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SP.383 If $a, b, c > 0$ then:

$$\frac{a^{10}c^5 + b^{10}a^5 + c^{10}b^5}{a^2b + b^2c + c^2a} \geq a^4b^4c^4$$

Proposed by Daniel Sitaru-Romania

Solution 1 by proposer, Solution 2 by Fayssal Abdelli-Bejaia-Algerie, Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 4 by Adrian Popa-Romania, Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

Solution 1 by proposer

$$\text{Let } f:(0,\infty) \to (0,\infty), f(x) = \left(\frac{a}{b}\right)^x + \left(\frac{b}{c}\right)^x + \left(\frac{c}{a}\right)^x$$

$$f'(x) = \left(\frac{a}{b}\right)^x \log\left(\frac{a}{b}\right) + \left(\frac{b}{c}\right)^x \log\left(\frac{b}{c}\right) + \left(\frac{c}{a}\right)^x \log\left(\frac{c}{a}\right)$$

$$f''(x) = \left(\frac{a}{b}\right)^x \log^2\left(\frac{a}{b}\right) + \left(\frac{b}{c}\right)^x \log^2\left(\frac{b}{c}\right) + \left(\frac{c}{a}\right)^x \log^2\left(\frac{c}{a}\right) \geq 0$$

$$\Rightarrow f''(x) \geq 0 \Rightarrow f' - \text{increasing, then } f'(x) \geq f'(0), \forall x \geq 0$$

$$f'(x) \geq \log\left(\frac{a}{b}\right) + \log\left(\frac{b}{c}\right) + \log\left(\frac{c}{a}\right) = \log\left(\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}\right) = \log 1 = 0$$

$$f'(x) \geq 0, \text{then } f - \text{increasing, so } f(5) \geq f(2).$$

Hence,

$$\left(\frac{a}{b}\right)^5+\left(\frac{b}{c}\right)^5+\left(\frac{c}{a}\right)^5\ge\left(\frac{a}{b}\right)^2+\left(\frac{b}{c}\right)^2+\left(\frac{c}{a}\right)^2\ge\frac{a}{b}\cdot\frac{b}{c}+\frac{b}{c}\cdot\frac{c}{a}+\frac{c}{a}\cdot\frac{a}{b}=$$

$$=\frac{a}{c}+\frac{b}{c}+\frac{c}{b}$$

Therefore,

$$\frac{a^5}{b^5}+\frac{b^5}{c^5}+\frac{c^5}{a^5}\ge\frac{a}{c}+\frac{b}{a}+\frac{c}{b}$$

$$\frac{a^{10}c^5+b^{10}a^5+c^{10}b^5}{a^4b^4c^4}\ge a^2b+b^2c+c^2a$$

Finally, it follows that:

$$\frac{a^{10}c^5+b^{10}a^5+c^{10}b^5}{a^2b+b^2c+c^2a}\ge a^4b^4c^4$$

Equality holds for $a=b=c$.

Solution 2 by Fayssal Abdelli-Bejaia-Algerie

$$\frac{a^{10}c^5+b^{10}a^5+c^{10}b^5}{3}\ge\sqrt[3]{a^{15}b^{15}c^{15}}\Rightarrow a^{10}c^5+b^{10}a^5+c^{10}b^5\ge 3a^5b^5c^5$$

$$\frac{3a^5b^5c^5}{a^2b+b^2c+c^2a}\overset{(1)}{\ge}a^4b^4c^4\Rightarrow abc\ge a^2b+b^2c+c^2a;(A)$$

$$\frac{a^2b+b^2c+c^2a}{3}\ge\sqrt[3]{a^3b^3c^3}\Rightarrow a^2b+b^2c+c^2a\ge 3abc;(B)$$

From $(A),(B)$, we have $a^2b+b^2c+c^2a=3abc\Rightarrow\frac{3a^5b^5c^5}{3abc}\ge a^4b^4c^4$ true.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$a^{10}c^5+b^{10}a^5+c^{10}b^5=(a^2c)^5+(b^2a)^5+(c^2b)^5\overset{Hölder}{\gtrapprox}\frac{(a^2c+b^2a+c^2b)^5}{3^4}=$$

$$=\frac{(a^2c+b^2a+c^2b)^3}{3^4}.(a^2c+b^2a+c^2b)^2\overset{\substack{AM-GM\\(\sum x)^2\ge3\sum xy}}{\gtrapprox}$$

$$\frac{\left(3\sqrt[3]{a^2c.b^2a.c^2b}\right)^3}{3^4}.3(a^2c.b^2a+b^2a.c^2b+c^2b.a^2c)=$$

$$=(abc)^3.abc(a^2b+b^2c+c^2a)=a^4b^4c^4(a^2b+b^2c+c^2a).$$

$$Therefore, \quad \frac{a^{10}c^5+b^{10}a^5+c^{10}b^5}{a^2b+b^2c+c^2a} \geq a^4b^4c^4.$$

Solution 4 by Adrian Popa-Romania

We must prove that:

$$a^{10}c^5+b^{10}a^5+c^{10}b^5 \geq a^5b^4c^6+b^5c^4a^6+c^5a^4b^6$$

$(10,0,5) > (5,4,6)$ **because** $10 > 5, 10+0 > 5+4, 10+0+5 = 5+4+6.$

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

We have:

$$\frac{a^6c}{b^4}+\frac{c^6b}{a^4}+\frac{b^6a}{c^4} \geq \frac{a^3+b^3+c^3}{3}\left(\frac{a^3c}{b^4}+\frac{c^3b}{a^4}+\frac{b^3a}{c^4}\right) \geq a^3+b^3+c^3$$

$$\geq a^2b+b^2c+c^2a$$

Hence,

$$\frac{a^{10}c^5+c^{10}b^5+b^{10}a^5}{(abc)^4} \geq a^2b+b^2c+c^2a$$

Finally, it follows that:

$$\frac{a^{10}c^5+b^{10}a^5+c^{10}b^5}{a^2b+b^2c+c^2a} \geq a^4b^4c^4$$

Equality holds for $a=b=c$.