On a New Alternating Convolution Formula for the Super Catalan Numbers

Jovan Mikić
University of Banja Luka
Faculty of Technology
Republic of Srpska
jovan.mikic@tf.unibl.org

Abstract

We present a new alternating convolution formula for the super Catalan numbers which arises as a generalization of two known binomial identities. We prove a generalization of this formula by using auxiliary sums, recurrence relations, and induction. By using a new method, we prove one interesting divisibility result with super Catalan numbers.

1 Introduction

Let l be a fixed non-negative integer, and let n be an arbitrary non-negative integer. Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the nth Catalan number.

Catalan (1874) observed [4] that numbers $S(n,l) = \frac{\binom{2n}{n}\binom{2l}{l}}{\binom{n+l}{n}}$ are always integers. See also [10, p. 18]. In particular, $S(n,0) = \binom{2n}{n}$ and $S(n,1) = 2C_n$. Gessel [7, p. 11] referred to these numbers as the super Catalan numbers, since $\frac{1}{2}S(n,1)$ is the Catalan number C_n.

We shall call $S(n,l)$ as the nth super Catalan number of order l. By symmetry, $S(n,l) = S(l,n)$. Also $S(n,l)$ is equal to $\frac{(2n)!(2l)!}{n!\cdot l!\cdot (n+l)!}$.

It is known that $S(n,l)$ is always an even integer except for the case $n = l = 0$. See [1, Introduction] and [3, Eq. (1), p. 1]. More generally, for positive l, numbers $\frac{1}{2}S(n,l)$ can be viewed as special cases of super ballot numbers [7, p. 11].

For only a few values of l, there exist combinatorial interpretations of $S(n,l)$. See, for example, [1, 3, 5, 8, 16, 18]. The problem of finding a combinatorial interpretation for super Catalan numbers of an arbitrary order l is an intriguing open problem.

There are several binomial coefficient identities for super Catalan numbers. For example, the identity of von Szily (1894): [7, Eq. (29), p. 11]

$$S(n,l) = \sum_{k\in\mathbb{Z}} (-1)^k \binom{2n}{n+k}\binom{2l}{l+k}.$$

Note that the identity of von Szily gives another proof that the number $S(n,l)$ is always an integer. See also [7, Eq. (31); Eq. (32), p. 12].

We present the following alternating convolution formula for the super Catalan numbers:

Theorem 1. *For all non-negative integers n and l, we have:*

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}S(k,l)S(2n-k,l)=S(n,l)S(n+l,n). \tag{1}$$

For $l=0$, Eq. (1) reduces to a known binomial identity [15, Example 3.6.2, p. 45]:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2k}{k}\binom{4n-2k}{2n-k}=\binom{2n}{n}^2. \tag{2}$$

Recently, the following binomial coefficient identity involving the Catalan numbers [14] was discovered:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}C_k\,C_{2n-k}=C_n\binom{2n}{n}. \tag{3}$$

See also [13, Remark 23, p. 15] and [6, 17]. For $l=1$, Eq. (1) reduces to an identity equivalent to Eq. (3). Therefore, we can see Eq. (1) as a natural generalization of Eqns. (2) and (3).

Furthermore, Eq. (1) has the following generalization:

Theorem 2. *Let n, l, and t be non-negative integers such that $t\le n$. Then*

$$\sum_{k=t}^{2n-t}(-1)^k\binom{2n-2t}{k-t}S(k,l)S(2n-k,l)=(-1)^t\frac{\binom{2l}{l}\binom{2t}{t}\binom{2(n+l-t)}{n+l-t}\binom{2n}{n}\binom{2n-2t}{n-t}}{\binom{n+l}{n}\binom{2n+l-t}{n}\binom{n}{t}}. \tag{4}$$

When $t=0$, Eq. (4) reduces to Eq. (1).

For $l=0$, Theorem 2 reduces to a recently [13, Lemma 10, p. 7] discovered formula:

$$\sum_{k=t}^{2n-t}(-1)^k\binom{2n-2t}{k-t}\binom{2k}{k}\binom{4n-2k}{2n-k}=(-1)^t\frac{\binom{2n}{n}\binom{2t}{t}\binom{2n-2t}{n-t}}{\binom{2n-t}{t}}. \tag{5}$$

It is readily verified that for $l=1$, Theorem 2 is equivalent to another recently [13, Lemma 13, p. 7] discovered formula:

$$\sum_{k=t}^{2n-t}(-1)^k\binom{2n-2t}{k-t}C_kC_{2n-k}=(-1)^t\frac{C_n\binom{2t}{t}\binom{2n-2t}{n-t}}{\binom{2n+1-t}{t}}. \tag{6}$$

We prove Theorem 2 by using auxiliary sums, recurrence relations, induction, and Eq. (5).

Let us consider the following sum:

$$\Psi(n,m,l)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}^m S(k,l)S(n-k,l). \tag{7}$$

Obviously, $\Psi(2n-1,m,l)=0$.

By Eq. (1), it follows that $\Psi(2n,1,l)$ is divisible by $S(n,l)$. We establish the following theorem:

Theorem 3. *The sum $\Psi(2n,m,l)$ is divisible by $S(n,l)$ for all non-negative integers n and l and for all positive integers m.*

For proving Theorem 3, we use Theorem 2 and a method we call the "method of D sums".

Definition 4. Let n, j, l, and t be non-negative integers such that $j\le\lfloor\frac{n}{2}\rfloor$, and let m be a positive integer. Let $A(n,m,l)=\sum_{k=0}^{n}\binom{n}{k}^m F(n,k,l)$, where $F(n,k,l)$ is an integer-valued function that depends on n, k, and l (not on m). Then the D sums for $A(n,m,l)$ are

$$D_A(n,j,t;l)=\sum_{u=0}^{n-2j}\binom{n-j}{u}\binom{n-j}{j+u}\binom{n}{j+u}^t F(n,j+u,l). \tag{8}$$

First, note that sum (7) is an instance of $A(n,m,l)$. For $m\ge 2$, by Eq. (8), it follows that

$$A(n,m,l)=D_A(n,0,m-2;l). \tag{9}$$

Furthermore, D sums satisfy the following two recurrence relations [11, Thm. 2, Thm. 3, p. 2]:

$$D_A(n,j,t+1;l)=\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n}{j+u}\binom{n-j}{u}D_A(n,j+u,t;l), \tag{10}$$

$$D_A(n,j,0;l)=\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n-j}{j+u}\binom{n-2j-u}{u}\sum_{v=0}^{n-2j-2u}\binom{n-2j-2u}{v}F(n,j+u+v,l). \tag{11}$$

