

## SOLVED PROBLEMS

BY FLORICĂ ANASTASE

### **PP36719 by Pirkuliyev Rovsen**

If  $a, b > 0$  then prove:

$$(a+b) \cdot \frac{\sin x}{x} + \frac{2ab}{a+b} \cdot \frac{\tan x}{x} > \frac{4\sqrt{2}ab}{a+b}, \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

**Solution.**

$$(a+b) \cdot \frac{\sin x}{x} + \frac{2ab}{a+b} \cdot \frac{\tan x}{x} > \frac{4\sqrt{2}ab}{a+b}, \quad \forall x \in \left(0, \frac{\pi}{2}\right); (*)$$

$$(a+b)^2 \cdot \sin x + 2ab \cdot \tan x > 4\sqrt{2}abx \text{ and from } (a+b)^2 \geq ab$$

It is enough to prove that:

$$4ab \cdot \sin x + 2ab \cdot \tan x - 4\sqrt{2}ab \geq 0 \Leftrightarrow 2 \sin x + \tan x - 4\sqrt{2}x \geq 0, \quad \forall x \in \left(0, \frac{\pi}{2}\right); (1)$$

Let be the function:  $f : \left(0, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$ ,  $f(x) = 2 \sin x + \tan x - 2\sqrt{2}x$ .

$$f'(x) = \sec^2 x + 2 \cos x - 2\sqrt{2}, \text{ and}$$

$$\begin{aligned} f''(x) &= 2(\sec^2 x \tan x - \sin x) = 2(\tan x(1 + \tan^2 x) - \sin x) = 2(\tan x - \sin x + \tan^3 x) \geq \\ &\geq 2(\sin x - \sin x + \tan^3 x) = 2\tan^3 x, \quad f''(x) > 0, \quad \forall x \in \left(0, \frac{\pi}{2}\right) \end{aligned}$$

Hence,  $f'$ -increases and  $f'(x) > f'(0) = 3 - 2\sqrt{2} > 0 \Rightarrow f(x) > f(0) = 0$ .

So, (1) is true and then (\*) is proved.

Florică Anastase

### **PP37083 by Mihàly Bencze**

Prove that:

$$\sum_{k=1}^n \frac{1}{k} \left( \int_0^1 e^{x^k} dx - 1 \right) \leq \frac{3n}{n+1}$$

**Solution.**

We know that:  $\log(1+x) \geq \frac{x}{2}$ ,  $\forall x \in [0, 1] \Leftrightarrow e^x \leq (1+x)^2$ ,  $\forall x \in [0, 1]$

$$e^{x^k} - 1 \leq 2x^k + x^{2k}$$

$$\int_0^1 (e^{x^k} - 1) dx \leq \frac{2}{k+1} + \frac{1}{2k+1}$$

Hence,

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k} \left( \int_0^1 e^{x^k} dx - 1 \right) &\leq \sum_{k=1}^n \left( \frac{2}{k(k+1)} + \frac{1}{k(2k+1)} \right) \leq \\ &\leq \sum_{k=1}^n \left( \frac{2}{k(k+1)} + \frac{1}{k(k+1)} \right) = \frac{3n}{n+1} \end{aligned}$$

Florică Anastase

**PP37084 by Mihàly Bencze** Compute:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{\frac{1}{k(k+1)}}$$

**Solution.**

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1, \text{ then, } \forall \varepsilon > 0, (1 - \varepsilon)x < e^x - 1 < (1 + \varepsilon)x$$

Hence,

$$(1 - \varepsilon) \frac{1}{k(k+1)} \leq e^{\frac{1}{k(k+1)}} - 1 \leq (1 + \varepsilon) \frac{1}{k(k+1)}$$

$$1 + (1 - \varepsilon) \frac{1}{n} \sum_{k=1}^n \frac{1}{k(k+1)} \leq \frac{1}{n} \sum_{k=1}^n e^{\frac{1}{k(k+1)}} \leq 1 + (1 - \varepsilon) \frac{1}{n} \sum_{k=1}^n \frac{1}{k(k+1)}$$

$$1 + (1 - \varepsilon) \frac{1}{n} \cdot \frac{n}{n+1} \leq \frac{1}{n} \sum_{k=1}^n e^{\frac{1}{k(k+1)}} \leq 1 + (1 - \varepsilon) \frac{1}{n} \cdot \frac{n}{n+1}$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{\frac{1}{k(k+1)}} = 1$$

Florică Anastase

**PP37253 by Mihàly Bencze**

Prove that if  $0 < a \leq b$  then  $\int_a^b \left( \frac{x}{\sqrt{1+x^2} \tan^{-1} x} \right)^3 dx > \log \frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}$

