An approach to calculate $\sum_{n=1}^{\infty} \frac{1}{n^3 \sin\left(\sqrt{2}\pi n\right)}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3 \sin(\varphi\pi n)}$ and finding a connection with alternating double series

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Abstract

In this article, we will show an approach to prove the elegant results:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin\left(\sqrt{2}\pi n\right)} = -\frac{13\pi^3}{360\sqrt{2}} \approx -0.791727...$$
$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin\left(\varphi\pi n\right)} = -\frac{17\pi^3}{1440\sqrt{5}} - \frac{\pi^3}{32} \approx -1.132647..., \quad \varphi = \frac{1+\sqrt{5}}{2}$$

through the use of a powerful identity perhaps presented by *S. Ramanujan* in [1] in any of your papers, which will be proved in this paper only by using the expansion to $\cot(\pi z)$ and $\csc(\pi z)$ recalled in [4], and double sum symmetry.

1 Introdution

The current paper deals with the evaluation of two apparently complicated results to prove, however we are presenting a subtle approach without using contour integration, or complex numbers. We will explore a curious result and relationships with the φ *golden ratio*, where it will drive our development and certainly culminate in connections with alternating double series.

2 A key result

Let's start the approach by presenting a key result that will allow us to flow with the purpose of our work, so consider the following theorem:

Theorem 1 *Let α be a irrational number such that $\mu(\alpha)=2$[1], then*

$$\sum_{n=1}^{\infty}\frac{\cot\left(\alpha n\pi\right)}{n^3}+\alpha^2\sum_{n=1}^{\infty}\frac{\cot\left(\frac{\pi n}{\alpha}\right)}{n^3}=\frac{\pi^3}{90\alpha}\left(\alpha^4-5\alpha^2+1\right). \tag{2.0}$$

Proof: By the article in [4], we know that:

$$\sum_{n=0}^{\infty}\frac{1}{n^2-x^2}=-\frac{\pi\cot(\pi x)}{2x}-\frac{1}{2x^2}. \tag{2.1}$$

Then making the sum starting from $n=1$ and changing $n\mapsto k$, then rearranging the terms and highlighting the term $\cot(\pi x)$ at (2.1) we get:

$$\cot(\pi x)=-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{2x}{k^2-x^2}+\frac{1}{\pi x} \tag{2.2}$$

Now, set $x=\alpha n$ and connecting (2.2) on LHS of (2.0) follows that:

$$\begin{aligned}
&=\sum_{n=1}^{\infty}\frac{1}{n^3}\left(-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{2\alpha n}{k^2-\alpha^2n^2}+\frac{1}{\pi\alpha n}\right)+\alpha^2\sum_{n=1}^{\infty}\frac{1}{n^3}\left(-\frac{1}{\pi\alpha}\sum_{k=1}^{\infty}\frac{2n}{k^2-n^2/\alpha^2}+\frac{\alpha}{\pi n}\right)\\
&=\frac{\zeta(4)}{\pi\alpha}-\frac{2\alpha}{\pi}\sum_{n,k\in\mathbb{N}_1}\frac{1}{n^2\left(k^2-\alpha^2n^2\right)}+\frac{\alpha^3}{\pi}\zeta(4)-\frac{2\alpha}{\pi}\sum_{n,k\in\mathbb{N}_1}\frac{1}{n^2\left(k^2-n^2/\alpha^2\right)}\\
&=\frac{1}{\pi\alpha}\left(\left(1+\alpha^4\right)\zeta(4)-2\alpha^2\sum_{n,k\in\mathbb{N}_1}\left(\frac{1}{n^2\left(k^2-\alpha^2n^2\right)}+\frac{1}{n^2\left(k^2-n^2/\alpha^2\right)}\right)\right)\\
&=\frac{1}{\pi\alpha}\left(\left(1+\alpha^4\right)\zeta(4)+2\alpha^2T_{nk}(\alpha)\right),
\end{aligned} \tag{2.3}$$

where, we use [3]. Let's to evaluate $T_{nk}(\alpha)$. First, note that by symmetry $T_{nk}(\alpha)=T_{kn}(\alpha)$, then taking $2T(n,k)$, we have:

