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ABOUT AN INEQUALITY BY ELDENIZ HESENOV-VI

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1) Let ΔDEF be the orthic triangle of acute $\Delta ABC, H$ –orthocenter.

Prove that:

$$\frac{AH\cdot r_a}{EF}+\frac{BH\cdot r_b}{FD}+\frac{CH\cdot r_c}{DE}\le\frac{4(r_a^2+r_b^2+r_c^2)}{a+b+c}$$

Proposed by Eldeniz Hesenov-Georgia

Solution. **Lemma. 2) Let ΔDEF be the orthic triangle of acute $\Delta ABC, H$ –orthocenter. In these conditions:**

$$\sum_{cyc}\frac{AH\cdot r_a}{EF}=\frac{s}{2}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$$

Proof. Using $AH=2R\cdot\cos A, EF=a\cdot\cos A$ and $r_a=\frac{F}{s-a}$ we get:

$$\sum_{cyc}\frac{AH\cdot r_a}{EF}=\sum_{cyc}\frac{2R\cos A\cdot\frac{F}{s-a}}{a\cos A}=RF\sum_{cyc}\frac{1}{a(s-a)}=2Rrs\cdot\frac{1}{4Rr}\left[1+\left(\frac{4R+r}{s}\right)^2\right]=$$

$=\frac{s}{2}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$, which follows from $\sum\frac{1}{a(s-a)}=\frac{1}{4Rr}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$

Let's get back to the main problem, using Lemma inequality becomes

$$\frac{s}{2}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\le\frac{4(r_a^2+r_b^2+r_c^2)}{a+b+c}\Leftrightarrow\frac{s}{2}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\le\frac{4[(4R+r)^2-2s^2}{2p}$$

$\Leftrightarrow 3s^2\le(4R+r)^2$ $(Doucet)$. Equality holds if and only if triangle is equilateral.

Remark. Let's find an opposite inequality.

2) Let ΔDEF be the orthic triangle of acute $\Delta ABC, H$ –orthocenter.

Prove that

$$\frac{AH\cdot r_a}{EF}+\frac{BH\cdot r_b}{FD}+\frac{CH\cdot r_c}{DE}\ge\frac{4(r_ar_b+r_br_c+r_cr_a)}{a+b+c}$$

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Solution. **Lemma. 4) Let ΔDEF be the orthic triangle of acute $\Delta ABC, H$ −orthocenter. Prove that**

$$\sum_{cyc}\frac{AH\cdot r_a}{EF}=\frac{s}{2}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$$

Proof. Using $AH=2R\cdot\cos A, EF=a\cdot\cos A$ and $r_a=\frac{F}{s-a}$ we get:

$$\sum_{cyc}\frac{AH\cdot r_a}{EF}=\sum_{cyc}\frac{2R\cos A\cdot\frac{F}{s-a}}{a\cos A}=RF\sum_{cyc}\frac{1}{a(s-a)}=2Rrs\cdot\frac{1}{4Rr}\left[1+\left(\frac{4R+r}{s}\right)^2\right]=$$

$=\frac{s}{2}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$, which follows from $\sum\frac{1}{a(s-a)}=\frac{1}{4Rr}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$

Let's get back to the main problem, using Lemma inequality becomes

$$\frac{s}{2}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\ge\frac{4(r_ar_b+r_br_c+r_cr_a)}{a+b+c}\Leftrightarrow\frac{s}{2}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\ge\frac{4s^2}{2s}\Leftrightarrow$$

$\Leftrightarrow 3s^2\le(4R+r)^2\ (Doucet)$. Equality holds if and only if triangle is equilateral.

5) Let ΔDEF be the orthic triangle of acute $\Delta ABC, H$ −orthocenter.

Prove that

$$\frac{4(r_ar_b+r_br_c+r_cr_a)}{a+b+c}\le\sum_{cyc}\frac{AH\cdot r_a}{EF}\le\frac{4(r_a^2+r_b^2+r_c^2)}{a+b+c}$$

Solution. See inequalities 1) and 3). Equality holds if and only if triangle is equilateral.

Remark. Let's replace r_a with h_a.

6) Let ΔDEF be the orthic triangle of acute $\Delta ABC, H$ −orthocenter.

Prove that

$$\frac{2R}{r}\cdot\frac{h_ah_b+h_bh_c+h_ch_a}{a+b+c}\le\sum_{cyc}\frac{AH\cdot h_a}{EF}\le\frac{2R}{r}\cdot\frac{h_a^2+h_b^2+h_c^2}{a+b+c}$$

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Solution. **Lemma 7) Let ΔDEF be the orthic triangle of an acute $\Delta ABC, H$ –orthocenter. Prove that:**

$$\sum_{cyc}\frac{AH\cdot h_a}{EF}=\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs}$$

Proof. Using $AH=2R\cdot\cos A, EF=a\cdot\cos A$ and $h_a=\frac{2F}{a}$ we get:

$$\sum_{cyc}\frac{AH\cdot r_a}{EF}=\sum_{cyc}\frac{2R\cos A\cdot\frac{2F}{a}}{a\cos A}=4RF\sum_{cyc}\frac{1}{a^2}=4Rrs\cdot\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{(4Rrs)^2}$$

$=\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs}$, which follows from $\sum\frac{1}{a^2}=\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{(4Rrs)^2}$.

