2)a} - \sum_{cyc} \frac{b+c}{(\lambda-1)(\lambda+2)a} =$$

$$= \frac{3(\lambda+1)}{(\lambda-1)(\lambda+2)} - \frac{1}{(\lambda-1)(\lambda+2)} \sum_{cyc} \frac{b+c}{a} \overset{(*)}{\leq} \frac{3}{\lambda+2}$$

$$(*) \Leftrightarrow \sum_{cyc} \frac{b+c}{a} \geq 3\lambda + 3 - 3\lambda + 3 = 6 \text{ true from } AM - GM.$$

Therefore,

$$\frac{x}{\lambda x + y + z} + \frac{y}{x + \lambda y + z} + \frac{z}{x + y + \lambda z} \leq \frac{3}{\lambda + 2}$$

844. For $\frac{\sqrt{3}}{3} \leq a, b, c \leq 1$ then prove that:

$$\sqrt[3]{abc} \cdot \tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \leq \sqrt{\frac{ab+bc+ca}{3}} \cdot \tan^{-1}\left(\sqrt[3]{abc}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

Let $g: (0, \infty) \to \mathbb{R};\ g(x) = (1 + x^2) \cdot \tan^{-1} x - x \Rightarrow g'(x) = 2x \cdot \tan^{-1} x > 0, \forall x > 0$

$$\Rightarrow g(x) \nearrow, x \in (0, \infty) \Rightarrow g(x) > g(0) = 0, \forall x \in (0, \infty)$$

$$\Rightarrow (1 + x^2) \cdot \tan^{-1} x > x, \forall x \in (0, \infty)$$

Next, let $f(x) = \begin{cases} \frac{\tan^{-1} x}{x}, x \in (0, \infty) \\ 1,\ x = 0 \end{cases}$

f –is continuous on $[0, \infty)$ and $f'(x) = \frac{x - (1+x^2) \cdot \tan^{-1} x}{x^2(1+x^2)} < 0, \forall x \in (0, \infty)$

$\Rightarrow f$ –is strictly decreasing on $[0, \infty)$. Now,

$$\frac{ab+bc+ca}{3} \geq (abc)^{\frac{2}{3}} \Rightarrow \sqrt{\frac{ab+bc+ca}{3}} \geq \sqrt[3]{abc}$$

$$\Rightarrow f\left(\sqrt[3]{abc}\right) \geq f\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \Rightarrow \frac{\tan^{-1}\left(\sqrt[3]{abc}\right)}{\sqrt[3]{abc}} \geq \frac{\tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right)}{\sqrt{\frac{ab+bc+ca}{3}}}$$

$$\Rightarrow \sqrt[3]{abc} \cdot \tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \leq \sqrt{\frac{ab+bc+ca}{3}} \cdot \tan^{-1}\left(\sqrt[3]{abc}\right)$$

Equality holds when $\frac{ab+bc+ca}{3} = \sqrt[3]{abc} \Leftrightarrow a = b = c$.

845. If $x,y,z>0$ and $\lambda\ge 1$ then

$$\frac{x}{\lambda x+y+z}+\frac{y}{x+\lambda y+z}+\frac{z}{x+y+\lambda z}\le\frac{3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by George Florin Șerban-Romania

$$\text{If } \lambda=1\Rightarrow\sum\frac{x}{x+y+z}=1\le\frac{3}{3}=1 \text{ true.}$$

$$\text{If } \lambda>1 \text{ let }\begin{cases}\lambda x+y+z=a\\ x+\lambda y+z=b\\ x+y+\lambda z=c\end{cases}\Rightarrow x+y+z=\frac{a+b+c}{\lambda+2}$$

$$\lambda x+\frac{a+b+c}{\lambda+2}-x=a\Rightarrow(\lambda-1)x=a-\frac{a+b+c}{\lambda+2}=\frac{a(\lambda+1)-b-c}{\lambda+2}$$

$$\Rightarrow x=\frac{a(\lambda+1)-b-c}{(\lambda-1)(\lambda+2)};y=\frac{-a+b(\lambda+1)-c}{(\lambda-1)(\lambda+2)};z=\frac{-a-b+c(\lambda+1)}{(\lambda-1)(\lambda+2)}$$

$$\sum_{cyc}\frac{x}{\lambda x+y+z}=\sum_{cyc}\frac{a(\lambda+1)-b-c}{(\lambda-1)(\lambda+2)a}=\sum_{cyc}\frac{a(\lambda+1)}{(\lambda-1)(\lambda+2)a}-\sum_{cyc}\frac{b+c}{(\lambda-1)(\lambda+2)a}=$$

$$=\frac{3(\lambda+1)}{(\lambda-1)(\lambda+2)}-\frac{1}{(\lambda-1)(\lambda+2)}\sum_{cyc}\frac{b+c}{a}\overset{(*)}{\le}\frac{3}{\lambda+2}$$

$$(*)\Leftrightarrow\sum_{cyc}\frac{b+c}{a}\ge 3\lambda+3-3\lambda+3=6 \text{ true from } AM-GM.$$

Therefore,

$$\frac{x}{\lambda x+y+z}+\frac{y}{x+\lambda y+z}+\frac{z}{x+y+\lambda z}\le\frac{3}{\lambda+2}$$

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ s=x+y+z\ and\ f(x)=\frac{x}{\alpha x+s},x>0.(\because\alpha=\lambda-1\ge 0)$$

$$We\ have:\ f'(x)=\frac{s}{(\alpha x+s)^2}\ and\ f''(x)=-\frac{2\alpha s}{(\alpha x+s)^3}\le 0$$

$$\overset{Using\ Jensen}{\Rightarrow}\sum\frac{x}{\lambda x+y+z}=\sum f(x)\le 3f\left(\frac{s}{3}\right)=\frac{3}{\lambda+2}.$$

$$Therefore,\quad \sum \frac{x}{\lambda x+y+z} \le \frac{3}{\lambda+2}.$$

846. **If $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 3$ and $\lambda \ge 0$ then:**

$$\frac{a}{a+\lambda bc} + \frac{b}{b+\lambda ca} + \frac{c}{c+\lambda ab} \le \frac{3}{1+\lambda abc}$$

Proposed by Marin Chirciu-Romania

Solution 1 by George Florin Șerban-Romania

$$\sum_{cyc} \frac{a}{a+\lambda bc} = \sum_{cyc}\left(1 - \frac{\lambda bc}{a+\lambda bc}\right) = 3 - \lambda \sum_{cyc} \frac{bc}{a+\lambda bc} \overset{(*)}{\le} \frac{3}{1+\lambda abc}$$

$$(*) \Leftrightarrow \lambda \sum_{cyc} \frac{bc}{a+\lambda bc} \ge 3 - \frac{3}{1+\lambda abc} = \frac{3\lambda abc}{1+\lambda abc}$$

If $\lambda = 0 \Rightarrow \sum \frac{a}{a} = \sum 1 = 3 \ge \frac{3}{1}$ true.

If $\lambda > 0$ then

$$\frac{1}{abc}\sum_{cyc} \frac{bc}{a+\lambda bc} \overset{(?)}{\ge} \frac{3}{1+\lambda abc} \Rightarrow \sum_{cyc} \frac{1}{a(a+\lambda bc)} \ge \frac{1}{1+\lambda abc}$$

$$\Rightarrow \sum_{cyc} \frac{1}{a^2+\lambda abc} \overset{Bergstrom}{\ge} \frac{9}{\sum a^2 + \lambda \sum abc} = \frac{9}{3+3\lambda abc} = \frac{1}{1+\lambda abc}$$

Therefore,

$$\frac{a}{a+\lambda bc} + \frac{b}{b+\lambda ca} + \frac{c}{c+\lambda ab} \le \frac{3}{1+\lambda abc}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ r = \lambda abc \ge 0\ and\ f(x) = \frac{x}{x+r}, a > 0.$$

$$We\ have:\ f'(x) = \frac{r}{(x+r)^2}\ and\ f''(x) = -\frac{2r}{(x+r)^3} \le 0$$

$$\to \sum \frac{a}{a+\lambda bc} = \sum \frac{a^2}{a^2+r} = \sum f(a^2) \overset{Jensen}{\le} 3f\left(\frac{1}{3}\sum a^2\right) =$$

$$= 3f(1) = \frac{3}{1+r} = \frac{3}{1+\lambda abc} \quad \to \quad \sum \frac{a}{a+\lambda bc} \le \frac{3}{1+\lambda abc}.$$

847. If $x, y, z > 0$ such that $x + y + z = 3$ and $\lambda \geq 0$ then

$$\frac{1}{\sqrt{x} + \lambda\sqrt{y}} + \frac{1}{\sqrt{y} + \lambda\sqrt{z}} + \frac{1}{\sqrt{z} + \lambda\sqrt{x}} \geq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Fayssal Abdelli-Bejaia-Algerie

$$A = \frac{1}{\sqrt{x} + \lambda\sqrt{y}} + \frac{1}{\sqrt{y} + \lambda\sqrt{z}} + \frac{1}{\sqrt{z} + \lambda\sqrt{x}} \geq \frac{3^2}{\sqrt{x} + \sqrt{y} + \sqrt{z} + \lambda(\sqrt{x} + \sqrt{y} + \sqrt{z})} =$$

$$= \frac{9}{(\lambda + 1)(\sqrt{x} + \sqrt{y} + \sqrt{z})} \overset{(*)}{\geq} \frac{3}{\lambda + 1}$$

$$(*) \Leftrightarrow \frac{3}{\sqrt{x} + \sqrt{y} + \sqrt{z}} \geq 1 \Leftrightarrow \sqrt{x} + \sqrt{y} + \sqrt{z} \leq 3; (1)$$

$$x + y + z \geq \frac{(\sqrt{x} + \sqrt{y} + \sqrt{z})^2}{3} \Rightarrow (\sqrt{x} + \sqrt{y} + \sqrt{z})^2 \leq 3(x + y + z) \leq 9$$

$$\Rightarrow \sqrt{x} + \sqrt{y} + \sqrt{z} \leq 3 \Rightarrow (1) \text{ is true.}$$

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$\frac{1}{\sqrt{x} + \lambda\sqrt{y}} + \frac{1}{\sqrt{y} + \lambda\sqrt{z}} + \frac{1}{\sqrt{z} + \lambda\sqrt{x}} \overset{Bergstrom}{\geq} \frac{9}{\sum(\sqrt{x} + \lambda\sqrt{y})} =$$

$$= \frac{9}{(\lambda + 1)(\sqrt{x} + \sqrt{y} + \sqrt{z})} \overset{AM-GM}{\geq} \frac{9}{(\lambda + 1)\left(\frac{x+1}{2} + \frac{y+1}{2} + \frac{z+1}{2}\right)} =$$

$$= \frac{9}{3(\lambda + 1)} = \frac{3}{\lambda + 1}$$

Therefore,

$$\frac{1}{\sqrt{x} + \lambda\sqrt{y}} + \frac{1}{\sqrt{y} + \lambda\sqrt{z}} + \frac{1}{\sqrt{z} + \lambda\sqrt{x}} \geq \frac{3}{\lambda + 1}$$

848. If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{(b + c)^4}{a^3 + \lambda abc} \geq \frac{48}{\lambda + 1} \sqrt[2]{abc}, 2 \leq \lambda \leq 5$$

Proposed by Marin Chirciu-Romania

Solution 1 by Fayssal Abdelli-Bejaia-Algerie

$$\sum_{cyc}\frac{(b+c)^4}{a^3+\lambda abc}=\sum_{cyc}\frac{[(b+c)^2]^2}{a^3+\lambda abc}\overset{Bergstrom}{\geq}\frac{(\sum(b+c)^2)^2}{\sum(a^3+\lambda abc)}=$$

$$=\frac{(2a^2+2b^2+2c^2+2ab+2bc+2ca)^2}{a^3+b^3+c^3+3\lambda abc}\geq\frac{16(ab+bc+ca)^2}{a^3+b^3+c^3+3\lambda abc}$$

$$\frac{ab+bc+ac}{3}\geq\sqrt[3]{ab\cdot bc\cdot ca}=\sqrt[3]{a^2b^2c^2}$$

$$\Rightarrow(ab+bc+ca)^2\geq 9(abc)^{\frac{4}{3}}=9abc\cdot\sqrt[3]{abc}$$

Hence,

$$\sum_{cyc}\frac{(b+c)^4}{a^3+\lambda abc}\geq\frac{16\cdot 9abc\cdot\sqrt[3]{abc}}{a^3+b^3+c^3+3\lambda abc}\overset{(*)}{\geq}\frac{48}{\lambda+1}\sqrt[2]{abc}$$

$$(*)\Leftrightarrow\frac{3abc}{a^3+b^3+c^3+3\lambda abc}\geq\frac{1}{\lambda+1}$$

$$\Leftrightarrow 3abc\lambda+3abc-a^3-b^3-c^3-3\lambda abc\geq 0$$

$$\Leftrightarrow 3abc\geq a^3+b^3+c^3;(1)$$

But $\frac{a^3+b^3+c^3}{3}\geq abc\Rightarrow a^3+b^3+c^3\geq 3abc;(2)$

From $(1),(2)\Rightarrow a^3+b^3+c^3=3abc\Rightarrow$

$$\frac{16\cdot 9abc\cdot\sqrt[3]{abc}}{a^3+b^3+c^3+3\lambda abc}\geq\frac{16\cdot 9abc\cdot\sqrt[3]{abc}}{(\lambda+1)\cdot 3abc}=\frac{48}{\lambda+1}\sqrt[2]{abc}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum\frac{(b+c)^4}{a^3+\lambda abc}=\sum\frac{(b+c)^4}{a(a^2+\lambda bc)}\overset{Hölder}{\geq}\frac{[\sum(a+b)]^4}{3(\sum a)(\sum(a^2+\lambda bc))}$$

$$=\frac{16(\sum a)^3}{3[(\sum a)^2+(\lambda-2)\sum ab]}\geq$$

$$\overset{3\sum ab\leq(\sum a)^2}{\geq}\frac{16(\sum a)^3}{3(\sum a)^2+(\lambda-2)(\sum a)^2}=\frac{16}{\lambda+1}\sum a\overset{AM-GM}{\geq}\frac{16}{\lambda+1}.3\sqrt[3]{abc}$$

$$\textit{Therefore,}\quad\sum\frac{(b+c)^4}{a^3+\lambda abc}\geq\frac{48}{\lambda+1}\sqrt[3]{abc},2\leq\lambda\leq 5$$

□

849. If $x,y,z\in\mathbb{R}_+^*$ then prove that:

$$\frac{xy}{x^3+y^3+xyz}+\frac{yz}{y^3+z^3+xyz}+\frac{zx}{z^3+x^3+xyz}\le\frac{3}{x+y+z}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by George Florin Șerban-Romania

$$a^3+b^3\ge ab(a+b),\forall a,b>0\Leftrightarrow(a+b)(a^2-ab+b^2)\ge ab(a+b)$$

$\Leftrightarrow(a+b)(a-b)^2\ge 0,\forall a,b>0$ which is clearly true.

$$\sum_{cyc}\frac{xy}{x^3+y^3+xyz}\le\sum_{cyc}\frac{xy}{xyx+y)+xyz}=\sum_{cyc}\frac{xy}{xy(x+y+z)}=$$

$$=\sum_{cyc}\frac{1}{x+y+z}=\frac{3}{x+y+z}$$

850. If $a,b,c>0$ then:

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}+\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}+\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\ge 3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}=\frac{\sqrt{a^2+b^2}\cdot\sqrt{a^2+c^2}}{a^2+bc}=\frac{|(a+ib)(a-ic)|}{a^2+bc}=$$

$$=\frac{|(a^2+bc)+i(b-c)a|}{a^2+bc}=\frac{\sqrt{(a^2+bc)^2+(b-c)^2a^2}}{a^2+bc}\ge 1;(1)$$

Similarly,

$$\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}\ge 1;(2)\text{and }\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\ge 1;(3)$$

Adding $(1),(2),(3)$ it follows that:

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}+\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}+\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\ge 3$$

Equality holds for $a=b=c$.

Solution 2 by Michael Sterghiou-Greece

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}+\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}+\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\ge 3;(1)$$

$$(a^2+b^2)(a^2+c^2)\overset{CBS}{\ge}(a^2+bc)^2\Rightarrow$$

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}\geq 1;(1)$$

$$\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}\geq 1;(2)\text{and }\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\geq 1;(3)$$

Adding $(1),(2),(3)$ it follows that:

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}+\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}+\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\geq 3$$

Equality holds for $a=b=c$.

Solution 3 by Florentin Vişescu-Romania

We want to prove that:

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}\geq 1;(1)\Leftrightarrow\sqrt{(a^2+b^2)(a^2+c^2)}\geq a^2+bc\Leftrightarrow$$

$$a^4+a^2c^2+b^2a^2+b^2c^2\geq a^4+2a^2bc+b^2c^2\Leftrightarrow$$

$$a^2c^2+b^2a^2\geq 2a^2bc\Leftrightarrow b^2+c^2\geq 2bc\Leftrightarrow(b-c)^2\geq 0 \text{ true.}$$

Similarly,

$$\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}\geq 1;(2)\text{and }\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\geq 1;(3)$$

Adding $(1),(2),(3)$ it follows that:

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}+\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}+\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\geq 3$$

Equality holds for $a=b=c$.

Solution 4 by Adrian Popa-Romania

$$(a^2+b^2)(a^2+c^2)\overset{CBS}{\geq}(a^2+bc)^2\Rightarrow$$

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}\geq 1;(1)$$

$$\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}\geq 1;(2)\text{and }\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\geq 1;(3)$$

Adding $(1),(2),(3)$ it follows that:

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}+\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}+\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\ge 3$$

Equality holds for $a=b=c$.

Solution 5 by Khaled Abd Imouti-Damascus-Syria

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}=\frac{\sqrt{a^2\left(1+\frac{b^2}{a^2}\right)a^2\left(1+\frac{c^2}{a^2}\right)}}{a^2\left(1+\frac{b}{a}\cdot\frac{c}{a}\right)}=\frac{\sqrt{\left(1+\left(\frac{b}{a}\right)^2\right)\left(1+\left(\frac{c}{a}\right)^2\right)}}{1+\frac{b}{a}\cdot\frac{c}{a}}=$$

$$\overset{x=\frac{b}{a},y=\frac{c}{a}}{=}\frac{\sqrt{(1+x^2)(1+y^2)}}{1+xy}\overset{?}{\ge}1\Leftrightarrow$$

$$\sqrt{(1+x^2)(1+y^2)}\overset{?}{\ge}1+xy\Leftrightarrow 1+x^2+y^2+x^2y^2\overset{?}{\ge}1+x^2y^2+2xy\Leftrightarrow$$

$$x^2-2xy+y^2\ge 0\Leftrightarrow (x-y)^2\ge 0 \text{ true.}$$

Therefore,

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}+\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}+\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\ge 3$$

Equality holds for $a=b=c$.

Solution 6 by Fayssal Abdelli-Bejaia-Algerie

$$A=\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc};B=\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca};C=\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}$$

$$A^2=\frac{(a^2+b^2)(a^2+c^2)}{(a^2+bc)^2}=\frac{a^4+a^2c^2+b^2a^2+b^2c^2}{a^4+b^2c^2+2a^2bc}=$$

$$=1+\frac{a^2c^2-2a^2bc+a^2b^2}{a^4+b^2c^2+2a^2bc}=1+\frac{a^2(c-b)^2}{a^4+b^2c^2+2a^2bc}\text{ and }\frac{a^2(c-b)^2}{a^4+b^2c^2+2a^2bc}\ge 0$$

Hence, $A^2\ge 1\Rightarrow A\ge 1;(1)$

Similarly,

$$B^2=1+\frac{b^2(a-c)^2}{b^4+2ab^2c+a^2c^2}\Rightarrow B^2\ge 1\Rightarrow B\ge 1;(2)$$

$$C^2=1+\frac{c^2(b-a)^2}{c^4+2abc^2+a^2b^2}\Rightarrow C^2\ge 1\Rightarrow C\ge 1;(3)$$

Adding $(1),(2),(3)$ it follows that:

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}+\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}+\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\geq 3$$

Equality holds for $a=b=c$.

