ABOUT SPECIAL LIMITS AND SUMS

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Theorem 1.
If $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ then

$$\sum_{k=1}^{n} \frac{a + f(k)}{f(k) + 2a + f(n-k)} = \frac{n-1}{2} + \frac{a + f(n)}{f(0) + 2a + f(n)}$$

Proof.
We consider the notation: $a_n = \frac{a+f(k)}{f(k)+2a+f(n-k)}$ therefore,

$$a_k + a_{n-k} = \frac{a + f(k)}{f(k) + 2a + f(n-k)} + \frac{a + f(n-k)}{f(n-k) + 2a + f(k)} = 1$$

Hence,

$$2\sum_{k=1}^{n-1} a_k = \sum_{k=1}^{n-1} (a_k + a_{n-k}) = n - 1$$

And then,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n-1} a_k + a_n = \frac{n-1}{2} + \frac{a + f(n)}{f(0) + 2a + f(n)}$$

□

Application 1.
For $a \geq 0$ find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{a + k^a}{k^a + 2a + (n-k)^a} \cdot \tan^{-1}\left(\frac{n^2}{n^2 - kn + k^2}\right)$$

Solution. Using the following identity

$$\tan^{-1}\left(\frac{k}{n}\right) + \tan^{-1}\left(\frac{n-k}{n}\right) = \tan^{-1}\left(\frac{n^2}{n^2 - nk + k^2}\right)$$

and put in Theorem 1 $f(x) = \tan^{-1}\left(\frac{k}{x}\right)$ it follows that:

$$S_n = \frac{1}{n} \sum_{k=1}^{n} \frac{a + k^a}{k^a + 2a + (n-k)^a} \cdot \tan^{-1}\left(\frac{n^2}{n^2 - kn + k^2}\right) =$$

$$= \sum_{k=1}^{n-1} \tan^{-1}\left(\frac{k}{n}\right) + \frac{a + n^a}{2a + n^a} \cdot \frac{\pi}{4} =$$

$$= \sum_{k=1}^{n} \tan^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{4} \cdot \frac{a}{2a+n^a}$$

Therefore,

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{a+k^a}{k^a+2a+(n-k)^a} \cdot \tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right) =$$

$$\lim_{n\to\infty} \frac{1}{n} \cdot \left(\sum_{k=1}^{n} \tan^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{4} \cdot \frac{a}{2a+n^a}\right) = \int_0^1 \tan^{-1} x dx \overset{IBP}{=}$$

$$= x \cdot \tan^{-1} x|_0^1 - \int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \frac{1}{2}\log 2$$

□

Application 2. Find:

$$\Omega = \lim_{p\to\infty} \frac{1}{p^a} \cdot \sum_{m=1}^{p} \sum_{n=1}^{m} \sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2}$$

Solution. Let $a_k = \frac{2k^2}{2k^2-2nk+n^2}$ hence,

$$a_k + a_{n-k} = \frac{2k^2}{2k^2-2nk+n^2} + \frac{(n-k)^2}{2(n-k)^2-2n(n-k)+n^2}$$

$$= \frac{k^2}{2k^2-2nk+n^2} + \frac{(n-k)^2}{-2k(n-k)+n^2} = \frac{k^2}{2k^2-2nk+n^2} + \frac{(n-k)^2}{n^2-2kn+2k^2} = 1$$

Hence,

$$\sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2} = \frac{n+1}{2}$$

and we get:

$$\sum_{m=1}^{p} \sum_{n=1}^{m} \sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2} = \sum_{m=1}^{p} \sum_{n=1}^{m} \frac{n+1}{2} =$$

$$\sum_{m=1}^{p} \left(\frac{m(m+1)}{4} + \frac{m}{2}\right) = \frac{p(p+1)(2p+7)}{24}$$

Therefore,

$$\Omega = \lim_{p\to\infty} \frac{1}{p^a} \cdot \sum_{m=1}^{p} \sum_{n=1}^{m} \sum_{k=1}^{n} \frac{k^2}{2k^2-2nk+n^2} =$$

$$= \lim_{p\to\infty} \frac{1}{p^a} \cdot \frac{p(p+1)(2p+7)}{24} = \begin{cases} 0, & \text{if } a > 3 \\ \frac{1}{12}, & \text{if } a = 3 \\ +\infty, & \text{if } a < 3 \end{cases}$$