Definition 5. Let F be from Definition 4, and let n, l, t be non-negative integers such that $t\le\lfloor\frac{n}{2}\rfloor$. Then $A_t(n,l)$ denotes

$$\sum_{k=t}^{n-t}\binom{n-2t}{k-t}F(n,k,l). \tag{12}$$

By substitution $k=u+j+v$, the inner sum of the right-side of (11) becomes

$$\sum_{v=0}^{n-2j-2u}\binom{n-2j-2u}{v}F(n,j+u+v,l)=A_{j+u}(n,l). \tag{13}$$

It is readily verified [10, Eq. (1.4), p. 5] that $\binom{n-j}{j+u}\binom{n-2j-u}{u}=\binom{n-j}{u}\binom{n-j-u}{j+u}$. By using this fact and Eq. (13), Relation 11 becomes

$$D_A(n,j,0;l)=\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n-j}{u}\binom{n-j-u}{j+u}A_{j+u}(n,l). \tag{14}$$

From now on, for calculating $D_A(n,j,0;l)$ sum, we use Eq. (14) instead of Relation 11.

It turns out that $\Psi_t(2n,l)$ is the left-side of Theorem 2. By Theorem 2, it follows that

$$\Psi_t(2n,l)=(-1)^t\frac{\binom{2l}{l}\binom{2t}{t}\binom{2(n+l-t)}{n+l-t}\binom{2n}{n}\binom{2n-2t}{n-t}}{\binom{n+l}{n}\binom{2n+l-t}{n}\binom{n}{t}}. \tag{15}$$

2 Motivation

In 1998, Calkin proved that the alternating binomial sum $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. In 2007, Guo, Jouhet, and Zeng proved, among other things, two generalizations of Calkin's result [9, Thm. 1.2, Thm. 1.3, p. 2]. Also, they presented an interesting congruence involving super Catalan numbers [9, Corollary 4.9., p. 11].

The first application of D sums [11, Section 8] was for proving Calkin's result [2, Thm. 1]. Also, by using D sums, it was proved [11] that $\sum_{k=0}^{2n}\binom{2n}{k}^m|n-k|$ is divisible by $n\binom{2n}{n}$ for all non-negative integers n and all positive integers m.

Recently, by the same method, it was proved [13, Thm. 1] that $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m\binom{2k}{k}\binom{4n-2k}{2n-k}$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. This result confirms Theorem 3 for $l=0$.

Furthermore, it was proved [13, Corollary 4] that $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m C_kC_{2n-k}$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. This result is sharper than Theorem 3 for $l=1$. By Theorem 3 and Remark 12, it can be shown that $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m C_kC_{2n-k}$ is divisible by C_n.

In order to prove Theorem 3, we give a slight generalization of a notion D sum in Definition 4. The function F, from Definition 4, depends on n, k, and on new parametar l. In all previous cases [11, 13], the function F depends only on n and k.

Recently, Chu [6, Theorem 5] presented, among other, formula similar to Theorem 2. It is readily verified that his formula implies Eq. (2) for the case $\lambda=\mu=t=0$ and $n:=2n$. Theorems 1 and 2 are not consequences of this formula.

This paper consists of two parts.

In the first part, we prove Theorem 2 by using recurrence relations, induction, and Eq. (5). This proof of Theorem 2 is interesting itself for two reasons. Firstly, the choice of auxiliary sums. We use four auxiliary sums similarly as in the proof of Eq. (5) [13, Corollary 22, p. 8]. Secondly, we obtain a relation between $\Psi_t(2n-2, l+1)$ and $\Psi_t(2n, l)$ and then use induction on l.

In the second part, we prove Theorem 3 by using method of D sums.

Our proof of Theorem 3 consists of two parts. Firstly, we show that $\Psi(2n, 2, l)$ is divisible by $S(n, l)$ by calculating $D_\Psi(2n, j, 0; l)$ and by using Eq. (9). Then we show that $D_\Psi(2n, j, 1; l)$ is divisible by $S(n, l)$ for all non-negative integers j and n such that $j \leq n$. This result is sufficient to prove Theorem 2 for $m \geq 3$. Namely, then by Relation 10 and induction, it can be shown that all $D_\Psi(2n, j, t; l)$ are divisible by $S(n, l)$ for all integers $t \geq 1$. By Relation 9, it follows that $\Psi(2n, m, l)$ is divisible by $S(n, l)$ for all $m \geq 3$. See [11, Section 5]. In order to calculate $D_\Psi(2n, j, 0; l)$, we use Eq. (15) and Relation (14).

The rest of the paper is structured as follows. In Section 3, we present our main lemma 6, auxiliary sums, and recurrence relations between these sums. These recurrences are given by lemmas 7, 8, and 9. For auxiliary sums, see [12, 13]. In Section 4, we prove all lemmas from Section 3. In Section 5, we give a proof of Theorem 2 by using the main lemma 6, induction, and Eq. (5). In Section 6, we give a proof of Theorem 1 by using Theorem 2. In Section 7, we give a proof of Theorem 3 by using Theorem 2 and a method of D sums.

3 Main Lemma, Auxiliary Sums, and Other Lemmas

Let n, t, and l be non-negative integers such that $t \leq \lfloor \frac{n}{2} \rfloor$.

By Definition 4 and Eq. (7), we know that, for the sum $\Psi(n, m, l)$,

$$F(n, k, l) = (-1)^k S(k, l) S(n-k, l).$$

Therefore, by Eq. (7), Definition 4, Definition 5, and Eq. (12), it follows that

$$\Psi_t(n, l) = \sum_{k=t}^{n-t} (-1)^k \binom{n-2t}{k-t} S(k, l) S(n-k, l). \tag{16}$$

By changing k to $n-k$ in Eq. (16), it follows that

$$\Psi_t(2n-1, l) = 0. \tag{17}$$

For $l = 0$, by Eq. (5), it follows that

$$\Psi_t(2n, 0) = (-1)^t \frac{\binom{2n}{n}\binom{2t}{t}\binom{2n-2t}{n-t}}{\binom{2n-t}{t}}. \tag{18}$$

For $t = n$, by Eq. (16), it follows that

$$\Psi_n(2n, l) = (-1)^n S(n, l)^2. \tag{19}$$

Now we are ready for the main lemma.