Solution 7 by Amrit Awasthi-India

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}\overset{AM-GM}{\geq}\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+\frac{b^2+c^2}{2}}=$$

$$=\frac{2\sqrt{(a^2+b^2)(a^2+c^2)}}{2a^2+(b^2+c^2)}=\frac{2\sqrt{(a^2+b^2)(a^2+c^2)}}{(a^2+b^2)+(a^2+c^2)}=$$

$$=\frac{2}{\sqrt{\frac{a^2+b^2}{a^2+c^2}}+\sqrt{\frac{a^2+c^2}{a^2+b^2}}}\overset{HM-GM}{\geq}\frac{1}{\sqrt{\sqrt{\frac{a^2+b^2}{a^2+c^2}}\cdot\sqrt{\frac{a^2+c^2}{a^2+b^2}}}}=1;(1)$$

Similarly,

$$\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}\geq 1;(2)\text{and }\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\geq 1;(3)$$

Adding $(1),(2),(3)$ it follows that:

$$\frac{\sqrt{(a^2+b^2)(a^2+c^2)}}{a^2+bc}+\frac{\sqrt{(b^2+c^2)(b^2+a^2)}}{b^2+ca}+\frac{\sqrt{(c^2+b^2)(c^2+a^2)}}{c^2+ab}\geq 3$$

Equality holds for $a=b=c$.

851. If $a,b,c\in(0,\infty),\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+(a+b+c)^2\leq 12$ then:

$$ab+bc+ca+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\leq 6$$

Proposed by Dan Radu Seclăman-Romania

Solution by Aggeliki Papaspyropoulou-Greece

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}=3$$

$$12\geq(a+b+c)^2+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq(a+b+c)^2+3\Rightarrow$$

$$12\geq(a+b+c)^2+3\Rightarrow(a+b+c)^2\leq 9\Rightarrow a+b+c\leq 3;(1)$$

$$3(ab+bc+ca)\leq(a+b+c)^2\Rightarrow ab+bc+ca\leq 3;(2)$$

$$a+b+c=\sqrt{a^2}+\sqrt{b^2}+\sqrt{c^2}\geq\sqrt{a}\cdot\sqrt{b}+\sqrt{b}\cdot\sqrt{c}+\sqrt{c}\cdot\sqrt{a}\Rightarrow$$

$$3\geq a+b+c\geq\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\Rightarrow\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\leq 3;(3)$$

By $(2),(3)$ we conclude that:

$$ab+bc+ca+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\leq 6$$

852. If $a,b,c>0, a+b+c=1$ then:

$$\prod_{cyc}\left(\frac{1-\sin a}{1+\sin a}\right)^a\leq\frac{1-\sin(a^2+b^2+c^2)}{1+\sin(a^2+b^2+c^2)}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since\ a,b,c>0\ and\ a+b+c=1\to 0<a,b,c<1<\frac{\pi}{2}.$$

$$Let\ f(x)=\log\left(\frac{1-\sin x}{1+\sin x}\right), x\in\left(0,\frac{\pi}{2}\right)$$

We have :

$$f'(x)=\frac{-\cos x.(1+\sin x)-\cos x.(1-\sin x)}{(1+\sin x)^2}.\frac{1+\sin x}{1-\sin x}=\frac{-2\cos x}{1-\sin^2 x}=-\frac{2}{\cos x}$$

$$And:\ f''(x)=-\frac{2\sin x}{\cos^2 x}<0\to f-concave\ on\ \left(0,\frac{\pi}{2}\right)$$

$$\overset{Using\ Jensen}{\Rightarrow}\sum a.f(a)\leq f\left(\sum a^2\right)\quad (\because\ a+b+c=1)$$

$$\leftrightarrow\log\prod_{cyc}\left(\frac{1-\sin a}{1+\sin a}\right)^a\leq\log\left(\frac{1-\sin(a^2+b^2+c^2)}{1+\sin(a^2+b^2+c^2)}\right)$$

$$Therefore,\quad\prod_{cyc}\left(\frac{1-\sin a}{1+\sin a}\right)^a\leq\frac{1-\sin(a^2+b^2+c^2)}{1+\sin(a^2+b^2+c^2)}$$

853. If $a,b,c>0, a+b+c=1$ then:

$$\prod_{cyc}\left(\frac{1-\cos a}{1+\cos a}\right)^{a^2+2bc}\leq\frac{1-\cos(a^3+b^3+c^3+6abc)}{1+\cos(a^3+b^3+c^3+6abc)}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since\ a,b,c>0\ and\ a+b+c=1\to 0<a,b,c<1<\frac{\pi}{2}.$$

$$Let\ f(x)=\log\left(\frac{1-\cos x}{1+\cos x}\right), x\in\left(0,\frac{\pi}{2}\right)$$

$$We\ have$$

$$:f'(x)=\frac{\sin x.(1+\cos x)+\sin x.(1-\cos x)}{(1+\cos x)^2}.\frac{1+\cos x}{1-\cos x}=\frac{2\sin x}{1-\cos^2 x}=\frac{2}{\sin x}$$

$$And:\ f''(x)=-\frac{2\cos x}{\sin^2 x}<0\to f-concave\ on\ \left(0,\frac{\pi}{2}\right)$$

$$Also, we\ have:\ \sum_{cyc}(a^2+2bc)=\left(\sum_{cyc}a\right)^2=1$$

$$\overset{Using\ Jensen}{\Rightarrow}\ \sum_{cyc}(a^2+2bc)f(a)\le f\left(\sum_{cyc}(a^2+2bc)a\right)=f(a^3+b^3+c^3+6abc)$$

$$\leftrightarrow \log\left(\prod_{cyc}\left(\frac{1-\cos a}{1+\cos a}\right)^{a^2+2bc}\right)\le\log\left(\frac{1-\cos(a^3+b^3+c^3+6abc)}{1+\cos(a^3+b^3+c^3+6abc)}\right)$$

$$Therefore,\quad \prod_{cyc}\left(\frac{1-\cos a}{1+\cos a}\right)^{a^2+2bc}\le\frac{1-\cos(a^3+b^3+c^3+6abc)}{1+\cos(a^3+b^3+c^3+6abc)}$$

854. If $a,b,c,d,e>0$ such that: $abcde=1$ then:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}\ge\sum\sqrt{a}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

Let $a=x^2, b=y^2, c=z^2 d=t^2, e=k^2$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}\ge\sum\sqrt{a}\Leftrightarrow$$

$$\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{t^2}+\frac{t^2}{k^2}+\frac{x^2}{x^2}\ge x+y+z+t+k;(xyztk)^2=1\Rightarrow xyztk=1,$$

$$\frac{1}{5}(x^2+y^2+z^2+t^2+k^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2}+\frac{1}{k^2}\right)\geq x+y+z+t+k$$

$$x^2+y^2+z^2+t^2+k^2\geq\frac{x+y+z+t+k}{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2}+\frac{1}{k^2}}\geq 5$$

$$\frac{(x+y+z+t+k)^2}{5}\geq x+y+z+t+k\Leftrightarrow x+y+z+t+k\geq 5 \text{ true.}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: 4.\frac{a}{b}+3.\frac{b}{c}+2.\frac{c}{d}+\frac{d}{e}\overset{Weighted\ AM-GM}{\geq}10\sqrt[10]{\left(\frac{a}{b}\right)^4\left(\frac{b}{c}\right)^3\left(\frac{c}{d}\right)^2\left(\frac{d}{e}\right)}=$$

$$=10\sqrt[10]{\frac{a^4}{bcde}}=10\sqrt[10]{\frac{a^5}{abcde}}=10\sqrt{a}\to 4.\frac{a}{b}+3.\frac{b}{c}+2.\frac{c}{d}+\frac{d}{e}\geq 10\sqrt{a}\ (1)$$

$$Similarly, we\ have:\ 4.\frac{b}{c}+3.\frac{c}{d}+2.\frac{d}{e}+\frac{e}{a}\geq 10\sqrt{b}\ \ (2), 4.\frac{c}{d}+3.\frac{d}{e}+2.\frac{e}{a}+\frac{a}{b}\geq 10\sqrt{c}\ (3)$$

$$4.\frac{d}{e}+3.\frac{e}{a}+2.\frac{a}{b}+\frac{b}{c}\geq 10\sqrt{d}\ \ (4), 4.\frac{e}{a}+3.\frac{a}{b}+2.\frac{b}{c}+\frac{c}{d}\geq 10\sqrt{e}\ (5)$$

$$\to(1)+(2)+(3)+(4)+(5)\to 10\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}\right)\geq 10\sum\sqrt{a}$$

$$Therefore,\quad \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}\geq\sum\sqrt{a}$$

Solution 3 by Michael Sterghiou-Greece

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}\geq\sum\sqrt{a};(1)$$

$$\sqrt{a}=\sqrt{\frac{a}{b}\cdot b}\overset{AGM}{\leq}\frac{1}{2}\left(\frac{a}{b}+b\right)\Rightarrow 2\sum_{cyc}\sqrt{a}\leq\sum_{cyc}a+\sum_{cyc}\frac{a}{b};(2)$$

Now, we homogenize the inequality

$$\sum_{cyc}\sqrt{a}\leq\sum_{cyc}a;(abcde)^{\frac{1}{10}}=1\Rightarrow LHS:\sum_{cyc}a^{\frac{6}{10}}b^{\frac{1}{10}}c^{\frac{1}{10}}d^{\frac{1}{10}}e^{\frac{1}{10}}\leq\sum_{cyc}a$$

Which is true as $(1,0,0,0,0)$ **majories** $\left(\frac{6}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}.,\frac{1}{10}\right)$**. Now,**

$$\sum_{cyc} a + \sum_{cyc} \frac{a}{b} \geq 2\sum_{cyc} \sqrt{a} \Rightarrow$$

$$\sum_{cyc} a + \sum_{cyc} \frac{a}{b} \geq 2\sum_{cyc} \sqrt{a} \Rightarrow \sum_{cyc} a - \sum_{cyc} \sqrt{a} + \sum_{cyc} \frac{a}{b} \geq \sum_{cyc} \sqrt{a} \text{ true.}$$

Equality holds for $a = b = c = 1$.

855. If $x, y \geq 0$ then:

$$x^3y^3(x+y)^3 \leq (x^2+y^2)(x^3+y^3)(x^4+y^4)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$(x^2+y^2)(x^3+y^3)(x^4+y^4) \overset{power-mean}{\geq} \frac{(x+y)^2}{2} \cdot \frac{(x+y)^3}{2^2} \cdot \frac{(x+y)^4}{2^3} = \frac{(x+y)^9}{2^6}$$

Need to show:

$$\frac{(x+y)^9}{2^6} \geq (x+y)^3x^3y^3 \Rightarrow (x+y)^6 \geq 2^6(xy)^3 \Rightarrow \frac{x+y}{2} \geq \sqrt{xy} \text{ (true by } AGM)$$

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$\sqrt{\frac{x^2+y^2}{2}} \overset{AQM}{\geq} \frac{x+y}{2} \Leftrightarrow \frac{x^2+y^2}{2} \geq \frac{(x+y)^2}{4} \Leftrightarrow x^2+y^2 \geq \frac{(x+y)^2}{2}; (1)$$

$$\sqrt{\frac{x^4+y^4}{2}} \overset{AQM}{\geq} \sqrt{x^2y^2} \Leftrightarrow x^4+y^4 \geq 2x^2y^2; (2)$$

$$\begin{cases} x^3+y^3 = (x+y)(x^2-xy+y^2) \\ x^2+y^2 \overset{AGM}{\geq} 2xy \end{cases} \Rightarrow x^3+y^3 \geq (x+y)\cdot xy; (3)$$

From $(1), (2)$ and (3) it follows that:

$$(x^2+y^2)(x^3+y^3)(x^4+y^4) \geq \frac{(x+y)^2}{2} \cdot (x+y) \cdot xy \cdot 2x^2y^2 \Leftrightarrow$$

$$(x^2+y^2)(x^3+y^3)(x^4+y^4) \geq x^3y^3(x+y)^3$$

Therefore,

$$x^3y^3(x+y)^3 \leq (x^2+y^2)(x^3+y^3)(x^4+y^4)$$

Solution 3 by Abdallah El Farissi-Bechar-Algerie

$$\frac{x^3+y^3}{2} \overset{x^3-convexe}{\geq} \left(\frac{x+y}{3}\right)^3 \Rightarrow x^3+y^3 \geq \frac{1}{4}(x+y)^3$$

$$\begin{cases} x^4 + y^4 \geq 2x^2y^2 \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{2}{xy} \end{cases}$$

$$(x^3 + y^3)(x^4 + y^4)\left(\frac{1}{x^2} + \frac{1}{y^2}\right) \geq (x + y)^3 xy$$

Therefore,

$$x^3y^3(x + y)^3 \leq (x^2 + y^2)(x^3 + y^3)(x^4 + y^4)$$

856. If $a, b, c > 0$, $[*]$ −GIF, then:

$$3 + ([a])^a + ([b])^b + ([c])^c \geq a^{[a]} + b^{[b]} + c^{[c]}$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

We shall show that: $1 + [x]^x \geq x^{[x]}, \forall x > 0$

For $0 < x < 1, [x] = 0$ and $1 + [x]^x = 1 + 0^x = 1 = x^0 = x^{[x]}$.

For $1 \leq x < 1, [x] = 1$ and $1 + [x]^x = 1 + 1^x = 2 > x = x^1 = x^{[x]}$.

For $2 \leq x < 3$ we have to show that: $1 + 2^x \geq x^2 \Leftrightarrow 2^x \geq x^2 - 1$ which is true for geometric representation.For $x \geq 3$, let $f(x) = x^{\frac{1}{x}} \Leftrightarrow \log f(x) = \frac{1}{x}\log x$. Differentiating by x, we get $\frac{f'(x)}{f(x)} = \frac{1}{x^2}(1 - \log x) < 0, \forall x > 3 \Rightarrow f \searrow \forall x \in [3, \infty)$

$$3 \leq [x] \leq x \Rightarrow x^{\frac{1}{x}} \leq [x]^{\frac{1}{[x]}} \Rightarrow x^{[x]} \leq [x]^x < [x]^x + 1.$$

Thus, for $x > 0$: $x + [x]^x \geq x^{[x]}$. Hence, for $a, b, c > 0$ we have:

$$(1 + ([a])^a) + \left(1 + ([b])^b\right) + (1 + ([c])^c) \geq a^{[a]} + b^{[b]} + c^{[c]}$$

Therefore,

$$([a])^a + ([b])^b + ([c])^c \geq a^{[a]} + b^{[b]} + c^{[c]}$$

857. If $a, b, c > 0$ such that $abc = 1$ and $\lambda \geq 0$ then:

$$\sum_{cyc} \frac{a^4}{b^2 + b + \lambda bc} \geq \frac{3}{\lambda + 2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Fayssal Abdelli-Bejaia-Algerie

$$\sum_{cyc}\frac{a^4}{b^2+b+\lambda bc}\overset{Bergstrom}{\geq}\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+a+b+c+\lambda(ab+bc+ca)}$$

$$\frac{a^2+b^2+c^2}{3}\geq\sqrt[3]{a^2b^2c^2}\geq 1\Rightarrow a^2+b^2+c^2\geq 3\Rightarrow(a^2+b^2+c^2)^2\geq 9\Rightarrow$$

$$\sum_{cyc}\frac{a^4}{b^2+b+\lambda bc}\geq\frac{9}{a^2+b^2+c^2+a+b+c+\lambda(ab+bc+ca)}\geq\frac{3}{\lambda+2};(1)$$

$$i)\ a^2+b^2+c^2\geq 3$$

$$ii)\frac{a+b+c}{3}\geq\sqrt[3]{abc}\geq 1\Rightarrow a+b+c\geq 3$$

$$iii)\frac{ab+bc+ca}{3}\geq\sqrt[3]{(abc)^2}\geq 1\Rightarrow ab+bc+ca\geq 3$$

$$a^2+b^2+c^2+a+b+c+\lambda(ab+bc+ca)\geq 6+3\lambda\Rightarrow$$

$$\frac{9}{a^2+b^2+c^2+a+b+c+\lambda(ab+bc+ca)}\leq\frac{3}{\lambda+2};(2)$$

From $(1),(2)$ it follows that:

$$\frac{9}{a^2+b^2+c^2+a+b+c+\lambda(ab+bc+ca)}=\frac{3}{\lambda+2}$$

Therefore,

$$\sum_{cyc}\frac{a^4}{b^2+b+\lambda bc}\geq\frac{3}{\lambda+2}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\sum a^2\overset{CBS}{\geq}\frac{1}{3}\left(\sum a\right)^2\overset{AM-GM}{\geq}\frac{1}{3}.3\sqrt[3]{abc}.\sum a\overset{abc=1}{=}\sum a\ (1)\ and\ \sum a^2$$

$$\geq\sum ab\ (2)$$

$$\to\sum\frac{a^4}{b^2+b+\lambda bc}\overset{CBS}{\geq}\frac{(\sum a^2)^2}{\sum b^2+\sum b+\lambda\sum bc}\overset{(1),(2)}{\geq}\frac{(\sum a^2)^2}{\sum a^2+\sum a^2+\lambda\sum a^2}=\frac{\sum a^2}{\lambda+2}\geq$$

$$\overset{AM-GM}{\geq}\frac{3\sqrt[3]{abc}^2}{\lambda+2}\overset{abc=1}{=}\frac{3}{\lambda+2}.$$

$$\textit{Therefore,}\quad \sum\frac{a^4}{b^2+b+\lambda bc}\ge\frac{3}{\lambda+2},\forall\lambda\ge 0$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum_{cyc}\frac{a^4}{b^2+b+\lambda bc}\overset{Bergstrom}{\ge}\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+a+b+c+\lambda(ab+bc+ca)}\ge\frac{3}{\lambda+2}$$

$$2(a^4+b^4+c^4)+4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\ge 3(a^2+b^2+c^2)+3abc+3\lambda\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$+\lambda(a^4+b^4+c^4)+2\lambda\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right),\qquad \text{which is true, because}$$

$$a^4+a^4+\frac{1}{a^2}\ge 3a^2\ (\textit{and analogs})$$

$$\lambda(a^4+b^4+c^4)+2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\ge 3\lambda\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$a^4+b^4+c^4+2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\ge 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$a^4+b^4+c^4\ge\frac{1}{a}+\frac{1}{b}+\frac{1}{c};\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

858. If $x,y,z>0, xyz=1$ then:

$$\sum_{cyc}\frac{z(x+y)^3}{(\sqrt{x}+\sqrt{y})(\sqrt[3]{x}+\sqrt[3]{y})(\sqrt[6]{x}+\sqrt[6]{y})}\ge 3$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