$$\begin{aligned}
T_{nk}(\alpha)&=\frac{1}{2}\sum_{n,k\in\mathbb{N}_1}\left(\frac{1}{n^2\left(k^2-\alpha^2n^2\right)}+\frac{1}{n^2\left(k^2-n^2/\alpha^2\right)}+\frac{1}{k^2\left(n^2-\alpha^2k^2\right)}+\frac{1}{k^2\left(n^2-k^2/\alpha^2\right)}\right)\\
&=\frac{1}{2}\sum_{n,k\in\mathbb{N}_1}\left(\frac{1}{k^2-\alpha^2n^2}\left(\frac{1}{n^2}-\frac{\alpha^2}{k^2}\right)+\frac{1}{n^2-\alpha^2k^2}\left(\frac{1}{k^2}-\frac{\alpha^2}{n^2}\right)\right)\\
&=\frac{1}{2}\sum_{n,k\in\mathbb{N}_1}\left(\frac{1}{k^2-\alpha^2n^2}\left(\frac{k^2-\alpha^2n^2}{k^2n^2}\right)+\frac{1}{n^2-\alpha^2k^2}\left(\frac{n^2-\alpha^2k^2}{n^2k^2}\right)\right)\\
&=\frac{1}{2}\cdot2\sum_{n,k\in\mathbb{N}_1}\frac{1}{n^2k^2}=\zeta^2(2)=\frac{5}{2}\zeta(4)
\end{aligned}$$

[1] See [5] (*www.mathworld.wolfram.com/IrrationalityMeasure.html*)

$$\therefore \ T_{nk}(\alpha) = \frac{5}{2}\zeta(4). \tag{2.4}$$

Therefore, leading (2.4) to (2.3), follows that:

$$= \frac{1}{\pi\alpha}\left(\left(1+\alpha^4\right)\zeta(4) - 2\alpha^2\cdot\frac{5}{2}\zeta(4)\right) = \frac{\zeta(4)}{\pi\alpha}\left(\alpha^4 - 5\alpha^2 + 1\right) = \frac{\pi^3}{90\alpha}\left(\alpha^4 - 5\alpha^2 + 1\right).$$

where we used $\zeta(4) = \dfrac{\pi^4}{90}$ ([3]). Thus, the proof for **Theorem 1** is complete. ■

3 First main result

From now on, we will address our goal for this article. First of all, let's assign a more compact notation to **Theorem 1**, as follows:

$$\mathcal{F}(\alpha) + \alpha^2\mathcal{F}\left(\frac{1}{\alpha}\right) = \frac{\pi^3}{90\alpha}\left(\alpha^4 - 5\alpha^2 + 1\right), \tag{3.0}$$

here $\mathcal{F}(\alpha) = \displaystyle\sum_{n=1}^{\infty}\frac{\cot(\alpha\pi n)}{n^3}$. Now, we want to prove:

Theorem 2 (Main Result 1) *The following equality holds:*

$$\sum_{n=1}^{\infty}\frac{1}{n^3\sin\left(\sqrt{2}\pi n\right)} = -\frac{13\pi^3}{360\sqrt{2}}. \tag{3.1}$$

Proof: Notice that:

$$\frac{1}{\sin(z)} + \frac{\cos(z)}{\sin(z)} = \frac{1+\cos(z)}{\sin(z)} = \frac{2\cos^2\left(\frac{z}{2}\right)}{2\sin\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right)} = \cot\left(\frac{z}{2}\right)$$

$$\therefore \ \csc(z) + \cot(z) = \cot\left(\frac{z}{2}\right). \tag{3.2}$$

Applying (3.2) with $z = \sqrt{2}\pi n$ in LHS of (3.1), we have:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^3\sin\left(\sqrt{2}\pi n\right)} &= \sum_{n=1}^{\infty}\frac{1}{n^3}\left(\cot\left(\frac{\sqrt{2}\pi n}{2}\right) - \cot\left(\sqrt{2}\pi n\right)\right) \\
&= \sum_{n=1}^{\infty}\frac{\cot\left(\frac{\pi n}{\sqrt{2}}\right)}{n^3} - \sum_{n=1}^{\infty}\frac{\cot\left(\sqrt{2}\pi n\right)}{n^3} \\
&= \mathcal{F}\left(\frac{1}{\sqrt{2}}\right) - \mathcal{F}\left(\sqrt{2}\right).
\end{aligned} \tag{3.3}$$