□

Application 3. For $a > 0b > 1$ find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{a+b^k}{b^k+2a+b^{n-k}} \cdot \tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)$$

Solution. Using the following identity

$$\tan^{-1}\left(\frac{k}{n}\right) + \tan^{-1}\left(\frac{n-k}{n}\right) = \tan^{-1}\left(\frac{n^2}{n^2-nk+k^2}\right)$$

and taking $a_k = \frac{a+b^k}{b^k+2a+b^{n-k}} \cdot \tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right)$ we get

$$a_k = \frac{a+b^k}{b^k+2a+b^{n-k}} \left(\tan^{-1}\left(\frac{k}{n}\right) + \tan^{-1}\left(\frac{n-k}{n}\right)\right)$$

Hence,

$$a_k + a_{n-k} = \frac{a+b^k}{b^k+2a+b^{n-k}} \left(\tan^{-1}\left(\frac{k}{n}\right) + \tan^{-1}\left(\frac{n-k}{n}\right)\right) +$$

$$+\frac{a+b^{n-k}}{b^k+2a+b^{n-k}} \left(\tan^{-1}\left(\frac{n-k}{n}\right) + \tan^{-1}\left(\frac{k}{n}\right)\right) = \tan^{-1}\left(\frac{k}{n}\right) + \tan^{-1}\left(\frac{n-k}{n}\right)$$

Thus,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n-1} a_k + a_n = \sum_{k=1}^{n-1} \tan^{-1}\left(\frac{k}{n}\right) + \frac{\pi}{4} \cdot \frac{a+b^n}{b^n+2a+1}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} \frac{a+b^k}{b^k+2a+b^{n-k}} \cdot \tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right) = \frac{1}{n}\sum_{k=1}^{n-1} \tan^{-1}\left(\frac{k}{n}\right) + \frac{\pi}{4n} \cdot \frac{a+b^n}{b^n+2a+1}$$

$$= \frac{1}{n}\sum_{k=1}^{n} \tan^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{4n} \cdot \frac{a+1}{1+2a+b^n}$$

Therefore,

$$\Omega = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{a+b^k}{b^k+2a+b^{n-k}} \cdot \tan^{-1}\left(\frac{n^2}{n^2-kn+k^2}\right) =$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} \tan^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{4n} \cdot \frac{a+1}{1+2a+b^n}\right) = \int_0^1 \tan^{-1} x\, dx = \frac{\pi}{4} - \log\sqrt{2}$$

Observation. If in Theorem 1 we take $a = p, p \in \mathbb{N}$ and summing after $p \in \mathbb{N}$ it follows that:

$$\sum_{p=1}^{n}\sum_{k=1}^{n} \frac{p+f(k)}{f(k)+2p+f(n-k)} = \sum_{p=1}^{n}\left(\frac{n-1}{2} + \frac{p+f(n)}{f(0)+2p+f(n)}\right) =$$

$$= \frac{p(n-1)}{2} + \sum_{p=1}^{n} \frac{p+f(n)}{f(0)+2p+f(n)}$$

□

Application 4. For $a \in \mathbb{R}$ find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n^a} \cdot \sum_{m=1}^{n}\sum_{k=1}^{n} \frac{m+k}{2m+k}$$

Solution. In Theorem 1, we take $f(x) = x$ and $a = m, m \in \{1, 2, ..., n\}$ then:

$$\sum_{k=1}^{n} \frac{m+k}{2m+k} = \frac{n-1}{2} + \frac{m+n}{2m+n}$$

$$\sum_{m=1}^{n}\sum_{k=1}^{n} \frac{m+k}{2m+k} = \frac{n(n-1)}{2} + \sum_{k=1}^{n} \frac{m+n}{2m+n}$$

$$\Omega = \lim_{n\to\infty} \frac{1}{n^a} \sum_{m=1}^{n}\sum_{k=1}^{n} \frac{m+k}{2m+k} = \lim_{n\to\infty} \frac{1}{n^a} \left(\frac{n(n-1)}{2} + \sum_{k=1}^{n} \frac{m+n}{2m+n} \right) =$$