Let: $x=a^6,y=b^6,z=c^6$ hence,

$$\sum_{cyc}\frac{z(x+y)^3}{(\sqrt{x}+\sqrt{y})(\sqrt[3]{x}+\sqrt[3]{y})(\sqrt[6]{x}+\sqrt[6]{y})}=\sum_{cyc}\frac{c^6(a^6+b^6)^3}{(a^3+b^3)(a^2+b^2)(a+b)}\ge$$

$$\ge\sum_{cyc}\frac{c^6(a^3+b^3)^6}{2^3(a^3+b^3)(a^2+b^2)(a+b)}\ge\frac{1}{2^3}\sum_{cyc}\frac{c^6(a^3+b^3)^5}{(a^2+b^2)(a+b)}\ge$$

$$\ge\frac{1}{2^4}\sum_{cyc}\frac{c^6(a^3+b^3)^5}{a^3+b^3}=\frac{1}{2^4}\sum_{cyc}c^6(a^3+b^3)^4\ge$$

$$\geq \frac{2^4}{2^4}\sum_{cyc} c^6\left((ab)^{\frac{3}{2}}\right)^4 = \sum_{cyc}(abc)^6 = 3$$

Solution 2 by Asmat Qatea-Afghanistan

$$\begin{cases} \sqrt{1+1}\cdot\sqrt{x+y} \geq \sqrt{x}+\sqrt{y} \\ \sqrt[3]{(1+1)^2}\cdot\sqrt[3]{x+y} \geq \sqrt[3]{x}+\sqrt[3]{y} \\ \sqrt[6]{(1+1)^5}\cdot\sqrt[6]{x+y} \geq \sqrt[6]{x}+\sqrt[6]{y} \end{cases}$$

$$\sum_{cyc}\frac{z(x+y)^3}{(\sqrt{x}+\sqrt{y})(\sqrt[3]{x}+\sqrt[3]{y})(\sqrt[6]{x}+\sqrt[6]{y})} \overset{\substack{Holder \\ (*)}}{\geq} \sum_{cyc}\frac{z(x+y)^3}{4(x+y)} =$$

$$= \frac{1}{4}\sum_{cyc} z(x+y)^2 \overset{(?)}{\geq} \frac{1}{4}\sum_{cyc} z(x+y)^2 =$$

$$= \frac{1}{4}(zx^2+zy^2+xy^2+xz^2+yx^2+yz^2+6xyz) \overset{AGM}{\geq} 3$$

859. If $a,b>0, n\geq\frac{1}{4}$ then:

$$\frac{1}{a^2+b^2}+n\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\geq\frac{4n+1}{2ab}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Fayssal Abdelly-Bejaia-Algerie

Let ssuppose that: $\frac{1}{a^2+b^2}+n\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\geq\frac{4n+1}{2ab}$ is true

$$\Leftrightarrow n\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\frac{4n}{2ab}\geq\frac{1}{2ab}-\frac{1}{a^2+b^2}, n\left(\frac{a^2+b^2}{a^2b^2}\right)-\frac{2n}{ab}\geq\frac{a^2+b^2-2ab}{2ab(a^2+b^2)}$$

$$n\left(\frac{a^2+b^2-2ab}{a^2b^2}\right)\geq\frac{(a-b)^2}{2ab(a^2+b^2)},\qquad n\left(\frac{(a-b)^2}{a^2b^2}\right)\geq\frac{(a-b)^2}{2ab(a^2+b^2)}$$

For $a=b$ is true. Let $a\neq b\Rightarrow\frac{n}{a^2b^2}\geq\frac{1}{2ab(a^2+b^2)}\Rightarrow\frac{n}{ab}\geq\frac{1}{2(a^2+b^2)}\Rightarrow 2n(a^2+b^2)\geq ab$

But $a^2+b^2\geq 2ab\Rightarrow 2n(a^2+b^2)\geq 4nab\geq ab$ because $n\geq\frac{1}{4}$.

Therefore,

$$\frac{1}{a^2+b^2}+n\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\geq\frac{4n+1}{2ab}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\frac{1}{a^2+b^2}+n\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\ge\frac{4n+1}{2ab}\Leftrightarrow\frac{n}{a^2}+\frac{n}{b^2}-\frac{2n}{ab}\ge\frac{1}{2ab}-\frac{1}{a^2+b^2}$$

$$\frac{n(a^2+b^2)-2nab}{(ab)^2}\ge\frac{a^2+b^2-2ab}{2ab(a^2+b^2)}$$

$$(a^2+b^2-2ab)\left(\frac{n}{a^2b^2}-\frac{1}{2ab(a^2+b^2)}\right)\ge 0$$

$$n\left(\frac{1}{(ab)^2}-\frac{1}{2ab(a^2+b^2)}\right)\ge 0$$

Because $a^2+b^2\ge 2ab\Rightarrow\frac{1}{4(ab)^2}-\frac{1}{2ab(a^2+b^2)}\ge 0, n\ge\frac{1}{4}\Rightarrow$

$$\frac{1}{2ab}-\frac{1}{ab(a^2+b^2)}\ge 0\Leftrightarrow\frac{\left((a^2+b^2)-2\right)ab}{2ab(a^2+b^2)}\ge 0\ \text{ true.}$$

Therefore,

$$\frac{1}{a^2+b^2}+n\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\ge\frac{4n+1}{2ab}$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{1}{a^2+b^2}+n\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=\frac{1}{a^2+b^2}+\frac{1}{4}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\left(n-\frac{1}{4}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\ge$$

$$\overset{AM-GM}{\ge} 2\sqrt{\frac{1}{a^2+b^2}\cdot\frac{1}{4}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}+\left(n-\frac{1}{4}\right).2\sqrt{\frac{1}{a^2}\cdot\frac{1}{b^2}}=\frac{1}{ab}+\left(n-\frac{1}{4}\right).\frac{2}{ab}=\frac{4n+1}{2ab}.$$

$$\textit{Therefore,}\quad \frac{1}{a^2+b^2}+n\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\ge\frac{4n+1}{2ab},\forall n\ge\frac{1}{4}$$

Solution 4 by Abdallah El Farissi-Bechar-Algerie

For $a,b>0$ and $n\ge\frac{1}{4}$ we have:

$$\frac{(a-b)^2}{2ab(a^2+b^2)}\le\frac{(a-b)^2}{4a^2b^2}=\frac{1}{4}\left(\frac{1}{a}-\frac{1}{b}\right)^2\le n\left(\frac{1}{a}-\frac{1}{b}\right)^2$$

It follows that:

$$\frac{(a-b)^2}{2ab(a^2+b^2)}=\frac{1}{2ab}-\frac{1}{a^2+b^2}\le n\left(\frac{1}{a}-\frac{1}{b}\right)^2=\frac{n}{a^2}+\frac{n}{b^2}-\frac{4n}{2ab}$$

$$\frac{4n+1}{2ab} \le \frac{1}{a^2+b^2} + \frac{n}{a^2} + \frac{n}{b^2}$$

860. If $a,b,c>0$ such that $abc=1$ and $n\in\mathbb{N}_{n\ge3}$ then :

$$\sum \frac{(b+c)^n}{a^3+\lambda} \ge \frac{3.2^n}{\lambda+1}, \lambda \ge 2.$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum \frac{(b+c)^n}{a^3+\lambda} = \sum \frac{(b+c)^n}{a^3+\lambda abc} = \sum \frac{(b+c)^n}{a(a^2+\lambda bc)} \overset{\text{Hölder}}{\ge} \frac{[\sum(b+c)]^n}{3^{n-3}.(\sum a)[\sum(a^2+\lambda bc)]}$$

$$= \frac{2^n}{3^{n-3}} \cdot \frac{(\sum a)^{n-1}}{(\sum a)^2+(\lambda-2)\sum ab} \ge$$

$$\overset{3\sum ab \le (\sum a)^2}{\ge} \frac{2^n}{3^{n-4}} \cdot \frac{(\sum a)^{n-1}}{3(\sum a)^2+(\lambda-2)(\sum a)^2}$$

$$= \frac{2^n}{3^{n-4}} \cdot \frac{(\sum a)^{n-3}}{\lambda+1} \overset{AM-GM}{\ge} \frac{2^n}{3^{n-4}} \cdot \frac{\left(3.\sqrt[3]{abc}\right)^{n-3}}{\lambda+1} = \frac{2^n}{3^{n-4}} \cdot \frac{3^{n-3}}{\lambda+1}$$

$$\textit{Therefore,} \quad \sum \frac{(b+c)^n}{a^3+\lambda} \ge \frac{3.2^n}{\lambda+1}, \lambda \ge 2$$

861. If $a,b,c>0$ such that $a+b+c=3$ and $n\in\mathbb{N}, n\ge 2$ then:

$$\frac{a^2}{a+\sqrt[n]{bc}} + \frac{b^2}{b+\sqrt[n]{ca}} + \frac{c^2}{c+\sqrt[n]{ab}} \ge \frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Asmat Qatea-Afghanistan

$$\frac{a^2}{a+\sqrt[n]{bc}} + \frac{b^2}{b+\sqrt[n]{ca}} + \frac{c^2}{c+\sqrt[n]{ab}} \overset{Bergstrom}{\ge} \frac{(a+b+c)^2}{(a+b+c)+\sum\sqrt[n]{ab}} = \frac{9}{3+\sum\sqrt[n]{ab}} \overset{(1)}{\ge} \frac{3}{2}$$

$$(1) \Leftrightarrow \frac{9}{3+\sum\sqrt[n]{ab}} \overset{AGM}{\ge} \frac{9}{3+\sum\frac{a+b+n-2}{n}} = \frac{3}{2}$$

Therefore,

$$\frac{a^2}{a+\sqrt[n]{bc}} + \frac{b^2}{b+\sqrt[n]{ca}} + \frac{c^2}{c+\sqrt[n]{ab}} \ge \frac{3}{2}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\text{Let } a = x^n, b = y^n, c = z^n; x^n + y^n + z^n = 3 \Rightarrow$$

$$3 = x^n + y^n + z^n \ge (x^2 + y^2 + z^2)\frac{x^{n-2} + y^{n-2} + z^{n-2}}{3} \ge x^2 + y^2 + z^2; n \ge 2$$

Hence,

$$\frac{a^2}{a + \sqrt[n]{bc}} + \frac{b^2}{b + \sqrt[n]{ca}} + \frac{c^2}{c + \sqrt[n]{ab}} = \frac{x^{2n}}{x^n + yz} + \frac{y^{2n}}{y^n + zx} + \frac{z^{2n}}{z^n + xy} \ge$$

$$\ge \frac{(x^n + y^n + z^n)^2}{x^n + y^n + z^n + xy + yz + zx} = \frac{3^2}{3 + xy + yz + zx} = \frac{3}{2}$$

Therefore,

$$\frac{a^2}{a + \sqrt[n]{bc}} + \frac{b^2}{b + \sqrt[n]{ca}} + \frac{c^2}{c + \sqrt[n]{ab}} \ge \frac{3}{2}$$

Solution 3 by Abdelli Fayssal-Bejaia-Algerie

$$A = \frac{a^2}{a + \sqrt[n]{bc}} + \frac{b^2}{b + \sqrt[n]{ca}} + \frac{c^2}{c + \sqrt[n]{ab}} \overset{Bergstrom}{\ge} \frac{(a + b + c)^2}{(a + b + c) + \left(\sqrt[n]{ab} + \sqrt[n]{bc} + \sqrt[n]{ca}\right)} \ge$$

$$\ge \frac{9}{3 + \left(\sqrt[n]{ab} + \sqrt[n]{bc} + \sqrt[n]{ca}\right)} \overset{(1)}{\ge} \frac{3}{2}$$

$$(1) \Leftrightarrow 3 + \sqrt[n]{ab} + \sqrt[n]{bc} + \sqrt[n]{ca} \le 6 \Leftrightarrow \sqrt[n]{ab} + \sqrt[n]{bc} + \sqrt[n]{ca} \le 3; (B)$$

$$\text{But: } \sqrt[n]{ab} + \sqrt[n]{bc} + \sqrt[n]{ca} \le \sqrt[n]{(a + b + c)(a + b + c)} \Rightarrow$$

$$\sqrt[n]{ab} + \sqrt[n]{bc} + \sqrt[n]{ca} \le (a + b + c)^{\frac{2}{n}} \le 3^{\frac{2}{n}} \le 3; n \ge 2 \Rightarrow (B) \Rightarrow (A) \text{ it's true.}$$

Therefore,

$$\frac{a^2}{a + \sqrt[n]{bc}} + \frac{b^2}{b + \sqrt[n]{ca}} + \frac{c^2}{c + \sqrt[n]{ab}} \ge \frac{3}{2}$$

862. If $a, b, c > 1$, then:

$$\sum_{cyc} log_{a+b}(1 + b^{b+1})(1 + c^{c+1}) \ge 6(a + b)^{c-b}(b + c)^{a-c}(c + a)^{b-a}$$

Proposed by Florică Anastase-Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum_{cyc} log_{a+b}(1 + b^{b+1})(1 + c^{c+1}) \ge \sum_{cyc} log_{a+b}(1 + b^2)(1 + c^2) \ge$$

$$\geq \sum_{cyc} log_{a+b}(b+c)^2 \geq 3 \cdot 2 \cdot \sqrt[3]{\prod_{cyc} log_{a+b}(b+c)} = 6$$

We need to prove that: $6 \geq 6(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a} \Leftrightarrow$

$$(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a} \leq 1 \Leftrightarrow$$

$$(a+b)^b(b+c)^c(c+a)^a \geq (a+b)^c(b+c)^a(c+a)^b$$

Let: $a+b=x, b+c=y, c+a=z \Rightarrow a=\frac{x-y+z}{2}, b=\frac{x+y-z}{2}, c=\frac{-x+y+z}{2}$.

$$\Rightarrow x^{x+y-z} \cdot y^{-x+y+z} \cdot z^{x-y+z} \geq x^{-x+y+z} \cdot y^{x-y+z} \cdot z^{x+y-z}$$

$$\Rightarrow x^{2x} \cdot y^{2y} \cdot z^{2z} \geq x^{2z} \cdot y^{2x} \cdot z^{2y}$$

$$\Rightarrow \left(\frac{x}{y}\right)^x \left(\frac{y}{z}\right)^y \left(\frac{z}{x}\right)^z \geq \left(\frac{x+y+z}{\frac{x}{\frac{x}{y}}+\frac{y}{\frac{y}{z}}+\frac{z}{\frac{z}{x}}}\right)^{x+y+z} = \left(\frac{x+y+z}{x+y+z}\right)^{x+y+z} = 1$$

Solution 2 by proposer

From Bernoulli's inequality, we have:

$$\begin{cases} a^{a+1} = (1+a-1)^{a+1} \geq a^2 \\ b^{b+1} = (1+b-1)^{b+1} \geq b^2 \\ c^{c+1} = (1+c-1)^{c+1} \geq c^2 \end{cases}$$

$$\Rightarrow \begin{cases} (1+a^{a+1})(1+b^{b+1}) \geq (1+a^2)(1+b^2) \geq (a+b)^2 \\ (1+b^{b+1})(1+c^{c+1}) \geq (1+b^2)(1+c^2) \geq (b+c)^2 \\ (1+c^{c+1})(1+a^{a+1}) \geq (1+c^2)(1+a^2) \geq (c+a)^2 \end{cases} \Rightarrow$$

$$\sum_{cyc} log_{a+b}(1+b^{b+1})(1+c^{c+1}) \geq \sum_{cyc} log_{a+b}(b+c)^2 =$$

$$= 2\sum_{cyc} log_{a+b}(b+c) \overset{Am-Gm}{\gtreqqless} 6 \cdot \sqrt[3]{\prod_{cyc} log_{a+b}(b+c)} = 6 \quad (i)$$

$\therefore x^x y^y z^z \geq x^z y^x z^y, \forall x,y,z > 1 \leftrightarrow (z-x)lnx + (x-y)lny + (y-z)lnz \leq 0$

$$If\ 1 \leq x \leq y \leq z \to (lnx \leq lny \leq lnz, z-x \geq x-y) \overset{Chebyshev's}{\Rightarrow}$$

$$(z-x)lnx + (x-y)lny \leq \frac{1}{2}(z-y)\, ln(xy) = (z-y)ln\sqrt{xy}$$

$$\to (z-x)lnx+(x-y)lny+(y-z)lnz \le (z-y)ln\sqrt{xy}+(y-z)lnz$$

$$=(z-y)ln\frac{\sqrt{xy}}{z}\le 0$$

From:$x=b+c, y=c+a, z=a+b \to (a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a}\le 1 \quad (ii)$

From $(i),(ii)$ it follows that:

$$\sum_{cyc} log_{a+b}(1+b^{b+1})(1+c^{c+1})\ge 6(a+b)^{c-b}(b+c)^{a-c}(c+a)^{b-a}$$

863. If $x,y,z\in\mathbb{R}, \sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}=3\sqrt{2}$ then:

$$x^2+y^2+z^2+6\ge 3(x+y+z)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Asmat Qatea-Afghanistan

$$(1+1+1)(x^2+1+y^2+1+z^2+1)\overset{Holder}{\ge}\left(\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}\right)^2$$

$$=\left(3\sqrt{2}\right)^2$$

$$x^2+y^2+z^2+3\ge 6\Rightarrow x^2+y^2+z^2+6\ge 9\overset{(?)}{\ge}3(x+y+z)$$

$$x+y+z\overset{?}{\le}3;(1)$$

We have: $\sqrt{x^2+1}\overset{RMS-AM}{\ge}\frac{x+1}{\sqrt{2}}$

$$\sum_{cyc}\sqrt{x^2+1}\ge\frac{x+y+z+3}{\sqrt{2}}\Rightarrow 3\sqrt{2}\ge\frac{x+y+z+3}{\sqrt{2}}$$

$x+y+z\le 3\Rightarrow (1)$ is true.

Solution 2 by Fayssal Abdelli-Bejaia-Algerie

$$x^2+y^2+z^2+6=(x^2+1)+(y^2+1)+(z^2+1)+3\ge$$

$$\ge\frac{\left(\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}\right)^2}{3}+3=9$$

$\Rightarrow x^2+y^2+z^2+6\ge 9\overset{?}{\ge}3(x+y+z)\Leftrightarrow 3(x+y+z)\overset{?}{\le}9\Leftrightarrow x+y+z\le 3$ true.

$$x^2-2x+1\ge 0\Rightarrow 2x^2+2\ge x^2+1+2x\Leftrightarrow 2(x^2+1)\ge(x+1)^2$$

$$\Rightarrow \sqrt{2}\cdot\sqrt{x^2+1} \ge |x+1| \Rightarrow \sqrt{x^2+1} \ge \frac{|x+1|}{\sqrt{2}}$$

Similarly, $\sqrt{y^2+1} \ge \frac{|y+1|}{\sqrt{2}}$ and $\sqrt{z^2+1} \ge \frac{|z+1|}{\sqrt{2}} \Rightarrow$

$$\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1} \ge \frac{|x+1|+|y+1|+|z+1|}{\sqrt{2}}$$

$$\Rightarrow 3\sqrt{2} \ge \frac{|x+1|+|y+1|+|z+1|}{\sqrt{2}} \Rightarrow 6 \ge |x+1|+|y+1|+|z+1|$$

$$\ge |x+y+z+3|$$

$$\Rightarrow |x+y+z+3| \le 6 \Rightarrow -6 \le x+y+z+3 \le 6 \Rightarrow x+y+z \le 3$$

Hence, $x^2+y^2+z^2+6 \ge 3(x+y+z)$

Solution 3 by Abdul Aziz-Semarang-Indonesia

$$\because (ad-bc)^2 \ge 0 \Rightarrow \sqrt{a^2+b^2}+\sqrt{c^2+d^2} \ge \sqrt{(a+c)^2+(b+d)^2}$$

$$3\sqrt{2} = \sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1} \ge \sqrt{(x+y+z)^2+9}$$

$$\Leftrightarrow 9 \ge 3(x+y+z)$$

By CBS inequality, we have:

$$\left(\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}\right)^2 \le 3(x^2+y^2+z^2+3) \Leftrightarrow$$

$$3 \le x^2+y^2+z^2 \Leftrightarrow x^2+y^2+z^2+6 \ge 9 \ge 3(x+y+z)$$

Equality holds if and only if $x=y=z=1$.