Lemma 6. *Let n be a positive integer, and let t and l be non-negative integers such that $t < n$. Then the following recurrence relation is true:*

$$\Psi_t(2n-2, l+1) = \frac{(2n+l-t)(2l+1)}{2(2n-1-2t)(2n-1)}\Psi_t(2n, l). \tag{20}$$

In order to prove lemma 6, we introduce four auxiliary sums, as follows:

$$P_t(n,l) = \sum_{k=t}^{n-t}(-1)^k(n-t-k)\binom{n-2t}{k-t}S(k,l)S(n-k,l), \tag{21}$$

$$R_t(n,l) = \sum_{k=t}^{n-t}(-1)^k\frac{2l+1}{k+l+1}\binom{n-2t}{k-t}S(k,l)S(n-k,l), \tag{22}$$

$$R'_t(n,l) = \sum_{k=t}^{n-t}(-1)^k\frac{2l+1}{(k+l+1)(n-k+l+1)}\binom{n-2t}{k-t}S(k,l)S(n-k,l), \tag{23}$$

$$T_t(n,l) = \sum_{k=t}^{n-t}(-1)^k\frac{(n-t-k)(2l+1)}{k+l+1}\binom{n-2t}{k-t}S(k,l)S(n-k,l). \tag{24}$$

It is readily verified that

$$P_t(2n,l) = (n-t)\Psi_t(2n,l), \tag{25}$$

$$R'_t(2n,l) = \frac{1}{n+l+1}R_t(2n,l). \tag{26}$$

For example, the proof of Eq. (25) is as follows.
By Eq. (21),

$$P_t(2n,l) = \sum_{k=t}^{2n-t}(-1)^k(2n-t-k)\binom{2n-2t}{k-t}S(k,l)S(2n-k,l).$$

By changing k to $2n-k$ in the above equation and use symmetry of binomial coefficients, we obtain

$$P_t(2n,l) = \sum_{k=t}^{2n-t}(-1)^k(k-t)\binom{2n-2t}{k-t}S(k,l)S(2n-k,l).$$

If we add two above equations and use Eq. (16), Eq. (25) follows. The proof of Eq. (26) is similar to the proof of Eq. (25).

We present other three lemmas.

Lemma 7. *Let n, l, and t be non-negative integers. Then the following recurrence relation is true:*

$$R_t(2n,l) = \frac{n+l+1}{4(2l+1)}\Psi_t(2n,l+1). \tag{27}$$

Lemma 8. *Let n be a positive integer, and let l and t be non-negative integers such that $t < n$. Then the following recurrence relation is true:*

$$\Psi_t(2n, l) = 4R_t(2n-1, l). \tag{28}$$

Lemma 9. *Let n be a positive integer, and let l and t be non-negative integers such that $t < n$. Then the following recurrence relation is true:*

$$(2n+l-t)R_t(2n-1, l) = \frac{2(2n-1-2t)(2n-1)}{n+l} R_t(2n-2, l). \tag{29}$$

By lemmas 7, 8, and 9, the main lemma 6 follows. We shall use $T_t(n, l)$ only for the proof of lemma 9.

Note that, for $l = 0$, Eqns. (28) and (29) simplify to [13, Corrolary 17, Corollary 20; p. 8], respectively.

4 Proofs of Four Lemmas

We begin with the proof of Lemma 7. We shall use well-known [10, The Central Binomial Coefficient, p. 26] recurrence relation for the central binomial coefficient:

$$\binom{2(l+1)}{l+1} = \frac{2(2l+1)}{l+1}\binom{2l}{l}. \tag{30}$$

4.1 Proof of Lemma 7

We start from Eq. (26).

By Eq. (23), Eq. (26) becomes

$$\begin{aligned} R_t(2n, l) &= (n+l+1)\sum_{k=t}^{2n-t}(-1)^k \frac{2l+1}{(k+l+1)(2n-k+l+1)}\binom{2n-2t}{k-t} S(k,l)S(2n-k,l), \\ &= (n+l+1)\sum_{k=t}^{2n-t}(-1)^k (2l+1)\binom{2n-2t}{k-t}\frac{\binom{2k}{k}\binom{2l}{l}}{(k+l+1)\binom{k+l}{k}}\frac{\binom{2(2n-k)}{2n-k}\binom{2l}{l}}{(2n-k+l+1)\binom{2n-k+l}{2n-k}}. \end{aligned}$$

It is readily verified [10, Eq. (1.4), p. 5] that

$$\begin{aligned} (k+l+1)\binom{k+l}{k} &= (l+1)\binom{k+l+1}{k}, \\ (2n-k+l+1)\binom{2n-k+l}{2n-k} &= (l+1)\binom{2n-k+l+1}{2n-k}. \end{aligned}$$

By using the two equations above, we obtain the equation for the $R_t(2n,l)$ sum:

$$R_t(2n,l) = (n+l+1)\sum_{k=t}^{2n-t}(-1)^k\binom{2n-2t}{k-t}\frac{\binom{2k}{k}(2l+1)\binom{2l}{l}}{(l+1)\binom{k+l+1}{k}}\frac{\binom{2(2n-k)}{2n-k}\binom{2l}{l}}{(l+1)\binom{2n-k+l+1}{2n-k}}. \tag{31}$$

By using Eq. (30), Eq. (31) becomes

$$\begin{aligned} R_t(2n,l) &= \frac{(n+l+1)}{2}\sum_{k=t}^{2n-t}(-1)^k\binom{2n-2t}{k-t}\frac{\binom{2k}{k}\binom{2(l+1)}{l+1}}{\binom{k+l+1}{k}}\frac{\binom{2(2n-k)}{2n-k}\binom{2l}{l}}{(l+1)\binom{2n-k+l+1}{2n-k}}, \\ &= \frac{(n+l+1)}{2}\sum_{k=t}^{2n-t}(-1)^k\binom{2n-2t}{k-t}S(k,l+1)\frac{\binom{2(2n-k)}{2n-k}\binom{2l}{l}}{(l+1)\binom{2n-k+l+1}{2n-k}}. \end{aligned} \tag{32}$$

Eq. (32) is equivalent to the following equation:

$$R_t(2n,l) = \frac{(n+l+1)}{4(2l+1)}\sum_{k=t}^{2n-t}(-1)^k\binom{2n-2t}{k-t}S(k,l+1)\frac{\binom{2(2n-k)}{2n-k}2(2l+1)\binom{2l}{l}}{(l+1)\binom{2n-k+l+1}{2n-k}}. \tag{33}$$

Again, by Eq. (30), Eq. (33) becomes

$$\begin{aligned} R_t(2n,l) &= \frac{(n+l+1)}{4(2l+1)}\sum_{k=t}^{2n-t}(-1)^k\binom{2n-2t}{k-t}S(k,l+1)\frac{\binom{2(2n-k)}{2n-k}\binom{2(l+1)}{l+1}}{\binom{2n-k+l+1}{2n-k}}, \\ &= \frac{(n+l+1)}{4(2l+1)}\sum_{k=t}^{2n-t}(-1)^k\binom{2n-2t}{k-t}S(k,l+1)S(2n-k,l+1). \end{aligned} \tag{34}$$

Eqns. (16) and (34) complete the proof of Lemma 7.