**Solution.**

We know that:  $\tan^{-1} x < x, \forall x > 0$ . Thus,

$$\int_a^b \left( \frac{x}{\sqrt{1+x^2} \tan^{-1} x} \right)^3 dx > \int_a^b \frac{1}{\sqrt{1+x^2}} dx = \log \frac{b+\sqrt{1+b^2}}{a+\sqrt{1+a^2}}$$

Florică Anastase

bfPP37408 by Mihàly Bencze Prove that:

$$\sum_{m=2}^n \prod_{k=2}^m \frac{k^3 - 1}{k^3 + 1} = \frac{(n-1)(2n+3)}{3(n+1)}$$

**Solution.**

$$\prod_{k=2}^m \frac{k^3 - 1}{k^3 + 1} = \prod_{k=2}^m \frac{k-1}{k+1} \cdot \prod_{k=2}^m \frac{k^2+k+1}{k^2-k+1} = \frac{2}{m^2+m} \cdot \frac{m^2+m+1}{3}$$

Hence,

$$\begin{aligned} \sum_{m=2}^n \prod_{k=2}^m \frac{k^3 - 1}{k^3 + 1} &= \frac{2}{3} \sum_{m=2}^n \left( 1 + \frac{1}{m(m+1)} \right) = \frac{2}{3} \sum_{m=2}^n \left( 1 + \frac{1}{m} - \frac{1}{m+1} \right) = \\ &= \frac{2}{3} \left( n-1 + \frac{1}{2} - \frac{1}{n+1} \right) = \frac{(n-1)(2n+3)}{3(n+1)} \end{aligned}$$

Florică Anastase

**PP37448 by Mihàly Bencze**

$$\text{Prove that: } \tan \left( \sum_{k=1}^n \tan^{-1} \frac{1}{k^2+k+1} \right) = \frac{n}{n+2}$$

**Solution.**

$$\tan^{-1} \frac{1}{k^2+k+1} = \tan^{-1} \frac{(k+1)-k}{1+k(k+1)} = \tan^{-1}(k+1) - \tan^{-1} k$$

Hence,

$$\tan \left( \sum_{k=1}^n \tan^{-1} \frac{1}{k^2+k+1} \right) = \tan \left( \tan^{-1}(n+1) - \frac{\pi}{4} \right) = \frac{n}{n+2}$$

Florică Anastase

**PP37324 by D.M. Bătinețu-Giurgiu-Daniel Sitaru**

In any  $\Delta ABC$  the following relationship holds:

$$\left( \sum_{cyc} \frac{w_a^3}{h_b + h_c} \right) \left( \sum_{cyc} \frac{1}{(h_a + h_b)^2} \right) \geq \frac{9}{8}$$

**Solution.**

$$\begin{aligned} \sum_{cyc} \frac{w_a^3}{h_b + h_c} &\geq \sum_{cyc} \frac{h_a^3}{h_b + h_c} = \sum_{cyc} \frac{(h_a^2)^2}{h_a h_b + h_a h_c} \stackrel{\text{Bergstrom}}{\geq} \frac{\left( \sum_{cyc} h_a^2 \right)^2}{\sum_{cyc} (h_a h_b + h_a h_c)} = \end{aligned}$$

$$= \frac{\left(\sum_{cyc} h_a^2\right)^2}{2 \sum_{cyc} h_a h_b} = \frac{1}{2} \sum_{cyc} h_a h_b$$

Therefore,

$$\left( \sum_{cyc} \frac{w_a^3}{h_b + h_c} \right) \left( \sum_{cyc} \frac{1}{(h_a + h_b)^2} \right) \geq \frac{1}{2} \left( \sum_{cyc} h_a h_b \right) \left( \sum_{cyc} \frac{1}{(h_a + h_b)^2} \right) \stackrel{Ji\ Chen}{\geq} \frac{1}{2} \cdot \frac{9}{4} = \frac{9}{8}$$

Florică Anastase

**PP37313 by Daniel Sitaru-Claudia Nănuță**

If  $a, b, c, d \in \mathbb{R}$  then:

$$(ad - bc)^8(a^2 + b^2)(c^2 + d^2) + (ac + bd)^{10} \leq (a^2 + b^2)^5(c^2 + d^2)^5$$

**Solution.**

$$\begin{aligned} (a^2 + b^2)(c^2 + d^2) &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 = (ac + bd)^2 + (ad - bc)^2 \\ (ad - bc)^8(a^2 + b^2)(c^2 + d^2) + (ac + bd)^{10} &= \\ (ad - bc)^8[(ac + bd)^2 + (ad - bc)^2] + (ac + bd)^{10} &= \\ = (ac + bd)^{10} + (ad - bc)^{10} + (ad - bc)^8(ac + bd)^2 &= \\ [(ac - bd)^2]^5 + [(ac - bd)^2]^5 + [(ad - bc)^2]^4(ac + bd)^2 & \end{aligned}$$