864. If $a,b,c>0$ such that $abc=1$ and $n\in\mathbb{N}, \lambda \ge 1$ then:

$$\frac{1}{a^n+\lambda}+\frac{1}{b^n+\lambda}+\frac{1}{c^n+\lambda} \le \frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

We must show: $\frac{1}{a^n+\lambda}+\frac{1}{b^n+\lambda}+\frac{1}{c^n+\lambda} \le \frac{3}{\lambda+1} \Leftrightarrow \frac{\lambda}{a^n+\lambda}+\frac{\lambda}{b^n+\lambda}+\frac{\lambda}{c^n+\lambda} \le \frac{3\lambda}{\lambda+1}$

$$\Leftrightarrow \sum_{cyc}\frac{a^n+\lambda-a^n}{a^n+\lambda} \le \frac{3\lambda}{\lambda+1} \Leftrightarrow 3-\sum_{cyc}\frac{a^n}{a^n+\lambda} \le \frac{3\lambda}{\lambda+1} \Leftrightarrow \sum_{cyc}\frac{a^n}{a^n+\lambda} \ge \frac{3}{\lambda+1};(1)$$

Because $abc=1 \Rightarrow \exists x,y,z \in \mathbb{R}_+^*$ such that $a=\frac{x}{y}, b=\frac{y}{z}, c=\frac{z}{x};(2)$

From (1),(2) we must show that:

$$\sum_{cyc} \frac{\frac{x^n}{y^n}}{\frac{x^n}{y^n}+\lambda} \ge \frac{3}{\lambda+1} \Leftrightarrow \sum_{cyc} \frac{x^n}{x^n+\lambda y^n} \ge \frac{3}{\lambda+1}; (3)$$

Let $x^n = u, y^n = v, z^n = w; u, v, w > 0; (4)$

From (3),(4) we must show:

$$\sum_{cyc} \frac{u}{u+\lambda v} \ge \frac{3}{\lambda+1}; (5)$$

$$\sum_{cyc} \frac{u}{u+\lambda v} = \sum_{cyc} \frac{u^2}{u^2+\lambda uv} \overset{Bergstrom}{\ge} \frac{(u+v+w)^2}{\sum u^2 + \lambda \sum uv}; (6)$$

From (5),(6) we must show:

$$\frac{(u+v+w)^2}{\sum u^2 + \lambda \sum uv} \ge \frac{3}{\lambda+1} \Leftrightarrow (\lambda+1)(u+v+w)^2$$

$$\ge 3(u^2+v^2+w^2) + 3\lambda(uv+vw+wu)$$

$$\Leftrightarrow (\lambda+1)\sum u^2 + 2(\lambda+1)\sum uv > 3\sum u^2 + 3\lambda \sum uv \Leftrightarrow$$

$(\lambda-2)\sum u^2 \ge (\lambda-2)\sum uv \Leftrightarrow \sum u^2 \ge \sum uv$ which is clearly true.

Solution 2 by Fayssal Abdelli-Bejaia-Algerie

$$\frac{1}{a^n+\lambda} + \frac{1}{b^n+\lambda} + \frac{1}{c^n+\lambda} \le \frac{3}{\lambda+1}; (A) \Leftrightarrow \frac{\lambda}{a^n+\lambda} + \frac{\lambda}{b^n+\lambda} + \frac{\lambda}{c^n+\lambda} \le \frac{3\lambda}{\lambda+1}$$

$$\Leftrightarrow \sum_{cyc} \frac{a^n+\lambda-a^n}{a^n+\lambda} \le \frac{3\lambda}{\lambda+1} \Leftrightarrow 3 - \sum_{cyc} \frac{a^n}{a^n+\lambda} \le \frac{3\lambda}{\lambda+1} \Leftrightarrow \sum_{cyc} \frac{a^n}{a^n+\lambda} \overset{(?)}{\ge} \frac{3}{\lambda+1}$$

$$\sum_{cyc} \frac{a^n}{a^n+\lambda} \ge \frac{\left(\sum \sqrt{a^n}\right)^2}{3\lambda + \sum a^n}$$

But: $\frac{1}{3}\sum \sqrt{a^n} \ge \sqrt[3]{\sqrt{a^n b^n c^n}} = 1 \Rightarrow \sum \sqrt{a^n} \ge 3 \Rightarrow \left(\sum \sqrt{a^n}\right)^2 \ge 9$, hence

$$\sum_{cyc} \frac{a^n}{a^n+\lambda} \ge \frac{9}{3\lambda + \sum a^n} \overset{(?)}{\ge} \frac{3}{\lambda+1}$$

$$\frac{3}{3\lambda + \sum a^n} \ge \frac{1}{\lambda+1} \Leftrightarrow 3\lambda + 3 \ge 3\lambda + a^n + b^n + c^n \Leftrightarrow a^n + b^n + c^n \le 3; (1)$$

But: $\frac{a^n+b^n+c^n}{3} \geq \sqrt[3]{a^n b^n c^n} \Rightarrow a^n + b^n + c^n \geq 3; (2)$

From $(1), (2)$ it follows that $\frac{9}{3\lambda+3} \geq \frac{3}{\lambda+1}$ true.

Therefore,

$$\frac{1}{a^n+\lambda} + \frac{1}{b^n+\lambda} + \frac{1}{c^n+\lambda} \leq \frac{3}{\lambda+1}$$

865. If $x, y, z > 0, \lambda \geq 1, then:$

$$4\sum \frac{yz}{(x+y)(x+z)} \leq 3 + \lambda\left(\frac{x^2+y^2+z^2}{xy+yz+zx} - 1\right)$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Since\ \lambda \geq 1 \to It's\ suffices\ to\ prove:\ 4\sum \frac{yz}{(x+y)(x+z)}$$
$$\leq 3 + \left(\frac{x^2+y^2+z^2}{xy+yz+zx} - 1\right) = \frac{(\sum x)^2}{\sum xy}$$
$$Let\ x = s-a, y = s-b,$$
$$z = s-c, where\ a, b, c\ are\ the\ side$$
$$-\ lengths\ of\ a\ triangle\ and\ s\ the\ semiperimeter.$$
$$We\ have:\ \sum x = s, \sum yz = \sum (s-b)(s-c) = \sum \frac{sr^2}{s-a} = r(4R+r)$$
$$And\ 4\sum \frac{yz}{(x+y)(x+z)} = 4\sum \frac{(s-b)(s-c)}{bc} = 4\sum \sin^2\frac{A}{2} = 2\sum(1-\cos A)$$
$$= 2\left(2 - \frac{r}{R}\right) = \frac{4R-2r}{R}$$
$$\to (*) \leftrightarrow \frac{4R-2r}{R} \leq \frac{s^2}{r(4R+r)} \leftrightarrow s^2 \geq \frac{2r(2R-r)(4R+r)}{R}\ (Blundon)$$
$$\to (*)\ is\ true. Therefore, 4\sum \frac{yz}{(x+y)(x+z)} \leq 3 + \lambda\left(\frac{x^2+y^2+z^2}{xy+yz+zx} - 1\right), \forall \lambda \geq 1.$$

866. If $a, b, c > 0$ such that $a + b + c = 1$ and $\lambda \geq 0$ then

$$\sum_{cyc} \frac{(b+c)\sqrt{b^2-bc+c^2}}{a+\lambda bc} \geq \frac{6}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Șerban-Romania

First, we prove that: $b^2 - bc + c^2 \geq \frac{(b+c)^2}{4}, \forall b, c \in \mathbb{R} \Leftrightarrow$

$4b^2 - 4bc + 4c^2 - b^2 - 2bc - c^2 \geq 0 \Leftrightarrow 3b^2 - 6bc + 3c^2 \geq 0 \Leftrightarrow$

$3(b-c)^2 \geq 0$, which is true for all $b, c \in \mathbb{R}$.

$$\sum_{cyc} \frac{(b+c)\sqrt{b^2 - bc + c^2}}{a + \lambda bc} \geq \sum_{cyc} \frac{(b+c)\sqrt{\frac{(b+c)^2}{4}}}{a + \lambda bc} = \frac{1}{2}\sum_{cyc} \frac{(b+c)^2}{a + \lambda bc} \overset{Bergstrom}{\geq}$$

$$\geq \frac{1}{2} \cdot \frac{\left(\sum(b+c)\right)^2}{\sum(a + \lambda bc)} = \frac{1}{2} \cdot \frac{(2\sum a)^2}{\sum a + \lambda \sum bc} = \frac{1}{2} \cdot \frac{4(\sum a)^2}{3 + \lambda \sum ab} = \frac{2 \cdot 9}{3 + \lambda \sum ab} \geq \frac{18}{3 + \frac{\lambda}{3}(\sum a)^2} =$$

$$= \frac{18}{3 + 3\lambda} = \frac{6}{\lambda + 1}$$

Therefore,

$$\sum_{cyc} \frac{(b+c)\sqrt{b^2 - bc + c^2}}{a + \lambda bc} \geq \frac{6}{\lambda + 1}$$

867. If $a, b, c > 0, n \geq 0$, then:

$$\sum_{cyc} \frac{a^2 - ab + b^2}{\sqrt{(n+1)(na+b)c}} \geq \frac{a+b+c}{n+1}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum \frac{a^2 - ab + b^2}{\sqrt{(n+1)(na+b)c}}$$

$$= \sum \frac{a^3 + b^3}{(a+b)\sqrt{(n+1)(nac+bc)}} \overset{Hölder}{\geq} \sum \frac{(a+b)^3}{4(a+b)\sqrt{(n+1)(nac+bc)}} =$$

$$= \frac{1}{4\sqrt{n+1}} \sum \frac{(a+b)^2}{\sqrt{nac+bc}} \overset{CBS}{\geq} \frac{1}{4\sqrt{n+1}} \cdot \frac{[\sum(a+b)]^2}{\sum \sqrt{nac+bc}} \overset{CBS}{\geq} \frac{4}{4\sqrt{n+1}} \cdot \frac{(a+b+c)^2}{\sqrt{3\sum(nac+bc)}}$$

$$= \frac{1}{n+1} \cdot \frac{(a+b+c)^2}{\sqrt{3\sum ab}} \geq$$

$$\overset{3\sum ab\le(\sum a)^2}{\ge}\frac{1}{n+1}\cdot\frac{(a+b+c)^2}{a+b+c}=\frac{a+b+c}{n+1}\to\boxed{\sum\frac{a^2-ab+b^2}{\sqrt{(n+1)(na+b)c}}\ge\frac{a+b+c}{n+1}}$$

Solution 2 by George Florin Șerban-Romania

$$\because a^2-ab+b^2\ge\frac{(a+b)^2}{4},\forall a,b\in\mathbb{R}\Leftrightarrow 3(a-b)^2\ge 0\Rightarrow$$

$$\sum_{cyc}\frac{a^2-ab+b^2}{\sqrt{(n+1)(na+b)c}}\ge\sum_{cyc}\frac{(a+b)^2}{4\sqrt{n+1}\cdot\sqrt{(na+b)c}}=\frac{1}{4\sqrt{n+1}}\cdot\sum_{cyc}\frac{(a+b)^2}{\sqrt{(na+b)c}}\ge$$

$$\overset{Bergstrom}{\ge}\frac{1}{4\sqrt{n+1}}\cdot\frac{(2a+2b+2c)^2}{\sum\left(\sqrt{(na+b)c}\right)}\overset{CBS}{\ge}\frac{(a+b+c)^2}{(n+1)(\sum a)}=\frac{a+b+c}{n+1}$$

Therefore,

$$\sum_{cyc}\frac{a^2-ab+b^2}{\sqrt{(n+1)(na+b)c}}\ge\frac{a+b+c}{n+1}$$

868. If $a,b,c>0$ such that $abc=1$ and $\lambda\ge 0$ then:

$$\sum_{cyc}\frac{(b^2+\lambda c^2)^2}{a+bc}\ge\frac{3}{2}(\lambda+1)^2$$

Proposed by Marin Chirciu-Romania

Solution 1 by Fayssal Abdelli-Bejaia-Algerie

$$\sum_{cyc}\frac{(b^2+\lambda c^2)^2}{a+bc}\ge\frac{3}{2}(\lambda+1)^2;\ (*)$$

$$A=\sum_{cyc}\frac{(b^2+\lambda c^2)^2}{a+bc}\ge\frac{[\sum(b^2+\lambda c^2)]^2}{\sum a+\sum bc}$$

$$\frac{a^2+b^2+c^2}{3}\ge\sqrt[3]{(abc)^2}=1\Rightarrow a^2+b^2+c^2\ge 3\Rightarrow\lambda(a^2+b^2+c^2)\ge 3\lambda$$

$$a^2+b^2+c^2+\lambda(a^2+b^2+c^2)\ge 3+3\lambda\ge 3(\lambda+1)$$

$$\Rightarrow\left(a^2+b^2+c^2+\lambda(a^2+b^2+c^2)\right)^2\ge 9(\lambda+2)^2$$

$$\Rightarrow\sum_{cyc}\frac{(b^2+\lambda c^2)^2}{a+bc}\ge\frac{9(\lambda+1)^2}{a+b+c+ab+bc+ca}\ge\frac{3}{2}(\lambda+1)^2;\ (B)$$

$$\frac{3}{a+b+c+ab+bc+ca} \ge \frac{1}{2} \Leftrightarrow a+b+c+ab+bc+ca \le 6; (1)$$

$$a+b+c \ge 3\sqrt[3]{abc} \Rightarrow a+b+c \ge 3$$

$$ab+bc+ca \ge 3\sqrt[3]{(abc)^2} \Rightarrow ab+bc+ca \ge 3 \Rightarrow$$

$$a+b+c+ab+bc+ca \ge 6; (2)$$

From (1),(2) it follows that $a+b+c+ab+bc+ca = 6$

(B)$\Rightarrow \frac{9(\lambda+1)^2}{6} \ge \frac{3}{2}(\lambda+1)^2$ which is true$\Rightarrow (*)$ is true.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\sum_{cyc} \frac{(b^2+\lambda c^2)^2}{a+bc} \ge \frac{[\sum(b^2+\lambda c^2)]^2}{\sum a + \sum bc} = \frac{(\lambda+1)^2(\sum a^2)^2}{\sum a + \sum bc} =$$

$$= \frac{(\lambda+1)^2(\sum a^4 + 2\sum a^2b^2)}{\sum a + \sum bc} = 3(\lambda+1)^2 \frac{\sum(ab)^2}{\sum a + \sum bc} = 3(\lambda+1)^2 \frac{\sum\left(\frac{1}{a}\right)^2}{\sum a + \sum\left(\frac{1}{a}\right)} \overset{abc=1}{\ge}$$

$$\ge \frac{3}{2}(\lambda+1)^2 \frac{\sum\left(\frac{1}{a}\right)^2}{\sum\left(\frac{1}{a}\right)^2} = \frac{3}{2}(\lambda+1)^2$$

869. If $a,b,c>0, n \in N^*$ then:

$$n \cdot \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}} + \sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}} \ge n+1$$

Proposed by Marin Chirciu-Romania

Solution 1 by Fayssal Abdelli-Bejaia-Algerie

$$A = n \cdot \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}; B = \sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}$$

$$a^2+b^2+c^2 \ge \frac{(a+b+c)^2}{3}, a^2+b^2+c^2 \ge ab+bc+ca$$

$$\Rightarrow \frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 \Rightarrow \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}} \ge 1 \Rightarrow A = n \cdot \sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}} \ge n$$

$$n\cdot\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}+\sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}\ge n+1\Leftrightarrow$$

$$n+\sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}\ge n+1\Leftrightarrow\sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}\ge 1;(1)$$

$$a^2+b^2+c^2\ge\frac{(a+b+c)^2}{3}\Rightarrow 3(a^2+b^2+c^2)\ge(a+b+c)^2\Rightarrow$$

$$\frac{1}{3(a^2+b^2+c^2)}\le\frac{1}{(a+b+c)^2}\Rightarrow\frac{3(ab+bc+ca)}{3(a^2+b^2+c^2)}\le\frac{3(ab+bc+ca)}{(a+b+c)^2}\Rightarrow$$

$$B=\sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}\ge 1$$

Therefore,

$$n\cdot\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}+\sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}\ge n+1$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ know\ that:\left(\sum a\right)^2\ge 3\sum ab\rightarrow 1\ge\frac{3(ab+bc+ca)}{(a+b+c)^2}\rightarrow\sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}$$

$$\ge\sqrt{\frac{3(ab+bc+ca)}{(a+b+c)^2}}=x>0\ (1)$$

$$Also,we\ have:3\sum a^2\overset{CBS}{\ge}\left(\sum a\right)^2\rightarrow\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}\ge\sqrt{\frac{(a+b+c)^2}{3(ab+bc+ca)}}=\frac{1}{x}\ (2)$$

$$From\ (1),(2),we\ have:\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}+\sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}\ge x+\frac{1}{x}\overset{AM-GM}{\ge}2$$

$$\rightarrow n\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}+\sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}$$

$$=(n-1)\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}+\left(\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}+\sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}\right)\ge$$

$$\overset{\sum a^2\ge\sum ab}{\ge}(n-1).1+2=n+1\ \to\ \boxed{n\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}+\sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}\ge n+1}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\text{Let}a^2+b^2+c^2=x;ab+bc+ca=y$$

$$n\cdot\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}+\sqrt[3]{\frac{3(ab+bc+ca)}{(a+b+c)^2}}\ge n+1\Leftrightarrow$$

$$n\cdot\sqrt{\frac{x}{y}}+\sqrt[3]{\frac{3y}{x+2y}}\ge n+1\Leftrightarrow\sqrt{\frac{x}{y}}+\sqrt[3]{\frac{3y}{x+2y}}\ge 2;(\because x\ge y)\Leftrightarrow$$

$$\sqrt{\frac{x}{y}}+\sqrt[3]{\frac{y}{x}}\ge 2\Leftrightarrow 2\cdot\sqrt{\sqrt{\frac{x}{y}}\cdot\sqrt[3]{\left(\frac{y}{x}\right)}}\ge 2\Leftrightarrow\sqrt[6]{\frac{x}{y}}\ge 1\Leftrightarrow\frac{x}{y}\ge 1\text{ true.}$$

870. If $a,b,c>0.k\ge 3$ then:

$$\frac{a}{\sqrt{ka+b}}+\frac{b}{\sqrt{kb+c}}+\frac{c}{\sqrt{kc+a}}\le\sqrt{\frac{3(a+b+c)}{k+1}}$$

Proposed by Marin Chirciu-Romania

Solution by Asmat Qatea-Afghanistan

$$(a+b+c)^{\frac{1}{2}}\left(\frac{a}{ak+b}+\frac{b}{bk+c}+\frac{c}{ck+a}\right)^{\frac{1}{2}}\overset{Holder}{\ge}\frac{a}{\sqrt{ka+b}}+\frac{b}{\sqrt{kb+c}}+\frac{c}{\sqrt{kc+a}}$$