From here we must find a linear system that allows us to discover the two terms on the right side of (3.3). For this, it is interesting to note that $\mathcal{F}(\alpha) = \mathcal{F}(\alpha + p)$, such that p is an integer number. Therefore, consider at (3.0) $\alpha = \sqrt{2} + 1$, therefore **Theorem 1** tells us that:

$$\mathcal{F}\left(\sqrt{2}+1\right) + \left(3+2\sqrt{2}\right)\mathcal{F}\left(\frac{1}{\sqrt{2}+1}\right) = \frac{\pi^3}{90}\left(1+\sqrt{2}\right),$$

conjugating the argument of the second function on the left side of the equality above, we have briefly the first equation of the linear system:

$$\mathcal{F}\left(\sqrt{2}+1\right) + \left(3+2\sqrt{2}\right)\mathcal{F}\left(\sqrt{2}-1\right) = \frac{\pi^3}{90}\left(1+\sqrt{2}\right). \tag{3.4}$$

For the second equation, just make $\alpha = \sqrt{2}$ in (3.0) and then the second equation follows immediately:

$$\mathcal{F}\left(\sqrt{2}\right) + 2\mathcal{F}\left(\frac{1}{\sqrt{2}}\right) = -\frac{\pi^3}{18\sqrt{2}}. \tag{3.5}$$

With (3.4) and (3.5) we compose the linear system:

$$\begin{cases} \mathcal{F}\left(\sqrt{2}+1\right) + \left(3+2\sqrt{2}\right)\mathcal{F}\left(\sqrt{2}-1\right) &= \dfrac{\pi^3}{90}\left(1+\sqrt{2}\right) \\ \mathcal{F}\left(\sqrt{2}\right) + 2\mathcal{F}\left(\dfrac{1}{\sqrt{2}}\right) &= -\dfrac{\pi^3}{18\sqrt{2}} \end{cases},$$

using the periodicity of $\mathcal{F}$, the system is reduced to:

$$\begin{cases} \mathcal{F}\left(\sqrt{2}\right) + \left(3+2\sqrt{2}\right)\mathcal{F}\left(\sqrt{2}\right) &= \dfrac{\pi^3}{90}\left(1+\sqrt{2}\right) \\ \mathcal{F}\left(\sqrt{2}\right) + 2\mathcal{F}\left(\dfrac{1}{\sqrt{2}}\right) &= -\dfrac{\pi^3}{18\sqrt{2}} \end{cases},$$

whose solution is:

$$\therefore\ \mathcal{F}\left(\sqrt{2}\right) = \frac{\pi^3}{180\sqrt{2}}, \quad \mathcal{F}\left(\frac{1}{\sqrt{2}}\right) = -\frac{11\pi^3}{360\sqrt{2}}.$$

Applying the system solution in (3.3):

$$\sum_{n=1}^{\infty}\frac{1}{n^3\sin\left(\sqrt{2}\pi n\right)} = -\frac{11\pi^3}{360\sqrt{2}} - \frac{\pi^3}{180\sqrt{2}} = -\frac{\pi^3}{\sqrt{2}}\left(\frac{11}{360}+\frac{1}{180}\right)$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^3 \sin\left(\sqrt{2}\pi n\right)} = -\frac{13\pi^3}{360\sqrt{2}} \approx -0.791727....$$

(Q.E.D.)

Therefore, the proof of our first main result is complete.

4 Second main result

We are now evaluating our second main result. The strategy here follows similar to the previous section. Therefore, we want to prove that:

Theorem 3 (Main result 2) *The following equaity holds:*

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin(\varphi\pi n)} = -\frac{17\pi^3}{1440\sqrt{5}} - \frac{\pi^3}{32}, \tag{4.0}$$

where $\varphi = \dfrac{1+\sqrt{5}}{2}$ *is the Golden Ratio.*

Proof: Using equality (3.2) again, we have briefly that the left side of (4.0) takes the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin(\varphi\pi n)} = \mathcal{F}\left(\frac{\varphi}{2}\right) - \mathcal{F}(\varphi). \tag{4.1}$$

Our next step is to find the equivalent linear system, such as provide us the result of $\mathcal{F}\left(\frac{\varphi}{2}\right)$ and $\mathcal{F}(\varphi)$. Therefore, note that when we take $\alpha = 2\varphi - 1$ in (3.0) this gives us the first system:

$$\mathcal{F}(2\varphi - 2) + \left(\frac{4}{\varphi^2}\right)\mathcal{F}\left(\frac{\varphi}{2}\right) = \frac{\pi^3}{90}\left(\frac{8}{\varphi^3} - \frac{10}{\varphi} + \frac{\varphi}{2}\right). \tag{4.2}$$

where we used the manipulation $\dfrac{1}{\varphi} = \varphi - 1$.