4.2 Proof of Lemma 8

We start from Eq. (21). Note that the last term in Eq. (21) is equal to zero. Therefore, Eq. (21) is equivalent to

$$P_t(n,l) = \sum_{k=t}^{n-1-t}(-1)^k(n-t-k)\binom{n-2t}{k-t}S(k,l)S(n-k,l). \tag{35}$$

It is readily verified [10, Eq. (1.2), p. 5] that

$$(n-t-k)\binom{n-2t}{k-t} = (n-2t)\binom{n-1-2t}{k-t}.$$

By the equation above, Eq. (35) becomes

$$P_t(n,l) = (n-2t)\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}S(k,l)S(n-k,l). \tag{36}$$

Note that, in Eq. (36), $k < n$.

Eq. (36) is equivalent to

$$P_t(n,l) = (n-2t)\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}S(k,l)\frac{\binom{2n-2k}{n-k}\binom{2l}{l}}{\binom{n-k+l}{n-k}}. \tag{37}$$

It is known that, for positive n, $\binom{2n}{n}$ is always an even integer. It is readily verified [10, Central Binomial Coefficient, p. 15] that

$$\binom{2n}{n} = 2\binom{2n-1}{n}.$$

By the equation above, Eq. (37) becomes

$$P_t(n,l) = 2(n-2t)\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}S(k,l)\frac{\binom{2n-2k-1}{n-k}\binom{2l}{l}}{\binom{n-k+l}{n-k}}. \tag{38}$$

Since $n-k>0$, Eq. (38) is equivalent to

$$P_t(n,l) = 2(n-2t)\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}S(k,l)\frac{(n-k)\binom{2n-2k-1}{n-k}\binom{2l}{l}}{(n-k)\binom{n-k+l}{n-k}}. \tag{39}$$

It is readily verified [10, Eq. (1.1), p. 5] that

$$(n-k)\binom{2n-2k-1}{n-k} = (2n-2k-1)\binom{2(n-1-k)}{n-1-k},$$
$$(n-k)\binom{n-k+l}{n-k} = (n-k+l)\binom{n-1+k+l}{n-1-k}.$$

By using the two equations above, Eq. (39) becomes

$$P_t(n,l) = 2(n-2t)\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}S(k,l)\frac{2n-2k-1}{n-k+l}\frac{\binom{2(n-1-k)}{n-1-k}\binom{2l}{l}}{\binom{n-1+k+l}{n-1-k}}. \tag{40}$$

Eq. (40) is equivalent to

$$\begin{aligned} P_t(n,l) &= 2(n-2t)\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}\frac{2(n-1-k)+1}{n-k+l}S(k,l)S(n-1-k,l), \\ &= 2(n-2t)\sum_{k=t}^{n-1-t}(-1)^k\binom{n-1-2t}{k-t}(2-\frac{2l+1}{n-k+l})S(k,l)S(n-1-k,l). \end{aligned} \tag{41}$$

Note that changing k to $n-k$ in Eq. (22), Eq. (22) becomes

$$R_t(n,l) = (-1)^n \sum_{k=t}^{n-t} (-1)^k \frac{2l+1}{n-k+l+1} \binom{n-2t}{k-t} S(k,l) S(n-k,l). \tag{42}$$

By setting $n := n-1$ in Eq. (42), we obtain that

$$R_t(n-1,l) = (-1)^{n-1} \sum_{k=t}^{n-1-t} (-1)^k \frac{2l+1}{n-k+l} \binom{n-1-2t}{k-t} S(k,l) S(n-1-k,l). \tag{43}$$

By using Eqns. (16), and (43), Eq. (41) implies an interesting recurrence relation:

$$P_t(n,l) = 4(n-2t)\Psi_t(n-1,l) + (-1)^n 2(n-2t) R_t(n-1,l). \tag{44}$$

By setting $n := 2n$, Eq. (44) becomes

$$P_t(2n,l) = 4(2n-2t)\Psi_t(2n-1,l) + 2(2n-2t) R_t(2n-1,l). \tag{45}$$

By Eq. (17), Eq. (45) becomes

$$P_t(2n,l) = 4(n-t) R_t(2n-1,l). \tag{46}$$

Finally, by Eq. (25), it follows that

$$(n-t)\Psi_t(2n,l) = 4(n-t) R_t(2n-1,l). \tag{47}$$

Since $t < n$, by canceling the factor $n-t$ in Eq. (47), Lemma 8 follows, as desired.

4.3 Proof of Lemma 9

For the proof of Lemma 9, we use $T_t(n,l)$ sums.

Eq. (24) is equivalent to

$$T_t(n,l) = (2l+1) \sum_{k=t}^{n-t} (-1)^k \left(\frac{n+l+1-t}{k+l+1} - 1\right) \binom{n-2t}{k-t} S(k,l) S(n-k,l).$$

By the equation above, it follows that

$$T_t(n,l) = (n+l+1-t) R_t(n,l) - (2l+1)\Psi_t(n,l). \tag{48}$$

By setting $n := 2n-1$ in Eq. (48) and by using Eq. (17), Eq. (48) becomes

$$T_t(2n-1,l) = (2n+l-t) R_t(2n-1,l). \tag{49}$$

On the other side, it can be shown that $T_t(n,l)$ is equal to

$$2(n-2t)(2l+1)\sum_{k=t}^{n-1-t}\frac{2n-2k-1}{(k+l+1)(n-k+l)}\binom{n-1-2t}{k-t}S(k,l)S(n-1-k,l). \tag{50}$$

The proof of Eq. (50) is similar to the proof of Eq. (40) in the previous subsection. For the sake of brevity, the proof of Eq. (50) is omitted.