Let's get back to the main problem, using Lemma inequality becomes:

$$\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs}\le\frac{2R}{r}\cdot\frac{h_a^2+h_b^2+h_c^2}{a+b+c}\Leftrightarrow$$

$$\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs}\le\frac{2R}{r}\cdot\frac{\left[\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-2\cdot\frac{2s^2r}{R}\right]}{2s}\Leftrightarrow$$

$$s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2\le\frac{R}{2r}\cdot 8Rr\left[\frac{s^4+2s^2(r^2+4Rr)+r^2(4R+r)^2}{4R^2}-2\cdot\frac{2s^2r}{R}\right]$$

$$s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2\le\frac{R}{2r}\cdot 2r\left[\frac{s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2}{R}\right]$$

$$s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2\le s^4+s^2(2r^2-8Rr)+r^2(4R+r)^2$$

For LHS, using Lemma, we have:

$$\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs}\ge\frac{2R}{r}\cdot\frac{h_ah_b+h_bh_c+h_ch_a}{a+b+c}$$

$$\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs}\ge\frac{2R}{r}\cdot\frac{\frac{2s^2r}{R}}{2s}\Leftrightarrow$$

$$s^2(s^2+2r^2-16Rr)+r^2(4R+r)^2\ge 0$$

Distinguish the following cases:

Case 1) If $(s^2+2r^2-16Rr)\ge 0$, inequality is obviously true.

Case 2) If $(s^2+2r^2-16Rr)<0$ inequality can be written as:

$r^2(4R+r)^2 \geq s^2(16Rr-2r^2-s^2)$, which follows from Gerretsen inequality

$$16Rr-5r^2 \leq s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$$

Remains to prove that $r^2(4R+r)^2 \geq \frac{R(4R+r)^2}{2(2R-r)}(16Rr-2r^2-16Rr+5r^2) \Leftrightarrow$

$R \geq 2r(Euler)$. Equality holds if and only if triangle is equilateral.

8) Let ΔDEF be the orthic triangle of acute $\Delta ABC, H$ –orthocenter.

Prove that:

$$\sum_{cyc}\frac{AH\cdot r_a}{EF} \leq \frac{R}{2r}\sum_{cyc}\frac{AH\cdot h_a}{EF}$$

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Solution. Using Lammas we have:

$$\sum_{cyc}\frac{AH\cdot h_a}{EF} = \frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs}; \sum_{cyc}\frac{AH\cdot r_a}{EF} = \frac{s}{2}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$$

Inequality becomes: $\frac{s}{2}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \leq \frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs} \Leftrightarrow$

$$Rs^2(s^2+2r^2-8Rr)+r^2(4R+r)^2 \geq 4Rr^2s^2+4Rr^2(4R+r)^2 \Leftrightarrow$$

$Rs^2(s^2+2r^2-10Rr) \geq (4R+r)^2(4Rr^2-2r^3)$, which follows from Gerretsen inequality

$$s^2 \geq 16Rr-5r^2 \geq \frac{r(4R+r)^2}{R+r}$$

Remains to prove that $\frac{r(4R+r)^2}{R+r}R(16Rr-5r^2+2r^2-10Rr) \geq (4R+r)^2(4Rr^2-2r^3) \Leftrightarrow$

$$R(6R-3r) \geq (R+r)(4R-2r) \Leftrightarrow 6R^2-3Rr \geq 4R^2+2Rr-2r^2 \Leftrightarrow$$

$2R^2-5Rr+2r^2 \geq 0 \Leftrightarrow (R-2r)(2R-r) \geq 0$ true from $R \geq 2r\ (Euler)$.

Equality holds if and only if triangle is equilateral.

Remark. It can be write the following identity:

9) Let ΔDEF be the orthic triangle of acute $\Delta ABC, H$ –orthocenter.

Prove that:

$$\frac{AH\cdot r_a}{EF}+\frac{BH\cdot r_b}{FD}+\frac{CH\cdot r_c}{DE}=\frac{2R}{r}\cdot\frac{h_a^2+h_b^2+h_c^2}{a+b+c}$$

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Solution. **Lemma. 10) Let ΔDEF be the orthic triangle of an acute $\Delta ABC, H$ –orthocenter. Prove that:**

$$\sum_{cyc}\frac{AH\cdot h_a}{EF}=\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs}$$

Proof. Using $AH=2R\cdot\cos A, EF=a\cdot\cos A$ and $h_a=\frac{2F}{a}$ we get:

$$\sum_{cyc}\frac{AH\cdot r_a}{EF}=\sum_{cyc}\frac{2R\cos A\cdot\frac{2F}{a}}{a\cos A}=4RF\sum_{cyc}\frac{1}{a^2}=4Rrs\cdot\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{(4Rrs)^2}$$

$=\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs}$, which follows from $\sum\frac{1}{a^2}=\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{(4Rrs)^2}$.

Using Lemma we have:

$$Lhs=\sum_{cyc}\frac{AH\cdot h_a}{EF}=\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs};(1)$$

$$Rhs=\frac{2R}{r}\cdot\frac{h_a^2+h_b^2+h_c^2}{a+b+c}=\frac{2R}{r}\cdot\frac{(\sum h_a)^2-2\sum h_bh_c}{2s}=\frac{2R}{r}\cdot\frac{\left[\left(\frac{s^2+r^2+4Rr}{2R}\right)^2-2\cdot\frac{2s^2r}{R}\right]}{2s}$$

$$=\frac{2R}{r}\cdot\left[\frac{s^4+2s^2(r^2+4Rr)+r^2(4R+r)^2}{4R^2}-2\cdot\frac{2s^2r}{R}\right]=$$

$$=\frac{2R}{r}\cdot\frac{1}{2s}\cdot\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{8R^2s}=\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs};(2)$$

From (1),(2) it follows that

$$\sum_{cyc}\frac{AH\cdot h_a}{EF}=\frac{s^2(s^2+2r^2-8Rr)+r^2(4R+r)^2}{4Rrs}$$

REFERENCES:

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