To find the second equation, just take $\alpha = 2\varphi + 1$ then apply **Theorem 1** again, this produces:

$$\mathcal{F}(2\varphi + 1) + \varphi^6 \mathcal{F}(2\varphi - 3) = \frac{\pi^3}{90}\left(\varphi^9 - 5\varphi^3 + \frac{1}{\varphi^3}\right), \tag{4.3}$$

where we used the manipulation $\varphi^2 - \varphi - 1 = 0$, $2\varphi + 1 = \varphi^3$ and $\dfrac{1}{2\varphi + 1} = 2\varphi - 3$ (see [2]).

(4.2) and (4.3) are necessary to find the term $\mathcal{F}\left(\frac{\varphi}{2}\right)$, however we need to insert a new equation to obtain the term $\mathcal{F}(\varphi)$. Therefore, taking $\alpha=\varphi$ and by using the φ manipulations, from **Theorem 1** it follows that:

$$\mathcal{F}(\varphi)+\varphi^2\mathcal{F}(\varphi-1)=\frac{\pi^3}{90}\left(\varphi^3-5\varphi+\frac{1}{\varphi}\right). \tag{4.4}$$

(4.2), (4.3) and (4.4) make up the following system:

$$\begin{cases} \mathcal{F}(2\varphi-2)+\left(\frac{4}{\varphi^2}\right)\mathcal{F}\left(\frac{\varphi}{2}\right) & =\frac{\pi^3}{90}\left(\frac{8}{\varphi^3}-\frac{10}{\varphi}+\frac{\varphi}{2}\right) \\ \mathcal{F}(2\varphi+1)+\varphi^6\mathcal{F}(2\varphi-3) & =\frac{\pi^3}{90}\left(\varphi^9-5\varphi^3+\frac{1}{\varphi^3}\right) \\ \mathcal{F}(\varphi)+\varphi^2\mathcal{F}(\varphi-1) & =\frac{\pi^3}{90}\left(\varphi^3-5\varphi+\frac{1}{\varphi}\right) \end{cases}$$

Again, with the use of $\mathcal{F}$ periodicity, we reconfigure the above system in:

$$\begin{cases} \mathcal{F}(2\varphi)+\left(\frac{4}{\varphi^2}\right)\mathcal{F}\left(\frac{\varphi}{2}\right) & =\frac{\pi^3}{90}\left(\frac{8}{\varphi^3}-\frac{10}{\varphi}+\frac{\varphi}{2}\right) \\ \mathcal{F}(2\varphi)+\varphi^6\mathcal{F}(2\varphi) & =\frac{\pi^3}{90}\left(\varphi^9-5\varphi^3+\frac{1}{\varphi^3}\right) \\ \mathcal{F}(\varphi)+\varphi^2\mathcal{F}(\varphi) & =\frac{\pi^3}{90}\left(\varphi^3-5\varphi+\frac{1}{\varphi}\right) \end{cases}.$$

Note that the second and third equations of the linear equation system, respectively, contain only one variable to be obtained. Therefore, by obtaining the term $\mathcal{F}(2\varphi)$ and plugging into the first equation we obtain the solutions:

$$\therefore\ \mathcal{F}(\varphi)=\frac{\pi^3}{450}\left(-\varphi^4+3\varphi^3+5\varphi^2-15\varphi-1+\frac{3}{\varphi}\right) \tag{4.5}$$

$$\mathcal{F}(2\varphi)=\frac{\pi^3}{900}\left(-4\varphi^{10}+7\varphi^9+20\varphi^4-35\varphi^3-\frac{4}{\varphi^2}+\frac{7}{\varphi^3}\right)$$

$$\mathcal{F}\left(\frac{\varphi}{2}\right)=\frac{\pi^3}{3600}\left(4\varphi^{12}-7\varphi^{11}-20\varphi^6+35\varphi^5+5\varphi^3-100\varphi+4+\frac{73}{\varphi}\right). \tag{4.6}$$