The inner sum in Eq. (50) can be written as

$$\frac{2(n-1-k)+1}{(k+l+1)(n-k+l)}\binom{n-1-2t}{k-t}S(k,l)S(n-1-k,l). \tag{51}$$

Let

$$R''_t(n,l)=(2l+1)\sum_{k=t}^{n-t}(-1)^k\frac{n-k}{(k+l+1)(n-k+l+1)}\binom{n-2t}{k-t}S(k,l)S(n-k,l). \tag{52}$$

By Eqns. (52) and (51), Eq. (50) becomes

$$T_t(n,l)=4(n-2t)R''_t(n-1,l)+2(n-2t)R'_t(n-1,l). \tag{53}$$

By setting $n:=2n-1$ in Eq. (53), it follows that

$$T_t(2n-1,l)=4(2n-1-2t)R''_t(2(n-1),l)+2(2n-1-2t)R'_t(2(n-1),l). \tag{54}$$

By Eq. (52), changing k to $n-k$, it is readily verified that

$$R''_t(2n,l)=nR'_t(2n,l). \tag{55}$$

By Eq. (55), Eq. (54) becomes

$$T_t(2n-1,l)=2(2n-1)(2n-1-2t)R'_t(2(n-1),l). \tag{56}$$

By Eq. (26), Eq. (56) becomes

$$T_t(2n-1,l)=\frac{2(2n-1)(2n-1-2t)}{n+l}R_t(2n-2,l). \tag{57}$$

Eqns. (49) and (57) complete the proof of Lemma 9.

4.4 Proof of the Main Lemma

By Lemmas (7) and (8), Eq. (29) of Lemma 9 becomes

$$(2n+l-t)\frac{\Psi_t(2n,l)}{4} = \frac{2(2n-1-2t)(2n-1)}{n+l}\frac{n+l}{4(2l+1)}\Psi_t(2n-2,l+1). \tag{58}$$

Eq. (58) is equivalent to

$$\begin{aligned}(2n+l-t)\Psi_t(2n,l) &= \frac{2(2n-1-2t)(2n-1)\Psi_t(2n-2,l+1)}{2l+1},\\ \frac{2(2n-1-2t)(2n-1)\Psi_t(2n-2,l+1)}{2l+1} &= (2n+l-t)\Psi_t(2n,l),\\ \Psi_t(2n-2,l+1) &= \frac{(2n+l-t)(2l+1)}{2(2n-1-2t)(2n-1)}\Psi_t(2n,l).\end{aligned}$$

The last equation above is Eq. (20). This completes the proof of the main lemma 6.

5 Proof of Theorem 2

Let n, l, and t be non-negative integers such that $t \le n$; and let

$$\varphi(2n,l,t) = (-1)^t \frac{\binom{2l}{l}\binom{2t}{t}\binom{2(n+l-t)}{n+l-t}\binom{2n}{n}\binom{2n-2t}{n-t}}{\binom{n+l}{n}\binom{2n+l-t}{n}\binom{n}{t}}. \tag{59}$$

Let us prove that

$$\Psi_t(2n,l) = \varphi(2n,l,t). \tag{60}$$

We shall use induction on l by using the main lemma 6. By setting $n := n+1$ in Eq. (20), it follows that

$$\Psi_t(2n,l+1) = \frac{(2n+2+l-t)(2l+1)}{2(2n+1-2t)(2n+1)}\Psi_t(2n+2,l). \tag{61}$$

For the proof of Theorem 2, we shall use Eq. (61), instead of Eq. (20).
By setting $l=0$ in Eq. (59), it follows that

$$\varphi(2n,0,t) = (-1)^t \frac{\binom{2t}{t}\binom{2n}{n}\binom{2n-2t}{n-t}^2}{\binom{2n-t}{n}\binom{n}{t}}. \tag{62}$$

It is readily verified [10, Eq. (1.4), p. 5] that

$$\binom{2n-t}{n}\binom{n}{t} = \binom{2n-t}{t}\binom{2n-2t}{n-t}. \tag{63}$$

By Eq. (63), Eq. (62) becomes as follows:

$$\varphi(2n,0,t) = (-1)^t \frac{\binom{2t}{t}\binom{2n}{n}\binom{2n-2t}{n-t}^2}{\binom{2n-t}{t}\binom{2n-2t}{n-t}},$$
$$\varphi(2n,0,t) = (-1)^t \frac{\binom{2t}{t}\binom{2n}{n}\binom{2n-2t}{n-t}}{\binom{2n-t}{t}}. \tag{64}$$

By Eqns. (18) and (64), it follows that Eq. (60) is true for $l = 0$. This confirms the induction base.

Furthermore, by setting $t := n$ in Eq. (59), we obtain that

$$\varphi(2n,l,n) = (-1)^n \frac{\binom{2l}{l}\binom{2n}{n}\binom{2l}{l}\binom{2n}{n}}{\binom{n+l}{n}\binom{n+l}{n}},$$
$$\varphi(2n,l,n) = (-1)^n S(n,l)^2. \tag{65}$$

By Eqns. (19) and (65), it follows that Eq. (60) is true for $t = n$. For all non-negative integers n and l, the equation

$$\Psi_n(2n,l) = \varphi(2n,l,n) \tag{66}$$

holds.

Let us prove that the function $\varphi(2n,l,t)$ satisfies recurrence relation (61). More precisely, let us prove that

$$\varphi(2n,l+1,t) = \frac{(2n+2+l-t)(2l+1)}{2(2n+1-2t)(2n+1)}\varphi(2n+2,l,t); \tag{67}$$

where n, l, and t are non-negative integers such that $t < n$.

By Eq. (59), the right-side of Eq. (67) becomes

$$\frac{(2n+2+l-t)(2l+1)}{2(2n+1-2t)(2n+1)}(-1)^t \frac{\binom{2l}{l}\binom{2t}{t}\binom{2((n+1)+l-t)}{(n+1)+l-t}\binom{2n+2}{n+1}\binom{2n+2-2t}{n+1-t}}{\binom{n+1+l}{n+1}\binom{2n+2+l-t}{n+1}\binom{n+1}{t}}. \tag{68}$$

By Eq. (30), we know that

$$\binom{2n+2}{n+1} = \frac{2(2n+1)}{n+1}\binom{2n}{n}, \tag{69}$$
$$\binom{2n+2-2t}{n+1-t} = \frac{2(2n+1-2t)}{n+1-t}\binom{2n-2t}{n-t}. \tag{70}$$

By Eq. (69), Eq. (68) becomes

$$\frac{(2n+2+l-t)(2l+1)}{2(2n+1-2t)(2n+1)}(-1)^t \frac{\binom{2l}{l}\binom{2t}{t}\binom{2(n+(l+1)-t)}{n+(l+1)-t}\frac{2(2n+1)}{n+1}\binom{2n}{n}\binom{2n+2-2t}{n+1-t}}{\binom{n+1+l}{n+1}\binom{2n+2+l-t}{n+1}\binom{n+1}{t}}. \tag{71}$$