$$= \lim_{n\to\infty} \frac{1}{n^{a-1}} \left(\frac{n-1}{2} + \frac{1}{n} \sum_{m=1}^{n} \frac{1+\frac{m}{n}}{1+2\frac{m}{n}} \right) = \frac{1}{2} \lim_{n\to\infty} \frac{n-1}{n^{a-1}} + \int_0^1 \frac{1+x}{1+2x} dx =$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\log 3 + \lim_{n\to\infty} \frac{n-1}{n^{a-1}}\right) = \begin{cases} \frac{1}{2}(1+\log\sqrt{3}); & a > 0 \\ \frac{1}{2}(2+\log\sqrt{3}; & a = 2 \\ \infty; & a > 2 \end{cases}$$

Application 5. For $a \in \mathbb{R}$ find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n^a} \cdot \sum_{p=1}^{n}\sum_{k=1}^{n} \frac{p+k^2}{2p^2+2k^2-2nk+n^2}$$

Solution. In Theorem 1, we take $f(x) = x^2$ and $a = p, p \in \{1, 2, ..., n\}$ then:

$$\sum_{k=1}^{n} \frac{p+k^2}{2p^2+2k^2-2nk+n^2} = \frac{n-1}{2} + \frac{p+n^2}{2p+n^2}$$

$$\sum_{p=1}^{n}\sum_{k=1}^{n} \frac{p+k^2}{2p^2+2k^2-2nk+n^2} = \frac{n(n-1)}{2} + \sum_{p=1}^{n} \frac{p+n^2}{2p+n^2} =$$

$$= \frac{n(n+1)}{2} - \sum_{p=1}^{n} \frac{p}{2p+n^2}$$

$$\sum_{p=1}^{n} \frac{p}{2p+n^2} \leq \sum_{p=1}^{n} \frac{p}{2+n^2} = \frac{n^2+n}{2(n^2+2)} \to \frac{1}{2}; (n\to\infty)$$

$$\sum_{p=1}^{n} \frac{p}{2p+n^2} \geq \sum_{p=1}^{n} \frac{p}{2n+n^2} = \frac{n^2+n}{2(n^2+2} \to \frac{1}{2}; (n\to\infty)$$

Therefore,

$$\Omega = \lim_{n\to\infty} \frac{1}{n^a} \sum_{p=1}^{n}\sum_{k=1}^{n} \frac{p+k^2}{2p^2+2k^2-2nk+n^2} = \lim_{n\to\infty} \frac{1}{n^a} \left(\frac{n(n+1)}{2} - \sum_{p=1}^{n} \frac{p}{2p+n^2} \right) =$$

$$= \begin{cases} 0, & a \le 2 \\ \infty, & a > 2 \end{cases}$$

□

Theorem 2.
If $f, g : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ then

$$\sum_{k=1}^{n} \frac{g(k)g(n-k)(a+f(k))}{f(k)+2a+f(n-k)} = \frac{1}{2}\sum_{k=1}^{n-1} g(k)g(n-k) + \frac{g(0)g(n)(a+f(n))}{f(0)+2a+f(n)}$$

Proof. Let $a_k = g(k)g(n-k) + \frac{g(0)g(n)(a+f(n))}{f(0)+2a+f(n)}$ and we get

$$a_k + a_{n-k} = g(k)g(n-k) + \frac{g(0)g(n)(a+f(n))}{f(0)+2a+f(n)} + \frac{g(n-k)g(k)(a+f(n-k))}{f(n-k)+2a+f(k)} = g(k)g(n-k)$$

Therefore,

$$2\sum_{k=1}^{n-1} a_k = \sum_{k=1}^{n-1}(a_k + a_{n-k}) = \sum_{k=1}^{n-1} g(k)g(n-k)$$

$$\sum_{k=1}^{n} = \frac{1}{2}\sum_{k=1}^{n-1} g(k)g(n-k) + \frac{g(0)g(n)(a+f(n))}{f(0)+2a+f(n)}$$

Application 6. Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k+2}{(k+1)(n-k+1)}$$

Solution. In Theorem 2, we take $f(x) = x$, $g(x) = \frac{1}{x+1}$ and $a = 2$, then

$$\frac{1}{n+4}\sum_{k=1}^{n} \frac{k+2}{(k+1)(n-k+1)} = \frac{1}{2}\sum_{k=1}^{n-1} \frac{1}{(k+1)(n-k+1)} + \frac{n+2}{(n+1)(n+4)} =$$

$$= \frac{1}{2(n+2)} \sum_{k=1}^{n-1} \left(\frac{1}{k+1} - \frac{1}{n-k+1} \right) + \frac{n+2}{(n+1)(n+4)}$$

Hence,

$$\sum_{k=1}^{n} \frac{k+2}{(k+1)(n-k+1)} = \frac{n+4}{2(n+2)} \sum_{k=1}^{n-1} \left(\frac{1}{k+1} - \frac{1}{n-k+1} \right) + \frac{n+2}{n+1}$$

Therefore,

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k+2}{(k+1)(n-k+1)} = 1$$

□

Theorem 3. If $f, g : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ then

$$\sum_{k=1}^{n} \frac{(g(k)+g(n-k))(a+f(k))}{f(k)+2a+f(n-k)} = \sum_{k=1}^{n-1} g(k) + \frac{(g(0)+g(n))(a+f(n))}{f(0)+2a+f(n)}$$

Proof. Let $a_k = \frac{(g(k)+g(n-k))(a+f(k))}{f(k)+2a+f(n-k)}$ and we get

$$a_k + a_{n-k} = \frac{(g(k)+g(n-k))(a+f(k))}{f(k)+2a+f(n-k)} - \frac{(g(n-k)+g(k))(a+f(n-k))}{f(n-k)+2a+f(k)} =$$

$$= g(k) + g(n-k)$$

Thus,

$$2\sum_{k=1}^{n-1} = \sum_{k=1}^{n-1} (a_k + a_{n-k}) = \sum_{k=1}^{n-1} (g(k)+g(n-k)) = 2\sum_{k=1}^{n-1} g(k)$$

and then

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n-1} g(k) + \frac{(g(0)+g(n))(a+f(n))}{f(0)+2a+f(n)}$$

□

Application 7. Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{(k+1)(n-k+1)(2n^2-2kn+k^2)}$$

Solution. In Theorem 3, we take $f(x) = x^2$, $g(x) = \frac{1}{x+1}$ and $a = 0$, we get

$$\sum_{k=1}^{n} \frac{(n+2)k^2}{(k+1)(n-k+1)(n^2+(n-k)^2)} = \sum_{k=1}^{n-1} \frac{1}{k+1} + \frac{n+2}{n+1}$$

Hence,

$$\sum_{k=1}^{n} \frac{k^2}{(k+1)(n-k+1)(2n^2-2kn+k^2)} = \frac{1}{n+2} \sum_{k=1}^{n-1} \frac{1}{k+1} + 1$$

$$\sum_{k=1}^{n} \frac{k^2}{(k+1)(n-k+1)(2n^2-2kn+k^2)} = \frac{1}{n+2}(\gamma_n + \log n - 1) + 1$$

Therefore,

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{(k+1)(n-k+1)(2n^2-2kn+k^2)} = 1 + \lim_{n\to\infty} \frac{\gamma_n + \log n - 1}{n+2} =$$

$$= 1 + \lim_{n\to\infty} \frac{\log n}{n+2} \stackrel{L.C-S}{=} 1 + \lim_{n\to\infty} \frac{\log(n+1) - \log n}{n+3-n-2} = 1$$

□

Theorem 4. If $S_n = \sum\limits_{k=1}^{n} x_k$ then

$$1)\ \sum_{k=1}^{n}(n-k+1)x_k = \sum_{k=1}^{n} s_k$$

$$2)\ \sum_{k=1}^{n}\frac{(n-k+1)(n-k+2)}{2}x_k = \sum_{k=1}^{n}(n-k+1)S_k$$

1) We have the following relations,

$$\begin{cases} S_1 = x_1 \\ S_2 = x_1 + x_2 \\ \dots\dots\dots\dots\dots\dots \\ S_n = x_1 + x_2 + \dots + x_n \end{cases}$$