Let us denote:  $\alpha = (ad - bc)^2, \beta = (ac + bd)^2$ . Hence,

$$\alpha^5 + \beta^5 + \alpha^4\beta \leq (\alpha + \beta)^5 \Leftrightarrow$$

$$\alpha\beta[4\alpha^3 + 10\alpha\beta(\alpha + \beta) + 5\beta^3] \geq 0$$

which is true for all  $\alpha, \beta > 0$  we have  $4\alpha^3 + 10\alpha\beta(\alpha + \beta) + 5\beta^3 \geq 0$

Florică Anastase

**PP37316 by Daniel Sitaru-Claudia Nănuță**

If  $a, b, c > 0, \sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 12$  then:

$$\frac{(a+b+\sqrt{ab})^3}{(a+b)^2} + \frac{(b+c+\sqrt{bc})^3}{(b+c)^2} + \frac{(c+a+\sqrt{ca})^3}{(c+a)^2} \geq 81$$

**Solution.**

$$\begin{aligned} \frac{(a+b+\sqrt{ab})^3}{(a+b)^2} + \frac{(b+c+\sqrt{bc})^3}{(b+c)^2} + \frac{(c+a+\sqrt{ca})^3}{(c+a)^2} &\stackrel{\text{Radon}}{\geq} \\ \geq \frac{(2a+2b+2c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2}{(2a+2b+2c)^3} &= \frac{(2a+2b+2c+12)^3}{(2a+2b+2c)^2} = \\ &= \frac{2(a+b+c+6)^3}{(a+b+c)^2} \end{aligned}$$

Let us denote  $x = a + b + c \stackrel{AM-GM}{\geq} \sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 12$ . Remains to prove that:

$$\frac{2(x+6)^3}{x^2} \geq 81 \Leftrightarrow 2(x+6)^3 - 81x^2 \geq 0 \Leftrightarrow (x-12)^2(2x+3) \geq 0, \text{ true for all } x \geq 12$$

Florica Anastase

**PP37380 by Daniel Sitaru**

If  $f : \mathbb{R} \rightarrow (0, \infty)$ ,  $f$ -continuous,  $a > 0$ ,  $f(x) = f(-x)$ ,  $\forall x \in \mathbb{R}$  then:

$$\int_{\frac{1}{a}}^a \frac{x + \log x}{xf\left(x - \frac{1}{x}\right)} dx = \frac{1}{2} \int_{\frac{1+\sqrt{1+a^2}}{2a}}^{\frac{a+\sqrt{4+a^2}}{2}} \frac{dx}{f(x)}$$

**Solution.**

$$\begin{aligned} \Omega &= \int_{\frac{1}{a}}^a \frac{x + \log x}{xf\left(x - \frac{1}{x}\right)} dx \stackrel{x=\frac{1}{u}}{=} \int_{\frac{1}{a}}^a \frac{\frac{1}{u} - \log u}{f\left(u - \frac{1}{u}\right)u} du = \\ &= \int_{\frac{1}{a}}^a \frac{\frac{1}{u^2} + 1 - \left(\frac{\log u}{u} + 1\right)}{f\left(u - \frac{1}{u}\right)} = \int_{\frac{1}{a}}^a \frac{\frac{1}{u^2} + 1}{f\left(u - \frac{1}{u}\right)} du - \Omega \\ \Omega &= \frac{1}{2} \int_{\frac{1}{a}}^a \frac{\frac{1}{u^2} + 1}{f\left(u - \frac{1}{u}\right)} du = \frac{1}{2} \int_{\frac{1+\sqrt{1+a^2}}{2a}}^{\frac{a+\sqrt{4+a^2}}{2}} \frac{dx}{f(x)} \end{aligned}$$

Florica Anastase

**PP37357 by Daniel Sitaru-Claudia Nănuță**

Solve for real numbers:

$$|\cos x| + |\cos y| = \sqrt{(2 + \sin x + \sin y)(2 - \sin x - \sin y)}$$

**Solution.**

$$|\cos x| + |\cos y| = \sqrt{(2 + \sin x + \sin y)(2 - \sin x - \sin y)}; (*)$$

It follows,

$$\begin{aligned} (|\cos x| + |\cos y|)^2 &= 4 - (\sin x + \sin y)^2 \\ \cos^2 x + \cos^2 y + 2|\cos x||\cos y| &= 4 - 2\sin x \sin y - \sin^2 x \sin^2 y \\ |\cos x \cos y| + \sin x \sin y &= 1 \end{aligned}$$

If  $\cos x \cos y \geq 0$ , we get:  $\cos(x - y) = 1 \Leftrightarrow x = 2k\pi + y$ ,  $k \in \mathbb{Z}$

If  $\cos x \cos y \leq 0$ , we get:  $\cos x \cos y + \sin x \sin y = 1 \Leftrightarrow x + y = (2k + 1)\pi \Leftrightarrow y = (2p + 1)\pi - x$ ,  $p \in \mathbb{Z}$

Florica Anastase