Eq. (71) simplifies to

$$(-1)^t \frac{(2n+2+l-t)(2l+1)}{2n+1-2t} \frac{\binom{2l}{l}\binom{2t}{t}\binom{2(n+(l+1)-t)}{n+(l+1)-t}\binom{2n}{n}\binom{2n+2-2t}{n+1-t}}{(n+1)\binom{n+1+l}{n+1}\binom{2n+2+l-t}{n+1}\binom{n+1}{t}}. \tag{72}$$

By Eq. (70), Eq. (72) becomes

$$(-1)^t \frac{(2n+2+l-t)(2l+1)}{2n+1-2t} \frac{\binom{2l}{l}\binom{2t}{t}\binom{2(n+(l+1)-t)}{n+(l+1)-t}\binom{2n}{n}\frac{2(2n+1-2t)}{n+1-t}\binom{2n-2t}{n-t}}{(n+1)\binom{n+1+l}{n+1}\binom{2n+2+l-t}{n+1}\binom{n+1}{t}}. \tag{73}$$

Eq. (73) is equivalent to

$$(-1)^t(2n+2+l-t) \frac{2(2l+1)\binom{2l}{l}\binom{2t}{t}\binom{2(n+(l+1)-t)}{n+(l+1)-t}\binom{2n}{n}\binom{2n-2t}{n-t}}{(n+1)\binom{2n+2+l-t}{n+1}\binom{n+1+l}{n+1}(n+1-t)\binom{n+1}{t}}. \tag{74}$$

It is readily verified [10, Eq. (1.1), Eq. (1.2), p. 5] that

$$(n+1)\binom{2n+2+l-t}{n+1} = (2n+2+l-t)\binom{2n+(l+1)}{n}, \tag{75}$$

$$(n+1-t)\binom{n+1}{t} = (n+1)\binom{n}{t}. \tag{76}$$

By Eqns. (75) and (76), Eq. (74) becomes

$$(-1)^t(2n+2+l-t) \frac{2(2l+1)\binom{2l}{l}\binom{2t}{t}\binom{2(n+(l+1)-t)}{n+(l+1)-t}\binom{2n}{n}\binom{2n-2t}{n-t}}{(2n+2+l-t)\binom{2n+(l+1)-t}{n}\binom{n+1+l}{n+1}(n+1)\binom{n}{t}}, \text{ or}$$

$$(-1)^t \frac{2(2l+1)\binom{2l}{l}\binom{2t}{t}\binom{2(n+(l+1)-t)}{n+(l+1)-t}\binom{2n}{n}\binom{2n-2t}{n-t}}{\binom{2n+(l+1)-t}{n}\binom{n+1+l}{n+1}(n+1)\binom{n}{t}}. \tag{77}$$

It is readily verified [10, Eq. (1.5), p. 5] that

$$\binom{n+1+l}{n+1}(n+1) = (l+1)\binom{n+l+1}{n}. \tag{78}$$

By Eq. (78), Eq. (77) becomes

$$(-1)^t \frac{\frac{2(2l+1)}{l+1}\binom{2l}{l}\binom{2t}{t}\binom{2(n+(l+1)-t)}{n+(l+1)-t}\binom{2n}{n}\binom{2n-2t}{n-t}}{\binom{2n+(l+1)-t}{n}\binom{n+(l+1)}{n}\binom{n}{t}}. \tag{79}$$

By Eq. (30), it follows that

$$\frac{2(2l+1)}{l+1}\binom{2l}{l} = \binom{2(l+1)}{l+1}.$$

By the last equation above, Eq. (79) becomes

$$(-1)^t \frac{\binom{2(l+1)}{l+1}\binom{2t}{t}\binom{2(n+(l+1)-t)}{n+(l+1)-t}\binom{2n}{n}\binom{2n-2t}{n-t}}{\binom{2n+(l+1)-t}{n}\binom{n+(l+1)}{n}\binom{n}{t}}. \tag{80}$$

By Eq. (59), it follows that Eq. (80) is equal to $\varphi(2n, l+1, t)$.

Therefore, Eqns. (68), (71), (72), (73), (74), (77), (79), and (80) complete the proof of Eq. (67).

Let l_0 be a fixed non-negative integer. Let us assume that the following equation is true for all non-negative integers n and t such that $t \leq n$:

$$\Psi_t(2n, l_0) = \varphi(2n, l_0, t). \tag{81}$$

Eq. (81) is our induction hypothesis.

Let us prove that Eq. (81) implies the following equation:

$$\Psi_t(2n, l_0+1) = \varphi(2n, l_0+1, t); \tag{82}$$

where n and t are non-negative integers such that $t \leq n$.

We treat two cases: $t = n$ and $t < n$ separately.

By setting $l = l_0 + 1$ in Eq. (66), it follows that Eq. (82) is true for $t = n$. This proves the first case.

Let us assume that $t < n$.

By setting $l := l_0$ in Eq. (61), we obtain that

$$\Psi_t(2n, l_0+1) = \frac{(2n+2+l_0-t)(2l_0+1)}{2(2n+1-2t)(2n+1)} \Psi_t(2n+2, l_0). \tag{83}$$

By setting $l := l_0$ in Eq. (67), we obtain that

$$\varphi(2n, l_0+1, t) = \frac{(2n+2+l_0-t)(2l_0+1)}{2(2n+1-2t)(2n+1)} \varphi(2n+2, l_0, t). \tag{84}$$

By setting $n := n+1$ in Eq. (81) (the induction hypothesis), it follows that

$$\Psi_t(2n+2, l_0) = \varphi(2n+2, l_0, t). \tag{85}$$

For $t < n$, by Eqns. (83), (84), and (85), Eq. (82) follows.

This proves the second case.

Therefore, we conclude that Eq. (81) implies Eq. (82). This proves the induction step. By induction, Theorem 2 follows, as desired. Also this completes the proof of Eq. (15).

6 Proof of Theorem 1

By setting $t = 0$ in Theorem 2, we obtain that

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}S(k,l)S(2n-k,l) = \frac{\binom{2l}{l}\binom{2(n+l)}{n+l}\binom{2n}{n}^2}{\binom{n+l}{n}\binom{2n+l}{n}}, \tag{86}$$

$$= \frac{\binom{2n}{n}\binom{2l}{l}}{\binom{n+l}{n}}\frac{\binom{2(n+l)}{n+l}\binom{2n}{n}}{\binom{2n+l}{n}},$$

$$= S(n,l)S(n+l,n).$$

The last equation above proves Theorem 1.