We need to prove:

$$\sqrt{\frac{3(a+b+c)}{k+1}}\overset{(?)}{\ge}(a+b+c)^{\frac{1}{2}}\left(\frac{a}{ak+b}+\frac{b}{bk+c}+\frac{c}{ck+a}\right)^{\frac{1}{2}}$$

$$\frac{3}{k+1}\ge\frac{a}{ak+b}+\frac{b}{bk+c}+\frac{c}{ck+a}\Leftrightarrow$$

$$\frac{3}{k+1}\ge\frac{1}{k}\left(\frac{ak+b-b}{ak+b}+\frac{bk+c-c}{bk+c}+\frac{ck+a-a}{ck+a}\right)\Leftrightarrow$$

$$\frac{3}{k+1}\ge 3-\left(\frac{b}{ak+b}+\frac{c}{bk+c}+\frac{a}{ck+a}\right)\Leftrightarrow$$

$$\frac{b}{ak+b}+\frac{c}{bk+c}+\frac{a}{ck+a}\geq\frac{3}{k+1}$$

$$\frac{b}{ak+b}+\frac{c}{bk+c}+\frac{a}{ck+a}=\frac{b^2}{abk+b^2}+\frac{c^2}{bck+c^2}+\frac{a^2}{cak+a^2}\geq$$

$$\geq\frac{(a+b+c)^2}{a^2+b^2+c^2+(ab+bc+ca)k}\overset{(?)}{\geq}3(a^2+b^2+c^2)+3k(ab+bc+ca)$$

$$(k-2)(a^2+b^2+c^2)-(k-2)(ab+bc+ca)\overset{(?)}{\geq}0$$

$(k-2)\left(a^2+b^2+c^2-(ab+bc+ca)\right)\geq 0$, which is true because $k\geq 2$ and

$$\sum a^2\geq\sum ab$$

871. If $a,b,c>0$ and $\lambda\geq\frac{1}{2}, n\in\mathbb{N}, n\geq 2$ then

$$\sum_{cyc}\frac{a}{\lambda b+\sqrt[n]{ab^{n-1}}}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Şerban-Romania

$$\sqrt[n]{ab^{n-1}}\leq\frac{a+(n-1)b}{n}\Rightarrow\sum_{cyc}\frac{a}{\lambda b+\sqrt[n]{ab^{n-1}}}\geq\sum_{cyc}\frac{a}{\lambda b+\frac{a+(n-1)b}{n}}=$$

$$=n\sum_{cyc}\frac{a}{b(\lambda n+n-1)+a}=n\sum_{cyc}\frac{a^2}{ab(\lambda n+n-1)+a^2}\overset{Bergstrom}{\geq}$$

$$\geq\frac{n(\sum a)^2}{(\lambda n+n-1)\sum ab+\sum a^2}=\frac{n(\sum a)^2}{(\lambda n+n-1)\sum ab+(\sum a)^2-2\sum ab}=$$

$$=\frac{n(\sum a)^2}{(\lambda n+n-3)\sum ab+(\sum a)^2}\overset{(1)}{\geq}\frac{3}{\lambda+1}$$

$$(1)\Leftrightarrow(n\lambda+n)(\sum a)^2\geq 3(\lambda n+n-3)\sum ab+3(\sum a)^2\Rightarrow$$

$$(n\lambda+n-3)(\sum a)^2\geq 3(\lambda n+n-3)\sum ab$$

$\lambda n+n-3\geq 0$ and $(\sum a)^2>3\sum ab\Leftrightarrow\sum(a-b)^2\geq 0$ true.

872. If $a,b,c>0$ then:

$$abc(a+b)^2(b+c)^2(c+a)^2\leq 64\left(\frac{a+b+c}{3}\right)^9$$

Proposed by Daniel Sitaru-Romania

Solution by George Florin Șerban-Romania

$$abc \le \left(\frac{a+b+c}{3}\right)^3$$

$$(a+b)(b+c)(c+a) \le \left[\frac{2(a+b+c)}{3}\right]^3 \Rightarrow (a+b)^2(b+c)^2(c+a)^2 \le 64\left(\frac{a+b+c}{3}\right)^6.$$

Therefore,

$$abc(a+b)^2(b+c)^2(c+a)^2 \le 64\left(\frac{a+b+c}{3}\right)^9$$

873. If $x, y, z > 0$ such that

$$\sum x^2 = 3 \text{ and } \lambda \ge 2 \text{ then}$$

$$\sum \frac{x}{\lambda - yz} \le \frac{3}{\lambda - 1}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\lambda - yz \overset{AM-GM}{\ge} \lambda - \frac{y^2+z^2}{2} = \lambda - \frac{3-x^2}{2} = \frac{2(\lambda-2)+(x^2+1)}{2} \overset{AM-GM}{\ge} \frac{2(\lambda-2)+2x}{2}$$
$$= x + (\lambda - 2) > 0.$$

$$\to \sum \frac{x}{\lambda - yz} \le \sum \frac{x}{x+(\lambda-2)} = \sum\left(1 - \frac{\lambda-2}{x+(\lambda-2)}\right) = 3 - (\lambda-2)\sum \frac{1}{x+(\lambda-2)} \le$$

$$\overset{CBS}{\le} 3 - (\lambda-2).\frac{9}{\sum[x+(\lambda-2)]} = 3 - \frac{9(\lambda-2)}{(\sum x)+3(\lambda-2)} \overset{CBS}{\le} 3 - \frac{9(\lambda-2)}{\sqrt{3\sum x^2}+3(\lambda-2)}$$

$$= 3 - \frac{9(\lambda-2)}{3+3(\lambda-2)} =$$

$$= 3 - \frac{3(\lambda-2)}{\lambda-1} = \frac{3}{\lambda-1}. \quad \text{Therefore, } \sum \frac{x}{\lambda-yz} \le \frac{3}{\lambda-1}.$$

874. If $x, y, z > 0$ and $2n + 3k = 12, 0 \le k \le 4$ then

$$\sum_{cyc} \frac{y+z}{x} \ge n + k\sum_{cyc} \frac{x}{y+z}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Dincă-Romania

$$\sum_{cyc}\frac{y+z}{x}\ge n+k\sum_{cyc}\frac{x}{y+z}\Leftrightarrow\sum_{cyc}\left(\frac{y+z}{x}+1-1\right)\ge n+k\sum_{cyc}\left(\frac{x}{y+z}+1-1\right)\Leftrightarrow$$

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-3\ge\frac{12-3k}{2}+k\left(\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}\right)-3k\Leftrightarrow$$

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)(x+y+z)\ge\frac{18-9k}{2}+k\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)(x+y+z);$$

$$f(k)=\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)(x+y+z)-\frac{18-9k}{2}-k\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)(x+y+z);$$

$k\in[0,4]$. It is enough to prove that $f(0)\ge 0, f(4)\ge 0$

$f(0)=\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)(x+y+z)-9\ge 0$, well known (harmonic inequality).

$$f(4)=\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)(x+y+z)+9-4\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)(x+y+z)\ge 0\Leftrightarrow$$

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{9}{x+y+z}\ge 4\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)$ which is Popoviciu's inequality for convex function

$f(t)=\frac{1}{t}, t\ge 0.$

875. If $a,b,c>0$ such that $a^2+b^2+c^2=3$ and $\lambda\ge 0$ then

$$\frac{a^2}{bc(a+\lambda b)}+\frac{b^2}{ca(b+\lambda c)}+\frac{c^2}{ab(c+\lambda a)}\ge\frac{3}{(\lambda+1)abc}$$

Proposed by Marin Chirciu-Romania

Solution 1 by George Florin Şerban-Romania

$$\sum_{cyc}\frac{a^2}{bc(a+\lambda b)}=\sum_{cyc}\frac{a^4}{a^2bc(a+\lambda b)}=\frac{1}{abc}\sum_{cyc}\frac{(a^2)^2}{a^2+\lambda ab}\overset{Bergstrom}{\ge}$$

$$\ge\frac{1}{abc}\cdot\frac{(a^2+b^2+c^2)^2}{\sum a^2+\lambda\sum bc}=\frac{9}{abc(3+\lambda\sum ab)}\ge\frac{9}{abc(3+3\lambda)}=\frac{3}{(\lambda+1)abc}$$

Solution 2 by Amrit Awasthi-India

$$\sum_{cyc}\frac{a^2}{bc(a+\lambda b)}=\sum_{cyc}\frac{a^4}{a^2bc(a+\lambda b)}=\frac{1}{abc}\sum_{cyc}\frac{(a^2)^2}{a^2+\lambda ab}\overset{Bergstrom}{\ge}$$

$$\geq\frac{(a^2+b^2+c^2)^2}{\sum a^3bc+\lambda\sum a^2b^2c}=\frac{9}{abc(\sum a^2+\lambda\sum ab)}\overset{Chebyschev}{\geq}\frac{9}{abc(3+3\lambda)}=\frac{3}{(\lambda+1)abc}$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$abc\cdot\sum_{cyc}\frac{a^2}{bc(a+\lambda b)}=\sum_{cyc}\frac{a^3}{ca+\lambda b}=\sum_{cyc}\frac{a^4}{a^2+\lambda ab}\geq\frac{(\sum a^2)^2}{\sum a^2+\lambda\sum ab}=$$

$$=\frac{9}{3+\lambda\sum ab}=\frac{3}{\lambda+1}$$

Therefore,

$$\frac{a^2}{bc(a+\lambda b)}+\frac{b^2}{ca(b+\lambda c)}+\frac{c^2}{ab(c+\lambda a)}\geq\frac{3}{(\lambda+1)abc}$$

876. For $x,y,z>0$.Prove that:

$$\left(\sum_{cyc}xy\right)^2\sum_{cyc}x+3xyz\sum_{cyc}xy\geq 4xyz\left(\sum_{cyc}x\right)^2$$

Proposed by Nikos Ntorvas-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum_{cyc}xy\right)^2\sum_{cyc}x+3xyz\sum_{cyc}xy\geq 4xyz\left(\sum_{cyc}x\right)^2\ ;(*)$$

$Let\ x=s-a,y=s-b,z$

$=s-c\ where\ a,b,c\ are\ the\ side\ lenghts\ of\ a\ triangle,s\ the\ semiperimeter.$

$$We\ have:\sum_{cyc}x=\sum_{cyc}(s-a)=s,\qquad xyz=(s-a)(s-b)(s-c)=sr^2.$$

$$\sum_{cyc}yz=\sum_{cyc}(s-b)(s-c)=\sum_{cyc}\frac{(s-a)(s-b)(s-c)}{(s-a)}=\sum_{cyc}\frac{sr^2}{(s-a)}=r\sum_{cyc}r_a$$

$$=r(4R+r)$$

$$\rightarrow\ (*)\ \leftrightarrow\ r^2(4R+r)^2.s+3sr^2.r(4R+r)\geq 4sr^2.s^2\ \leftrightarrow\ (4R+r)^2+3r(4R+r)\geq 4s^2$$

$$\leftrightarrow 4(4R^2+4Rr+3r^2-s^2)+4r(R-2r)\geq 0$$

$$\text{Which is true from Gerretsen's inequality } (\because 4R^2+4Rr+3r^2 \geq s^2) \text{ and Euler's inequality } (\because R \geq 2r).$$

$$\text{Therefore,}\quad \left(\sum_{cyc} xy\right)^2 \sum_{cyc} x + 3xyz\sum_{cyc} xy \geq 4xyz\left(\sum_{cyc} x\right)^2.$$

877. If $a,b,c \geq 0, a+b+c=2$ then:

$$\sum_{cyc}\frac{1}{1+a^2} \geq 2$$

Proposed by Mehmet Şahin-Ankara-Turkyie

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{1}{1+a^2} \overset{?}{\gtrsim} 1-\frac{a}{2} \leftrightarrow 2 \overset{?}{\gtrsim} (1+a^2)(2-a) = -a^3+2a^2-a+2 \leftrightarrow a(a-1)^2 \overset{?}{\gtrsim} 0$$

$$\text{Which is true} \rightarrow \frac{1}{1+a^2} \geq 1-\frac{a}{2}\ (\text{And analogs})$$

$$\rightarrow \sum_{cyc}\frac{1}{1+a^2} \geq \sum_{cyc}\left(1-\frac{a}{2}\right) = 3-\frac{1}{2}(a+b+c) = 3-\frac{1}{2}.2 = 2$$

$$\text{Therefore,}\quad \sum_{cyc}\frac{1}{1+a^2} \geq 2.$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\sum_{cyc}\frac{1}{1+a^2} \geq 2\ ;(*)$$

$(\tan x - 1)^2 \geq 0 \Leftrightarrow \tan^2 x - 2\tan x + 1 \geq 0$. Suppose $\tan x \geq 0 \Rightarrow$

$$2\tan x - 1 - \tan^2 x \leq 0 \Leftrightarrow 2\tan^2 x - \tan x - \tan^3 x \leq 0 \Leftrightarrow$$

$$2 \geq 2(1+\tan^2 x) - \tan x\,(1+\tan^2 x) \Leftrightarrow$$

$$2 \geq 2\cdot\frac{1}{\cos^2 x} - \frac{\sin x}{\cos x}\cdot\frac{1}{\cos^2 x} \Leftrightarrow 2\cos^2 x \geq 2 - \frac{\sin x}{\cos x}$$

$$\cos^2 x \geq 1 - \frac{1}{2}\tan x\ ;(**)$$

Let $\tan x = a, \tan y = b, \tan z = c \Rightarrow$

$$\sum_{cyc}\frac{1}{1+a^2}=\cos^2 x+\cos^2 y+\cos^2 z\overset{(**)}{\geq}1-\frac{1}{2}\tan x+1-\frac{1}{2}\tan y+1-\frac{1}{2}\tan z\geq$$

$$\geq 3-\frac{1}{2}(\tan x+\tan y+\tan z)\geq 3-\frac{1}{2}\cdot 2=2$$

Solution 3 by Ihsan Yucel-Turkiye

$$\frac{1}{1+b^2}=1-\frac{b^2}{1+b^2}\geq 1-\frac{b^2}{2b}$$

$$\frac{1+b^2}{2}\geq\sqrt{1\cdot b^2}\Rightarrow\frac{1}{2b}\geq\frac{1}{1+b^2}\Rightarrow-\frac{1}{2b}\leq-\frac{1}{1+b^2}\Leftrightarrow 1-\frac{1}{1+b^2}\geq 1-\frac{1}{2b}$$

$$\frac{1}{1+b^2}=1-\frac{b^2}{b^2+1}\geq 1-\frac{b^2}{2b}=1-\frac{b}{2}$$

Analogously, $\frac{1}{1+a^2}\geq 1-\frac{a}{2}$ and $\frac{1}{1+c^2}\geq 1-\frac{c}{2}$.

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}\geq 1-\frac{b}{2}+1-\frac{c}{2}+1-\frac{a}{2}=3-\frac{a+b+c}{2}=3-\frac{2}{2}=2$$

878. If $a,b,c>0$ and $\lambda\geq -2$ then

$$\sum_{cyc}\frac{(a^2-\lambda ab+b^2)^n}{(a+b)^{2n}}\geq 3\left(\frac{2-\lambda}{4}\right)^n$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Şerban-Romania

$$\frac{a^2-\lambda ab+b^2}{(a+b)^2}\geq\frac{2-\lambda}{4}\Leftrightarrow 4a^2-4\lambda ab+4b^2\geq(2-\lambda)a^2+(2-\lambda)b^2+(4-2\lambda)ab$$

$\Leftrightarrow(\lambda+2)(a-b)^2\geq 0$, which is true for $\lambda\geq -2$ and for all $a,b\in\mathbb{R}$

$$\sum_{cyc}\frac{(a^2-\lambda ab+b^2)^n}{(a+b)^{2n}}=\sum_{cyc}\left(\frac{a^2-\lambda ab+b^2}{(a+b)^2}\right)^n\overset{Holder}{\geq}\frac{1}{3^{n-1}}\left(\sum_{cyc}\frac{a^2-\lambda ab+b^2}{(a+b)^2}\right)^n\geq$$

$$\geq\frac{1}{3^{n-1}}\left(\sum_{cyc}\frac{2-\lambda}{4}\right)^n=3\left(\frac{2-\lambda}{4}\right)^n$$

879. If $a,b>0$ such that $a+b=1$ and $n\in N^*$. Prove that :

$$\sum_{k=1}^{n}\frac{1}{a^k+b^k}\leq 2^n-1$$

Proposed by Amrit Awasthi-India

Solution 1 by Kamel Gandouli Rezgui-Tunisia

$$P(n):\sum_{k=1}^{n}\frac{1}{a^k+b^k}\le 2^n-1$$

$$n=1:\frac{1}{a+b}=1\le 2^1-1$$

$$\text{Suppose that: } P(n):\sum_{k=1}^{n}\frac{1}{a^k+b^k}\le 2^n-1$$

$$\sum_{k=1}^{n+1}\frac{1}{a^k+b^k}=\sum_{k=1}^{n}\frac{1}{a^k+b^k}+\frac{1}{a^{n+1}+b^{n+1}}$$

$$a^{n+1}+b^{n+1}\overset{(?)}{\ge}\left(\frac{1}{2}\right)^n$$

$$\text{Let } f(x)=x^{n+1}+(1-x)^{n+1}, f'(x)=(n+1)(x^n+(1-x)^n)$$

$$f'(x)=0\Leftrightarrow x=\frac{1}{2}.$$

$$\max\{f(x)\}=\left(\frac{1}{2}\right)^{n+1}+\left(\frac{1}{2}\right)^{n+1}=\left(\frac{1}{2}\right)^n\Rightarrow\frac{1}{a^{n+1}+b^{n+1}}\le 2^n$$

$$\Rightarrow\sum_{k=1}^{n+1}\frac{1}{a^k+b^k}\le 2^n-1+2^n=2^{n+1}-1$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

From Power Mean inequality, we have : $\frac{a^k+b^k}{2}\ge\left(\frac{a+b}{2}\right)^k=\left(\frac{1}{2}\right)^k, \forall k\in N^*$

$\to\frac{1}{a^k+b^k}\le 2^{k-1}, \forall k\in N^*$. *Therefore,* $\sum_{k=1}^{n}\frac{1}{a^k+b^k}\le\sum_{k=1}^{n}2^{k-1}=2^n-1.$

880. If $a,b,c>0$ such that $a+b+c\ge 3$ and $\lambda\ge 0$ then :

$$\sum\frac{a}{\sqrt{1+\lambda b}}\ge\frac{3}{\sqrt{1+\lambda}}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{From Hölder, we have:}$$

$$\left(\sum \frac{a}{\sqrt{1+\lambda b}}\right)^2\left(\sum a(1+\lambda b)\right) \ge \left(\sum a\right)^3$$

$$\to \left(\sum \frac{a}{\sqrt{1+\lambda b}}\right)^2 \ge \frac{(a+b+c)^3}{(a+b+c)+\lambda(ab+bc+ca)}$$

$$\overset{(\sum a)^2\ge 3\sum ab}{\ge} \frac{3(a+b+c)^3}{3(a+b+c)+\lambda(a+b+c)^2} =$$

$$= \frac{3(a+b+c)^2}{3+\lambda(a+b+c)} = \frac{3(a+b+c)}{\frac{3}{a+b+c}+\lambda} \overset{a+b+c\ge 3}{\ge} \frac{3.3}{\frac{3}{3}+\lambda} = \frac{9}{1+\lambda}.$$

$$\textit{Therefore,} \quad \sum \frac{a}{\sqrt{1+\lambda b}} \ge \frac{3}{\sqrt{1+\lambda}}.$$