Doing (4.6) - (4.5) and taking this to (4.1), we get:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin(\varphi \pi n)} = \frac{\pi^3}{3600} \left(4\varphi^{12} - 7\varphi^{11} - 20\varphi^6 + 35\varphi^5 + 8\varphi^4 - 19\varphi^3 + 20\varphi + 12 + \frac{49}{\varphi} \right), \tag{4.7}$$

where, with the aid of the *Fibonacci Recurrence Formula* $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$ and $\varphi = \frac{1+\sqrt{5}}{2}$, the term:

$$\left(4\varphi^{12} - 7\varphi^{11} - 20\varphi^6 + 35\varphi^5 + +8\varphi^4 - 19\varphi^3 + 20\varphi + 12 + \frac{49}{\varphi} \right) = -\frac{85}{2\sqrt{5}} - \frac{225}{2}.$$

Connecting this to (4.7), we finally get the desired result:

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^3 \sin(\varphi \pi n)} = -\frac{17\pi^3}{1440\sqrt{5}} - \frac{\pi^3}{32} \approx -1.132647....$$

(Q.E.D.)

5 A possible connection with alternate double series

To find such a connection, we will use the third boxed result on pg. 7 of [4]:

$$\therefore \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 - x^2} = -\frac{\pi \csc(\pi x)}{2x} - \frac{1}{2x^2}. \tag{5.0}$$

So, putting in evidence $\csc(\pi x)$, changing $n \to k$ and then making k start from 1, we get briefly, that:

$$\frac{1}{\sin(\pi x)} = \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2 - x^2} + \frac{1}{\pi x}. \tag{5.1}$$

Now, doing $x = \sqrt{2}n$ and taking this to the sum in (3.1), we have:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin(\sqrt{2}\pi n)} = \frac{2\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{n^2 (k^2 - 2n^2)} + \frac{\pi^3}{90\sqrt{2}},$$

by equating with the right side of (3.1), we won the following:

$$\frac{2\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{n^2 (k^2 - 2n^2)} + \frac{\pi^3}{90\sqrt{2}} = -\frac{13\pi^3}{360\sqrt{2}}, \tag{5.2}$$

solving this, we get the first result:

$$\therefore \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n^2\left(k^2-2n^2\right)}=-\frac{17\pi^4}{1440}. \tag{5.3}$$

On the other hand, by applying partial fractions in the double series in (5.3), it produces:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n^2\left(k^2-2n^2\right)}=\frac{5\pi^4}{360}+2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{k^2\left(k^2-2n^2\right)},$$

applying (5.3) and solving for the double series, we come to the second result:

$$\therefore \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{k^2\left(k^2-2n^2\right)}=-\frac{37\pi^4}{2880}. \tag{5.4}$$

Therefore, we have the sums (5.3) and (5.4) via main result 1:

$$\therefore \quad \begin{aligned}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n^2\left(k^2-2n^2\right)}&=-\frac{17\pi^4}{1440},\\ \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{k^2\left(k^2-2n^2\right)}&=-\frac{37\pi^4}{2880}.\end{aligned}$$

It is convenient for us to also find the alternating double series associated with the main result 2. Therefore, we continue making the same steps:

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^3\sin\left(\varphi\pi n\right)}=\frac{\left(1+\sqrt{5}\right)}{\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n^2\left(k^2-\varphi n^2-n^2\right)}+\frac{\pi^3}{36\sqrt{5}}-\frac{\pi^3}{180},$$

where we again use the properties of the *Golden Ratio* found in [2]. Thus, equating with (4.0), we get:

$$\therefore \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n^2\left(k^2-\varphi n^2-n^2\right)}=-\frac{\pi^4}{288}-\frac{\pi^4}{45\sqrt{5}}. \tag{5.5}$$

On the other hand, applying partial fractions in (5.5) we have:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n^2\left(k^2-\varphi n^2-n^2\right)}=-(\varphi+1)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n^2k^2}+(\varphi+1)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2(k^2-\varphi n^2-n^2)}$$

$$-\frac{\pi^4}{288}-\frac{\pi^4}{45\sqrt{5}}=-(\varphi+1)\frac{\pi^4}{72}+(\varphi+1)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2(k^2-\varphi n^2-n^2)}. \tag{5.6}$$