Remark 10. Theorem 1 can be written as

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}S(k,l)S(2n-k,l) = \frac{\binom{2l}{l}\binom{2(n+l)}{n+l}\binom{2n}{n}}{\binom{2n+l}{l}}. \tag{87}$$

By symmetry of binomial coefficients and [10, Eq. (1.4), p. 5], it follows that

$$\binom{2n+l}{n}\binom{n+l}{n} = \binom{2n+l}{l}\binom{2n}{n}. \tag{88}$$

By Eqns. (86) and (88), Eq. (87) follows, as desired.

7 Proof of Theorem 3

We consider the following sum:

$$\Psi(2n,m,l) = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m S(k,l)S(2n-k,l). \tag{89}$$

By Theorem 1 and Eq. (89), we obtain that $\Psi(2n,1,l) = S(n,l)S(n+l,n)$. Hence Theorem 3 is true for $m = 1$.

Our proof of Theorem 3 consists from two parts.

In the first part, we show that Theorem 3 is true for $m = 2$. In the second part we prove that $D_\Psi(2n,j,1;l)$ is divisible by $S(n,l)$ for all non-negative integers n, j, and l such that $j \le n$. Then, by Eq. (9), it follows that Theorem 3 is true for all integers m greater than 2.

7.1 The First Part

By setting $A := \Psi$ and $n := 2n$ in Relation (14), we obtain that

$$D_\Psi(2n,j,0;l) = \sum_{u=0}^{n-j}\binom{2n-j}{u}\binom{2n-j-u}{j+u}\Psi_{j+u}(2n,l). \tag{90}$$

Let t be a non-negative integer such that $t \leq n$. Let us prove the following equation

$$\binom{2n-t}{t}\Psi_t(2n,l) = (-1)^t \frac{\binom{2l}{l}\binom{2t}{t}\binom{2(n+l-t)}{n+l-t}\binom{2n}{n}\binom{2n-t}{n}}{\binom{n+l}{n}\binom{2n+l-t}{n}}. \tag{91}$$

By Eq. (15), it follows that

$$\binom{2n-t}{t}\Psi_t(2n,l) = \binom{2n-t}{t}(-1)^t \frac{\binom{2l}{l}\binom{2t}{t}\binom{2(n+l-t)}{n+l-t}\binom{2n}{n}\binom{2n-2t}{n-t}}{\binom{n+l}{n}\binom{2n+l-t}{n}\binom{n}{t}}. \tag{92}$$

Eq. (92) is equivalent to

$$\binom{2n-t}{t}\Psi_t(2n,l) = (-1)^t \frac{\binom{2l}{l}\binom{2t}{t}\binom{2(n+l-t)}{n+l-t}\binom{2n}{n}}{\binom{n+l}{n}\binom{2n+l-t}{n}} \frac{\binom{2n-t}{t}\binom{2n-2t}{n-t}}{\binom{n}{t}}. \tag{93}$$

By symmetry of binomial coefficients and [10, Eq. (1.4), p. 5], it follows that

$$\frac{\binom{2n-t}{t}\binom{2n-2t}{n-t}}{\binom{n}{t}} = \binom{2n-t}{n}. \tag{94}$$

By Eqns. (93) and (94), Eq. (91) follows.
By setting $t := j+u$ in Eq. (91), we obtain that

$$\binom{2n-j-u}{j+u}\Psi_{j+u}(2n,l) = (-1)^{j+u} \frac{\binom{2l}{l}\binom{2(j+u)}{j+u}\binom{2(n+l-j-u)}{n+l-j-u}\binom{2n}{n}\binom{2n-j-u}{n}}{\binom{n+l}{n}\binom{2n+l-j-u}{n}}. \tag{95}$$

By Eq. (95), Eq. (90) becomes as follows

$$\begin{aligned} D_\Psi(2n,j,0;l) &= \sum_{u=0}^{n-j}\binom{2n-j}{u}(-1)^{j+u}\frac{\binom{2l}{l}\binom{2(j+u)}{j+u}\binom{2(n+l-j-u)}{n+l-j-u}\binom{2n}{n}\binom{2n-j-u}{n}}{\binom{n+l}{n}\binom{2n+l-j-u}{n}}, \\ &= \frac{\binom{2n}{n}\binom{2l}{l}}{\binom{n+l}{n}}\sum_{u=0}^{n-j}(-1)^{j+u}\frac{\binom{2(j+u)}{j+u}\binom{2(n+l-j-u)}{n+l-j-u}\binom{2n-j}{u}\binom{2n-j-u}{n}}{\binom{2n+l-j-u}{n}}, \\ &= S(n,l)\sum_{u=0}^{n-j}(-1)^{j+u}\frac{\binom{2(j+u)}{j+u}\binom{2(n+l-j-u)}{n+l-j-u}\binom{2n-j}{u}\binom{2n-j-u}{n}}{\binom{2n+l-j-u}{n}}. \end{aligned} \tag{96}$$

By symmetry of binomial coefficients and [10, Eq. (1.4), p. 5], it follows that

$$\binom{2n-j}{u}\binom{2n-j-u}{n} = \binom{2n-j}{n}\binom{n-j}{u}. \tag{97}$$

By Eq. (97), Eq. (96) is equal to

$$D_\Psi(2n,j,0;l) = S(n,l)\sum_{u=0}^{n-j}(-1)^{j+u}\frac{\binom{2(j+u)}{j+u}\binom{2(n+l-j-u)}{n+l-j-u}\binom{2n-j}{n}\binom{n-j}{u}}{\binom{2n+l-j-u}{n}}. \tag{98}$$

By setting $j := 0$ in Eq. (98), we have that

$$D_\Psi(2n,0,0;l) = S(n,l)\sum_{u=0}^{n}(-1)^u \frac{\binom{2u}{u}\binom{2(n+l-u)}{n+l-u}\binom{2n}{n}\binom{n}{u}}{\binom{2n+l-u}{n}}, \text{ or}$$

$$D_\Psi(2n,0,0;l) = S(n,l)\sum_{u=0}^{n}(-1)^u \binom{2u}{u} S(n,n+l-u)\binom{n}{u}. \tag{99}$$

By Eq. (9), it follows that

$$\Psi(2n,2,l) = D_\Psi(2n,0,0;l). \tag{100}$$

Hence, by Eqns. (99) and (100), it follows that

$$\Psi(2n,2,l) = S(n,l)\sum_{u=0}^{n}(-1)^u \binom{2u}{u} S(n,n+l-u)\binom{n}{u}. \tag{101}$$

By Eq. (101), it follows that Theorem 3 is true for $m = 2$.

Remark 11. If l is a positive integer then $\Psi(2n,2,l)$ is divisible by $2S(n,l)$.

Since l is a positive integer and $u \leq n$ in Eq. (101) , it follows that $n-u+l$ is a positive integer also. Then the integer $S(n,n+l-u)$ must be even. By Eq. (101), it follows that $\Psi(2n,2,l)$ is divisible by $2S(n,l)$.

7.2 The Second Part

Let us calculate $D_\Psi(2n,j,1;l)$; where j is a non-negative integer such that $j \leq n$. We shall use Eq. (10).

By setting $t := 0$, $n := 2n$, and $A := \Psi$ in Eq. (10), we obtain that

$$D_\Psi(2n,j,1;l) = \sum_{u=0}^{n-j}\binom{2n}{j+u}\binom{2n-j}{u} D_\Psi(2n,j+u,0;l). \tag{102}$$

We use Eq. (98) from the previous subsection. Eq. (98) is equivalent to

$$D_\Psi(2n,j,0;l) = (-1)^j S(n,l)\binom{2n-j}{n}\sum_{v=0}^{n-j}(-1)^v \frac{\binom{2(j+v)}{j+v}\binom{2(n+l-j-v)}{n+l-j-v}\binom{n-j}{v}}{\binom{2n+l-j-v}{n}}. \tag{103}$$

By setting $j := j+u$ in Eq. (103), it follows that $D_\Psi(2n,j+u,0;l)$ is equal to

$$(-1)^{j+u} S(n,l)\binom{2n-j-u}{n}\sum_{v=0}^{n-j-u}(-1)^v \frac{\binom{2(j+u+v)}{j+u+v}\binom{2(n+l-j-u-v)}{n+l-j-u-v}\binom{n-j-u}{v}}{\binom{2n+l-j-u-v}{n}}. \tag{104}$$

Let $Q(n, j+u, l)$ denote the sum

$$\sum_{v=0}^{n-j-u} (-1)^v \frac{\binom{2(j+u+v)}{j+u+v}\binom{2(n+l-j-u-v)}{n+l-j-u-v}\binom{n-j-u}{v}}{\binom{2n+l-j-u-v}{n}}. \tag{105}$$

By Eq. (105), Eq. (104) becomes

$$D_\Psi(2n, j+u, 0; l) = (-1)^{j+u} S(n, l)\binom{2n-j-u}{n} Q(n, j+u, l). \tag{106}$$

By Eqns. (102) and Eq. (106), we obtain that $D_\Psi(2n, j, 1; l)$ is equal to

$$(-1)^j S(n, l) \sum_{u=0}^{n-j} (-1)^u \binom{2n-j}{u}\binom{2n}{j+u}\binom{2n-j-u}{n} Q(n, j+u, l). \tag{107}$$

By symmetry of binomial coefficients and [10, Eq. (1.4), p. 5], it follows that

$$\binom{2n}{j+u}\binom{2n-j-u}{n} = \binom{2n}{n}\binom{n}{j+u}. \tag{108}$$

By Eqns. (107) and (108), it follows that

$$D_\Psi(2n, j, 1; l) = (-1)^j S(n, l) \sum_{u=0}^{n-j} (-1)^u \binom{2n-j}{u}\binom{n}{j+u}\binom{2n}{n} Q(n, j+u, l). \tag{109}$$

Note that the number $\binom{2n}{n} Q(n, j+u, l)$ in Eq. (109) is an integer.
By Eq. (105), it follows that

$$\binom{2n}{n} Q(n, j+u, l) = \sum_{v=0}^{n-j-u} (-1)^v \binom{2(j+u+v)}{j+u+v}\binom{n-j-u}{v} \frac{\binom{2n}{n}\binom{2(n+l-j-u-v)}{n+l-j-u-v}}{\binom{2n+l-j-u-v}{n}}. \tag{110}$$

Namely, by Eq. (110), it follows that $\binom{2n}{n} Q(n, j+u, l)$ is equal to

$$\sum_{v=0}^{n-j-u} (-1)^v \binom{2(j+u+v)}{j+u+v}\binom{n-j-u}{v} S(n, n+l-j-u-v). \tag{111}$$

By Eq. (111), the number $\binom{2n}{n} Q(n, j+u, l)$ is always an integer. By Eq. (109), it follows that $D_\Psi(2n, j, 1; l)$ is divisible by $S(n, l)$ for all non-negative integers n, j, and l such that $j \leq n$. Then, by Eq. (10) and induction, it can be shown that $D_\Psi(2n, j, t; l)$ is divisible by $S(n, l)$ for all positive integers t and for all non-negative integers n, j, and l such that $j \leq n$. See [11, How Does This Method Work, p. 7].

By setting $m := t + 2$ in Eq. (9), we obtain that

$$\Psi(2n, t+2, l) = D_{\Psi}(2n, 0, t; l). \tag{112}$$

By Eq. (112), it follows that $\Psi(2n, t+2, l)$ is divisible by $S(n, l)$ for all positive integers t and for all non-negative integers n, j, and l such that $j \leq n$. Since $t \geq 1$, it follows $t+2 \geq 3$. Therefore, Theorem 3 is true for all non-negative integers n and for all positive integers m such that $m \geq 3$. This completes the second part and proves Theorem 3.

Remark 12. Let l be a positive integer. Then $S(n, n+l-j-u-v)$ is always an even integer.

By Eq. (111), it follows that number $\binom{2n}{n}Q(n, j+u, l)$, is an even integer. By Eq (109), it follows that $D_{\Psi}(2n, j, 1; l)$ is divisible by $2S(n, l)$ for all non-negative integers n, j, and l such that $j \leq n$. By the method of D sums, it can be shown that $\Psi(2n, m, l)$ is divisible by $2S(n, l)$ for $m \geq 3$. Finally, by Theorem 1 and Remark 11, it follows that $\Psi(2n, m, l)$ is divisible by $2S(n, l)$ for all non-negative integers n and for all positive integers m and l.

Remark 13. Let n be a non-negative integer and let m be a positive integer. Then the sum $\Psi(2n, m, l)$ is divisible [13, Thm. 1, Corollary 4; p. 2] by $\binom{2n}{n}$ for $l = 0$ and $l = 1$. This is not true for all positive integers l. For example, take $l = 2$, $n = 4$, and $m = 1$.

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