By adding up these relations, we have:

$$\sum_{k=1}^{n}(n-k+1)x_k = \sum_{k=1}^{n} S_k$$

For 2), we have:

$$\sum_{k=1}^{n}\frac{(n-k+1)(n-k+2)}{2}x_k = \sum_{k=1}^{n}(nx_1 + (n-1)x_2 + \dots + (n-k+1)x_k) =$$

$$\sum_{k=1}^{n}(S_1 + S_2 + \dots + S_k) = \sum_{k=1}^{n}(n-k+1)S_k$$

□

Application 8. For $a \in \mathbb{R}$ find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n^a}\cdot\sum_{k=1}^{n}\left(\frac{(n-k+1)((n-k+2)}{2}\cdot\sum_{p=1}^{k}\frac{p^2}{2p^2-2kp+k^2}\right)$$

Solution. In Theorem 1, we take $f(x) = x^2$, $a = 0$ we get

$$x_k = \sum_{p=1}^{k}\frac{p^2}{2p^2-2kp+k^2} = \frac{k+1}{2}$$

In Theorem 4, we take

$$S_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n}\sum_{p=1}^{k}\frac{p^2}{2p^2-2kp+k^2} = \sum_{k=1} n\frac{k+1}{2} = \frac{n(n+1)}{4} + \frac{n}{2} = \frac{n(n+2)}{2}$$

Hence,

$$\sum_{k=1}^{n}\left(\frac{(n-k+1)(n-k+2)}{2}\cdot\sum_{p=1}^{k}\frac{p^2}{2p^2-2kp+k^2}\right) = \sum_{k=1}^{n}(n-k+1)\frac{k(k+2)}{2} =$$

$$\sum_{k=1}^{n}\frac{k(k+2)(n-k+1)}{2}=\sum_{k=1}^{n}k^3+(n+1)\sum_{k=1}^{n}k^2+2(n-1)\sum_{k=1}^{n}k=$$
$$=\frac{n(n+1)(7n^2+21n-10)}{12}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{n^a}\cdot\sum_{k=1}^{n}\left(\frac{(n-k+1)((n-k+2)}{2}\cdot\sum_{p=1}^{k}\frac{p^2}{2p^2-2kp+k^2}\right)=$$
$$=\lim_{n\to\infty}\frac{1}{n^a}\cdot\frac{n(n+1)(7n^2+21n-10)}{12}=\begin{cases}=0, & a>4\\ \frac{7}{12}, & a=4\\ \infty, & a<4\end{cases}$$

□

Application 9. For $a\in\mathbb{R}$ find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n^a}\cdot\sum_{k=1}^{n}\left((n-k+1)\cdot\sum_{p=1}^{k}\frac{p^3(k-p)}{2p^2-2kp+p^2}\right)$$

Solution. In Theorem 2, we take $f(x)=x^2$, $g(x)=x$ it follows that

$$x_k=\sum_{p=1}^{n}\frac{p^3(k-p)}{2p^2-2kp+p^2}=\frac{k(k^2-1}{12}$$

In Theorem 4, we take

$$S_n=\sum_{k=1}^{n}x_k=\sum_{k=1}^{n}\frac{k(k^2-1)}{12}=\frac{(n^2+1)(n^2+n-2}{48}$$

Hence,

$$\frac{1}{n^a}\cdot\sum_{k=1}^{n}\left((n-k+1)\cdot\sum_{p=1}^{k}\frac{p^3(k-p)}{2p^2-2kp+p^2}\right)=\sum_{k=1}^{n}(n-k+1)x_k=\sum_{k=1}^{n}S_k=$$
$$=\sum_{k=1}^{n}\frac{(k^2+1)(k^2+k-2)}{48}=$$
$$=\frac{1}{48}\cdot\frac{n(n+1)(2n+1)(3n^2+3n+1)}{30}+\frac{1}{48}\cdot\frac{n^2(n+1)^2}{4}-\frac{1}{48}\cdot\frac{n(n+1)(2n+1)}{6}$$
$$+\frac{1}{48}\cdot\frac{n(n+1)}{2}-\frac{n}{24}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\frac{1}{n^a}\cdot\sum_{k=1}^{n}\left((n-k+1)\cdot\sum_{p=1}^{k}\frac{p^3(k-p)}{2p^2-2kp+p^2}\right)=\begin{cases}0, & a>5\\ \frac{1}{1440}, & a=5\\ \infty, & a>5\end{cases}$$

□

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