Solving (5.6) for the double series and, again, using the properties of *Golden Ratio*, we won:

$$\therefore \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2(k^2-\varphi n^2-n^2)}=\frac{19\pi^4}{960}-\frac{71\pi^4}{2880\sqrt{5}}. \tag{5.7}$$

Therefore, we obtain the following results via main result 2:

$$\therefore \quad \begin{aligned}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n^2\left(k^2-\varphi n^2-n^2\right)}&=-\frac{\pi^4}{288}-\frac{\pi^4}{45\sqrt{5}},\\ \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2(k^2-\varphi n^2-n^2)}&=\frac{19\pi^4}{960}-\frac{71\pi^4}{2880\sqrt{5}}.\end{aligned}$$

6 Discussion of results

With this approach, in the first instance we can assume the hypothesis (subject to verification) that, when α is of the form $p+\frac{q}{\sqrt{r}}$ suh that $\sqrt{r}\in\mathbb{I}$, where $\mathbb{I}$ denote the set of irrational numbers and $p,q\in\mathbb{Q}$, then the result for $\mathcal{F}\left(\frac{\alpha}{2}\right)-\mathcal{F}\left(\alpha\right)$ can be of the form $\left(k\cdot p+\frac{a\cdot q}{\sqrt{r}}\right)\pi^3$, for $a,k\in\mathbb{Q}$. That is , the existence of a term that does not depend on $\sqrt{r}$.

Another curiosity concerns the continuous fractions. Note that in the presentation of the calculations to prove the two main results, the use of linear systems to obtain the $\mathcal{F}(\alpha)$ functions was characterized, where in both we solve linear systems of 2×2. This evidence can be justified by observing the decomposition into continuous fractions of the $\sqrt{2}$ and φ values, as follows:

$$\varphi=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}=[1;\overline{1}],\quad \sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}=[1;\overline{2}] \tag{5.0}$$

The relationship to be observed between the number of equations and the decomposition coefficients of α, is between the periodicity of the coefficients when applied to the functional equation of **Theorem 1**, this allowed us to compose a 2×2 linear system already that the sum of the amount of non-periodic coefficients with the periodicals is equal to 2. In short, this applies to the most generalized case in which, from $\alpha=\left[a_0;a_1,a_2,...,a_i,\overline{b_1,b_2,b_3,...,b_j}\right]$

, we can write the generalized system:

$$\begin{bmatrix} 1 & \alpha_0^2 & 0 & \cdots & 0 \\ 1 & \alpha_1^2 & 0 & \cdots & 0 \\ 1 & \alpha_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{i+j}^2 & 0 & \cdots & 0 \\ 1 & \alpha_{i+j+1}^2 & 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathcal{F}(\alpha_1) \\ \mathcal{F}(\alpha_0) \\ \mathcal{F}(\alpha_2) \\ \mathcal{F}(\alpha_3) \\ \vdots \\ \mathcal{F}(\alpha_{i+j+1}) \end{bmatrix} = \begin{bmatrix} r(\alpha_1) \\ r(\alpha_2) \\ r(\alpha_3) \\ r(\alpha_4) \\ \vdots \\ r(\alpha_{i+j+1}) \end{bmatrix}, \tag{5.1}$$

such that $\alpha_{i+j+1} = \alpha_{i+1}$, where $\alpha_0 = \alpha$, $\alpha_1 = [a_1; a_2, ...]$, $\alpha_{i+j+1} = [a_{i+j+1}, a_{i+j+2}, ...]$ and $r(\alpha_p) = \frac{\pi^3}{90}\left(\alpha_p^4 - 5\alpha_p^2 + 1\right)$. This implies a repetition of values α_i, such that from a previous equation, it is possible to determine the subsequent equation thus making the system of equations with $(i + j + 1) \times (i + j + 1)$ possible and determined.

7 Conclusion

We can conclude that it is possible to determine, through the method discussed in this paper, a family of infinite series of $\sum_{n=1}^{\infty} \frac{1}{n^3 \sin(\alpha \pi n)}$ nature varying α and calculating the equivalent systems with a preview of the behavior of the decomposition coefficients in continuous fractions of the observed parameter. Surprising results can jump in front of us with the right choice of the values of α. Grateful for the attention and dedication to this reading. we receive good reviews with good feelings.

To my wife and my son. God be praised!

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