881. If $a, \mathrm{b}, \mathrm{c}, x, \mathrm{y}, \mathrm{z} > 0$ *such that* $x + \mathrm{y} + \mathrm{z} = 3$ *and* $\lambda \ge 2$, then

$$\frac{a}{a+\lambda \mathrm{b}x} + \frac{\mathrm{b}}{\mathrm{b}+\lambda \mathrm{cy}} + \frac{\mathrm{c}}{\mathrm{c}+\lambda a\mathrm{z}} \ge \frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{a+\lambda bx} + \frac{b}{b+\lambda cy} + \frac{c}{c+\lambda az} = \frac{\left(\sqrt{\frac{a}{b}}\right)^2}{\frac{a}{b}+\lambda x} + \frac{\left(\sqrt{\frac{b}{c}}\right)^2}{\frac{b}{c}+\lambda y} + \frac{\left(\sqrt{\frac{c}{a}}\right)^2}{\frac{c}{a}+\lambda z} \overset{\text{Bergstrom}}{\ge}$$

$$\frac{\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}\right)^2}{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\lambda(x+y+z)} \overset{x+y+z=3}{=}$$

$$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2\left(\left(\sqrt{\frac{a}{b}}\right)\left(\sqrt{\frac{b}{c}}\right)+\left(\sqrt{\frac{b}{c}}\right)\left(\sqrt{\frac{c}{a}}\right)+\left(\sqrt{\frac{c}{a}}\right)\left(\sqrt{\frac{a}{b}}\right)\right)}{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\lambda}$$

$$= \frac{u+v+w+2\left(\sqrt{uv}+\sqrt{vw}+\sqrt{wu}\right)}{u+v+w+3\lambda}\left(u=\frac{a}{b}, v=\frac{b}{c}, w=\frac{c}{a}\right) \overset{?}{\ge} \frac{3}{\lambda+1}$$

$$\Leftrightarrow (\lambda+1)\sum u + 2(\lambda+1)\sum \sqrt{uv} \overset{?}{\ge} 9\lambda + 3\sum u$$

$$\Leftrightarrow (\lambda-2)\sum u+2(\lambda-2+3)\sum\sqrt{uv}\overset{?}{\geq}9(\lambda-2+2)$$

$$\Leftrightarrow (\lambda-2)\left(\sum u+2\sum\sqrt{uv}-9\right)+6\left(\sum\sqrt{uv}-3\right)\overset{?}{\geq}0$$

$$\Leftrightarrow (\lambda-2)\left(\left(\sum\sqrt{u}\right)^2-9\right)+6\left(\sum\sqrt{uv}-3\right)\underset{(*)}{\overset{?}{\geq}}0$$

$$\text{Now, } uvw=\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}=1\ \therefore\ \sum\sqrt{u}\overset{A-G}{\geq}3\sqrt[6]{uvw}=3$$

$$\Rightarrow\left(\sum\sqrt{u}\right)^2-9\overset{(i)}{\geq}0\ \textit{and also}\ \sum\sqrt{uv}\overset{A-G}{\geq}3\sqrt[3]{uvw}=3\Rightarrow\sum\sqrt{uv}-3\overset{(ii)}{\geq}0$$

$\therefore$ via (i), (ii) *and* $\because \lambda-2\geq 0$

$$\therefore (\lambda-2)\left(\left(\sum\sqrt{u}\right)^2-9\right)+6\left(\sum\sqrt{uv}-3\right)\geq 0\Rightarrow(*)\text{ is true}\Rightarrow\text{if } a,b,c,x,y,z>0,\text{ then}$$

$$:\frac{a}{a+\lambda bx}+\frac{b}{b+\lambda cy}+\frac{c}{c+\lambda az}\geq\frac{3}{\lambda+1}\ \forall\,\lambda\geq 2\text{ for } x+y+z=3\ (QED)$$

882. If $x,y,z>0$ and $-1\leq\lambda\leq 2$ then:

$$(\lambda+1)[(\textstyle\sum x^2)^2+xyz\sum x]\geq\lambda[(\sum yz)^2+3\sum y^2z^2]$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Șerban-Romania

$$x^4+x^4+y^4+z^4\geq 4\sqrt{x^4x^4y^4z^4}=4x^2yz\Rightarrow 4\textstyle\sum x^4\geq 4\sum x^2yz$$

$$\Rightarrow\textstyle\sum x^4\geq xyz(x+y+z)\Rightarrow\sum x^4\geq\sum x^2y^2\Leftrightarrow$$

$$2\textstyle\sum x^4\geq 2\sum x^2y^2\Leftrightarrow\sum(x^2-y^2)^2\geq 0\text{ true.}$$

$$(\lambda+1)(\textstyle\sum x^4+2\sum x^2y^2+xyz\sum x)\geq\lambda(\sum y^2z^2+2xyz\sum x+3\sum y^2z^2)$$

$$(\lambda+1)\textstyle\sum x^4+(2\lambda+2)\sum x^2y^2+(\lambda+1)xyz\sum x\geq 4\lambda\sum y^2z^2+2\lambda xyz\sum x$$

$$(\lambda+1)\textstyle\sum x^4\geq(\lambda-1)(2\sum x^2y^2+xyz\sum x)$$

$$\lambda\leq 1\Rightarrow\lambda\in[-1,1]\text{ true. }\ \textstyle\sum x^4>0,\ 2\sum x^2y^2+xyz\sum x>0.$$

$$\text{If }\lambda>1\Rightarrow\lambda\in(1,2]\Rightarrow(\lambda-1)(2\textstyle\sum x^2y^2+xyz\sum x)\leq(\lambda-1)(2\sum x^4+\sum x^4)=$$

$$=(\lambda-1)\cdot 3\textstyle\sum x^4=(3\lambda-3)\sum x^4\leq(\lambda+1)\sum x^4\Leftrightarrow$$

$$3\lambda-3\leq\lambda+1\Leftrightarrow\lambda\leq 2\text{ true.Therefore,}$$

$$(\lambda+1)[(\textstyle\sum x^2)^2+xyz\sum x]\geq\lambda[(\sum yz)^2+3\sum y^2z^2]$$

883. If $x, y, z \geq 0$, $\frac{x+1}{z+1} + \frac{z+1}{y+1} + \frac{y+1}{x+1} = 3$ then:

$$\frac{x+1}{z^2 + \sqrt[3]{1+3z}} + \frac{y+1}{x^2 + \sqrt[3]{1+3x}} + \frac{z+1}{y^2 + \sqrt[3]{1+3y}} \leq 3$$

Proposed by Daniel Sitaru-Romania

Solution by Nguyen Van Canh-BenTre-Vietnam

Lemma: If $x \geq 0$ then: $x^2 + \sqrt[3]{1+3x} \geq x+1 \ (*)$

Proof:

Let us denote $t = \sqrt[3]{1+3x} \to x = \frac{t^3-1}{3}, t \geq 1$

$$(*) \leftrightarrow \left(\frac{t^3-1}{3}\right)^2 + t - \left(\frac{t^3-1}{3}\right) - 1 \geq 0$$

$$\leftrightarrow \frac{t^6 - 2t^3 + 1}{9} + t - \frac{t^3}{3} + \frac{1}{3} - 1 \geq 0$$

$$\leftrightarrow t^6 - 5t^3 + 9t - 5 \geq 0 \leftrightarrow (t-1)^3(t^3 + 3t^2 + 6t + 5) \geq 0$$

(Which is true since $t \geq 1$)

Therefore, $(*)$ is true. Equality $\leftrightarrow t = 1 \leftrightarrow x = 0$

Now,

$$\frac{x+1}{z^2 + \sqrt[3]{1+3z}} + \frac{y+1}{x^2 + \sqrt[3]{1+3x}} + \frac{z+1}{y^2 + \sqrt[3]{1+3y}} \leq \frac{x+1}{z+1} + \frac{z+1}{y+1} + \frac{y+1}{x+1} = 3.$$

Equality $\leftrightarrow x = y = z = 0$. Q.E.D

884. For $x, y, z > 0$.Prove that:

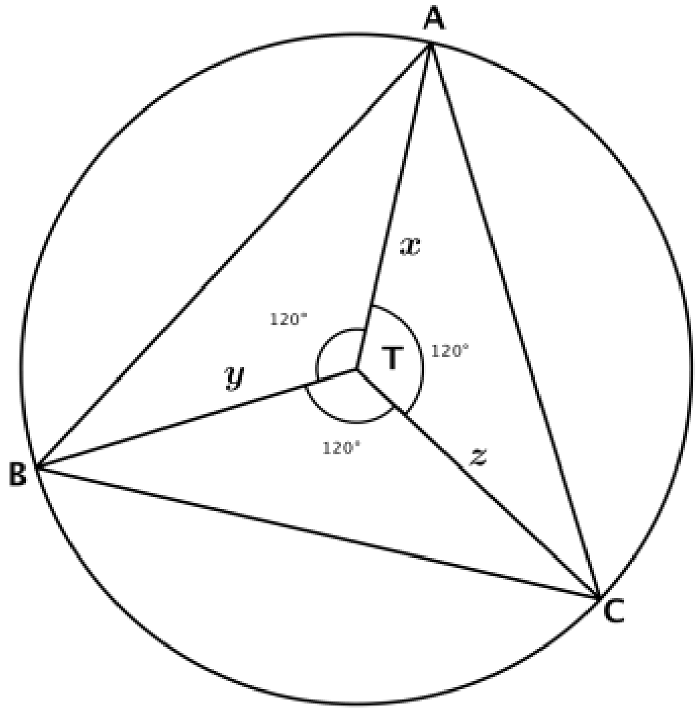
$$1 + 2 \cdot \frac{x^2 + y^2 + z^2}{xy + yz + zx} \geq \sqrt{3 \sum \frac{x^2 + xy + y^2}{y^2 + yz + z^2}}$$

Proposed by Bogdan Fuştei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$1 + 2.\frac{x^2 + y^2 + z^2}{xy + yz + zx} \overset{(*)}{\geq} \sqrt{3 \sum \frac{x^2 + xy + y^2}{y^2 + yz + z^2}}$$

$$Let's\ T,A,B,C\ be\ four\ points\ of\ the\ plane\ such\ that\ TA = x, TB = y, TC = z, \mu(CTA) = \mu(ATB) = \mu(BTC) = 120^{\circ}$$



$$Applying\ the\ Law\ of\ cosines\ in\ \Delta ATB, we\ have:\ c^2 = x^2 + y^2 - 2xy\cos 120^{\circ} = x^2 + xy + y^2$$

$$Similarly, we\ have:\ b^2 = z^2 + zx + x^2\ and\ a^2 = y^2 + yz + z^2.$$

$$Area\ of\ \Delta ABC : S = \sum [ATB] = \sum \frac{1}{2} xy \sin 120^{\circ} = \frac{\sqrt{3}}{4} \sum xy$$

$$(*) \leftrightarrow \frac{1}{xy + yz + zx} \sum (y^2 + yz + z^2) \geq \sqrt{3 \sum \frac{x^2 + xy + y^2}{y^2 + yz + z^2}} \leftrightarrow \frac{\sqrt{3}}{4S} \sum a^2$$

$$\geq \sqrt{3 \sum \frac{a^2}{b^2}} \leftrightarrow \sum a^2 \geq 4S \sqrt{\sum \frac{c^2}{a^2}}$$

$$We\ know\ that:\ \forall u, v, w > 0, \sum u.a^2 \geq 4S \sqrt{\sum uv} \quad (Oppenheim)$$

$$Let\ u=\frac{b^2}{a^2},v=\frac{c^2}{b^2},w=\frac{a^2}{c^2}\to\sum\frac{b^2}{a^2}.a^2\geq 4S\sqrt{\sum\frac{b^2}{a^2}.\frac{c^2}{b^2}}\leftrightarrow\sum a^2\geq 4S\sqrt{\sum\frac{c^2}{a^2}}$$

$$\to(*)\ is\ true.$$

$$Therefore,\quad 1+2.\frac{x^2+y^2+z^2}{xy+yz+zx}\geq\sqrt{3\sum\frac{x^2+xy+y^2}{y^2+yz+z^2}}.$$

885. If $x,y,z>0$, then:

$$\sum\left(\frac{x}{y}\right)^3+\sum\left(\frac{y}{x}\right)^3\geq\frac{2(x^3+y^3+z^3)}{xyz}\geq 6$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum\left(\frac{x}{y}\right)^3=\sum\frac{1}{3}\left[\left(\frac{x}{y}\right)^3+\left(\frac{x}{y}\right)^3+\left(\frac{y}{z}\right)^3\right]\overset{AM-GM}{\geq}\sum\left(\frac{x}{y}\right).\left(\frac{x}{y}\right).\left(\frac{y}{z}\right)=\sum\frac{x^2}{yz}$$

$$=\frac{x^3+y^3+z^3}{xyz}$$

$$Similarly,we\ have:\sum\left(\frac{y}{x}\right)^3\geq\frac{x^3+y^3+z^3}{xyz}$$

$$\to\sum\left(\frac{x}{y}\right)^3+\sum\left(\frac{y}{x}\right)^3\geq\frac{2(x^3+y^3+z^3)}{xyz}\overset{AM-GM}{\geq}\frac{2.3xyz}{xyz}=6.$$

$$Therefore,\quad\sum\left(\frac{x}{y}\right)^3+\sum\left(\frac{y}{x}\right)^3\geq\frac{2(x^3+y^3+z^3)}{xyz}\geq 6.$$

886. If $a,b,c\geq 0$ such that $ab+bc+ca=3$ then:

$$\sum_{cyc}\frac{bc}{a^2+1}\geq\frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Fayssal Abdelli-Bejaia-Algerie

$$\sum_{cyc}\frac{bc}{a^2+1}\overset{Bergstrom}{\geq}\frac{\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2}{3+a^2+b^2+c^2}=$$

$$=\frac{ab+bc+ca+2\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})}{3+a^2+b^2+c^2}\ge\frac{3+2\sqrt{abc}\cdot 3\sqrt[3]{\sqrt{abc}}}{3+a^2+b^2+c^2}=$$

$$=\frac{3+6\sqrt[3]{(abc)^2}}{3+a^2+b^2+c^2}\overset{(1)}{\ge}\frac{3}{2}$$

$$(1)\Leftrightarrow 6+12\sqrt[3]{(abc)^2}\ge 9+3(a^2+b^2+c^2);(2)$$

But $a^2+b^2+c^2\ge 3\sqrt[3]{(abc)^2}$

$$4\sqrt[3]{(abc)^2}\ge 1+3\sqrt[3]{(abc)^2}\Rightarrow\sqrt[3]{(abc)^2}\ge 1$$

But $ab+bc+ca\ge 3\sqrt[3]{(abc)^2}\Rightarrow\sqrt[3]{(abc)^2}\le 1\Rightarrow abc\le 1$ then from (2), it follows that

$a^2+b^2+c^2\le 3$, but $a^2+b^2+c^2\ge 3\sqrt[3]{(abc)^2}\ge 3$

Therefore,

$$\sum_{cyc}\frac{bc}{a^2+1}\ge\frac{3}{2}$$

887. If $a,b,c,d,x,y,z,t>0$ such that $x+y+z+t=4$ and $\lambda\ge 3$, then

$$\frac{a}{a+\lambda bx}+\frac{b}{b+\lambda cy}+\frac{c}{c+\lambda dz}+\frac{d}{d+\lambda at}\ge\frac{4}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{a+\lambda bx}+\frac{b}{b+\lambda cy}+\frac{c}{c+\lambda dz}+\frac{d}{d+\lambda at}$$

$$=\frac{\left(\sqrt{\frac{a}{b}}\right)^2}{\frac{a}{b}+\lambda x}+\frac{\left(\sqrt{\frac{b}{c}}\right)^2}{\frac{b}{c}+\lambda y}+\frac{\left(\sqrt{\frac{c}{d}}\right)^2}{\frac{c}{d}+\lambda z}$$

$$+\frac{\left(\sqrt{\frac{d}{a}}\right)^2}{\frac{d}{a}+\lambda t}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{d}}+\sqrt{\frac{d}{a}}\right)^2}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}+\lambda(x+y+z+t)}\overset{x+y+z+t=4}{=}$$

$$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}+2\left(\left(\sqrt{\frac{a}{b}}\right)\left(\sqrt{\frac{b}{c}}\right)+\left(\sqrt{\frac{a}{b}}\right)\left(\sqrt{\frac{c}{d}}\right)+\left(\sqrt{\frac{a}{b}}\right)\left(\sqrt{\frac{d}{a}}\right)+\left(\sqrt{\frac{b}{c}}\right)\left(\sqrt{\frac{c}{d}}\right)+\left(\sqrt{\frac{b}{c}}\right)\left(\sqrt{\frac{d}{a}}\right)+\left(\sqrt{\frac{c}{d}}\right)\left(\sqrt{\frac{d}{a}}\right)\right)}{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}+4\lambda}$$

$$=\frac{u+v+w+\mu+2(\sqrt{uv}+\sqrt{uw}+\sqrt{u\mu}+\sqrt{vw}+\sqrt{v\mu}+\sqrt{w\mu})}{u+v+w+4\lambda}$$

$$\left(u=\frac{a}{b},v=\frac{b}{c},w=\frac{c}{d},\mu=\frac{d}{a}\right)\overset{?}{\gtrsim}\frac{4}{\lambda+1}\Leftrightarrow(\lambda+1)\sum_{cyc}u+2(\lambda+1)\sum_{sym}\sqrt{uv}\overset{?}{\gtrsim}16\lambda+4\sum_{cyc}u$$

$$\Leftrightarrow(\lambda-3)\sum_{cyc}u+2(\lambda-3+4)\sum_{sym}\sqrt{uv}\overset{?}{\gtrsim}16(\lambda-3+3)$$

$$\Leftrightarrow(\lambda-3)\left(\sum_{cyc}u+2\sum_{sym}\sqrt{uv}-16\right)+8\left(\sum_{sym}\sqrt{uv}-6\right)\overset{?}{\gtrsim}0$$

$$\Leftrightarrow(\lambda-3)\left(\left(\sum_{cyc}\sqrt{u}\right)^2-16\right)+8\left(\sum_{sym}\sqrt{uv}-6\right)\underset{(*)}{\overset{?}{\gtrsim}}0$$

$$\text{Now, } uvw\mu=\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}\cdot\frac{d}{a}=1\therefore\sum_{cyc}\sqrt{u}\overset{A-G}{\gtrsim}4\sqrt[4]{\sqrt{uvw\mu}}=4$$

$$\Rightarrow\left(\sum_{cyc}\sqrt{u}\right)^2-16\overset{(i)}{\gtrsim}0\text{ and also }\sum_{sym}\sqrt{uv}\overset{A-G}{\gtrsim}6\sqrt[6]{(\sqrt{uv})(\sqrt{uw})(\sqrt{u\mu})(\sqrt{vw})(\sqrt{v\mu})(\sqrt{w\mu})}$$

$$=6\sqrt[6]{(uvw\mu)^{\frac{3}{2}}}=6$$

$$\Rightarrow\sum_{sym}\sqrt{uv}-6\overset{(ii)}{\gtrsim}0\therefore\text{ via (i), (ii) and }\because\lambda-3$$

$$\geq 0(\lambda-3)\left(\left(\sum_{cyc}\sqrt{u}\right)^2-16\right)+8\left(\sum_{sym}\sqrt{uv}-6\right)\geq 0\Rightarrow(*)\text{ is true}$$

$$\Rightarrow\forall a,b,c,d,x,y,z,t>0\text{ such that }x+y+z+t=4,$$

$$:\frac{a}{a+\lambda bx}+\frac{b}{b+\lambda cy}+\frac{c}{c+\lambda dz}+\frac{d}{d+\lambda at}\geq\frac{4}{\lambda+1}\ \forall\lambda\geq 3\text{ (QED)}$$

888. If $x,y,z>0$, such that : $\sum x^2\geq\sum\frac{x}{y}$. Prove that :

$$\sum x\geq 3$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let's\ prove\ that:\sum\frac{x}{y}\geq\frac{9(x^2+y^2+z^2)}{(x+y+z)^2}\quad(*)$$

$$(*)\leftrightarrow\left(\sum x^2+2\sum xy\right)\sum\frac{x}{y}\geq 9\sum x^2\leftrightarrow\sum\frac{x^3}{y}+\sum\frac{xz^2}{y}+3\sum xy+2\sum\frac{x^2z}{y}$$

$$\geq 7\sum x^2$$

$$\leftrightarrow\sum\left(\frac{x^3}{y}-2.\frac{x^2z}{y}+\frac{xz^2}{y}\right)+4\sum\left(\frac{x^2z}{y}-2zx+yz\right)-7\left(\sum x^2-\sum xy\right)\geq 0$$

$$\leftrightarrow\sum\frac{x}{y}(z-x)^2+4\sum\frac{z}{y}(x-y)^2-\frac{7}{2}\sum(x-y)^2\geq 0$$

$$\leftrightarrow\sum\left(\frac{y}{z}-4+\frac{4z}{y}\right)(x-y)^2+\frac{1}{2}\sum(x-y)^2\geq 0$$

$$\leftrightarrow\sum\frac{(y-2z)^2}{yz}(x-y)^2+\frac{1}{2}\sum(x-y)^2\geq 0$$

$$Which\ is\ true.$$

$$\rightarrow\sum x^2\geq\sum\frac{x}{y}\geq\frac{9(x^2+y^2+z^2)}{(x+y+z)^2}\leftrightarrow(x+y+z)^2\geq 9$$

$$Therefore,\quad \sum x\geq 3.$$

889. If $x,y,z>0$ such that $xy+yz+zx\geq\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$. Prove that:

$$\sum x^3+6xyz\geq\left(\sum x\right)^2$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$xy+yz+zx\geq\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\Rightarrow xy+yz+zx+xy+yz+zx$$

$$\geq\left(\frac{x}{y}+xy\right)+\left(\frac{y}{z}+yz\right)+\left(\frac{z}{x}+zx\right)\overset{A-G}{\geq}2x+2y+2z\Rightarrow\sum x\leq\sum xy$$

$$\Rightarrow\left(\sum x\right)^2\leq\left(\sum x\right)\left(\sum xy\right)$$

$$\overset{?}{\leq}\sum x^3+6xyz\Leftrightarrow\sum x^3+3xyz\overset{?}{\geq}\sum_{cyc}x^2y+\sum_{cyc}xy^2\rightarrow\text{true via Schur}\Rightarrow\left(\sum x\right)^2$$

$$\leq\sum x^3+6xyz\ (\text{QED})$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \geq 3(xy+yz+zx) \geq$$

$$\geq 3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right) \geq 9$$

Hence, $x+y+z \geq 3$. Thus,

$$x^3+y^3+z^3+6xyz = x^3+2xyz+y^3+2xyz+z^3+3xyz =$$

$$= x(x^2+2yz)+y(y^2+2zx)+z(z^2+2xy) \geq$$

$$\geq \frac{(x+y+z)\left(x^2+y^2+z^2+2(xy+yz+zx)\right)}{3} = x^2+y^2+z^2+2(xy+yz+zx) =$$

$$= (x+y+z)^2$$

890. If $x,y,z>0$, such that: $\sum xy \geq \sum \frac{x}{y}$. Prove that :

$$\left(\sum xy\right)\sqrt[3]{xyz} \geq 3$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$xy+yz+zx \geq \frac{x}{y}+\frac{y}{z}+\frac{z}{x} \Rightarrow xy+yz+zx+xy+yz+zx$$

$$\geq \left(\frac{x}{y}+xy\right)+\left(\frac{y}{z}+yz\right)+\left(\frac{z}{x}+zx\right) \overset{A-G}{\geq} 2x+2y+2z \Rightarrow \sum xy \geq \sum x$$

$$\Rightarrow \frac{xy+yz+zx}{x+y+z} \overset{(*)}{\geq} 1$$

$$\text{Again, } xy+yz+zx \geq \frac{x}{y}+\frac{y}{z}+\frac{z}{x} \Rightarrow xyz \geq \frac{1}{xy+yz+zx}\sum_{cyc} xy^2$$

$$= \frac{1}{xy+yz+zx}\sum_{cyc}\frac{x^2y^2}{x} \overset{Bergstrom}{\geq} \frac{(xy+yz+zx)^2}{(xy+yz+zx)(x+y+z)}$$

$$= \frac{xy+yz+zx}{x+y+z} \overset{via\ (*)}{\geq} 1 \Rightarrow xyz \overset{(**)}{\geq} 1$$

$$\text{Via A} - \text{G}, \left(\sum xy\right)\sqrt[3]{xyz} \geq 3\left(\sqrt[3]{x^2y^2z^2}\right)\left(\sqrt[3]{xyz}\right) = 3xyz \overset{via\ (**)}{\geq} 3 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum xy \ge \sum \frac{x}{y} = \sum \frac{1}{3}\left(\frac{x}{y}+\frac{x}{y}+\frac{y}{z}\right) \overset{AM-GM}{\ge} \sum \sqrt[3]{\frac{x}{y}\cdot\frac{x}{y}\cdot\frac{y}{z}} = \sum \frac{x}{\sqrt[3]{xyz}} = \frac{x+y+z}{\sqrt[3]{xyz}}$$

$$\to \left(\sum xy\right)\sqrt[3]{xyz} \ge \sum x \ge \sqrt{3\sum xy} \ge \sqrt{3\sum \frac{x}{y}} \overset{AM-GM}{\ge} \sqrt{3.3} = 3$$

$$Therefore, \quad \left(\sum xy\right)\sqrt[3]{xyz} \ge 3.$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$xy+yz+zx \ge \frac{x}{y}+\frac{y}{z}+\frac{z}{x} = \frac{x^2z+z^2y+y^2x}{xyz}$$

$$xyz\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{xy+yz+zx}{xyz}xyz \ge \frac{x^2z+z^2y+y^2x}{xyz}$$

$$xyz \ge \frac{x^2z+z^2y+y^2x}{xy+yz+zx}$$

Hence,

$$(xy+yz+zx)\sqrt[3]{xyz} \le \sqrt[3]{(xy+yz+zx)^3}\cdot\sqrt[3]{\frac{x^2z+z^2y+y^2x}{xy+yz+zx}} =$$

$$= \sqrt[3]{(xy+yz+zx)^2(x^2z+z^2y+y^2x)} \ge \sqrt[3]{\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)(x^2z+z^2y+y^2x)}$$

$$\ge \sqrt[3]{(x+y+z)^3} = x+y+z = 3$$

891. If $x,y,z>0$ such that $\frac{y}{x}+\frac{z}{y}+\frac{x}{z}=3$ then:

$$\sqrt{\frac{y}{x}}+\sqrt{\frac{z}{y}}+\sqrt{\frac{x}{z}} \ge \frac{9(x^2+y^2+z^2)}{(x+y+z)^2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by George Florin Şerban-Romania

Lemma. $\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\geq\frac{9(x^2+y^2+z^2)}{(x+y+z)^2}\Rightarrow 3\geq\frac{9\sum x^2}{(\sum x)^2}\Rightarrow(\sum x)^2\geq 3\sum x^2$

But: $(\sum x)^2\overset{CBS}{\leq}3\sum x^2\Rightarrow(\sum x)^2=3\sum x^2\Rightarrow x=y=z.$

$$LHS=\sqrt{\frac{y}{x}}+\sqrt{\frac{z}{y}}+\sqrt{\frac{x}{z}}=1+1+1=3$$

$$RHS=\frac{9(x^2+y^2+z^2)}{(x+y+z)^2}=\frac{27x^2}{9x^2}=3\Rightarrow\sqrt{\frac{y}{x}}+\sqrt{\frac{z}{y}}+\sqrt{\frac{x}{z}}\geq\frac{9(x^2+y^2+z^2)}{(x+y+z)^2}$$

Solution 2 by George Florin Şerban-Romania

$$\frac{y}{x}+\frac{z}{y}+\frac{x}{z}=3\geq 3\sqrt[3]{\frac{y}{x}\cdot\frac{z}{y}\cdot\frac{x}{z}}=3\Rightarrow\frac{y}{x}=\frac{z}{y}=\frac{x}{z}\Rightarrow$$

$$\begin{cases}x^2=yz\\ y^2=xz\\ z^2=xy\end{cases}\Rightarrow\sum x^2=\sum xy\Rightarrow 2\sum x^2-2\sum xy=0\Rightarrow\sum(x-y)^2=0$$

$$\Rightarrow x-y=y-z=z-x=0\Rightarrow x=y=z.$$

$$\sqrt{\frac{y}{x}}+\sqrt{\frac{z}{y}}+\sqrt{\frac{x}{z}}=3\geq\frac{9(x^2+y^2+z^2)}{(x+y+z)^2}=\frac{9\cdot 3x^2}{9x^2}=3\ true.$$

Solution 3 by Fayssal Abdelli-Bejaia-Algerie

$$\sqrt{\frac{y}{x}}+\sqrt{\frac{z}{y}}+\sqrt{\frac{x}{z}}\geq\frac{9(x^2+y^2+z^2)}{(x+y+z)^2};(A)$$

$$\sqrt{\frac{y}{x}}+\sqrt{\frac{z}{y}}+\sqrt{\frac{x}{z}}\geq 3\sqrt[3]{\sqrt{\frac{y}{x}\cdot\frac{z}{y}\cdot\frac{x}{z}}}=3;(B)$$

$$3\overset{(?)}{\geq}\frac{9(x^2+y^2+z^2)}{(x+y+z)^2}\Leftrightarrow(x+y+z)^2\overset{(?)}{\geq}3(x^2+y^2+z^2)\Rightarrow$$

$$xy+yz+zx\geq x^2+y^2+z^2;(1)$$

We have: $x^2+y^2+z^2\geq\frac{(x+y+z)^2}{3}\Rightarrow 3(x^2+y^2+z^2)\geq(x+y+z)^2\Rightarrow$

$$x^2+y^2+z^2\ge xy+yz+zx;\ (2)$$

From (1),(2)⇒ $x^2+y^2+z^2=xy+yz+zx$.

$$(A)\Rightarrow\sqrt{\frac{y}{x}}+\sqrt{\frac{z}{y}}+\sqrt{\frac{x}{z}}\ge\frac{9(xy+yz+zx)}{x^2+y^2+z^2+2xy+2yz+2zx}\ge\frac{9(xy+yz+zx)}{3(xy+yz+zx)}=3$$

Proved by (B).

892. If $a,b,c>0$ such that $ab+bc+ca=abc$, then prove:

$$\frac{a+b+c}{3}\cdot\sqrt{\frac{a+b+c+9}{6abc}}\ge 1$$

Proposed by Neculai Stanciu-Romania

Solution by Fayssal Abdelli-Bejaia-Algerie

$$\frac{a+b+c}{3}\cdot\sqrt{\frac{a+b+c+9}{6abc}}\ge 1;\ (1)$$

$$\because\ a+b+c\ge 3\sqrt[3]{abc}$$

$$\frac{a+b+c}{3}\cdot\sqrt{\frac{a+b+c+9}{6abc}}\overset{AM-GM}{\ge}\sqrt[3]{abc}\cdot\sqrt{\frac{\sqrt[3]{abc}+3}{2abc}};\ (2)$$

Let $x=\sqrt[3]{abc}$, then

$$x^2\cdot\frac{x+3}{2x^3}\ge 1\Leftrightarrow\frac{x+3}{2x}\ge 1\Leftrightarrow x+3\ge 2x\Leftrightarrow x\le 3\Leftrightarrow\sqrt[3]{abc}\le 3;\ (3)$$

But $ab+bc+ca=abc$ and $ab+bc+ca\ge 3\sqrt[3]{(abc)^2}\Leftrightarrow abc\ge 3\sqrt[3]{(abc)^2}\Leftrightarrow$

$$\sqrt[3]{abc}\ge 3;\ (4)$$

From (3),(4) it follows that $\sqrt[3]{abc}=3$

Therefore,

$$\frac{a+b+c}{3}\cdot\sqrt{\frac{a+b+c+9}{6abc}}\ge 3\cdot\sqrt{\frac{3}{27}}=1$$

893. Let $x,y,z>0$, such that $\sum x=3$. Prove that:

$$\left(\sum\left(\frac{1+x}{yz+x}\right)^{2021}\right)\left(\sum\frac{x^2}{y}\right)\ge 9$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ AM-GM, we\ have:\ \frac{1+x}{yz+x}+\frac{yz+x}{2}+\frac{(1+x)^2}{4}\ge 3\sqrt[3]{\frac{1+x}{yz+x}\cdot\frac{yz+x}{2}\cdot\frac{(1+x)^2}{4}}$$

$$=\frac{3(1+x)}{2}$$

$$\to \frac{1+x}{yz+x}\ge -\frac{1}{4}(x^2+2yz)+\frac{2x+5}{4}, \forall x,y,z>0$$

$$\to \sum\frac{1+x}{yz+x}\ge -\frac{1}{4}\left(\sum x\right)^2+\frac{1}{2}\sum x+3.\frac{5}{4}=-\frac{9}{4}+\frac{3}{2}+\frac{15}{4}=3$$

By Power Mean inequality,

$$\sum\left(\frac{1+x}{yz+x}\right)^{2021}\ge 3\left(\frac{1}{3}.\sum\frac{1+x}{yz+x}\right)^{2021}\ge 3\left(\frac{1}{3}.3\right)^{2021}=3.$$

$$Also, we\ have:\ \sum\frac{x^2}{y}\overset{Bergstrom}{\ge}\frac{(x+y+z)^2}{y+z+x}=x+y+z=3$$

$$Therefore,\quad \left(\sum\left(\frac{1+x}{yz+x}\right)^{2021}\right)\left(\sum\frac{x^2}{y}\right)\ge 9.$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\left(\sum\left(\frac{1+x}{yz+x}\right)^{2021}\right)\left(\sum\frac{x^2}{y}\right)=\left(\sum\left(1+\frac{1-2x}{yz+x}\right)^{2021}\right)\left(\sum\frac{x^2}{y}\right)\ge$$

$$\left[(1+1+1)+2021\left(\frac{1-yz}{zx+y}+\frac{1-zx}{zx+y}+\frac{1-xy}{xy+z}\right)\right]\left(\sum\frac{x^2}{y}\right)\ge$$

$$\ge 3\sum\frac{x^2}{y}\ge 3\cdot\frac{(x+y+z)^2}{x+y+z}=3(x+y+z)=9$$

Solution 3 by Michael Sterghiou-Greece

$$\left(\sum\left(\frac{1+x}{yz+x}\right)^{2021}\right)\left(\sum\frac{x^2}{y}\right)\ge 9;(1)$$

Let $(p,q,r)=(\sum x,\sum xy,\prod x), p=3, q\le 3, r\le 1$ and $q\ge 3pq\ge 9r$. Now,

$$\sum\frac{x^2}{y}\overset{CBS}{\ge}\frac{p^2}{p}=3;\ \sum\left(\frac{1+x}{yz+x}\right)^{2021}\overset{Jensen}{\ge}3\left[\sum\frac{1+x}{3(yz+x)}\right]^{2021}=$$

$$= 3\left[\frac{1}{3}\sum\frac{(x+1)^2}{(x+yz)(1+x)}\right]^{2021} \overset{CBS}{\geq} 3\left[\frac{\frac{1}{3}\left(\sum(x+1)\right)^2}{\sum(x^2+x+yz+r)}\right]^{2021} =$$

$$= 3\left[\frac{1}{3}\cdot\frac{(3+3)^2}{(9-2q)+3+q+3r}\right]^{2021} = 3\left(\frac{12}{12-(q-3r)}\right)^{2021} = 3\left(\frac{12}{12}\right)^{2021} = 3$$

Therefore,

$$\left(\sum\left(\frac{1+x}{yz+x}\right)^{2021}\right)\left(\sum\frac{x^2}{y}\right) \geq 9$$

894. Let $x, y, z > 0$ such that $\sum xy = 3$. Prove that:

$$\frac{3}{4}\left(\sum x - xyz\right)^2 \geq \sum x^2y^2 \geq 3xyz$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

$x+y+z \geq xy+yz+zx = 3; (1)$. Hence, $x^2+y^2+z^2 \geq (xy)^2+(yz)^2+(zx)^2$

$$3(xy+yz+zx) = (xy+yz+zx)^2 \geq x^2yz+xy^2z+xyz^2 \geq 3xyz(x+y+z)$$

$$\Rightarrow xy+yz+zx \geq xyz(x+y+z) \text{ and}$$

$$x^2-(xy)^2+y^2-(yz)^2+z^2-(zx)^2 \geq$$

$$\geq (xy)^2-(xyz)^2+(yz)^2-(xyz)^2+(zx)^2-(xyz)^2$$

$$x^2(1-y^2)+y^2(1-z^2)+z^2(1-x^2) \geq$$

$$\geq (xy)^2(1-z^2)+(yz)^2(1-x^2)+(zx)^2(1-y^2)$$

$$z^2(1+x)(1+y)(1-x)(1-y)+x^2(1+y)(1+z)(1-y)(1-z)+$$

$$+y^2(1+z)(1+x)(1-z)(1-x) \geq 0$$

$3\sqrt[3]{x^2y^2z^2(1-x)^2(1-y)^2(1-z)^2(1+x)^2(1+y)^2(1+z)^2} \geq 0$ true. Hence,

$$\frac{3}{4}(x+y+z-xyz)^2 \geq ((xy)^2+(yz)^2+(zx)^2)$$

$$\frac{3}{4}(x+y+z-xyz)^2 \geq (xy)^2+(yz)^2+(zx)^2$$

$$\frac{3}{4}\left(x^2+y^2+z^2+2(xy+yz+zx)\right)+(xyz)^2-2xyz(x+y+z)$$

$$\geq (xy)^2+(yz)^2+(zx)^2$$

$$3(x^2+y^2+z^2)6(xy+yz+zx)+3(xyz)^2-6xyz(x+y+z)\geq$$

$$\geq 4((xy)^2+(yz)^2+(zx)^2)$$ **true, because:**

$$(xy)^2+(yz)^2+(zx)^2\geq 3xyz$$

$$\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\geq\frac{(xy+yz+zx)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}{3}\geq xy+yz+zx=3$$

Because: $xy+yz+zx=3$ **and** $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{3}{xyz}\geq 3;\left(xyz\leq 1\Leftrightarrow\frac{1}{xyz}\geq 1\right)$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{3}{4}\left(\sum x-xyz\right)^2\overset{(1)}{\geq}\sum x^2y^2\overset{(2)}{\geq}3xyz$$

$$(1)\leftrightarrow\left(3\sum x-3xyz\right)^2\geq 12\sum x^2y^2\leftrightarrow\left[\left(\sum xy\right)\left(\sum x\right)-3xyz\right]^2$$

$$\geq 4\left(\sum xy\right)\left(\sum x^2y^2\right)$$

$$\leftrightarrow\left(\sum x^2y+\sum xy^2\right)^2\geq 4\left(\sum xy\right)\left(\sum x^2y^2\right)$$

$$\textit{From } AM-GM, \textit{we have}: \left(\sum x^2y+\sum xy^2\right)^2$$

$$\geq 4\left(\sum x^2y\right)\left(\sum xy^2\right)\overset{?}{\geq}4\left(\sum xy\right)\left(\sum x^2y^2\right)$$

$$\leftrightarrow xyz\left(\sum x^3+3xyz\right)\geq xyz\left(\sum x^2y+\sum xy^2\right)\leftrightarrow\sum x^3+3xyz$$

$$\geq\sum x^2y+\sum xy^2\ (\textit{Schur inequality})$$

$$\rightarrow (1)\ \textit{is true}\rightarrow\frac{3}{4}\left(\sum x-xyz\right)^2\geq\sum x^2y^2$$

$$\sum x^2y^2\overset{\sum a^2\geq\sum ab}{\geq}\sum(xy)(zx)=xyz\sum x\geq xyz\sqrt{3\sum xy}=3xyz.$$

$$\textit{Therefore},\quad\frac{3}{4}\left(\sum x-xyz\right)^2\geq\sum x^2y^2\geq 3xyz$$

895. If $a,b,c,d>0, a+b+c+d=1$ then:

$$\frac{a}{b\sqrt[3]{1+b}}+\frac{b}{c\sqrt[3]{1+c}}+\frac{c}{d\sqrt[3]{1+d}}+\frac{d}{a\sqrt[3]{1+a}}\geq 4\cdot\sqrt[3]{\frac{4}{5}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Asmat Qatea-Afghanistan

$$\frac{a}{b\sqrt[3]{1+b}}+\frac{b}{c\sqrt[3]{1+c}}+\frac{c}{d\sqrt[3]{1+d}}+\frac{d}{a\sqrt[3]{1+a}}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq} 4\cdot\sqrt[4]{\frac{1}{\sqrt[3]{(1+a)(1+b)(1+c)(1+d)}}}\overset{AM-GM}{\geq} 4\cdot\sqrt[3]{\frac{1}{\frac{a+b+c+d+4}{4}}}=4\cdot\sqrt[3]{\frac{4}{5}}$$

Therefore,

$$\frac{a}{b\sqrt[3]{1+b}}+\frac{b}{c\sqrt[3]{1+c}}+\frac{c}{d\sqrt[3]{1+d}}+\frac{d}{a\sqrt[3]{1+a}}\geq 4\cdot\sqrt[3]{\frac{4}{5}}$$

Solution 2 by Khaled Abd Imouti-Damascus-Syria

$$\frac{a}{b\sqrt[3]{1+b}}+\frac{b}{c\sqrt[3]{1+c}}+\frac{c}{d\sqrt[3]{1+d}}+\frac{d}{a\sqrt[3]{1+a}}=$$

$$=\frac{a}{b}\cdot\frac{1}{\sqrt[3]{1+b}}+\frac{b}{c}\cdot\frac{1}{\sqrt[3]{1+c}}+\frac{c}{d}\cdot\frac{1}{\sqrt[3]{1+d}}+\frac{d}{a}\cdot\frac{1}{\sqrt[3]{1+a}}\overset{AM-GM}{\geq}$$

$$\overset{AM-GM}{\geq} 4\cdot\sqrt[4]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}\cdot\frac{d}{a}\cdot\frac{1}{\sqrt[3]{(a+b)(1+c)(1+d)(1+a)}}}=$$

$$=\frac{4}{\sqrt[12]{(1+a)(1+b)(1+c)(1+d)}};(1)$$

$$a+b+c+d=1\Rightarrow a+b+c+d+4=5\Rightarrow$$
$$(1+a)+(1+b)+(1+c)+(1+d)=5$$

$$\frac{5}{4}\geq\sqrt[4]{(1+a)(1+b)(1+c)(1+d)}\Rightarrow\sqrt[3]{\frac{5}{4}}\geq\sqrt[12]{(1+a)(1+b)(1+c)(1+d)};(2)$$

From (1),(2) it follows that:

$$\frac{a}{b\sqrt[3]{1+b}}+\frac{b}{c\sqrt[3]{1+c}}+\frac{c}{d\sqrt[3]{1+d}}+\frac{d}{a\sqrt[3]{1+a}}\geq 4\cdot\sqrt[3]{\frac{4}{5}}$$

896. If $x,y,z>0, xyz=1$ then:

$$(x-y)^4+(y-z)^4+(z-x)^4\geq 2\left(3-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\right)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by George Florin Șerban-Romania

$$a=x-y, b=y-z, c=z-x\Rightarrow a+b+c=0$$

$$\sum_{cyc}(x-y)^4=\sum_{cyc}a^4=\left(\sum_{cyc}a^2\right)^2-2\sum_{cyc}a^2b^2=$$

$$\left[\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\right]-2\left[\left(\sum_{cyc}ab\right)^2-2abc(a+b+c)\right]=$$

$$=4\left(\sum_{cyc}ab\right)^2-2\left(\sum_{cyc}ab\right)^2=2\left(\sum_{cyc}ab\right)^2=2\left(\sum_{cyc}xy-\sum_{cyc}x^2\right)^2\overset{(1)}{\geq}$$

$$\geq 2\left(3-\sum_{cyc}\frac{1}{x}\right)^2=2\left(3-\sum_{cyc}xy\right)^2$$

$$(1)\Leftrightarrow\left(\sum_{cyc}xy-\sum_{cyc}x^2\right)^2\overset{(2)}{\geq}\left(3-\sum_{cyc}xy\right)^2\Leftrightarrow$$

$$\left(\sum_{cyc}xy-\sum_{cyc}x^2\right)^2-\left(3-\sum_{cyc}xy\right)^2\geq 0\Leftrightarrow$$

$$\left(\sum_{cyc}xy-\sum_{cyc}x^2+3-\sum_{cyc}xy\right)\left(\sum_{cyc}xy-\sum_{cyc}x^2-3+\sum_{cyc}xy\right)\geq 0\Leftrightarrow$$

$$\left(3-\sum_{cyc}x^2\right)\left(2\sum_{cyc}xy-\sum_{cyc}x^2-3\right)\geq 0$$

$$\sum_{cyc}x^2\overset{AM-GM}{\geq}3\cdot\sqrt[3]{(xyz)^2}=3\Rightarrow 3-\sum_{cyc}x^2\leq 0$$

Now,

$$2\sum_{cyc}xy-\sum_{cyc}x^2-3\overset{(3)}{\leq}0$$

Let $(p,q,r)=(\sum x,\sum xy,\prod x), r=1$ **then (3) becomes as:**

$2q-(p^2-2q)-3\leq 0\Leftrightarrow p^2\geq 4q-3\Leftrightarrow p^3\geq 4pq-3p$, **which is true from**

$p^3-4pq+9r\geq 0\ (Schur)$

Now, $p^3 \geq 4pq - 9 \geq 4pq - 3p \Leftrightarrow p \geq 3$, which is true from

$p = \sum_{cyc} x \geq 3 \cdot \sqrt[3]{xyz} = 3$.Therefore,

$$(x-y)^4 + (y-z)^4 + (z-x)^4 \geq 2\left(3 - \frac{1}{x} - \frac{1}{y} - \frac{1}{z}\right)^2$$

Solution 2 by Asmat Qatea-Afghanistan

$$(x-y)^4 + (y-z)^4 + (z-x)^4 = \frac{((x-y)^2)^2}{1} + \frac{((y-z)^2)^2}{1} + \frac{((z-x)^2)^2}{1} \overset{Bergstrom}{\geq}$$

$$\geq \frac{4(x^2+y^2+z^2-xy-yz-zx)^2}{3} = \frac{4\left((x+y+z)^2 - 3(xy+yz+zx)\right)^2}{3} \overset{AM-GM}{\geq}$$

$$\geq \frac{4\left(\left(3\sqrt[3]{xyz}\right)^2 - 3(xy+yz+zx)\right)^2}{3} = \frac{4(9-3(xy+yz+zx))^2}{3} =$$

$$= 12\left(3 - \frac{xy+yz+zx}{xyz} \cdot xyz\right)^2 = 12\left(3 - \frac{1}{x} - \frac{1}{y} - \frac{1}{z}\right)^2$$

Therefore,

$$(x-y)^4 + (y-z)^4 + (z-x)^4 \geq 2\left(3 - \frac{1}{x} - \frac{1}{y} - \frac{1}{z}\right)^2$$

897. If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{a}{b+c} + \left(\sum_{cyc} \sqrt{\frac{a}{b+c}}\right)^2 \geq 6$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Michael Sterghiou-Greece

$$\sum_{cyc} \frac{a}{b+c} + \left(\sum_{cyc} \sqrt{\frac{a}{b+c}}\right)^2 \geq 6; (1)$$

(1) is homogeneous so, WLOG assume $p = a+b+c = 3, ab+bc+ca = q, abc = r$.

$$\sum_{cyc} \frac{a}{b+c} = \sum_{cyc} \frac{a^2}{ab+ac} \overset{Bergstrom}{\geq} \frac{p^2}{2q} = \frac{9}{2q}; (2)$$

$$\left(\sum_{cyc}\sqrt{\frac{a}{b+c}}\right)\left(\sum_{cyc}\sqrt{\frac{a}{b+c}}\right)\left(\sum_{cyc}a^2(b+c)\right)\overset{Holder}{\geq}(a+b+c)^3=27$$

$$\left(\sum_{cyc}\sqrt{\frac{a}{b+c}}\right)^2\geq\frac{27}{\sum(a^2b+a^2c)}=\frac{27}{pq-3r}=\frac{9}{q-r};(3)$$

So, it is suffices that $\frac{9}{2q}+\frac{9}{q-r}\geq 6;(4)$

From the 4^{th} degree Schur we have:

$6pr\geq(p^2-q)(4q-p^2)$ or $r\geq\frac{1}{18}(9-q)(4q-9)$ and as (3) is increasing function of r

we get the following inequality: $\frac{1}{18}(9-4q)^2(3-q)\geq 0$ which holds as $q\leq 3$.

Equality holds for $a=b=c$.

898. If $x,y,z>0$ such that $x+y+z=1$ and $\lambda\leq\frac{87}{28}$ then

$$\frac{x^2+\lambda xyz}{y+z}+\frac{y^2+\lambda xyz}{z+x}+\frac{z^2+\lambda xyz}{x+y}\geq\frac{\lambda+3}{6}$$

Proposed by Marin Chirciu-Romania

Solution by Fayssal Abdelli-Bejaia-Algerie

$$\frac{x^2+\lambda xyz}{y+z}+\frac{y^2+\lambda xyz}{z+x}+\frac{z^2+\lambda xyz}{x+y}\geq\frac{\lambda+3}{6};(*)$$

$$\frac{x^2+\lambda xyz}{y+z}+\frac{y^2+\lambda xyz}{z+x}+\frac{z^2+\lambda xyz}{x+y}=$$

$$=\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}+\lambda xyz\left(\frac{1}{x+z}+\frac{1}{z+y}+\frac{1}{y+x}\right)\overset{Bergstrom}{\geq}$$

$$\geq\frac{(x+y+z)^2}{x+y+z}+\lambda xyz\cdot\frac{9}{2(x+y+z)}\geq\frac{1}{2}+\frac{9\lambda}{2}xyz\overset{(1)}{\geq}\frac{\lambda+3}{6}$$

$$(1)\Leftrightarrow 27\lambda xyz\geq\lambda;(2)\Rightarrow\begin{cases}xyz\geq\frac{1}{27}, if\ \lambda>0;(3)\\ xyz\leq\frac{1}{27}, if\ \lambda<0;(4)\end{cases}$$

But $x+y+z=1$ and $x+y+z\geq 3\sqrt[3]{xyz}\Rightarrow 27xyz\leq 1\Rightarrow xyz\leq\frac{1}{27};(5)$

If $\lambda < 0 \Rightarrow (*)$ is proved by (3),(5).

If $\lambda > 0$ from (4)$\Rightarrow xyz = \frac{1}{27}$, then from (1) we get $\lambda = \lambda$ true.

If $\lambda = 0$: $(*) \Rightarrow \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{1}{2}$, which is true from

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \overset{Bergstrom}{\geq} \frac{(x+y+z)^2}{2(x+y+z)} \geq \frac{1}{2}$$

899. If $a, b, c > 0$ and $\lambda \geq 0$ then

$$\sum \frac{a^3}{a^2 + \lambda ab + b^2} \geq \frac{a+b+c}{\lambda + 2}$$

Proposed by Marin Chirciu-Romania

Solution by Nguyen Van Canh-BenTre-Vietnam

$$\sum \frac{a^3}{a^2 + \lambda ab + b^2}$$

$$= \sum \frac{(a^2)^2}{a^3 + \lambda a^2 b + ab^2} \overset{C-S}{\geq} \frac{(\sum a^2)^2}{\sum a^3 + \lambda(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2)} \overset{(1)}{\geq} \frac{a+b+c}{\lambda + 2};$$

$$(1) \leftrightarrow (\lambda + 2)\left(\sum a^2\right)^2$$

$$\geq \left(\sum a\right)\left(\sum a^3 + \lambda(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2)\right);$$

$\leftrightarrow \lambda(\sum a^2)^2 + 2(\sum a^2)^2 \geq (\sum a)\left(\sum a^3 + (ab^2 + bc^2 + ca^2)\right) + \lambda(\sum a)(a^2b + b^2c + c^2a)$;

Because:

- $(\sum a^2)^2 \geq (\sum a)(a^2b + b^2c + c^2a)$

$$\leftrightarrow \sum a^4 + 2\sum a^2b^2 \geq (a^3b + b^3c + c^3a) + \sum a^2b^2 + abc(a+b+c);$$

$$\leftrightarrow \sum a^4 + \sum a^2b^2 \geq (a^3b + b^3c + c^3a) + abc(a+b+c); (2)$$

Which is true since:

- $a^4 + a^4 + a^4 + b^4 \overset{AM-GM}{\geq} 4.\sqrt[4]{(a^3)^4 b^4} = 4a^3b;$

- $b^4+b^4+b^4+c^4 \overset{AM-GM}{\geq} 4.\sqrt[4]{(b^3)^4c^4}=4b^3c;$
- $c^4+c^4+c^4+a^4 \overset{AM-GM}{\geq} 4.\sqrt[4]{(c^3)^4a^4}=4c^3a;$

$$\to 4\sum a^4 \geq 4(a^3b+b^3c+c^3a)$$

$$\to \sum a^4 \geq (a^3b+b^3c+c^3a);(3)$$

- $\sum x^2 \geq \sum yz \overset{x=ab,y=bc,z=ca}{\Rightarrow} \sum (ab)^2 \geq abc\sum a;(4)$

$\overset{(3)+(4)}{\Rightarrow} \sum a^4+\sum a^2b^2 \geq (a^3b+b^3c+c^3a)+abc(a+b+c) \to (2)$ **is true.**

$$\to \lambda\left(\sum a^2\right)^2 \geq \lambda\left(\sum a\right)(a^2b+b^2c+c^2a),(\because\ \lambda \geq 0)$$

- $2(\sum a^2)^2 \geq (\sum a)\left(\sum a^3+(ab^2+bc^2+ca^2)\right);$

$$\leftrightarrow 2\left(\sum a^4+2\sum a^2b^2\right)$$

$$\geq \sum a^4+(a^3b+b^3c+c^3a)+2(ab^3+bc^3+ca^3)+\sum a^2b^2$$

$$+abc\sum a;$$

$$\leftrightarrow \sum a^4+3\sum a^2b^2+(a^3b+b^3c+c^3a)$$

$$\geq 2(a^3b+b^3c+c^3a)+2(ab^3+bc^3+ca^3)+abc\sum a;$$

$$\leftrightarrow \left(\sum a^4+3\sum a^2b^2-2\sum ab(a^2+b^2)\right)$$

$$+\left(a^3b+b^3c+c^3a-abc\sum a\right) \geq 0;$$

Which is true since:

- $\sum a^4+3\sum a^2b^2-2\sum ab(a^2+b^2) \geq 0;(5)$

  $\leftrightarrow \frac{1}{2}((a-b)^4+(b-c)^4+(c-a)^4) \geq 0;$ **(true)**

- $a^3b+b^3c+c^3a=\frac{a^2}{\frac{1}{ba}}+\frac{b^2}{\frac{1}{cb}}+\frac{c^2}{\frac{1}{ac}} \overset{C-S}{\geq} \frac{(a+b+c)^2}{\sum\frac{1}{ab}}=abc\sum a;(6)$

$$\overset{(5),(6)}{\Rightarrow} 2\left(\sum a^2\right)^2 \ge \left(\sum a\right)\left(\sum a^3 + (ab^2+bc^2+ca^2)\right);$$

Therefore,

$$\lambda\left(\sum a^2\right)^2 + 2\left(\sum a^2\right)^2$$
$$\ge \left(\sum a\right)\left(\sum a^3 + (ab^2+bc^2+ca^2)\right) + \lambda\left(\sum a\right)(a^2b+b^2c+c^2a)$$

So, (1) true . Proved.

900. If $x, y, z, t > 0$ then:

$$\frac{75x+36(y+z)}{y+z+t} + \frac{75y+36(z+t)}{z+t+x} + \frac{75z+36(t+x)}{t+x+y} + \frac{75t+36(x+y)}{x+y+z} \ge 196$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

Denote: $x+y+z=a; y+z+t=b; z+t+x=c; t+x+y=d$ then

$$3(x+y+z+t) = a+b+c+d \Leftrightarrow x+y+z+t = \frac{a+b+c+d}{3}$$

Hence, we get:

$$x = \frac{a+c+d-2b}{3}; y = \frac{a+b+d-2c}{3}; z = \frac{a+b+c-2d}{3}; t = \frac{b+c+d-2a}{3}$$

The inequality becomes :

$$\frac{13(a+c+d-2b)+36a}{b} + \frac{13(a+b+d-2c)+36b}{c} + \frac{13(a+b+c-2d)+36c}{d}$$
$$+ \frac{13(b+c+d-2a)+36d}{a} =$$
$$= 49\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right) + 13\left(\frac{c}{b}+\frac{d}{b}+\frac{a}{c}+\frac{d}{c}+\frac{a}{d}+\frac{c}{d}+\frac{b}{a}+\frac{c}{a}\right) - 104 \overset{AM-GM}{\ge}$$
$$\ge 49\cdot 4 + 13\cdot 8 - 104 = 196$$

Equality holds for: $x = y = z = t$.

Solution 2 by George Florin Şerban-Romania

$$\sum_{cyc}\frac{75x+36(y+z)}{y+z+t}=39\sum_{cyc}\frac{x}{y+z+t}+36\sum_{cyc}\frac{x+y+z}{y+z+t}\overset{(1)}{\geq}196$$

$$\sum_{cyc}\frac{x+y+z}{y+z+t}\overset{AM-GM}{\geq}4\cdot\sqrt[4]{\prod_{cyc}\frac{x+y+z}{y+z+t}}=4$$

We prove that:

$$\sum_{cyc}\frac{x}{y+z+t}\geq\frac{4}{3};(2)\Leftrightarrow\sum_{cyc}\frac{x+y+z+t}{y+z+t}\geq\frac{16}{3}\Leftrightarrow$$

$$\left(\sum_{cyc}x\right)\left(\sum_{cyc}\frac{1}{y+z+t}\right)\overset{AM-HM}{\geq}\left(\sum_{cyc}x\right)\left(\frac{16}{\sum(y+z+t)}\right)=\frac{16}{3}$$

Hence, (2) is true. From (1),(2) it follows thet:

$$\sum_{cyc}\frac{75x+36(y+z)}{y+z+t}\geq 39\cdot\frac{4}{3}+36\cdot 4=196$$

Equality holds for: $x=y=z=t$.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru