



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2) \cdot \dots \cdot (n+k)} = \frac{1}{m} \sum_{n=1}^{\infty} \frac{n+m-n}{n(n+1)(n+2) \cdot \dots \cdot (n+m)} = \\
 & = \frac{1}{m} \left( \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2) \cdot \dots \cdot (n+m-1)} - \frac{1}{(n+1)(n+2) \cdot \dots \cdot (n+m)} \right) = \\
 & = \frac{1}{m} \left( \sum_{n=1}^{\infty} f(n) - f(n+1) \right) = \frac{1}{m} (f(1) - f(n+1)) = \frac{1}{m} \left( \frac{1}{m!} - 0 \right) = \frac{1}{m \cdot m!} \\
 \Omega(n, k) & = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2) \cdot \dots \cdot (n+k)} = \sum_{k=1}^{\infty} \frac{1}{k! \cdot k} \Leftrightarrow 1 < \Omega < e - 1 \\
 \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} & \stackrel{\text{Bergstrom}}{\geq} \frac{a+b+c}{2} \stackrel{\text{AM-GM}}{\geq} \frac{3}{2} \text{ true } \because \frac{3}{2} > \frac{3}{2\Omega(n, k)}
 \end{aligned}$$

Therefore,

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} > \frac{3}{2\Omega(n, k)}$$

**Solution 2 by Fayssal Abdelli-Bejaia-Algerie**

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \stackrel{\text{Bergstrom}}{\geq} \frac{(a+b+c)^2}{2(a+b+c)} = \frac{a+b+c}{2} \stackrel{?}{\geq} \frac{3}{2\Omega(n, k)}$$

We have:

$$\begin{aligned}
 & \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2) \cdot \dots \cdot (n+k)} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(n+k)!} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(n-1)!}{(n+k)!} = \\
 & = \sum_{k=1}^{\infty} \frac{0!}{(k+1)!} + \frac{1!}{(k+2)!} + \frac{2!}{(k+3)!} + \dots
 \end{aligned}$$

We know that:

$$\sum_{k=1}^{\infty} \frac{0!}{(k+1)!} = \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right) - 2 = e - 2$$

$$\sum_{k=1}^{\infty} \frac{1!}{(k+2)!} = \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots = \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right) - 2 - \frac{1}{2} = e - \frac{5}{2}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sum_{k=1}^{\infty} \frac{2!}{(k+3)!} = 2! \sum_{k=1}^{\infty} \frac{1!}{(k+3)!} = 2 \left( \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \dots \right) = 2e - \frac{10}{3}$$

$$\sum_{k=1}^{\infty} \frac{0!}{(k+1)!} + \frac{1!}{(k+2)!} + \frac{2!}{(k+3)!} + \dots = 1,04 > 1 \rightarrow \Omega(k, n) > 1$$

**We need to prove that:**

$$\frac{a+b+c}{2} > \frac{3}{2\Omega(n, k)}$$

$$\therefore \frac{a+b+c}{3} \geq \sqrt[3]{abc} \text{ and } abc = 1 \rightarrow a+b+$$

**1447. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \sum_{i=1}^k i \left( i + \frac{1}{2} \right) (i+1) \right]^{-1}$$

*Proposed by Vasile Mircea Popa-Romania*

**Solution 1 by Kaushik Mahanta-Assam-India**

$$\begin{aligned} \sum_{i=1}^k i \left( i + \frac{1}{2} \right) (i+1) &= \sum_{i=1}^k \left( i^3 + \frac{i^2}{2} + i + \frac{i}{2} \right) = \sum_{i=1}^k \left( i^3 + \frac{3}{2}i^2 + \frac{i}{2} \right) = \\ &= \left( \frac{k(k+1)}{2} \right)^2 + \frac{3}{2} \cdot \frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{4} = \frac{k(k+1)^2(k+2)}{4} \end{aligned}$$

$$\Omega = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \sum_{i=1}^k i \left( i + \frac{1}{2} \right) (i+1) \right]^{-1} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{4}{k(k+1)^2(k+2)}$$

$$= \sum_{k=1}^{\infty} \frac{4}{k(k+1)^2(k+2)}$$

$$\begin{aligned} &= 4 \sum_{k=1}^{\infty} \left[ -\frac{1}{2(k+2)} - \frac{1}{(k+1)^2} + \frac{2}{2k} \right] = 2 \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k+2} \right) - 4 \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} \\ &= 2 \cdot \frac{3}{2} - 4 \left( \frac{\pi^2}{6} - 1 \right) = 7 - \frac{2\pi^2}{3} \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Solution 2 by Asmat Qatea-Afghanistan**

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \sum_{i=1}^k i \left( i + \frac{1}{2} \right) (i+1) \right]^{-1} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \sum_{i=1}^k \left( i^3 + \frac{3}{2} i^2 + \frac{i}{2} \right) \right]^{-1} = \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \sum_{i=1}^k i^3 + \frac{3}{2} \sum_{i=1}^k i^2 + \frac{1}{2} \sum_{i=1}^k i \right]^{-1} = \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \frac{k^2(k+1)^2}{4} + \frac{k(k+1)(2k+1)}{4} + \frac{k(k+1)}{4} \right]^{-1} = \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \frac{k(k+1)(k^2+3k+2)}{4} \right]^{-1} = 4 \cdot \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1)(k^2+3k+2)} = \\
 &= 4 \cdot \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1)^2(k+2)} = 4 \cdot \lim_{n \rightarrow \infty} \sum_{k=1}^n \left( \frac{\frac{1}{2}}{k} - \frac{\frac{1}{2}}{k+2} - \frac{1}{(k+1)^2} \right) = \\
 &= \lim_{n \rightarrow \infty} \left( 2 \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+2} \right) - 4 \sum_{k=1}^n \frac{1}{(k+1)^2} \right) = 3 - 4(\zeta(2) - 1) = 7 - 4\zeta(2)
 \end{aligned}$$

**Therefore,**

$$\Omega = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \sum_{i=1}^k i \left( i + \frac{1}{2} \right) (i+1) \right]^{-1} = 7 - \frac{2\pi^2}{3}$$

**Solution 3 by Mohammad Hamed Nasery-Afghanistan**

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \sum_{i=1}^k i \left( i + \frac{1}{2} \right) (i+1) \right]^{-1} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \sum_{i=1}^k \left( i^3 + \frac{3}{2} i^2 + \frac{i}{2} \right) \right]^{-1} = \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \sum_{i=1}^k i^3 + \frac{3}{2} \sum_{i=1}^k i^2 + \frac{1}{2} \sum_{i=1}^k i \right]^{-1} = \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \frac{k^2(k+1)^2}{4} + \frac{3}{2} \frac{k(k+1)(2k+1)}{6} + \frac{1}{2} \frac{k(k+1)}{2} \right]^{-1} =
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= 4 \cdot \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1)^2(k+2)} = 4 \cdot \lim_{n \rightarrow \infty} \sum_{k=1}^n \left( \frac{\frac{1}{2}}{k} - \frac{\frac{1}{2}}{k+2} - \frac{1}{(k+1)^2} \right) = \\
 &= \lim_{n \rightarrow \infty} \left( 2 \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+2} \right) - 4 \sum_{k=1}^n \frac{1}{(k+1)^2} \right) = 3 - 4(\zeta(2) - 1) = 7 - 4\zeta(2)
 \end{aligned}$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[ \sum_{i=1}^k i \left( i + \frac{1}{2} \right) (i+1) \right]^{-1} = 7 - \frac{2\pi^2}{3}$$

**1448. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{2} \cdot \sqrt[4]{4} \cdot \sqrt[8]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right)$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Asmat Qatea-Afghanistan**

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \left( \frac{\sqrt{2} \cdot \sqrt[4]{4} \cdot \sqrt[8]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right) = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}} \cdot 2^{\frac{2}{2^2}} \cdot 2^{\frac{3}{2^3}} \cdot \dots \cdot 2^{\frac{n}{2^n}}}{n} = \\
 &= \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n}}}{n} = \lim_{n \rightarrow \infty} \frac{2^2}{n} = 0 \\
 &\because x + x^2 + x^3 + \dots = \frac{x}{1-x} \\
 &1 + 2x + 3x^2 + \dots = \frac{1}{(1-x)^2} \\
 &x + 2x^2 + 3x^3 + \dots = \frac{x}{(1-x)^2} \\
 &\text{For } x = \frac{1}{2} \rightarrow \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2
 \end{aligned}$$

**Solution 2 by Surjeet Singhania-India**

$$\lim_{n \rightarrow \infty} \frac{1}{n} \prod_{k=1}^n (2^k)^{\frac{1}{2^k}} = \lim_{n \rightarrow \infty} \frac{1}{n} \prod_{k=1}^n 2^{\frac{k}{2^k}}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\text{Let } S_n = \sum_{k=1}^n \frac{k}{2^k} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n}; \quad (1)$$

$$\frac{1}{2}S_n = \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{n}{2^{n+1}}; \quad (2)$$

*From (1), (2) we get:*

$$S_n - \frac{1}{2}S_n = \frac{1}{2} + \frac{1}{2^2}(2-1) + \frac{1}{2^3}(3-2) + \cdots + \frac{1}{2^n}(n-n+1) + \frac{n}{2^{n+1}}$$

$$\frac{1}{2}S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \frac{n}{2^{n+1}} = \frac{1 - \left(\frac{1}{2}\right)^n}{\frac{1}{2}} + \frac{n}{2^n} = 2(1 - 2^{-n}) + \frac{n}{2^n}$$

$$\Omega = \lim_{n \rightarrow \infty} \frac{1}{n} \prod_{k=1}^n (2^k)^{\frac{1}{2^k}} = \lim_{n \rightarrow \infty} \frac{1}{n} \prod_{k=1}^n 2^{\frac{k}{2^k}} = \lim_{n \rightarrow \infty} \frac{1}{n} 2^{\sum_{k=1}^n \frac{k}{2^k}} = \lim_{n \rightarrow \infty} \underbrace{\frac{1}{n} 2^{2(1-2^{-n})+\frac{n}{2^n}}}_{x_n}$$

Now,  $2(1 - 2^{-n}) + \frac{n}{2^n} < 2 + 1 = 3$ , since  $\frac{1}{2^n} < 1$  and  $\frac{n}{2^n} < 1$ ,  $\forall n \in \mathbb{N}$

$$\text{So, } x_n = \frac{1}{n} 2^{2(1-2^{-n})+\frac{n}{2^n}} < \frac{1}{n} 2^3 = \frac{8}{n}, \text{ hence } 0 < x_n < \frac{8}{n}, \forall n \in \mathbb{N}$$

*Then:  $\lim_{n \rightarrow \infty} x_n = 0$*

*Therefore,*

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{2} \cdot \sqrt[4]{4} \cdot \sqrt[8]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right) = 0$$

**Solution 3 by Syed Shahabudeen-India**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{2} \cdot \sqrt[4]{4} \cdot \sqrt[8]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \prod_{k=1}^n (2^k)^{\frac{1}{2^k}} = \lim_{n \rightarrow \infty} \frac{1}{n} \prod_{k=1}^n 2^{\frac{k}{2^k}} = \lim_{n \rightarrow \infty} \frac{1}{n} 2^{\sum_{k=1}^n \frac{k}{2^k}}$$

$$\sum_{k=1}^{\infty} k^n x^k = \frac{x \mathcal{A}_n(x)}{(1-x)^{n+1}}, \text{ where } \mathcal{A}_n(x) \text{ is a eulerian polynomial.}$$

$$\text{For } n = 1 \text{ and } x = \frac{1}{2}, \sum_{k=1}^{\infty} \frac{k}{2^k} = 2.$$

*Therefore,*



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt[4]{2} \cdot \sqrt[8]{4} \cdot \sqrt[16]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right) = 0$$

**Solution 4 by Sire Ambrose-Albania**

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left( \frac{\sqrt[4]{2} \cdot \sqrt[8]{4} \cdot \sqrt[16]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right) = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdot \dots \cdot (2^n)^{\frac{1}{2^n}}}{n} \\ \frac{1 \cdot 1 \cdot 1 \cdot \dots \cdot 1}{n} < \Omega &< \frac{2^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdot \dots \cdot (2^n)^{\frac{1}{2^n}}}{n} \\ \frac{1^n}{n} < \Omega &< \frac{(2^n)^{\frac{n}{2^n}}}{n} \Leftrightarrow 0 < \frac{1}{n} < \Omega < \frac{2^{\frac{n^2}{2^n}}}{n} \rightarrow 0 \end{aligned}$$

*Therefore,*

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt[4]{2} \cdot \sqrt[8]{4} \cdot \sqrt[16]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right) = 0$$

**Solution 5 by Amrit Awasthi-India**

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left( \frac{\sqrt[4]{2} \cdot \sqrt[8]{4} \cdot \sqrt[16]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right) \\ \log \Omega &= \lim_{n \rightarrow \infty} \log 2 \sum_{k=1}^n k \cdot 2^{-k} - \log n; (*) \\ \therefore \sum_{k=1}^n k^m z^k &= \left( z \frac{d}{dz} \right)^m \frac{1 - z^{n+1}}{1 - z}, \text{For } m = 1 \text{ and } z = \frac{1}{2}, \text{we have:} \\ \sum_{k=1}^n k(2^{-1})^k &= \frac{1}{2} \frac{\left( 1 - (n+1) \left( \frac{1}{2} \right)^n + n \left( \frac{1}{2} \right)^{n+1} \right)}{\left( 1 - \frac{1}{2} \right)^2} = 2 \cdot \left( 1 + \underbrace{\frac{-n-2}{2^{n+1}}}_{\rightarrow 0} \right) = 2; (**) \end{aligned}$$

*Replacing (\*\*) in (\*), we get:  $\log(\Omega) = 2 \log 2 - \log n$*

$$\log(\Omega) = \lim_{n \rightarrow \infty} \log \left( \frac{4}{n} \right), \text{then: } \Omega = \lim_{n \rightarrow \infty} \frac{4}{n} = 0$$

**Solution 6 by Mohammad Rostami-Afghanistan**



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{2} \cdot \sqrt[4]{4} \cdot \sqrt[8]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right) = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}} \cdot 2^{\frac{2}{2^2}} \cdot 2^{\frac{3}{2^3}} \cdot \dots \cdot 2^{\frac{n}{2^n}}}{n} = \lim_{n \rightarrow \infty} \frac{2^{\sum_{k=1}^n \frac{k}{2^k}}}{n}$$

$$\text{We know: } \sum_{k=1}^n kz^k = z \left[ \frac{1 - (n+1)z^n + nz^{n+1}}{(1-z)^2} \right]$$

$$\Omega = \lim_{n \rightarrow \infty} \frac{\frac{1}{2} \cdot \frac{1 - \frac{n+1}{2^n} + \frac{n}{2^{n+1}}}{\left(\frac{1}{2}\right)^2}}{n} = \lim_{n \rightarrow \infty} \frac{2^{2\left(1 - \frac{2+n}{2^{n+1}}\right)}}{n} = 0$$

**Solution 7 by Fayssal Abdelli-Bejaia-Algerie**

$$\sqrt{2} \cdot \sqrt[4]{4} \cdot \sqrt[8]{8} \cdot \dots \cdot \sqrt[2^n]{2^n} = 2^{\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n}}$$

$$S_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} \rightarrow 2, \text{ because}$$

$$\begin{aligned} S &= \frac{1}{2} + \left( \frac{1}{2^2} + \frac{1}{2^2} \right) + \left( \frac{1}{2^3} + \frac{2}{2^3} \right) + \left( \frac{1}{2^4} + \frac{3}{2^4} \right) + \dots = \\ &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \frac{1}{2} \left( \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} \right) \end{aligned}$$

$$S_n = \frac{1}{2} S_n + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}$$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 1 \rightarrow \frac{1}{2} S_n = 1 \text{ then: } S_n \rightarrow 2$$

*Therefore,*

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{2} \cdot \sqrt[4]{4} \cdot \sqrt[8]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right) = \lim_{n \rightarrow \infty} \frac{2^S}{n} = 0$$

**Solution 8 by Samar Das-India**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{2} \cdot \sqrt[4]{4} \cdot \sqrt[8]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right) = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{2}} \cdot 2^{\frac{2}{2^2}} \cdot 2^{\frac{3}{2^3}} \cdot \dots \cdot 2^{\frac{n}{2^n}}}{n} = \lim_{n \rightarrow \infty} \frac{2^{\sum_{k=1}^n \frac{k}{2^k}}}{n}$$

$$\text{Let } R_n = \sum_{k=1}^n \frac{k}{2^k}; (1) \rightarrow \frac{1}{2} R_n = \sum_{k=1}^n \frac{k}{2^{k+1}}; (2)$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\text{Doing (1) - (2): } \frac{1}{2}R_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} - \frac{n}{2^{n+1}}$$

$$R_n = 2\left(1 - \frac{1}{2^n}\right) - \frac{2n}{2^{n+1}}$$

$$\text{Therefore, } \Omega = \lim_{n \rightarrow \infty} \frac{2^{2-\frac{2}{2^n}-\frac{2n}{2^{n+1}}}}{n} = \lim_{n \rightarrow \infty} \frac{4}{n} \cdot \frac{1}{2^{\frac{2}{2^n}}} \cdot \frac{1}{2^{\frac{2n}{2^{n+1}}}} = 0$$

**Solution 9 by Kaushik Mahanta-Assam-India**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt[4]{2} \cdot \sqrt[8]{4} \cdot \sqrt[16]{8} \cdot \dots \cdot \sqrt[2^n]{2^n}}{n} \right) = \lim_{n \rightarrow \infty} \frac{\frac{1}{2^2} \cdot 2^{\frac{2}{2^2}} \cdot 2^{\frac{3}{2^3}} \cdot \dots \cdot 2^{\frac{n}{2^n}}}{n} = \lim_{n \rightarrow \infty} \frac{2^{\sum_{k=1}^n \frac{k}{2^k}}}{n}$$

$$S = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \dots + \frac{n}{2^n}$$

$$\frac{S}{2} = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \dots + \frac{n}{2^{n+1}}$$

$$\frac{S}{2} = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \frac{n}{2^{n+1}}$$

$$S + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}\right) + \frac{n}{2^n} = \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} + \frac{n}{2^n} = \frac{n}{2^n} + 2(1 - 2^{-n})$$

$$\Omega = \lim_{n \rightarrow \infty} \frac{2^{\frac{n}{2^n} + 2\left(1 - \frac{1}{2^n}\right)}}{n} = 0$$

**1449. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n+1]{(n+1)! (2n+1)!!}}{n+1} - \frac{\sqrt[n]{n! (2n-1)!!}}{n} \right)$$

*Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania*

**Solution 1 by Kaushik Mahanta-Assam-India**

If  $k$  is odd:  $k!! = \frac{(k+1)!}{2^{\frac{k+1}{2}} \binom{k+1}{2}!}$ . Put:  $k = 2n - 1, k = 2n + 1$  then:

$$(2n-1)!! = \frac{(2n)!}{2^n \cdot n!} \text{ and } (2n+1)! = \frac{(2n+2)!}{2^{n+1} \cdot (n+1)!}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}\Omega &= \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n+1]{(n+1)! (2n+1)!!}}{n+1} - \frac{\sqrt[n]{n! (2n-1)!!}}{n} \right) = \\ &= \lim_{n \rightarrow \infty} \left( \frac{\left((n+1)!\right)^{\frac{1}{n+1}} \cdot ((2n+2)!)^{\frac{1}{n+1}}}{2((n+1)!)^{\frac{1}{n+1}} \cdot (n+1)} - \frac{(n!)^{\frac{1}{n}} \cdot ((2n)!)^{\frac{1}{n}}}{2 \cdot (n!)^{\frac{1}{n}} \cdot n} \right)\end{aligned}$$

By Stirling's :

$$\begin{aligned}(2n+2)! &= \sqrt{2\pi(2n+2)} \left(\frac{2n+2}{e}\right)^{2n+2} \\ (2n)! &= \sqrt{2\pi(2n)} \left(\frac{2n}{e}\right)^{2n} \\ \Omega &= \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{(2\pi(2n+2))^{\frac{1}{2n+2}} \cdot \left(\frac{2n+2}{e}\right)^2}{(n+1)} - \frac{(2\pi(2n))^{\frac{1}{2n}} \cdot \left(\frac{2n}{e}\right)^2}{n} \right) = \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{4(n+1)^2}{e^2(n+1)} = \frac{4}{e^2} \cdot \frac{n^2}{n} \right) = \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{4}{e^2} (n+1-n) = \frac{2}{e^2}\end{aligned}$$

**Solution 2 by Marian Ursărescu-Romania**

$$\begin{aligned}\Omega &= \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n+1]{(n+1)! (2n+1)!!}}{n+1} - \frac{\sqrt[n]{n! (2n-1)!!}}{n} \right) = \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n! (2n-1)!!}}{n} \left( \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{(n+1)! (2n+1)!!}}{\sqrt[n]{n! (2n-1)!!}} - 1 \right) = \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n! (2n-1)!!}}{n^2} \cdot n \left( \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{(n+1)! (2n+1)!!}}{\sqrt[n]{n! (2n-1)!!}} - 1 \right); (1) \\ \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n! (2n-1)!!}}{n^2} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n! (2n-1)!!}{n^{2n}}} \stackrel{C-D}{=} \lim_{n \rightarrow \infty} \frac{(n+1)! (2n+1)!!}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{n! (2n-1)!!} = \\ &= \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^{2n} \cdot \frac{2n+1}{n+1} = \frac{2}{e^2}; (2) \\ \lim_{n \rightarrow \infty} n \left( \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{(n+1)! (2n+1)!!}}{\sqrt[n]{n! (2n-1)!!}} - 1 \right) &= \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} n \left( \frac{e^{\log\left(\frac{n}{n+1} \cdot \frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{\sqrt[n]{n!(2n-1)!!}}\right)} - 1}{\log\left(\frac{n}{n+1} \cdot \frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{\sqrt[n]{n!(2n-1)!!}}\right)} \right) \cdot \log\left(\frac{n}{n+1} \cdot \frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{\sqrt[n]{n!(2n-1)!!}}\right) \\
 &= \lim_{n \rightarrow \infty} n \cdot \log\left(\frac{n}{n+1} \cdot \frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{\sqrt[n]{n!(2n-1)!!}}\right) = \\
 &= \lim_{n \rightarrow \infty} \log\left[\left(\frac{n}{n+1}\right)^n \left(\frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{\sqrt[n]{n!(2n-1)!!}}\right)^n\right] = \\
 &= \log\left[\lim_{n \rightarrow \infty} \left(\left(\frac{n}{n+1}\right)^n \cdot \frac{(n+1)!(2n+1)!!}{n!(2n-1)!! \sqrt[n+1]{(n+1)!(2n+1)!!}}\right)\right] = \\
 &= \log\left[\lim_{n \rightarrow \infty} \left(\left(\frac{n}{n+1}\right)^n \cdot \frac{(n+1)(2n+1)}{\sqrt[n+1]{(n+1)!(2n+1)!!}}\right)\right] = \\
 &= \log\left[\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{(n+1)(2n+1)}{(n+1)^2} \cdot \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!(2n+1)!!}}\right] \stackrel{(2)}{\cong} \\
 &= \log\left(\frac{1}{e} \cdot 2 \cdot \frac{e^2}{2}\right) = \log e = 1; (3)
 \end{aligned}$$

**From (1), (2), (3) it follows that:**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{n+1} - \frac{\sqrt[n]{n!(2n-1)!!}}{n} \right) = \frac{2}{e^2}$$

**1450. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n+1]{(n+1)!(2n+1)!!}}{\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \dots + (-1)^{n-1}\frac{1}{n}\binom{n}{n}} \right)$$

*Proposed by Daniel Sitaru-Romania*



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Solution 1 Asmat Qatea-Afghanistan**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{5 \sqrt{8 \sqrt{11 \sqrt{\dots \sqrt{3n-1}}}}}}{\binom{n}{1} - \frac{1}{2} \binom{n}{2} + \dots + (-1)^{n-1} \frac{1}{n} \binom{n}{n}} \right) = \lim_{n \rightarrow \infty} \frac{S}{P}$$

$$R = \sqrt{5 \sqrt{5^2 \sqrt{5^3 \sqrt{\dots \sqrt{5^n}}}}} = 5^{\frac{1}{2}} \cdot 5^{\frac{1}{4}} \cdot 5^{\frac{3}{8}} \cdot 5^{\frac{4}{16}} \cdot \dots = \underbrace{5^{\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \dots}}_{(*)} = 5^2 = 25; (S < R)$$

$$\begin{aligned} P &= \sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k-1}}{k} = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} \int_0^1 x^{k-1} dx = \int_0^1 \sum_{k=1}^n \binom{n}{k} (-x)^{k-1} dx = \\ &= \int_0^1 \frac{1}{x} \sum_{k=0}^n \binom{n}{k} (-x)^k dx = \int_0^1 \frac{1}{x} \left[ 1 - \sum_{k=0}^n \binom{n}{k} (-x)^k \right] dx = \int_0^1 \frac{1 - (1-x)^n}{x} dx = \\ &= \int_0^1 \frac{1 - x^n}{1-x} dx = H_n \end{aligned}$$

$$0 = \lim_{n \rightarrow \infty} \frac{\sqrt{5}}{H_n} < \Omega < \lim_{n \rightarrow \infty} \frac{25}{H_n} = 0$$

**Proof for (\*):**

$$x + x^2 + x^3 + \dots = \frac{x}{1-x}, x \in (0, 1) \rightarrow 1 + 2x + 3x^2 + \dots = \frac{1}{(1-x)^2}$$

$$\text{For } x = \frac{1}{2} \rightarrow \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$

**Solution 2 by Adrian Popa-Romania**

$$\begin{aligned} E &= \sqrt{5 \sqrt{8 \sqrt{11 \sqrt{\dots \sqrt{3n-1}}}}} = 5^{\frac{1}{2}} \cdot 8^{\frac{1}{2^2}} \cdot 11^{\frac{1}{2^3}} \cdot \dots \cdot (3n-1)^{\frac{1}{2^n}} = \\ &= e^{\log(3 \cdot 2 - 1)^{\frac{1}{2}} \cdot (3 \cdot 3 - 1)^{\frac{1}{2^3}} \cdot \dots \cdot (3n-1)^{\frac{1}{2^n}}} = e^{\sum_{k=2}^n \frac{1}{2^k} \log(3k-1)} \stackrel{\log(1+x) \leq x}{\leq} e^{\sum_{k=2}^n \frac{3k-2}{2^k}} \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\sum_{k=2}^n \frac{3k-2}{2^k} = 3 \sum_{k=2}^n \frac{k}{2^k} - \sum_{k=2}^n \frac{1}{2^{k-1}}$$

$$\text{Let } x \in (0, 1) \rightarrow 1 + x + x^2 + \dots + x^n = \frac{x^{n+1}-1}{x-1} = \frac{1}{1-x}$$

$$1 + 2x + 3x^2 + \dots + nx^{n-1} = \frac{1}{(1-x)^2} | \cdot x$$

$$\rightarrow x + 2x^2 + 3x^3 + \dots + nx^n = \frac{x}{(1-x)^2}$$

$$x = \frac{1}{2} \rightarrow \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = 2 \rightarrow \sum_{k=2}^{\infty} \frac{k}{2^k} = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\sum_{k=2}^{\infty} \frac{1}{2^{k-1}} = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} = \frac{1}{2} \cdot \frac{\left(\frac{1}{2}\right)^{n-1} - 1}{\frac{1}{2} - 1} = 1$$

$$\rightarrow \sum_{k=2}^n \frac{3k-2}{2^k} = \frac{9}{2} - 1 = \frac{7}{2} \rightarrow E < e^{\frac{7}{2}} = \sqrt{e^7}$$

$$S = \binom{n}{1} - \frac{1}{2} \binom{n}{2} + \dots + (-1)^{n-1} \frac{1}{n} \binom{n}{n}$$

$$(1-x)^n = \binom{n}{0} + x \binom{n}{1} + x^2 \binom{n}{2} + \dots + (-1)^n x^n \binom{n}{n} = 1 - x \binom{n}{1} + x^2 \binom{n}{2} + \dots + (-1)^n x^n \binom{n}{n}$$

$$\int_0^1 \frac{1 - (1-x)^n}{x} dx = \int_0^1 \frac{x \binom{n}{1} - x^2 \binom{n}{2} + \dots + (-1)^n x^n \binom{n}{n}}{x} dx =$$

$$= \int_0^1 \left( \binom{n}{1} - x \binom{n}{2} + x^2 \binom{n}{3} - \dots + (-1)^n x^{n-1} \binom{n}{n} \right) dx =$$

$$= \binom{n}{1} x \Big|_0^1 - \frac{\binom{n}{2} x^2}{2} \Big|_0^1 + \frac{\binom{n}{3} x^3}{3} \Big|_0^1 + \dots + \frac{(-1)^n \binom{n}{n} x^n}{n} \Big|_0^1 = \binom{n}{1} - \frac{1}{2} \binom{n}{2} + \frac{1}{3} \binom{n}{3} + \dots + \frac{(-1)^n}{n} \binom{n}{n}$$

On the other hand,

$$\int_0^1 \frac{1 - (1-x)^n}{x} dx = \int_0^1 \frac{[1 - (1-x)][1 + (1-x) + (1-x)^2 + \dots + (1-x)^{n-1}]}{x} dx =$$

$$= \int_0^1 (1 - (1-x) + (1-x)^2 + \dots + (1-x)^{n-1}) dx = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

$S = H_n \rightarrow S = \log n + \gamma$ . Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{5 \sqrt{8 \sqrt{11 \sqrt{\dots \sqrt{3n-1}}}}}}{\binom{n}{1} - \frac{1}{2} \binom{n}{2} + \dots + (-1)^{n-1} \frac{1}{n} \binom{n}{n}} \right) < \lim_{n \rightarrow \infty} \frac{\sqrt{e^7}}{\log n + \gamma} = 0$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**1451. If  $(a_n)_{n \geq 1}$ ,  $a_n \in \mathbb{R}$ ,  $\forall n \in \mathbb{N}^*$  such that  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n^2 \cdot a_n} = a \in \mathbb{R}_+$ , then find:**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n+1]{a_{n+1}}}{n+1} - \frac{\sqrt[n]{a_n}}{n} \right)$$

*Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania*

**Solution 1 by Marian Ursărescu-Romania**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n+1]{a_{n+1}}}{n+1} - \frac{\sqrt[n]{a_n}}{n} \right) = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n^2} \cdot n \left( \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} - 1 \right) =$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n^2} \cdot n \left( e^{\log \left( \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} \right)} - 1 \right); (1)$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n^2} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{a_n}{n^{2n}}} \stackrel{C-D}{=} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{a_n} =$$

$$= \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^{2n} \cdot \frac{n^2}{(n+1)^2} \cdot \frac{a_{n+1}}{n^2 \cdot a_n} = \frac{a}{e^2}; (2)$$

$$\lim_{n \rightarrow \infty} \frac{e^{\log \left( \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} \right)} - 1}{\log \left( \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} \right)} \cdot n \log \left( \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} \right) =$$

$$= \lim_{n \rightarrow \infty} \log \left( \frac{n}{n+1} \right)^n \cdot \frac{\sqrt[n+1]{a_{n+1}^n}}{a_n} =$$

$$= \log \left( \lim_{n \rightarrow \infty} \left( \left( \frac{n}{n+1} \right)^n \cdot \frac{a_{n+1}}{n^2 \cdot a_n} \cdot \frac{n^2}{(n+1)^2} \cdot \frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}^n}} \right) \right) = \log \left( \frac{1}{e} \cdot a \cdot 1 \cdot \frac{e^2}{a} \right) =$$

$$= \log e = 1; (3)$$

From (1), (2), (3) it follows that:

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n+1]{a_{n+1}}}{n+1} - \frac{\sqrt[n]{a_n}}{n} \right) = \frac{a}{e^2}$$

**Solution 2 by Remus Florin Stanca-Romania**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n+1]{a_{n+1}}}{n+1} - \frac{\sqrt[n]{a_n}}{n} \right) = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n} \cdot \left( \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} - 1 \right) =$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$= \lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n^2} \cdot n \left( \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} - 1 \right); \quad (1)$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{a_n}}{n^2} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{a_n} = \frac{1}{e^2} \cdot \lim_{n \rightarrow \infty} \frac{a_{n+1}}{n^2 a_n} = \frac{a}{e^2} \stackrel{(1)}{\Rightarrow}$$

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n+1]{a_{n+1}}}{n+1} - \frac{\sqrt[n]{a_n}}{n} \right) = \frac{a}{e^2} \cdot \lim_{n \rightarrow \infty} n \left( \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} - 1 \right) = \\ &= \frac{a}{e^2} \cdot \lim_{n \rightarrow \infty} n \left( \frac{\frac{\sqrt[n+1]{a_{n+1}}}{n+1} - 1}{\frac{\sqrt[n]{a_n}}{n}} \right) = \frac{a}{e^2} \cdot \lim_{n \rightarrow \infty} \log \left( \left( \frac{n}{n+1} \right)^n \cdot \frac{a_{n+1}^{\frac{n}{n+1}}}{a_n} \right) = \\ &= \frac{a}{e^2} \cdot \lim_{n \rightarrow \infty} \log \left( \frac{1}{e} \cdot \frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} \right) = \frac{a}{e^2} \cdot \lim_{n \rightarrow \infty} \left( \frac{1}{e} \cdot \frac{a_{n+1}}{n^2 a_n} \cdot \frac{n^2}{\sqrt[n]{a_n}} \right) = \\ &= \frac{a}{e^2} \cdot \log \left( \frac{a}{e} \cdot \lim_{n \rightarrow \infty} \frac{(n+1)^{2n+2}}{a_{n+1}} \cdot \frac{a_n}{n^{2n}} \right) = \frac{a}{e^2} \cdot \log \left( \frac{a}{e} \cdot \frac{e^2}{a} \right) = \frac{a}{e^2} \end{aligned}$$

**1452.**

$$x_1 = 4, x_{n+1} = \frac{(1-2n^2)x_n - 4n^2}{(2+x_n)n^2 + 1}, n \geq 1$$

**Find:**

$$\Omega = \lim_{n \rightarrow \infty} \left( (3+x_n)^{\sum_{k=1}^n \frac{k^2}{n}} \right)^{\sum_{k=1}^n \left( 1 - \frac{k}{n+k} \right)}$$

*Proposed by Ruxandra Daniela Tonilă-Romania*

**Solution by Adrian Popa-Romania**

$$x_{n+1} = \frac{(1-2n^2)x_n - 4n^2}{(2+x_n)n^2 + 1} = \frac{x_n - 2n^2(x_n + 2)}{(2+x_n)n^2 + 1} \rightarrow 2 + x_{n+2} = \frac{2 + x_n}{(2+x_n)n^2 + 1}$$

$$\text{Let } y_n = 2 + x_n \rightarrow y_{n+1} = \frac{y_n}{n^2 y_n + 1}$$

$$x_1 = 1 \rightarrow y_1 = 6$$

$$k = 1 \rightarrow y_2 = \frac{y_1}{1^2 y_1 + 1} = \frac{6}{7}$$

$$k = 2 \rightarrow y_3 = \frac{y_2}{2^2 y_2 + 1} = \frac{y_1}{y_1(1^2 + 2^2) + 1}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 k = n - 1 \rightarrow y_n &= \frac{y_1}{y_1(1^2 + 2^2 + \dots + (n-1)^2) + 1} = \frac{6}{6 \cdot \frac{n(n-1)(2n-1)}{6} + 1} = \\
 &= \frac{6}{n(n-1)(2n-1) + 1} \rightarrow x_n = y_n - 2 = \frac{4 - n(n-1)(2n-1)}{1 + n(n-1)(2n-1)} \\
 \Omega &= \lim_{n \rightarrow \infty} \left( (3 + x_n)^{\sum_{k=1}^n \frac{k^2}{n}} \right)^{\sum_{k=1}^n \left( 1 - \frac{k}{n+k} \right)} = \lim_{n \rightarrow \infty} \left[ (1 + 2 + x_n)^{\frac{n(n+1)(2n+1)}{6n}} \right]^{\sum_{k=1}^n \frac{n}{n+k}} = \\
 &= \lim_{n \rightarrow \infty} \left\{ \left[ (1 + y_n)^{\frac{1}{y_n}} \right]^{\frac{6}{n(n-1)(2n-1)+1} \cdot \frac{(n+1)(2n+1)}{6}} \right\}^{\sum_{k=1}^n \frac{1}{1+\frac{k}{n}}} = \\
 &= \exp \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + \frac{k}{n}} \right\} = \exp \left\{ \int_0^1 \frac{1}{x+1} dx \right\} = 2
 \end{aligned}$$

**1453. Find:**

$$\Omega = \lim_{x \rightarrow 3} \frac{x! - 2x}{x - 3}$$

*Proposed by Jaihon Obaidullah-Afghanistan*

**Solution by Ajentunmobi Abdulqooyum-Nigeria**

$$\Omega = \lim_{x \rightarrow 3} \frac{x! - 2x}{x - 3} \stackrel{x! = \Gamma(x+1)}{=} \lim_{x \rightarrow 3} \frac{\frac{d}{dx}(\Gamma(x+1) - 2x)}{\frac{d}{dx}(x-3)} =$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 3} \frac{\Gamma'(x+1) - 2}{1} = \lim_{x \rightarrow 3} \Gamma(x+1)\psi(x+1) - 2 = \\
 &= (\Gamma(4)\psi(4) - 2) = 6\psi(4) - 2
 \end{aligned}$$

$$\psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k}; |n = 3|$$

$$\psi(4) = \psi(3+1) = -\gamma + \sum_{k=1}^3 \frac{1}{k} = -\gamma + 1 + \frac{1}{2} + \frac{1}{3} = -\gamma + \frac{11}{6}$$

$$\Omega = 6 \left( -\gamma + \frac{11}{6} \right) - 2 = -6\gamma + 9$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**1454. Find:**

$$\Omega = \lim_{\substack{x \rightarrow 1 \\ n \rightarrow \infty}} \left( \sum_{k=0}^n \binom{n}{k} \frac{x^k - 1}{x - 1} \right) \left( \sum_{k=0}^{\left[ \frac{n}{2} \right]} \frac{(n-k-1)! \cdot (-1)^k}{(n-2k)! \cdot k! \cdot 4^k} \right)$$

*Proposed by Asmat Qatea-Afghanistan*

**Solution 1 by Ahmed Yackoube Chach-Mauritania**

$$\begin{aligned}
 \Omega &= \lim_{\substack{x \rightarrow 1 \\ n \rightarrow \infty}} \left( \sum_{k=0}^n \binom{n}{k} \frac{x^k - 1}{x - 1} \right) \left( \sum_{k=0}^{\left[ \frac{n}{2} \right]} \frac{(n-k-1)! \cdot (-1)^k}{(n-2k)! \cdot k! \cdot 4^k} \right) = \lim_{x \rightarrow 1} I(x) \cdot J_n \\
 I(x) &= \sum_{k=0}^n \binom{n}{k} \frac{x^k - 1}{x - 1} = \frac{1}{x-1} \left[ \sum_{k=0}^n \binom{n}{k} x^k - \sum_{k=0}^n \binom{n}{k} \right] = \frac{(x+1)^n - 2^n}{x-1} \xrightarrow{x \rightarrow 1} n \cdot 2^{n-1} \\
 J_n &= \sum_{k=0}^{\left[ \frac{n}{2} \right]} \frac{(n-k-1)! \cdot (-1)^k}{(n-2k)! \cdot k! \cdot 4^k} = \sum_{k=0}^{\left[ \frac{n}{2} \right]} \frac{(n-k)! \cdot (-1)^k}{(n-k)! (n-2k)! \cdot k! \cdot 4^k} = \\
 &= \sum_{k=0}^{\left[ \frac{n}{2} \right]} \frac{(-1)^k}{n-k} \binom{n}{k} 2^{-2k} = \sum_{k=0}^{\left[ \frac{n}{2} \right]} \frac{(-1)^k}{n} \left[ \binom{n}{k} + \binom{n-1}{k-1} \right] 2^{-2k} = \frac{1}{n} (A_n + B_n) \\
 A_n &= \sum_{k=0}^{\left[ \frac{n}{2} \right]} (-1)^k \binom{n}{k} 2^{-2k} = 2^{-n} \sum_{k=0}^{\left[ \frac{n}{2} \right]} (-1)^k \binom{n}{k} 2^{n-2k} = 2^{-n} U_n(1) \\
 &= 2^{-n} (n+1) \\
 B_n &= \sum_{k=0}^{\left[ \frac{n}{2} \right]} (-1)^k \binom{n-2}{k-1} 2^{-2k} = \sum_{k=0}^{\left[ \frac{n}{2} \right]} (-1)^k \binom{n-1}{k-1} 2^{-2k} \underset{k \rightarrow k+1}{=} \\
 &= \sum_{k=0}^{\left[ \frac{n-2}{2} \right]} (-1)^{k+1} \binom{n-2}{k} 2^{-2k-2} = -2^{-n} \sum_{k=0}^{\left[ \frac{n-2}{2} \right]} (-1)^k \binom{n-2}{k} 2^{n-2-2k} = \\
 &= -2^{-n} U_{n-2}(1) = -2^{-n} (n-1); \because U_n(X) - Chebyshev's\ polynomial \\
 J_n &= \frac{2^{-n}}{n} (n+1 - n+1) = \frac{2^{1-n}}{n}
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Therefore,**

$$\Omega = \lim_{\substack{x \rightarrow 1 \\ n \rightarrow \infty}} \left( \sum_{k=0}^n \binom{n}{k} \frac{x^k - 1}{x - 1} \right) \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k-1)! \cdot (-1)^k}{(n-2k)! \cdot k! \cdot 4^k} \right) = \lim_{n \rightarrow \infty} n 2^{n-1} \cdot \frac{2^{1-n}}{n} = 1$$

**Solution 2 by Amrit Awasthi-India**

$$\Omega = \lim_{\substack{x \rightarrow 1 \\ n \rightarrow \infty}} \left( \sum_{k=0}^n \binom{n}{k} \frac{x^k - 1}{x - 1} \right) \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k-1)! \cdot (-1)^k}{(n-2k)! \cdot k! \cdot 4^k} \right)$$

$$p = \sum_{k=0}^n \binom{n}{k} \frac{x^k - 1}{x - 1}; q = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k-1)! \cdot (-1)^k}{(n-2k)! \cdot k! \cdot 4^k}$$

$$\text{We know: } \lim_{x \rightarrow a} \frac{x^k - a^k}{x - a} = k a^{k-1}$$

**Putting  $a = 1$  and substituting we have:**

$$p = \sum_{k=0}^n \binom{n}{k} \cdot k$$

**Now, consider the expansion of**

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n$$

**Differentiating both sides respect to  $x$ , we have**

$$n(1+x)^{n-1} = \binom{n}{1} + \binom{n}{2} 2x + \cdots + \binom{n}{n} nx^{n-1}$$

$$x = 1 \Rightarrow n \cdot 2^{n-1} = \sum_{k=1}^n \binom{n}{k} \cdot k = \sum_{k=1}^n \binom{n}{k} \cdot k = p; (*)$$

**Now, consider the following expansion**

$$(1 + \sqrt{1+4x})^n + (1 - \sqrt{1+4x})^n = n \cdot 2^n \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\binom{n-k}{k} x^k}{n-k} \right)$$

**Putting  $x = -\frac{1}{4}$  and rearranging we have:**



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{2^{1-n}}{n} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\binom{n-k}{k} (-1)^k}{4^k (n-k)} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(n-k)! (-1)^k}{4^k (n-k)(n-2k)! k!} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(n-k-1)! (-1)^k}{4^k (n-2k)! k!} = q; (**)$$

Hence, using (\*), (\*\*) it follows that

$$\Omega = p \cdot q = \lim_{n \rightarrow \infty} \left( n \cdot 2^{n-1} \cdot \frac{2^{1-n}}{n} \right) = 1$$

**1455. Find z if:**

$$\frac{-\log(2 - 2 \cos(1))}{2z} + \frac{i(\pi - 1)}{2z} = {}_2F_1(1, 1; 2; z)$$

where  ${}_2F_1(a, b; c; z)$  – is Gaussian hypergeometric function and  $i = \sqrt{-1}$ .

*Proposed by Amrit Awasthi-India*

**Solution 1 by proposer**

We know that:  $\cos k + i \sin k = e^{ik}$ .

Dividing both sides with  $k$  and summing both sides from  $k = 1$  to  $k = \infty$  we have,

$$\sum_{k=1}^{\infty} \frac{\cos k}{k} + i \sum_{k=1}^{\infty} \frac{\sin k}{k} = \sum_{k=1}^{\infty} \frac{e^{ik}}{k}$$

Now, using two infinite famous sums, we have:

$$\sum_{k=1}^{\infty} \frac{\cos k}{k} = -\frac{1}{2} \log(2 - 2 \cos(1)), \quad \sum_{k=1}^{\infty} \frac{\sin k}{k} = \frac{\pi - 1}{2}$$

Thus,

$$\frac{-\log(2 - 2 \cos(1))}{2} + \frac{i(\pi - 1)}{2} = \sum_{k=1}^{\infty} \frac{e^{ik}}{k} = -\log(1 - e^i) = {}_2F_1(1, 1; 2; e^i) \cdot e^i$$

Hence rearranging to get it in the form of real statement, we have  $z = e^i$ .

**Solution 2 by Akerele Olofin-Nigeria**

$${}_2F_1(1, 1; 2; z) = \sum_{n=0}^{\infty} \frac{(1)_n (1)_n z^n}{(2)_n} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{\Gamma(1+n)\Gamma(1+n)}{\Gamma(2+n)} \frac{z^n}{n!} =$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \frac{z^n}{n+1} = -\frac{\log(1-z)}{z} \\
 \rightarrow &\frac{-\log(2-2\cos(1)) + i(\pi-1)}{2z} = -\frac{\log(1-z)}{z} \\
 \rightarrow &-\log(2-2\cos(1)) + i(\pi-1) = -2\log(1-z) \\
 \rightarrow &-\log(2-2\cos(1)) + \log(e^{i(\pi-1)}) = -2\log(1-z) \\
 \rightarrow &\log\left(\frac{e^{i(\pi-1)}}{2-2\cos(1)}\right) = \log(1-z)^2 \\
 \rightarrow &\frac{e^{i(\pi-1)}}{2-2\cos(1)} = (1-z)^2 \rightarrow z^2 - 2z + \left(1 - \frac{e^{i(\pi-1)}}{2-2\cos(1)}\right) \\
 \rightarrow &z = 1 + ie^{\frac{i}{2}}\sqrt{2-2\cos(1)}
 \end{aligned}$$

**Solution 3 by Syed Shahabudeen-India**

$$\begin{aligned}
 {}_2F_1(1, 1; 2; z) &= \frac{-\log(2-2\cos(1))}{2z} + \frac{i(\pi-1)}{2z} \\
 {}_2F_1(1, 1; 2; z) &= \frac{-\log(1-z)}{z} \quad (\text{by Euler hypergeometric Integral}) \\
 \frac{-\log(1-z)}{z} &= \frac{-\log(2-2\cos(1))}{2z} + \frac{i(\pi-1)}{2z} = \\
 &= \frac{1}{2}(-\log(2-2\cos(1)) + \log(e^{i(\pi-1)})) = \log\sqrt{\frac{e^{i(\pi-1)}}{2-2\cos(1)}} \\
 \log(1-z) &= \log\sqrt{\frac{e^{i(\pi-1)}}{2-2\cos(1)}}, \quad z = 1 - \sqrt{\frac{2-2\cos(1)}{e^{i(\pi-1)}}}
 \end{aligned}$$

**1456. Prove that:**

$$\begin{aligned}
 &\sum_{n=0}^{\infty} \frac{{}_2F_1(2n, n; n+1; -1)}{{}_1F_1(1; n+1; 2) - {}_1F_1(1; n+1; -2)} G_{3,5}^{1,2}\left(1 \left| \begin{matrix} n, n+\frac{1}{2}, n+1 \\ n+\frac{1}{2}, n, n+1, \frac{n}{2}, \frac{n+1}{2} \end{matrix} \right. \right) = \\
 &= \frac{1}{2\pi} \left( \sqrt{\frac{e}{2}} \operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{\pi}} \right)
 \end{aligned}$$

Where  ${}_1F_1(a; b; z)$  –confluent hypergeometric function



# ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$G_{p,q}^{m,n} \left( z \Big| \begin{matrix} a_p \\ b_q \end{matrix} \right) - \text{Meijer } G\text{-function}$$

${}_2F_1(a, b; c; z)$  – Gauss hypergeometric function,  $\text{erf}(x)$  – error function.

*Proposed by Izumi Ainsworth-Lima-Peru*

*Solution by Dawid Bialek-Poland*

$$\begin{aligned} G_{3,5}^{1,2} \left( 1 \left| \begin{matrix} n, n + \frac{1}{2}, n + 1 \\ n + \frac{1}{2}, n, n + 1, \frac{n}{2}, \frac{n+1}{2} \end{matrix} \right. \right) &= \\ = \frac{2^{2a-b} \cdot z^{b-1} \cdot \Gamma(2b - 2a - 1)}{\sqrt{\pi^3} \cdot \Gamma(b)} \cdot ( {}_1F_1(2b - 2a - 1; b; 2\sqrt{z}) \\ - {}_1F_1(2b - 2a - 1; b; -2\sqrt{z})) ; (*) \end{aligned}$$

We have:

$$\begin{aligned} G_{3,5}^{1,2} \left( 1 \left| \begin{matrix} n, n + \frac{1}{2}, n + 1 \\ n + \frac{1}{2}, n, n + 1, \frac{n}{2}, \frac{n+1}{2} \end{matrix} \right. \right) &= G_{3,5}^{1,2} \left( 1 \left| \begin{matrix} n, n + \frac{1}{2}, n + 1 \\ n + 1 - \frac{1}{2}, n + 1 - 1, n + 1, \frac{n+1-1}{2}, \frac{n+1}{2} \end{matrix} \right. \right) \\ \stackrel{(*)}{=} \frac{2^{n-1}}{\sqrt{\pi^3} \cdot \Gamma(n+1)} \cdot ( {}_1F_1(1; n+1; 2) - {}_1F_1(1; n+1; -2)) \\ \Omega &= \sum_{n=0}^{\infty} \frac{{}_2F_1(2n, n; n+1; -1)}{{}_1F_1(1; n+1; 2) - {}_1F_1(1; n+1; -2)} \cdot \frac{2^{n-1}}{\sqrt{\pi^3} \cdot \Gamma(n+1)} \\ &\quad \cdot ( {}_1F_1(1; n+1; 2) - {}_1F_1(1; n+1; -2)) = \\ &= \frac{1}{2\sqrt{\pi^3}} \sum_{n=0}^{\infty} \frac{{}_2F_1(2n, n; n+1; -1)}{\Gamma(n+1)} \cdot 2^n \end{aligned}$$

Note:  ${}_2F_1(2n, n; n+1; -1) = {}_2F_1(2n, n; 2n-n+1; -1)$ , so from Kummer's theorem

as  $z = -1$ :

$$\begin{aligned} {}_2F_1(a, b; a-b+1; -1) &= \frac{\Gamma(a-b+1) \cdot \Gamma\left(1 + \frac{a}{2}\right)}{\Gamma(a+1) \cdot \Gamma\left(1 + \frac{a}{2} - b\right)} \Rightarrow \\ {}_2F_1(2n, n; 2n-n+1; -1) &= {}_2F_1(2n, n; n+1; -1) = \frac{\Gamma(n+1) \cdot \Gamma(n+1)}{\Gamma(2n+1)} ; (**) \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 \Omega & \stackrel{(**)}{=} \frac{1}{2\sqrt{\pi^3}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1) \cdot \Gamma(n+1)}{\Gamma(2n+1)} \cdot \frac{2^n}{\Gamma(n+1)} = \frac{1}{2\sqrt{\pi^3}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(2n+1)} \cdot 2^n = \\
 & = \frac{1}{2\sqrt{\pi^3}} \sum_{n=0}^{\infty} \frac{\sqrt{\pi}}{4^n \Gamma\left(n + \frac{1}{2}\right)} \cdot 2^n = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{1}{\Gamma\left(n + \frac{3}{2}\right)} \cdot \left(\frac{1}{2}\right)^{n+1} \\
 \Omega & = \frac{1}{2\pi} \left( \sum_{n=0}^{\infty} \frac{1}{\Gamma\left(n + \frac{3}{2}\right)} \cdot \left(\frac{1}{2}\right)^{n+1} + \frac{1}{\Gamma\left(\frac{1}{2}\right)} \right) = \\
 & = \frac{1}{4\pi \cdot \Gamma\left(\frac{3}{2}\right)} \cdot \underbrace{\sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(1)} \cdot \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(n + \frac{3}{2}\right)} \cdot \frac{\left(\frac{1}{2}\right)^n}{\Gamma(n+1)}}_{_1F_1\left(1; \frac{3}{2}; \frac{1}{2}\right)} + \frac{1}{2\pi \cdot \Gamma\left(\frac{1}{2}\right)} = \\
 & = \frac{1}{2\pi} \cdot \left( \frac{1}{\sqrt{\pi}} \cdot {}_1F_1\left(1; \frac{3}{2}; \frac{1}{2}\right) + \frac{1}{\sqrt{\pi}} \right)
 \end{aligned}$$

**Recall error function definition:**

$$\begin{aligned}
 \operatorname{erf}(x) & \stackrel{\text{def}}{=} \frac{2xe^{-x^2}}{\sqrt{\pi}} \cdot {}_1F_1\left(1; \frac{3}{2}; x^2\right) \Rightarrow {}_1F_1\left(1; \frac{3}{2}; \frac{1}{2}\right) = {}_1F_1\left(1; \frac{3}{2}; \left(\frac{1}{\sqrt{2}}\right)^2\right) \\
 & = \sqrt{\pi} \cdot \sqrt{\frac{e}{2}} \cdot \operatorname{erf}\left(\frac{1}{\sqrt{2}}\right); (***) 
 \end{aligned}$$

$$\begin{aligned}
 \Omega & = \frac{1}{2\pi} \cdot \left( \frac{1}{\sqrt{\pi}} \cdot {}_1F_1\left(1; \frac{3}{2}; \frac{1}{2}\right) + \frac{1}{\sqrt{\pi}} \right) \stackrel{(***)}{=} \frac{1}{2\pi} \cdot \left( \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} \cdot \sqrt{\frac{e}{2}} \cdot \operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{\pi}} \right) = \\
 & = \frac{1}{2\pi} \cdot \left( \sqrt{\frac{e}{2}} \cdot \operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{\pi}} \right)
 \end{aligned}$$

**1457. Prove that:**

$$\int_0^\infty \left( \frac{\sin(\pi x^2) - x^3 \cos(\pi x^2)}{x^6 + 1} - \frac{\cos(\pi x^2)}{x^3 + 1} \right) dx = \frac{\pi e^{-\frac{\pi\sqrt{3}}{2}}}{3}$$

*Proposed by Angad Singh-India*

**Solution by proposer**

Consider the complex line integral,



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$J = \oint_C \frac{e^{inz^2}}{z^3 + a^3} dz$$

where  $a > 0$  and  $n \geq 0$  and  $C$  is a quarter circle of radius  $R$  in the first quadrant centered at the origin traversed anti-clockwise. Hence,

$$J = \int_0^R \frac{e^{inx^2}}{x^3 + a^3} dx + \int_0^{\frac{\pi}{2}} \frac{e^{in(Re^{i\theta})^2}}{(Re^{i\theta})^3 + a^3} Re^{i\theta} d\theta + \int_R^0 \frac{e^{in(iy)^2}}{(iy)^3 + a^3} idy$$

$$\text{Let } I_1 = \int_0^R \frac{e^{inx^2}}{x^3 + a^3} dx$$

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{e^{in(Re^{i\theta})^2}}{(Re^{i\theta})^3 + a^3} Re^{i\theta} d\theta$$

$$I_3 = \int_R^0 \frac{e^{in(iy)^2}}{(iy)^3 + a^3} idy$$

Now, using the residue theorem,

$$J = 2\pi i \frac{\omega e^{ina^2\omega^2}}{3a^2}, \text{ where } \omega = -\frac{1}{2} - \frac{i\sqrt{3}}{3}. \text{ Thus,}$$

$$Re(j) = \frac{\pi e^{-\frac{na^2\sqrt{3}}{2}}}{3a^2} \left( \sqrt{3} \cos\left(\frac{na^2}{2}\right) - \sin\left(\frac{na^2}{2}\right) \right)$$

Similarly,

$$Re(I_1) = \int_0^R \frac{\cos(nx^2)}{x^3 + a^3} dx$$

Now, observe that

$$\begin{aligned} |I_2| &= \left| \int_0^{\frac{\pi}{2}} \frac{e^{in(Re^{i\theta})^2}}{(Re^{i\theta})^3 + a^3} Re^{i\theta} d\theta \right| \leq + \int_0^{\frac{\pi}{2}} \left| \frac{e^{in(Re^{i\theta})^2}}{(Re^{i\theta})^3 + a^3} \right| |Re^{i\theta}| d\theta \leq \\ &\leq \frac{R}{R^3 - a^3} \int_0^{\frac{\pi}{2}} e^{-nR^2 \sin 2\theta} d\theta \end{aligned}$$

Thus,

$$|I_2| \leq \frac{\pi R}{2(R^3 - a^3)}, \text{ since } \lim_{R \rightarrow \infty} \frac{\pi R}{2(R^3 - a^3)} = 0, \quad \text{thus } I_2 \text{ vanishes.}$$

Now,



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$I_3 = \int_R^0 \frac{e^{in(iy)^2}}{(iy)^2 + a^3} idy \Rightarrow Re(I_3) = - \int_0^R \frac{a^3 \sin(n\pi^2) - x^3 \cos(nx^2)}{x^6 + a^6} dx$$

Finally, we have,

$$Re(J) = Re(I_1) + Re(I_2) + Re(I_3)$$

Thus,

$$\begin{aligned} & \frac{\pi e^{-\frac{na^2\sqrt{3}}{2}}}{3a^2} \left( \sqrt{3} \cos\left(\frac{na^2}{2}\right) - \sin\left(\frac{na^2}{2}\right) \right) = \\ & = \int_0^R \frac{\cos(nx^2)}{x^3 + a^3} dx - \int_0^R \frac{a^3 \sin(n\pi^2) - x^3 \cos(nx^2)}{x^6 + a^6} dx \end{aligned}$$

Substituting  $n = \pi$ ,  $a = 1$  and letting  $R \rightarrow \infty$  we complete the proof.

**1458. Prove that:**

$$\int_a^b \frac{dx}{x(x+2)(x+4)(x+6) \dots (x+2n)} = \frac{1}{2^n \cdot n!} \left( \sum_{k=1}^n \binom{n}{k} (-1)^k \log\left(\frac{b+2k}{a+2k}\right) \right)$$

*Proposed by Asmat Qatea-Afghanistan*

**Solution 1 by Ravi Prakash-New Delhi-India**

$$\begin{aligned} & \frac{1}{x(x+2)(x+4)(x+6) \dots (x+2n)} = \\ & = \sum_{k=0}^n \frac{1}{(-2k)(-2k+2) \dots (-2) \cdot 2 \cdot 4 \cdot \dots (2n-2k)} \cdot \frac{1}{x+k} = \\ & = \sum_{k=0}^n \frac{(-1)^k}{2^n k! (n-k)!} \cdot \frac{1}{x+2k} = \frac{1}{2^n} \cdot \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{x+2k} \end{aligned}$$

Thus,

$$\begin{aligned} & \int_a^b \frac{dx}{x(x+2)(x+4)(x+6) \dots (x+2n)} = \frac{1}{2^n \cdot n!} \sum_{k=0}^n (-1)^k \binom{n}{k} \int_a^b \frac{dx}{x+2k} = \\ & = \frac{1}{2^n \cdot n!} \left( \sum_{k=1}^n \binom{n}{k} (-1)^k \log\left(\frac{b+2k}{a+2k}\right) \right) \end{aligned}$$

**Solution 2 by Serlea Kabay-Liberia**

Let  $f_n(x) = x(x+2)(x+4) \dots (x+2n)$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\rightarrow \frac{1}{f_n(x)} = \prod_{k=0}^n \frac{1}{x+2k} = \frac{a_0}{x} + \frac{a_1}{x+2} + \cdots + \frac{a_n}{x+2n} = \sum_{k=0}^n \frac{a_k}{x+2k}$$

$$\rightarrow \frac{x+2m}{f_n(x)} = \sum_{k=0}^n \frac{(x+2m)a_k}{x+2k}, \text{ where } m \in [0, n]$$

$$\text{Putting } x = -2m, a_m = \prod_{k=0, k \neq m}^n \frac{1}{2k-2m} = \frac{1}{2^n} \prod_{k=0, k \neq m}^n \frac{1}{k-m}$$

$$\begin{aligned} \rightarrow a_m &= \frac{2^{-n}}{(-m)(1-m)(2-m) \dots ((m-1)-m)((m+1)-m) \dots (n-m)} = \\ &= \frac{2^{-n}}{(-m)(-m+1)(-m+2) \dots (-1) \cdot 1 \cdot 2 \dots (n-m)} = \\ &= \frac{(-1)^m}{2^n \cdot m(m-1)(m-2) \dots (-1) \cdot 1 \cdot 2 \dots (n-m)} = \\ &= \frac{(-1)^m}{2^n \cdot m! (n-m)!} = \frac{(-1)^m}{2^n \cdot n!} \cdot \frac{n!}{m! (n-m)!} \\ a_m &= \frac{(-1)^m}{2^n \cdot n!} \binom{n}{m} \end{aligned}$$

$$\begin{aligned} \text{Now, } \frac{1}{f_n(x)} &= \frac{1}{2^n \cdot n!} \cdot \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{x+2k} \Leftrightarrow \int_a^b \frac{1}{f_n(x)} dx = \frac{1}{2^n \cdot n!} \cdot \sum_{k=0}^n \binom{n}{k} (-1)^k \int_a^b \frac{dx}{x+2k} = \\ &= \frac{1}{2^n \cdot n!} \left( \sum_{k=1}^n \binom{n}{k} (-1)^k \log \left( \frac{b+2k}{a+2k} \right) \right) \end{aligned}$$

**1459. Prove that:**

$$\int_0^1 \sqrt[n]{\left(\frac{x}{1-x}\right)^x} \sin\left(\frac{\pi x}{n}\right) dx = \frac{\pi}{2n} \sqrt[n]{e}, \forall n > 1$$

*Proposed by Surjeet Singhania, Kaushik Mahanta-India*

**Solution by proposers**

$$\text{Define } f(z) = \exp\left(\frac{z \log z - z \log(1-z) + i\pi z}{n}\right)$$

We are define  $f(z)$  on its analytical branch with  $0 \leq \arg(1-z) > 2\pi, -\pi \leq \arg z < \pi$  it is easy to see that  $f(z)$ .

Analytic also we can easily find branch points and branch cut.



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Branch point at  $z = 0, 1$  and branch cut is  $[0, 1]$ . Now, integrate  $f(z)$  on Dogbone

contour  $f(z)$  is bounded function. Hence,

$$\int_{\gamma_1, \gamma_2} f(z) dz = 0 \text{ as radius shrinks.}$$

$$\begin{aligned} \int_A f(z) dz &= - \int_0^1 \exp\left(\frac{z \log z - z \log(1-z) - 2i\pi z + i\pi z}{n}\right) dx \\ &= - \int_0^1 \sqrt[n]{\left(\frac{x}{1-x}\right)^x} \exp\left(-\frac{\pi i x}{n}\right) dx \\ \int_B f(z) dz &= \int_0^1 f(x) dx \end{aligned}$$

Using Cauchy's Residue Theorem:

$$\begin{aligned} \int_A f(z) dz + \int_B f(z) dz &= -2\pi i \operatorname{Res}_{z=\infty} f(z) \\ 2i \int_0^1 \sqrt[n]{\left(\frac{x}{1-x}\right)^x} \sin\left(\frac{\pi x}{n}\right) dx &= 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) \end{aligned}$$

$$\text{Now, } \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = \operatorname{Res}_{z=0} \frac{1}{z^{2-n} \sqrt[n]{1-z}} = \frac{\pi}{2n} \sqrt[n]{e}$$

Therefore,

$$\int_0^1 \sqrt[n]{\left(\frac{x}{1-x}\right)^x} \sin\left(\frac{\pi x}{n}\right) dx = \frac{\pi}{2n} \sqrt[n]{e}$$

**1460. Prove that:**

$$\int_{2020}^{2021} (2021-x) \log(x-2020) \log(2021-x)^{2021} dx = 2021 \left(1 - \frac{\pi^2}{12}\right)$$

*Proposed by Muhammad Afzal-Pakistan*

**Solution 1 by Syed Shahabudeen-India**

$$\begin{aligned} \Omega &= \int_{2020}^{2021} (2021-x) \log(x-2020) \log(2021-x)^{2021} dx = \\ &\stackrel{t=x-2020}{=} 2021 \int_0^1 (1-t) \log t \log(1-t) dt = \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= 2021 \cdot \frac{\partial}{\partial b} \frac{\partial}{\partial a} \int_0^1 (1-t)^{b+1} t^a dt =$$

$$= 2021 \frac{\partial}{\partial a} \int_0^1 (1-t)^{b+1} \log(1-t) t^a dt =$$

$$= 2021 \frac{\partial}{\partial a} \left( \int_0^1 t^a \log(1-t) dt - \int_0^1 t^{a+1} \log(1-t) dt \right)$$

Where,  $\frac{\partial}{\partial a} \left( \frac{H_{a+1}}{a+1} \right) = \frac{\psi^1(a+2)}{a+1} + \frac{-\psi(a+2)-\gamma}{(a+1)^2}$

$$\frac{\partial}{\partial a} \left( \frac{H_{a+2}}{a+2} \right) = \frac{\psi^1(a+3)}{a+2} + \frac{-\psi(a+3)-\gamma}{(a+2)^2}$$

$$\Omega = 2021 \left( \frac{\psi^1(a+2)}{a+1} + \frac{-\psi(a+2)-\gamma}{(a+1)^2} + \frac{\psi^1(a+3)}{a+2} + \frac{-\psi(a+3)-\gamma}{(a+2)^2} \right) =$$

$$= 2021 \left( \psi(2) + \gamma - \psi^1(2) + \frac{\psi^1(3)}{2} - \frac{\psi(3)}{4} - \frac{\gamma}{4} \right)$$

We know that:

$$\psi(2) = 1 - \gamma, \psi^1(2) = \zeta(2) - 1, \psi^1(3) = \zeta(2) - \frac{5}{4}, \psi(3) = \frac{3}{2} - \gamma$$

Therefore,

$$\Omega = 2021 \left( 1 - \zeta(2) + 1 + \frac{\zeta(2)}{2} - \frac{5}{8} - \frac{3}{8} \right) = 2021 \left( 1 - \frac{\pi^2}{12} \right)$$

**Solution 2 by Ajentunmobi Abdulqooyum-Nigeria**

$$\Omega = \int_{2020}^{2021} (2021-x) \log(x-2020) \log(2021-x)^{2021} dx =$$

$$\stackrel{t=x-2020}{=} 2021 \int_0^1 (1-t) \log t \log(1-t) dt =$$

$$= 2021 \left\{ \underbrace{\int_0^1 \log(1-t) \log t dt}_A - \underbrace{\int_0^1 t \log t \log(1-t) dt}_B \right\}$$

$$A = \int_0^1 \log(1-t) \log t dt = - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial p} \Big|_{p=0} \int_0^1 t^{p+n} dt =$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= - \sum_{k=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial p} \Big|_{p=0} \frac{1}{p+n+1} = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{(p+n+1)^2} \Big|_{p=0} = \\
 &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) - \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \\
 &= 1 - \sum_{n=2}^{\infty} \frac{1}{n^2} = 1 - \left( \frac{\pi^2}{6} - 1 \right) = 2 - \frac{\pi^2}{6}
 \end{aligned}$$

Also,

$$\begin{aligned}
 B &= \int_0^1 t \log t \log(1-t) dt = - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 t \cdot t^n \log t dt = \\
 &= - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial m} \Big|_{m=0} \int_0^1 t^{m+n+1} dt = - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 t^{m+n+1} dt = \\
 &= - \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{\partial}{\partial m} \Big|_{m=0} \frac{1}{m+n+2} = \sum_{n=1}^{\infty} \frac{1}{n(n+2)^2} = 1 - \frac{\pi^2}{12}
 \end{aligned}$$

Thus,

$$\Omega = 2021(A - B) = 2021 \left[ 2 - \frac{\pi^2}{6} - \left( 1 - \frac{\pi^2}{12} \right) \right] = 2021 \left( 1 - \frac{\pi^2}{12} \right)$$

**1461. Prove that:**

$$\int_0^\infty e^{-x^2} \frac{4x \sin 4x - (x^2 - 3) \cos 4x}{x^4 + 10x^2 + 9} dx = \frac{\pi \operatorname{erf} 1}{2e^3}$$

*Proposed by Angad Singh-India*

**Solution by proposer**

Consider the complex line integral,

$$J = \oint_C \frac{e^{-z^2}}{1+z^2} dz$$

$C$  – is a rectangle having vertices  $R + i0, R + ia, -R + ia$  and  $-R + i0$  traversed anti-clockwise where  $a > 1$ . Hence,



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$J = \int_{-R}^R \frac{e^{-x^2}}{1+x^2} dx + \int_0^a \frac{e^{-(R+iy)^2}}{1+(R+iy)^2} idy + \int_R^{-R} \frac{e^{-(x+ia)^2}}{1+(x+ia)^2} dx \\ + \int_a^0 \frac{e^{-(R+iy)^2}}{1+(-R+iy)^2} idy$$

**Let:**  $I_1 = \int_{-R}^R \frac{e^{-x^2}}{1+x^2} dx$

$$I_2 = \int_0^a \frac{e^{-(R+iy)^2}}{1+(R+iy)^2} idy$$

$$I_3 = \int_R^{-R} \frac{e^{-(x+ia)^2}}{1+(x+ia)^2} dx$$

$$I_4 = \int_a^0 \frac{e^{-(R+iy)^2}}{1+(-R+iy)^2} idy$$

**Now, using the residue theorem,**

$$J = 2\pi i \frac{e^{-i^2}}{2i} = \pi e$$

$$\text{Now, } |I_2| = \left| \int_0^a \frac{e^{-(R+iy)^2}}{1+(R+iy)^2} idy \right| \leq e^{-R^2} \int_0^a \frac{e^{y^2}}{R^2+y^2-1} dy \leq \frac{e^{a^2-R^2}}{R^2-1}$$

$$\text{Since, } \lim_{R \rightarrow \infty} \frac{e^{a^2-R^2}}{R^2-1} = 0$$

**and  $I_4$  is the complex conjugate of  $I_2$ , therefore both  $I_2$  and  $I_4$  vanishes.**

**Also,**

$$Re(I_3) = 2e^{a^2} \int_0^R e^{-x^2} \frac{2ax \sin 2ax - (x^2 - a^2 + 1) \cos 2ax}{x^4 + 2(a^2 + 1)x^2 + a^4 - 2a^2 + 1} dx$$

**Since,  $Re(J) = Re(I_1) + Re(I_2) + Re(I_3) + Re(I_4)$  and it is known that,**

$$\int_0^\infty \frac{e^{-x^2}}{1+x^2} dx = \frac{1}{2}\pi e \operatorname{erfc}(1)$$

**Substituting  $a = 2$  and letting  $R \rightarrow \infty$ , completes the proof.**

**1462. If we define**



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sum_{n=0}^{\infty} \frac{\varphi^{3n+1} + \varphi^{3n-1}(-x)^n}{\varphi^{4n}} = f(x) \sum_{n=0}^{\infty} \frac{\varphi^{3n-1} + \varphi^{3n+1}(-x)^n}{\varphi^{4n}}$$

**then prove that:**

$$\int_{\frac{1}{\varphi}}^{\infty} \frac{f(x)}{x^2} dx = \frac{3\varphi}{2} - \frac{1}{4}(\varphi - 3) \log(2\varphi + 3)$$

**$\varphi$  –Golden Ratio.**

*Proposed by Srinivasa Raghava-AIRMC-India*

**Solution 1 by Syed Shahabudeen-India**

$$\begin{aligned} \varphi \sum_{n=0}^{\infty} \frac{1}{\varphi^n} + \frac{1}{\varphi} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{\varphi}\right)^n &= f(x) \left( \frac{1}{\varphi} \sum_{n=1}^{\infty} \frac{1}{\varphi^n} + \varphi \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{\varphi}\right)^n \right) \\ \frac{\varphi^2}{\varphi - 1} + \frac{1}{\varphi + x} &= f(x) \left( \frac{1}{\varphi - 1} + \frac{\varphi^2}{\varphi + x} \right) \\ \varphi^3 + \frac{1}{\varphi + x} &= f(x) \left( \varphi + \frac{\varphi^2}{\varphi + x} \right); \left( \because \frac{1}{\varphi - 1} = \varphi \right) \rightarrow f(x) = \frac{\varphi^4 + x\varphi^3 + 1}{2\varphi^2 + x\varphi} \end{aligned}$$

$$\text{Let } \Omega = \int_{\frac{1}{\varphi}}^{\infty} \frac{f(x)}{x^2} dx = \frac{1}{\varphi} \int_{\frac{1}{\varphi}}^{\infty} \frac{\varphi^4 + x\varphi^3 + 1}{(2\varphi + x)x^2} dx =$$

$$= \left( \varphi^3 \int_{\frac{1}{\varphi}}^{\infty} \frac{1}{(2\varphi + x)x^2} dx + \varphi^2 \int_{\frac{1}{\varphi}}^{\infty} \frac{x}{(2\varphi + x)x^2} dx + \frac{1}{\varphi} \int_{\frac{1}{\varphi}}^{\infty} \frac{1}{(2\varphi + x)x^2} dx \right)$$

$$A = \frac{\log(x + 2\varphi)}{4\varphi^2} - \frac{\log x}{4\varphi^2} - \frac{1}{2\varphi x} \quad (\text{indefinite Integration by partial function})$$

$$B = \frac{\log x}{2\varphi} - \frac{\log(x + 2\varphi)}{2\varphi}$$

$$\Omega = \frac{\log(x + 2\varphi)}{4\varphi^2} - \frac{\log x}{4\varphi^2} - \frac{1}{2\varphi x} + \frac{\log x}{2\varphi} - \frac{\log(x + 2\varphi)}{2\varphi} +$$

$$+ (\varphi - 1) \left( \frac{\log(x + 2\varphi)}{4\varphi^2} - \frac{\log x}{4\varphi^2} - \frac{1}{2\varphi x} \right) =$$

$$= \log \left( \frac{x + 2\varphi}{x} \right) \left( \frac{\varphi}{4} + \frac{1}{4\varphi} - \frac{\varphi}{2} - \frac{1}{4\varphi^2} \right) + \frac{1}{2x} \left( \frac{1}{\varphi} - \varphi^2 - 1 \right)$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$P = \frac{1}{4\varphi} - \frac{\varphi}{4} - \frac{1}{4\varphi^2} = \frac{1}{4}(\varphi - 1) - \frac{\varphi}{4} - \frac{1}{4}(2 - \varphi) = \frac{1}{4}(\varphi - 3)$$

$$Q = \frac{1}{\varphi} - \varphi^2 - 1 = (\varphi - 1) - (\varphi + 1) - 1 = -3$$

Thus,

$$\begin{aligned}\Omega &= \left( \frac{\varphi - 3}{4} \log \left( \frac{2x + \varphi}{x} \right) - \frac{3}{2x} \right)_{x=\frac{1}{\varphi}}^{\infty} = \frac{3\varphi}{2} - \frac{\varphi - 3}{4} \log(1 + 2\varphi^2); (\because \varphi^2 = \varphi + 1) \\ &= \frac{3\varphi}{2} - \frac{1}{4}(\varphi - 3) \log(2\varphi + 3)\end{aligned}$$

*Solution 2 by Mohammad Rostami-Afghanistan*

$$\begin{aligned}\sum_{n=0}^{\infty} [\varphi^{1-n} + \varphi^{-1-n}(-x)^n] &= f(x) \sum_{n=0}^{\infty} [\varphi^{-1-n} + \varphi^{1-n}(-x)^n] \\ \rightarrow \varphi \sum_{n=0}^{\infty} \left(\frac{1}{\varphi}\right)^n + \frac{1}{\varphi} \sum_{n=0}^{\infty} \left(-\frac{x}{\varphi}\right)^n &= f(x) \left[ \frac{1}{\varphi} \sum_{n=0}^{\infty} \left(\frac{1}{\varphi}\right)^n + \varphi \sum_{n=0}^{\infty} \left(-\frac{x}{\varphi}\right)^n \right] \\ \varphi \left( \frac{1}{1 - \frac{1}{\varphi}} \right) + \frac{1}{\varphi} \left( \frac{1}{1 + \frac{x}{\varphi}} \right) &= f(x) \left[ \frac{1}{\varphi} \left( \frac{1}{1 - \frac{1}{\varphi}} \right) + \varphi \left( \frac{1}{1 + \frac{x}{\varphi}} \right) \right] \\ \rightarrow \frac{\varphi^2}{\varphi - 1} + \frac{1}{\varphi + x} &= f(x) \left( \frac{1}{\varphi - 1} + \frac{\varphi^2}{\varphi + x} \right), (\varphi^2 = \varphi + 1, \frac{1}{\varphi} = \varphi - 1) \\ \frac{\varphi + 1}{\varphi - 1} + \frac{1}{\varphi + x} &= f(x) \left( \frac{1}{\varphi - 1} + \frac{\varphi + 1}{\varphi - x} \right) \\ \rightarrow \frac{\varphi^2 + (x + 1)\varphi + x + \varphi - 1}{(\varphi - 1)(\varphi + x)} &= f(x) \left( \frac{\varphi + x + \varphi^2 - 1}{(\varphi - 1)(\varphi + x)} \right) \\ \varphi + 1 + x\varphi + \varphi + x + \varphi - 1 &= f(x)(\varphi + x + \varphi) \\ f(x) &= \frac{3\varphi + (\varphi + 1)x}{2\varphi + x} = \frac{(x + 2\varphi) + \varphi(x + 1)}{2\varphi + x} \\ \rightarrow f(x) &= 1 + \varphi \left( \frac{x + 2\varphi + 1 - 2\varphi}{2\varphi + x} \right) = 1 + \varphi + \frac{\varphi - 2\varphi^2}{2\varphi + x} = 1 + \varphi - \frac{\varphi + 2}{2\varphi + x} \\ \rightarrow f(x) &= 1 + \varphi - \frac{\varphi + 2}{2\varphi + x}\end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 \int_{\frac{1}{\varphi}}^{\infty} \frac{f(x)}{x^2} dx &= (1 + \varphi) \int_{\frac{1}{\varphi}}^{\infty} \frac{1}{x^2} dx - (\varphi + 2) \int_{\frac{1}{\varphi}}^{\infty} \frac{1}{x^2(x + 2\varphi)} dx = \\
 &= (1 + \varphi) \left[ -\frac{1}{x} \right]_{\frac{1}{\varphi}}^{\infty} + \frac{\varphi + 2}{4\varphi^2} \int_{\frac{1}{\varphi}}^{\infty} \frac{1}{x} dx - \frac{\varphi + 2}{2\varphi} \int_{\frac{1}{\varphi}}^{\infty} \frac{1}{x^2} dx - \frac{\varphi + 2}{4\varphi^2} \int_{\frac{1}{\varphi}}^{\infty} \frac{1}{x + 2\varphi} dx = \\
 &= (1 + \varphi)\varphi + \frac{\varphi + 2}{4\varphi^2} [\log x]_{\frac{1}{\varphi}}^{\infty} + \frac{\varphi + 2}{2\varphi} \left[ \frac{1}{x} \right]_{\frac{1}{\varphi}}^{\infty} - \frac{\varphi + 2}{4\varphi^2} [\log(x + 2\varphi)]_{\frac{1}{\varphi}}^{\infty} = \\
 &= \varphi + \varphi^2 - \frac{\varphi}{2} - 1 + \frac{\varphi + 2}{4(\varphi + 1)} \left[ \log \left( \frac{x}{x + 2\varphi} \right) \right]_{\frac{1}{\varphi}}^{\infty} = \\
 &= \frac{\varphi}{2} + \varphi + \frac{(\varphi + 2)(\varphi - 1)}{4(\varphi^2 - 1) \left( -\log \frac{1}{1 + 2\varphi^2} \right)} = \\
 &= \frac{3\varphi}{2} + \frac{\varphi^2 + \varphi - 2}{4\varphi} \log(1 + 2\varphi + 2) = \frac{3\varphi}{2} + \frac{2\varphi - 1}{4\varphi} \log(2\varphi + 3) = \\
 &= \frac{3\varphi}{2} - \frac{1}{4} \left( \frac{1}{\varphi} - 2 \right) \log(2\varphi + 3) = \frac{3\varphi}{2} - \frac{1}{4} (\varphi - 3) \log(2\varphi + 3)
 \end{aligned}$$

Therefore,

$$\int_{\frac{1}{\varphi}}^{\infty} \frac{f(x)}{x^2} dx = \frac{3\varphi}{2} - \frac{1}{4} (\varphi - 3) \log(2\varphi + 3)$$

**1463. Evaluate the expression in a closed form:**

$$\Omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{7x^2 + 6} + \frac{1}{7x^2 + 1} \right) \left( \frac{1}{7x^2 + 5} + \frac{1}{7x^2 + 2} \right) \left( \frac{1}{7x^2 + 4} + \frac{1}{7x^2 + 3} \right) dx$$

*Proposed by Srinivasa Raghava-AIRMC-India*

**Solution 1 by Asmat Qatea-Afghanistan**

$$\begin{aligned}
 (a+b)(c+d)(f+e) &= \sum_{cyc} abc, \int_{-\infty}^{\infty} \frac{dx}{ax^2 + b} = \frac{\pi}{\sqrt{ab}}, 7x^2 \rightarrow x \\
 \frac{1}{(x+a_1)(x+a_2)(x+a_3)} &= \frac{A_1}{x+a_1} + \frac{A_2}{x+a_2} + \frac{A_3}{x+a_3} \\
 A_1 &= \lim_{x \rightarrow -a_1} \frac{x+a_1}{(x+a_1)(x+a_2)(x+a_3)} = \frac{1}{(-a_1+a_2)(-a_1+a_3)}
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$A_2 = \lim_{x \rightarrow -a_2} \frac{x + a_2}{(x + a_1)(x + a_2)(x + a_3)} = \frac{1}{(-a_2 + a_1)(-a_2 + a_3)}$$

$$A_3 = \lim_{x \rightarrow -a_3} \frac{x + a_3}{(x + a_1)(x + a_2)(x + a_3)} = \frac{1}{(-a_3 + a_1)(-a_3 + a_2)}$$

$$I_{acf} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{(7x^2 + 6)(7x^2 + 5)(7x^2 + 4)}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{\frac{1}{2}}{7x^2 + 6} + \frac{-1}{7x^2 + 5} + \frac{\frac{1}{2}}{7x^2 + 4} \right) dx = \frac{1}{\pi} \left( \frac{\frac{\pi}{2}}{\sqrt{42}} - \frac{\pi}{\sqrt{35}} + \frac{\frac{\pi}{2}}{\sqrt{28}} \right)$$

$$I_{acf} = \frac{1}{\sqrt{7}} \left( \frac{\frac{1}{2}}{\sqrt{6}} - \frac{1}{\sqrt{5}} + \frac{\frac{1}{2}}{\sqrt{4}} \right)$$

$$I_{ace} = \frac{1}{\sqrt{7}} \left( \frac{\frac{1}{3}}{\sqrt{6}} - \frac{\frac{1}{2}}{\sqrt{5}} + \frac{\frac{1}{6}}{\sqrt{3}} \right), I_{adf} = \frac{1}{\sqrt{7}} \left( \frac{\frac{1}{8}}{\sqrt{6}} + \frac{\frac{1}{8}}{\sqrt{2}} - \frac{\frac{1}{4}}{\sqrt{4}} \right)$$

$$I_{ade} = \frac{1}{\sqrt{7}} \left( \frac{\frac{1}{12}}{\sqrt{6}} + \frac{\frac{1}{4}}{\sqrt{2}} - \frac{\frac{1}{3}}{\sqrt{3}} \right), I_{bcf} = \frac{1}{\sqrt{7}} \left( \frac{\frac{1}{12}}{\sqrt{1}} + \frac{\frac{1}{4}}{\sqrt{5}} - \frac{\frac{1}{3}}{\sqrt{4}} \right)$$

$$I_{bce} = \frac{1}{\sqrt{7}} \left( \frac{\frac{1}{8}}{\sqrt{1}} + \frac{\frac{1}{8}}{\sqrt{5}} - \frac{\frac{1}{4}}{\sqrt{3}} \right), I_{bdf} = \frac{1}{\sqrt{7}} \left( \frac{\frac{1}{3}}{\sqrt{1}} - \frac{\frac{1}{2}}{\sqrt{2}} + \frac{\frac{1}{6}}{\sqrt{4}} \right)$$

$$I_{bde} = \frac{1}{\sqrt{7}} \left( \frac{\frac{1}{2}}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{\frac{1}{2}}{\sqrt{3}} \right)$$

$$\Omega = \sum_{cyc} I_{abc} = \frac{1}{\sqrt{7}} \left( \frac{\frac{25}{24}}{\sqrt{6}} - \frac{\frac{9}{8}}{\sqrt{5}} + \frac{\left(\frac{1}{12}\right)}{\sqrt{4}} + \frac{\frac{1}{12}}{\sqrt{3}} - \frac{\frac{9}{8}}{\sqrt{2}} + \frac{\frac{25}{24}}{\sqrt{1}} \right)$$

**Solution 2 by Syed Shahabudeen-India**

$$\Omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{7x^2 + 6} + \frac{1}{7x^2 + 1} \right) \left( \frac{1}{7x^2 + 5} + \frac{1}{7x^2 + 2} \right) \left( \frac{1}{7x^2 + 4} + \frac{1}{7x^2 + 3} \right) dx$$

$$\stackrel{t=7x^2}{=} \frac{2\sqrt{7}}{14\pi} \int_0^{\infty} \frac{(2t+t)^3}{\sqrt{t}(t+1)(t+2)(t+3)(t+4)(t+5)(t+6)} dt =$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$= \frac{1}{\pi\sqrt{7}} \int_0^\infty \frac{1}{\sqrt{t}} \left( \frac{A}{t+1} + \frac{B}{t+2} + \frac{C}{t+3} + \frac{D}{t+4} + \frac{E}{t+5} + \frac{F}{t+6} \right) dt$$

**By applying Vedic method for partial function, we get**

$$A = \frac{5^3}{5!}, B = \frac{-3^3}{4!}, C = \frac{1}{2(3!)}, D = \frac{1}{3!2!}, E = \frac{-3^3}{4!}, F = \frac{5^3}{5!}$$

**Hence,**

$$\begin{aligned} \Omega &= \frac{1}{\pi\sqrt{7}} \left( \frac{25}{24} \int_0^\infty \left( \frac{1}{\sqrt{t}(t+1)} + \frac{1}{\sqrt{t}(t+6)} \right) dt - \frac{9}{8} \int_0^\infty \left( \frac{1}{\sqrt{t}(t+2)} + \frac{1}{\sqrt{t}(t+5)} \right) dt \right. \\ &\quad \left. + \frac{1}{12} \int_0^\infty \left( \frac{1}{\sqrt{t}(t+3)} + \frac{1}{\sqrt{t}(t+4)} \right) dt \right) \\ &\quad \because \int_0^\infty \frac{1}{\sqrt{t}(t+a)} dt = \frac{\pi}{\sqrt{a}} \end{aligned}$$

**Therefore,**

$$\Omega = \frac{1}{\sqrt{7}} \left( \frac{25}{24} \left( \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{6}} \right) - \frac{9}{8} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} \right) + \frac{1}{12} \left( \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} \right) \right)$$

**Solution 3 by Ravi Prakash-New Delhi-India**

$$\Omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{7x^2+6} + \frac{1}{7x^2+1} \right) \left( \frac{1}{7x^2+5} + \frac{1}{7x^2+2} \right) \left( \frac{1}{7x^2+4} + \frac{1}{7x^2+3} \right) dx$$

**As the integrand is even,**

$$\Omega = \frac{2}{\pi} \int_0^{\infty} \frac{(14x^2+7)^3 dx}{(7x^2+1)(7x^2+2) \dots (7x^2+6)}$$

**To split the integrand into partial fraction, we put  $7x^2 = t$ .**

$$\begin{aligned} &\frac{(2t+7)^3}{(t+1)(t+2)(t+3)(t+4)(t+5)(t+6)} = \\ &= \frac{5^3}{1 \cdot 2 \cdot \dots \cdot 5} \frac{1}{t+1} + \frac{3^3}{(-1) \cdot 1 \cdot 2 \cdot \dots \cdot 4} \frac{1}{t+2} + \frac{1}{(-2) \cdot (-1) \cdot 1 \cdot 2 \cdot 3} \frac{1}{t+3} + \\ &\quad + \frac{-1}{(-3) \cdot (-2) \cdot (-1) \cdot 1 \cdot 2} \frac{1}{t+4} + \frac{(-3)^3}{(-4) \cdot (-3) \cdot (-2) \cdot (-1) \cdot 1} \frac{1}{t+5} + \\ &\quad + \frac{(-5)^3}{(-5) \cdot (-4) \cdot \dots \cdot (-1)} \frac{1}{t+6} \end{aligned}$$

**Thus,**



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned} \frac{\pi}{2}\Omega &= \frac{25}{24} \int_0^\infty \frac{dx}{7x^2+1} - \frac{9}{8} \int_0^\infty \frac{dx}{7x^2+2} + \frac{1}{12} \int_0^\infty \frac{dx}{7x^2+3} + \frac{1}{12} \int_0^\infty \frac{dx}{7x^2+4} - \\ &\quad - \frac{9}{8} \int_0^\infty \frac{dx}{7x^2+5} + \frac{25}{24} \int_0^\infty \frac{dx}{7x^2+6} \\ &\quad \therefore \int_0^\infty \frac{dx}{ax^2+b} = \frac{\pi}{2\sqrt{ab}} \end{aligned}$$

It follows that:

$$\Omega = \frac{25}{24} \left( 1 + \frac{1}{\sqrt{6}} \right) - \frac{9}{8} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} \right) + \frac{1}{12} \left( \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} \right)$$

**1464. Find:**

$$\Omega = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \int_{\varepsilon}^1 \frac{x\sqrt{x} \log x}{x^3 + x\sqrt{x} + 1} dx$$

*Proposed by Vasile Mircea Popa-Romania*

**Solution 1 by Ajetunmobi Abdulquyyoum-Nigeria**

$$\begin{aligned} I &= \int_0^1 \frac{x\sqrt{x} \log x}{x^3 + x\sqrt{x} + 1} dx \stackrel{\sqrt{x}=u}{=} 4 \int_0^1 \frac{u^4 \log u}{u^6 + u^3 + 1} du = 4 \int_0^1 \frac{u^4(1-u^3) \log u}{(1-u^3)(u^6 + u^3 + 1)} du = \\ &= 4 \int_0^1 \frac{u^4(1-u^3) \log u}{1-u^9} du = 4 \underbrace{\int_0^1 \frac{u^4 \log u}{1-u^9} du}_A - 4 \underbrace{\int_0^1 \frac{u^7 \log u}{1-u^9} du}_B \\ A &= 4 \int_0^1 \frac{u^4 \log u}{1-u^9} du = 4 \sum_{n=0}^{\infty} \int_0^1 u^{9n+4} \log u du = 4 \sum_{n=0}^{\infty} \frac{\partial}{\partial s} \Big|_{s=0} \int_0^1 u^{9n+4+s} du = \\ &= -4 \sum_{n=0}^{\infty} \frac{1}{9n+5+s} \Big|_{s=0} = -4 \cdot \frac{1}{9^2} \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{5}{9}\right)^2} = -\frac{4}{81} \psi^1\left(\frac{5}{9}\right) \\ B &= 4 \int_0^1 \frac{u^7 \log u}{1-u^9} du = 4 \sum_{n=0}^{\infty} \int_0^1 u^{9n+7} \log u du = 4 \sum_{n=0}^{\infty} \frac{\partial}{\partial s} \Big|_{s=0} \int_0^1 u^{9n+7+s} du = \\ &= -4 \sum_{n=0}^{\infty} \frac{1}{9n+8+s} \Big|_{s=0} = -4 \cdot \frac{1}{9^2} \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{8}{9}\right)^2} = -\frac{4}{81} \psi^1\left(\frac{8}{9}\right) \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$I = A - B = \frac{4}{81} \psi^1\left(\frac{8}{9}\right) - \frac{4}{81} \psi^1\left(\frac{5}{9}\right)$$

**Solution 2 by Syed Shahabudeen-India**

$$\begin{aligned} I &= \int_0^1 \frac{x\sqrt{x}\log x}{x^3 + x\sqrt{x} + 1} dx \stackrel{\sqrt{x}=t}{=} 4 \int_0^1 \frac{t^4 \log t}{t^6 + t^3 + 1} dt = 4 \int_0^1 \frac{t^4(t^3 - 1)\log t}{t^9 - 1} dt = \\ &= 4 \left( \underbrace{\int_0^1 \frac{t^7 \log t}{t^9 - 1} dt}_A - \underbrace{\int_0^1 \frac{t^4 \log t}{t^9 - 1} dt}_B \right) \end{aligned}$$

We known that:

$$\frac{(-1)^{m+2}}{p^2} \psi^m(z) = \int_0^1 \frac{t^{m-1} \log^m t}{t^p - 1} dt$$

Hence,

$$A = \frac{1}{9^2} \psi^1\left(\frac{8}{9}\right), B = \frac{1}{9^2} \psi^1\left(\frac{5}{9}\right)$$

Therefore,

$$\Omega = \frac{4}{81} \left( \psi^1\left(\frac{8}{9}\right) - \psi^1\left(\frac{5}{9}\right) \right) = \frac{4}{81} \left( \zeta\left(2, \frac{8}{9}\right) - \zeta\left(2, \frac{5}{9}\right) \right)$$

**Solution 3 by Mohammad Rostami-Afghanistan**

$$\begin{aligned} \Omega &= \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \int_{\varepsilon}^1 \frac{x\sqrt{x}\log x}{x^3 + x\sqrt{x} + 1} dx = \int_0^1 \frac{x\sqrt{x}\log x}{x^3 + x\sqrt{x} + 1} dx \stackrel{\sqrt{x}=u}{=} \\ &= 4 \int_0^1 \frac{u^4 \log u}{u^6 + u^3 + 1} du = 4 \int_0^1 \frac{u^4(1 - u^3) \log u}{(1 - u^3)(u^6 + u^3 + 1)} du 4 \int_0^1 \frac{u^4(1 - u^3) \log u}{1 - u^9} du = \\ &= 4 \left( \int_0^1 u^4 \sum_{n=0}^{\infty} (u^9)^n \frac{\partial}{\partial a} \Big|_{a=0} x^a du - \int_0^1 u^7 \sum_{n=0}^{\infty} (u^9)^n \frac{\partial}{\partial b} \Big|_{b=0} x^b du \right) = \\ &= 4 \left( \sum_{n=0}^{\infty} \frac{\partial}{\partial a} \Big|_{a=0} \int_0^1 u^{4+9n+a} du - \sum_{k=0}^{\infty} \frac{\partial}{\partial b} \Big|_{b=0} \int_0^1 u^{7+9k+b} du \right) = \\ &= 4 \left( \sum_{n=0}^{\infty} \left[ \frac{1}{9n+a+5} \right]_{a=0}' - \sum_{k=0}^{\infty} \left[ \frac{1}{9k+b+8} \right]_{b=0}' \right) = \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 &= 4 \left( \sum_{n=0}^{\infty} \left[ \frac{1}{9n+a+5} \right]_{a=0} - \sum_{k=0}^{\infty} \left[ \frac{1}{(9k+b+8)^2} \right]_{b=0} \right) = \\
 &= 4 \left[ -\frac{1}{81} \sum_{n=0}^{\infty} \left[ \frac{1}{\left(n + \frac{5}{9}\right)^2} \right]_{a=0} + \frac{1}{81} \sum_{k=0}^{\infty} \left[ \frac{1}{\left(k + \frac{8}{9}\right)^2} \right]_{b=0} \right] \\
 &\because \psi^{(1)}(z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}
 \end{aligned}$$

Therefore,

$$\Omega = \frac{4}{81} \left[ \psi^{(1)}\left(\frac{8}{9}\right) - \psi^{(1)}\left(\frac{5}{9}\right) \right]$$

*Solution 4 by Serlea Kabay-Liberia*

$$\begin{aligned}
 \text{Let } \Omega(n) &= \int_0^1 \frac{x^n}{1-x^9} dx \\
 \Omega(n) &= \sum_{k=0}^{\infty} \int_0^1 x^{9k+n} dx = \sum_{k=0}^{\infty} \left[ \frac{x^{9k+n+1}}{9k+n+1} \right]_0^1 = \sum_{k=0}^{\infty} \frac{1}{9k+n+1} = \frac{1}{9} \psi^{(0)}\left(\frac{n+1}{9}\right)
 \end{aligned}$$

Now,

$$\begin{aligned}
 \Omega &= \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \frac{x\sqrt{x} \log x}{x^3 + x\sqrt{x} + 1} dx \stackrel{x=u^2}{=} \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \frac{4u^4 \log u}{u^6 + u^3 + 1} du = \\
 &= \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \frac{4(u^4 - u^7) \log u}{1 - u^9} du = 4 \left( \int_0^1 \frac{u^4 \log u}{1 - u^9} du - \int_0^1 \frac{u^7 \log u}{1 - u^9} du \right) = \\
 &= 4 \left( \frac{\partial \Omega(n)}{\partial n} \Big|_{n=4} - \frac{\partial \Omega(n)}{\partial n} \Big|_{n=7} \right) = \frac{4}{81} \left[ \psi^{(1)}\left(\frac{8}{9}\right) - \psi^{(1)}\left(\frac{5}{9}\right) \right]
 \end{aligned}$$

**1465. If  $a, b \in \mathbb{R}$ ,  $a < b$  and  $f: [a, b] \rightarrow \mathbb{R}$  continuous function such that**

$$\int_a^b f(x) dx = 2021, \text{ then find:}$$

$$\Omega = \int_a^b x(f(x) + f(a+b-x)) dx$$

*Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania*



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Solution 1 by Adrian Popa-Romania**

$$\begin{aligned}
 & \because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \\
 \Omega &= \int_a^b x(f(x) + f(a+b-x)) dx = \int_a^b (a+b-x)(f(a+b-x) + f(x)) dx \\
 2\Omega &= \int_a^b (a+b)(f(a+b-x) + f(x)) dx = \\
 &= \int_a^b (a+b)f(a+b-x) dx + \int_a^b (a+b)f(x) dx = \\
 &= (a+b) \left( \underbrace{\int_a^b f(a+b-x) dx}_{\int_a^b f(x) dx} + \int_a^b f(x) dx \right) = 2(a+b) \int_a^b f(x) dx \\
 &= 2(a+b) \cdot 2021
 \end{aligned}$$

Therefore,

$$\Omega = \int_a^b x(f(x) + f(a+b-x)) dx = 2021(a+b)$$

**Solution 2 by Timson Azeez Folorunsho-Nigeria**

$$\begin{aligned}
 \int_a^b f(x) dx &= 2021; \quad \Omega = \int_a^b x(f(x) + f(a+b-x)) dx; \quad (1) \\
 x \rightarrow a+b-x &\Rightarrow \Omega = \int_a^b (a+b-x)(f(a+b-x) + f(x)) dx; \quad (2)
 \end{aligned}$$

From (1), (2) we get:

$$\begin{aligned}
 2\Omega &= \int_a^b x(f(x) + f(a+b-x)) dx + \int_a^b (a+b)(f(a+b-x) + f(x)) dx \\
 &\quad - \int_a^b x(f(x) + f(a+b-x)) dx \\
 \Rightarrow 2\Omega &= \int_a^b (a+b)(f(a+b-x) + f(x)) dx
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\therefore \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$2\Omega = (a+b) \int_a^b (f(x) + f(a+b-x)) dx = (a+b) \int_a^b (2f(x)) dx$$

Therefore,

$$\Omega = \int_a^b x(f(x) + f(a+b-x)) dx = 2021(a+b)$$

**Solution 3 by Fayssal Abdelli-Bejaia-Algerie**

$$\begin{aligned} \Omega &= \int_a^b x(f(x) + f(a+b-x)) dx = \\ &= \int_a^b xf(x) dx + \underbrace{\int_a^b xf(a+b-x) dx}_{\Omega_1} \end{aligned}$$

$$\begin{aligned} \Omega_1 &= \int_a^b xf(a+b-x) dx \stackrel{y=a+b-x}{=} - \int_b^a (a+b-y)f(y) dy = \int_a^b (a+b-y)f(y) dy = \\ &= \int_a^b (a+b)f(y) dy - \int_a^b f(y) dy \end{aligned}$$

$$\Omega = \int_a^b xf(x) dx + (a+b) \int_a^b f(x) dx - \int_a^b xf(x) dx = (a+b) \int_a^b f(x) dx$$

Therefore,

$$\Omega = \int_a^b x(f(x) + f(a+b-x)) dx = 2021(a+b)$$

**Solution 4 by Obaidullah Jaihon-Afghanistan**

$$\begin{aligned} \Omega &= \int_a^b x(f(x) + f(a+b-x)) dx = \int_a^b (a+b-x)(f(a+b-x) + f(x)) dx = \\ &= \int_a^b (a+b)(f(x) + f(a+b-x)) dx - \Omega \end{aligned}$$

$$2\Omega = (a+b) \int_a^b (f(x) + f(a+b-x)) dx = 2(a+b) \int_a^b f(x) dx$$

Therefore,



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Omega = \int_a^b x(f(x) + f(a+b-x))dx = 2021(a+b)$$

**1466.**

$$\int_0^\infty e^{-nx} \cos(ax) dx = \frac{n}{n^2 + a^2}$$

$$\int_0^\infty x e^{-nx^2} \cos(ax^2) dx = \frac{n}{2(n^2 + a^2)}$$

$$\int_0^\infty e^{-nx^2} \cos(ax) dx = \frac{1}{2} \sqrt{\frac{\pi}{n}} e^{-\frac{a^2}{4n}}$$

$$\int_0^\infty e^{-nx^2} \cos(ax^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{n + \sqrt{n^2 + a^2}}{n^2 + a^2}}$$

$$\int_0^\infty \frac{\cosh x \cdot \cosh(ax)}{\cosh(nx)} dx = \frac{\pi}{4n} \left( \sec\left(\frac{\pi(a-1)}{2n}\right) + \sec\left(\frac{\pi(a+1)}{2n}\right) \right)$$

*Proposed by Angad Singh-India*

**Solution by proposer**

Observe that,

$$\int_0^\infty e^{-nx} \cos(ax) dx = \operatorname{Re} \left( \int_0^\infty e^{-nx+iax} dx \right) = \operatorname{Re} \left( \frac{1}{n-ia} \right) = \frac{n}{n^2 + a^2}$$

This proves (i).

Substituting  $x = t^2$  and replacing  $t$  by  $x$ , proves (ii).

Let,  $I = I(n, a) = \int_0^\infty e^{-nx^2} \cos(ax) dx \stackrel{IBP}{=} \frac{2n}{a} \int_0^\infty x e^{-nx^2} \sin(ax) dx$  and

$$\frac{\partial}{\partial a} = - \int_0^\infty x e^{-nx^2} \sin(ax) dx = - \frac{aI}{2n} \Rightarrow \frac{\partial I}{\partial a} + \frac{a}{2n} I = 0$$

Solving this PDE, we obtain,

$$\log(I) = -\frac{a^2}{4n} + \log(c) \Rightarrow I(n, a) = ce^{-\frac{a^2}{4n}}$$

Since  $c = I(n, 0) = \frac{1}{2} \sqrt{\frac{\pi}{n}}$ , we have,  $I(n, a) = \frac{1}{2} \sqrt{\frac{\pi}{n}} e^{-\frac{a^2}{4n}}$  this proves (iii).



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Again, observe that,

$$\int_0^\infty e^{-nx^2} \cos(ax^2) dx = \operatorname{Re} \left( \int_0^\infty e^{-nx^2+iax^2} dx \right) = \frac{\sqrt{\pi}}{2} \operatorname{Re} \left( \frac{1}{\sqrt{n-ia}} \right)$$

Thus,  $\int_0^\infty e^{-nx^2} \cos(ax^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}} \sqrt{\frac{n+\sqrt{n^2+a^2}}{n^2+a^2}}$  and this proves (iv).

$$\begin{aligned} \int_0^\infty \frac{\cosh x \cdot \cosh(ax)}{\cosh(nx)} dx &= \frac{1}{2} \int_0^1 t^{n-a-2} \cdot \frac{(1+t^2)(1+t^{2a})}{1+t^{2n}} dt = \\ &= \frac{1}{2} \int_1^\infty t^{n-a-2} \cdot \frac{(1+t^2)(1+t^{2a})}{1+t^{2n}} dt \\ \Rightarrow \int_0^\infty \frac{\cosh x \cdot \cosh(ax)}{\cosh(nx)} dx &= \frac{1}{4} \int_0^\infty t^{n-a-2} \cdot \frac{(1+t^2)(1+t^{2a})}{1+t^{2n}} dt \end{aligned}$$

Since it is known that,

$$\begin{aligned} \int_0^\infty \frac{t^{a-1}}{1+t^{2n}} dt &= \frac{\pi}{2n} \csc\left(\frac{\pi a}{2n}\right) \Rightarrow \\ \int_0^\infty \frac{\cosh x \cdot \cosh(ax)}{\cosh(nx)} dx &= \frac{\pi}{4n} \left( \sec\left(\frac{\pi(a-1)}{2n}\right) + \sec\left(\frac{\pi(a+1)}{2n}\right) \right) \end{aligned}$$

**1467. Find:**

$$\Omega = \int_0^\infty \frac{x \log(1+x^2)}{1+x^2+x^4} dx$$

*Proposed by Vasile Mircea Popa-Romania*

**Solution 1 by Rana Ranino-Setif-Algerie**

$$\begin{aligned} \Omega &= \int_0^\infty \frac{x \log(1+x^2)}{1+x^2+x^4} dx = \frac{1}{2} \int_0^\infty \frac{\log(1+x)}{1+x+x^2} dx = \\ &= \frac{1}{2} \int_0^1 \int_0^\infty \frac{x}{(1+xy)(1+x+x^2)} dx dy = \\ &= \frac{1}{2} \int_0^1 \frac{1}{y^2-y+1} \int_0^\infty \left( \frac{x}{1+x+x^2} - \frac{y}{1+xy} + \frac{y}{1+x+x^2} \right) dx dy \\ \Omega &= \frac{1}{2} \int_0^1 \frac{1}{y^2-y+1} \left[ \log\left(\frac{\sqrt{x^2+x+1}}{xy+1}\right) + \frac{2y-1}{\sqrt{3}} \cdot \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \right]_0^\infty dy = \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 &= \frac{1}{2} \int_0^1 \frac{1}{y^2 - y + 1} \left( -\log y + \frac{\pi(2y-1)}{3\sqrt{3}} \right) dy \\
 \Omega &= \frac{\pi}{6\sqrt{3}} \int_0^1 \frac{2y-1}{y^2 - y + 1} dy - \frac{1}{2} \int_0^1 \frac{\log y}{y^2 - y + 1} dy = \\
 &= \frac{\pi}{6\sqrt{3}} \underbrace{(\log(y^2 - y + 1)|_0^1)}_0 - \frac{1}{2} \int_0^1 \frac{(y+1)\log y}{y^3 + 1} dy = -\frac{1}{18} \int_0^1 \frac{\left(y^{-\frac{1}{3}} + y^{-\frac{2}{3}}\right) \log y}{y+1} dy \\
 \Omega &= -\frac{1}{18} \cdot \frac{\partial}{\partial s} \Big|_{s=0} \int_0^1 \frac{y^{s-\frac{1}{3}} + y^{s-\frac{2}{3}}}{y+1} dy = \\
 &= -\frac{1}{36} \frac{\partial}{\partial s} \Big|_{s=0} \left\{ \psi\left(\frac{s}{2} + \frac{5}{6}\right) - \psi\left(\frac{s}{2} + \frac{1}{3}\right) - \psi\left(\frac{s}{2} + \frac{1}{6}\right) \right\} (*)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \Omega &= \int_0^\infty \frac{x \log(1+x^2)}{1+x^2+x^4} dx = \frac{1}{72} \left\{ \psi^{(1)}\left(\frac{1}{3}\right) + \psi^{(1)}\left(\frac{1}{6}\right) - \psi^{(1)}\left(\frac{2}{3}\right) - \psi^{(1)}\left(\frac{5}{6}\right) \right\} \\
 (*) &: \int_0^1 \frac{y^n}{y+1} dy = \frac{1}{2} \left\{ \psi\left(\frac{n}{2} + 1\right) - \psi\left(\frac{n}{2} + \frac{1}{2}\right) \right\}
 \end{aligned}$$

**Solution 2 by Ajetunmobi Abdulqooyum-Nigeria**

$$\begin{aligned}
 \Omega &= \int_0^\infty \frac{x \log(1+x^2)}{1+x^2+x^4} dx = \frac{1}{2} \int_0^\infty \frac{\log(1+x)}{1+x+x^2} dx \stackrel{y=\frac{1}{x+1} \rightarrow x=\frac{1-y}{y}}{=} \\
 &= \frac{1}{2} \int_0^1 \frac{\log\left(\frac{1}{y}\right)}{y^2 - y + 1} dy = -\frac{1}{2} \int_0^1 \frac{\log y}{y^2 - y + 1} dy = \\
 &= -\frac{1}{2} \int_0^1 \frac{(1+x)\log y}{(1+x)(1-y+y^2)} dy = -\frac{1}{2} \int_0^1 \frac{(1+y)\log y}{1+y^3} dy \\
 \int_0^1 \frac{(1+y)\log y}{1+y^3} dy &= \int_0^1 \frac{\log y}{1+y^3} dy + \int_0^1 \frac{y \log y}{1+y^3} dy = A + B; \\
 A &= \int_0^1 \frac{\log y}{1+y^3} dy = \sum_{n=0}^{\infty} (-1)^n \frac{\partial}{\partial s} \Big|_{s=0} \int_0^1 y^{3n+s} dy = \sum_{n=0}^{\infty} (-1)^n \frac{\partial}{\partial s} \Big|_{s=0} \frac{1}{3n+1+s} = \\
 &= -\frac{1}{36} \left( \psi^{(1)}\left(\frac{1}{6}\right) - \psi^{(1)}\left(\frac{2}{3}\right) \right)
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 B &= \int_0^1 \frac{y \log y}{1+y^3} dy = \sum_{n=0}^{\infty} (-1)^n \frac{\partial}{\partial s} \Big|_{s=0} \int_0^1 y^{3n+s+1} dy = - \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+2)^2} = \\
 &= -\frac{1}{36} \left( \psi^{(1)}\left(\frac{1}{3}\right) - \psi^{(1)}\left(\frac{5}{6}\right) \right)
 \end{aligned}$$

Thus,

$$\Omega = -\frac{1}{2}(A + B) = \frac{1}{72} \left\{ \psi^{(1)}\left(\frac{1}{3}\right) + \psi^{(1)}\left(\frac{1}{6}\right) - \psi^{(1)}\left(\frac{2}{3}\right) - \psi^{(1)}\left(\frac{5}{6}\right) \right\}$$

**Solution 3 by Mohammad Rostami-Afghanistan**

$$\begin{aligned}
 \Omega &= \int_0^\infty \frac{x \log(1+x^2)}{(1+x^2)+(x^2)^2} dx \stackrel{1+x^2=u}{=} \frac{1}{2} \int_1^\infty \frac{\log u}{u-(u-1)^2} du = \frac{1}{2} \int_1^\infty \frac{\log u}{u^2-u+1} du = \\
 &\stackrel{u=\frac{1}{t}}{=} \frac{1}{2} \int_1^0 \frac{\log\left(\frac{1}{t}\right)}{\frac{1}{t^2}-\frac{1}{t}+1} \left(-\frac{dt}{t^2}\right) = -\frac{1}{2} \int_0^1 \frac{\log t}{t^2-t+1} dt = -\frac{1}{2} \int_0^1 \frac{(1+t)\log t}{(1+t)(t^2-t+1)} dt = \\
 &= -\frac{1}{2} \left( \int_0^1 \frac{\log t}{1+t^3} dt + \int_0^1 \frac{t \log t}{1+t^3} dt \right) = \\
 &= -\frac{1}{2} \left( \int_0^1 \sum_{n=0}^{\infty} (-t^3) \frac{\partial}{\partial a} \Big|_{a=0} t^a dt + \int_0^1 t \sum_{k=0}^{\infty} (-t^3) \frac{\partial}{\partial b} \Big|_{b=0} t^b dt \right) = \\
 &= -\frac{1}{2} \left( \sum_{n=0}^{\infty} (-1)^n \frac{\partial}{\partial a} \Big|_{a=0} \int_0^1 t^{3n+a} dt + \sum_{k=0}^{\infty} (-1)^k \frac{\partial}{\partial b} \Big|_{b=0} \int_0^1 t^{3k+b+1} dt \right) = \\
 &= -\frac{1}{2} \left( \sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{3n+a+1} \right]_{a=0}' + \sum_{k=0}^{\infty} (-1)^k \left[ \frac{1}{3k+b+2} \right]_{b=0}' \right) = \\
 &= -\frac{1}{2} \left[ -\sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)^2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{(3k+2)^2} \right] = -\frac{1}{2} \left[ -\frac{1}{9} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{1}{3}\right)^2} - \frac{1}{9} \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(k+\frac{2}{3}\right)^2} \right]
 \end{aligned}$$

$$\text{note: } \Phi(-1, m+1, z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(z+k)^{m+1}} = \frac{1}{(-2)^{m+1} m!} \left[ \psi^{(m)}\left(\frac{z}{2}\right) - \psi^{(m)}\left(\frac{z+1}{2}\right) \right]$$

Hence,

$$\Omega = \frac{1}{72} \left\{ \psi^{(1)}\left(\frac{1}{3}\right) + \psi^{(1)}\left(\frac{1}{6}\right) - \psi^{(1)}\left(\frac{2}{3}\right) - \psi^{(1)}\left(\frac{5}{6}\right) \right\}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**1468.** Let  $J_n(x)$  –Bessel function and  $L_x[f](y)$  –Laplace Transform and if

$$S(n) = \int_0^1 L_x[e^{-x}J_n(x)](y) dy$$

then prove that:  $\int_0^\infty S(n) dn = \log\left(\frac{\log(2+\sqrt{5})}{\log(1+\sqrt{2})}\right)$ ,  $\sum_{n=1}^\infty S(n) = \log\left(\frac{\sqrt{5}-3}{\sqrt{2}-2}\right)$

*Proposed by Srinivasa Raghava-AIRMC-India*

*Solution by Syed Shahabudeen-Kerala-India*

$$S(n) = \int_0^1 L_x[e^{-x}J_n(x)](y) dy$$

$$\text{We know: } L_x\{e^{-x}J_n(x)\} = \frac{(\sqrt{(y+1)^2 + 1} - (y+1)^2)}{\sqrt{(y+1)^2 + 1}}$$

$$S(n) = \int_0^1 \frac{(\sqrt{(y+1)^2 + 1} - (y+1)^2)}{\sqrt{(y+1)^2 + 1}} dy \stackrel{t=\sqrt{(y+1)^2 + 1}-(y+1)}{=} \int_{\sqrt{5}-2}^{\sqrt{5}-1} t^{n-1} dt = \\ = \frac{1}{n} \left( (\sqrt{2}-1)^n - (\sqrt{5}-2)^n \right)$$

$$\text{Let: } \Omega = \int_0^\infty S(n) dn = \int_0^\infty \frac{1}{n} \left( (\sqrt{2}-1)^n - (\sqrt{5}-2)^n \right) dn = \\ = \int_0^\infty \frac{e^{\log(\sqrt{2}-1)} - e^{n \log(\sqrt{5}-2)}}{n} dn$$

Above Integral is a Frullani Integral form i.e.

$$\int_0^\infty \frac{(e^{at} - e^{bt})}{t} dt = \log\left(\frac{a}{b}\right) \Rightarrow \Omega = \log\left(\frac{\log(2+\sqrt{5})}{\log(1+\sqrt{2})}\right)$$

$$\text{Let: } \lambda = \sum_{n=1}^\infty S(n) = \sum_{n=1}^\infty \frac{(\sqrt{2}-1)^n - (\sqrt{5}-2)^n}{n} = \sum_{n=1}^\infty \frac{(\sqrt{2}-1)^n}{n} - \sum_{n=1}^\infty \frac{(\sqrt{5}-2)^n}{n} = \\ = -\log(1 - (\sqrt{2}-1)) + \log(1 - (\sqrt{5}-2)) = \log(3 - \sqrt{5}) - \log(2 - \sqrt{2}) = \\ = \log\left(\frac{3 - \sqrt{5}}{2 - \sqrt{2}}\right) = \log\left(\frac{\sqrt{5} - 3}{\sqrt{2} - 2}\right)$$

**1469.** Prove that:

$$\int_0^\infty \left( \int_0^1 L_x[e^{-\varphi x} J_z(x)](y) dy \right) dz = \log\left( \frac{\log\left(\frac{1}{2}\left(\sqrt{5} + \sqrt{6(\sqrt{5}+3)} + 3\right)\right)}{\log\left(\frac{1}{2}\left(\sqrt{5} + \sqrt{2(\sqrt{5}+5)} + 1\right)\right)} \right)$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$\varphi$  – Golden Ratio,  $J_n(x)$  – Bessel function and  $L_x[f](y)$  – Laplace Transform

*Proposed by Srinivasa Raghava-AIRMC-India*

**Solution by Syed Shahabudeen-Kerala-India**

$$\Omega = \int_0^\infty \int_0^1 L_x[e^{-\varphi z} J_z(x)](y) dy dz$$

$$\text{We know: } L_x\{e^{-\varphi z} J_z(x)\} = \frac{(\sqrt{(y+\varphi)^2 + 1} - (y+\varphi)^z)}{\sqrt{(y+\varphi)^2 + 1}}$$

$$\begin{aligned} S(n) &= \int_0^\infty \int_0^1 \frac{(\sqrt{(y+\varphi)^2 + 1} - (y+\varphi))^z}{\sqrt{(y+\varphi)^2 + 1}} dy \stackrel{t=\sqrt{(y+\varphi)^2+1}-(y+\varphi)}{=} \\ &= - \int_0^\infty \int_{\sqrt{\varphi^2+1}-\varphi}^{\sqrt{(1+\varphi)^2+1}-(1+\varphi)} t^{z-1} dt dz = \\ &= \int_0^\infty \frac{(\sqrt{\varphi^2+1}-\varphi)^z - (\sqrt{(1+\varphi)^2+1}-(1+\varphi))^z}{z} dz = \\ &= \int_0^\infty \frac{e^{z \log(\sqrt{\varphi^2+1}-\varphi)} - e^{z \log(\sqrt{(1+\varphi)^2+1}-(1+\varphi))}}{z} dz \end{aligned}$$

Above Integral is a Frullani Integral form i.e.

$$\int_0^\infty \frac{(e^{at} - e^{bt})}{t} dt = \log\left(\frac{a}{b}\right) \Rightarrow \Omega = \log\left(\frac{\log(2 + \sqrt{5})}{\log(1 + \sqrt{2})}\right)$$

$$\Omega = \log\left(\frac{\log\left(\sqrt{(1+\varphi)^2+1}-(1+\varphi)\right)}{\log\left(\sqrt{\varphi^2+1}-\varphi\right)}\right)$$

$$\therefore \left(\sqrt{(1+\varphi)^2+1}-(1+\varphi)\right) = \frac{1}{\sqrt{(1+\varphi)^2+1}+(1+\varphi)} \text{ and}$$

$$\sqrt{\varphi^2+1}-\varphi = \frac{1}{\sqrt{\varphi^2+1}+\varphi}$$

$$\Omega = \log\left(\frac{\log\left(\sqrt{(1+\varphi)^2+1}+(1+\varphi)\right)}{\log\left(\sqrt{\varphi^2+1}-\varphi\right)}\right)$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sqrt{(1+\varphi)^2 + 1} - (1+\varphi) = \frac{\sqrt{5} + \sqrt{6(\sqrt{5}+3)} + 3}{2}$$

$$\sqrt{\varphi^2 + 1} + \varphi = \frac{\sqrt{5} + \sqrt{2(\sqrt{5}+5)} + 1}{2}$$

Therefore,

$$\Omega = \log \left( \frac{\log \left( \frac{1}{2} \left( \sqrt{5} + \sqrt{6(\sqrt{5}+3)} + 3 \right) \right)}{\log \left( \frac{1}{2} \left( \sqrt{5} + \sqrt{2(\sqrt{5}+5)} + 1 \right) \right)} \right)$$

**1470. If  $0 < a \leq b, f : [a, b] \rightarrow (0, \infty), f$  –continuous, then :**

$$\int_a^b \int_a^b \int_a^b \frac{f^3(x) dx dy dz}{f(y)f(z) + f^2(x)} \geq \frac{(b-a)^2}{2} \int_a^b f(x) dx$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco**

Let  $x, y, z \in [a, b]$ .

$$\begin{aligned}
 \text{We have : } & \frac{f^3(x)}{f(y)f(z) + f^2(x)} = \\
 & = f(x) \cdot \left( 1 - \frac{f(y)f(z)}{f(y)f(z) + f^2(x)} \right) \stackrel{AM-GM}{\geq} f(x) - \frac{f(x)f(y)f(z)}{2\sqrt{f(y)f(z).f^2(x)}} = \\
 & = f(x) - \frac{1}{2}\sqrt{f(y)f(z)} \stackrel{AM-GM}{\geq} f(x) - \frac{1}{4}(f(y) + f(z)) \\
 \rightarrow & \frac{f^3(x)}{f(y)f(z) + f^2(x)} \geq f(x) - \frac{1}{4}(f(y) + f(z)), \forall x, y, z \in [a, b] \\
 \rightarrow & \int_a^b \int_a^b \int_a^b \frac{f^3(x) dx dy dz}{f(y)f(z) + f^2(x)} \geq \int_a^b \int_a^b \int_a^b \left( f(x) - \frac{1}{4}(f(y) + f(z)) \right) dx dy dz \\
 & = \left( 1 - \frac{2}{4} \right) (b-a)^2 \int_a^b f(x) dx
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

*Therefore,*

$$\int_a^b \int_a^b \int_a^b \frac{f^3(x) dx dy dz}{f(y)f(z) + f^2(x)} \geq \frac{(b-a)^2}{2} \int_a^b f(x) dx$$

**1471. If  $0 < a \leq b, f : [a, b] \rightarrow (0, \infty), f$  –continuous, then:**

$$\int_a^b \int_a^b \frac{f^3(x) dx dy}{f^2(x) + f(x)f(y) + f^2(y)} \geq \frac{b-a}{3} \int_a^b f(x) dx$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco**

Let  $x, y \in [a, b]$ .

$$\begin{aligned}
 \text{We have : } & \frac{f^3(x)}{f^2(x) + f(x)f(y) + f^2(y)} \stackrel{\text{AM-GM}}{\geq} \frac{f^3(x)}{f^2(x) + \frac{f^2(x) + f^2(y)}{2} + f^2(y)} \\
 & = \frac{2}{3} \cdot \frac{f^3(x)}{f^2(x) + f^2(y)} = \\
 & = \frac{2}{3} f(x) \cdot \left(1 - \frac{f^2(y)}{f^2(x) + f^2(y)}\right) \stackrel{\text{AM-GM}}{\geq} \frac{2}{3} f(x) - \frac{2}{3} \cdot \frac{f(x)f^2(y)}{2f(x)f(y)} = \frac{2}{3} f(x) - \frac{1}{3} f(y) \\
 & \rightarrow \frac{f^3(x)}{f^2(x) + f(x)f(y) + f^2(y)} \geq \frac{2}{3} f(x) - \frac{1}{3} f(y), \forall x, y \in [a, b] \\
 & \rightarrow \int_a^b \int_a^b \frac{f^3(x) dx dy}{f^2(x) + f(x)f(y) + f^2(y)} \geq \int_a^b \int_a^b \left(\frac{2}{3} f(x) - \frac{1}{3} f(y)\right) dx dy \\
 & = \left(\frac{2}{3} - \frac{1}{3}\right) (b-a) \int_a^b f(x) dx
 \end{aligned}$$

*Therefore,*

$$\int_a^b \int_a^b \frac{f^3(x) dx dy}{f^2(x) + f(x)f(y) + f^2(y)} \geq \frac{b-a}{3} \int_a^b f(x) dx$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**1472. If  $0 < a \leq b, f : [a, b] \rightarrow (0, \infty), f$  –continuous, then:**

$$\int_a^b \int_a^b \frac{dxdy}{\sqrt{[f(x) + f(y)]f(x)f(y)}} \leq \frac{b-a}{2} \int_a^b \frac{dx}{f(x)} + \frac{1}{4} \left( \int_a^b \frac{dx}{f(x)} \right)^2$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco**

Let  $x, y \in [a, b]$ .

$$\begin{aligned}
 \text{We have : } & \frac{1}{\sqrt{[f(x) + f(y)]f(x)f(y)}} = \\
 & = \frac{2}{\sqrt{[f(x) + f(y)][4f(x)f(y)]}} \stackrel{\text{AM-GM}}{\geq} \frac{1}{f(x) + f(y)} + \frac{1}{4f(x)f(y)} \leq \\
 & \stackrel{\text{CBS}}{\leq} \frac{1}{4} \left( \frac{1}{f(x)} + \frac{1}{f(y)} \right) + \frac{1}{4f(x)f(y)} \\
 & \rightarrow \frac{1}{\sqrt{[f(x) + f(y)]f(x)f(y)}} \leq \frac{1}{4} \left( \frac{1}{f(x)} + \frac{1}{f(y)} \right) + \frac{1}{4f(x)f(y)}, \forall x, y \in [a, b] \\
 & \rightarrow \int_a^b \int_a^b \frac{dxdy}{\sqrt{[f(x) + f(y)]f(x)f(y)}} \leq \int_a^b \int_a^b \left( \frac{1}{4} \left( \frac{1}{f(x)} + \frac{1}{f(y)} \right) + \frac{1}{4f(x)f(y)} \right) dxdy \\
 & = \frac{b-a}{2} \int_a^b \frac{dx}{f(x)} + \frac{1}{4} \left( \int_a^b \frac{dx}{f(x)} \right)^2
 \end{aligned}$$

**1473. If  $0 < a \leq b$  then:**

$$\int_a^b \int_a^b \int_a^b \frac{e^{2x^2+y^2} + e^{2y^2+z^2}}{e^{2(x^2+y^2)}} dx dy dz \geq 2(b-a)^2 \int_a^b e^{-x^2} dx$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Asmat Qatea-Afghanistan**

$$\int_a^b \int_a^b \int_a^b \frac{e^{2x^2+y^2} + e^{2y^2+z^2}}{e^{2(x^2+y^2)}} dx dy dz \geq 2(b-a)^2 \int_a^b e^{-x^2} dx$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Leftrightarrow \int_a^b \int_a^b \int_a^b \frac{e^{2x^2+y^2} + e^{2y^2+z^2}}{e^{2(x^2+y^2)}} dx dy dz \geq \int_a^b \int_a^b 2e^{-x^2} dx dy dz$$

**We need to prove that:**

$$\begin{aligned} \frac{e^{2x^2+y^2} + e^{2y^2+z^2}}{e^{2(x^2+y^2)}} &\geq 2e^{-x^2} \Leftrightarrow e^{2x^2+y^2} + e^{2y^2+z^2} \geq 2e^{x^2+2y^2} \\ e^{2x^2+y^2} + e^{2y^2+z^2} &\stackrel{AM-GM}{\geq} 2e^{2x^2+3y^2+z^2} \stackrel{?}{\geq} 2e^{x^2+2y^2} \\ &\Rightarrow 2x^2 + 3y^2 + z^2 \stackrel{?}{\geq} x^2 + 2y^2 \\ &\Rightarrow x^2 + y^2 + z^2 \geq 0 \text{ (true).} \end{aligned}$$

**1474. If  $0 < a \leq b < \frac{\pi}{2}$  then:**

$$3(b-a) + 3(\sin b - \sin a) \leq 6 \int_a^b \frac{\sin x}{x} dx \leq 4(b-a) + 2(\sin b - \sin a)$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Ravi Prakash-New Delhi-India**

$$\text{Let } f(x) = 3x + 3x \cos x - 6 \sin x, 0 \leq x \leq \frac{\pi}{2}$$

$$f'(x) = 3 + 3 \cos x - 3x \sin x - 6 \cos x = 3(1 - \cos x - x \sin x)$$

$$f''(x) = 3(\sin x - \sin x - x \cos x) = -3x \cos x < 0, \forall x \in (0, \frac{\pi}{2})$$

$$\rightarrow f'(x) \text{-decreasing on } (0, \frac{\pi}{2}) \rightarrow f'(x) < f'(0) \rightarrow f'(x) < 0, \forall x \in (0, \frac{\pi}{2})$$

$$\rightarrow f(x) \text{-decreasing on } (0, \frac{\pi}{2}) \rightarrow f(x) < f(0) = 0, \forall 0 < x \leq \frac{\pi}{2}$$

$$\rightarrow 3 + 3 \cos x < \frac{6 \sin x}{x}, \forall 0 < x < \frac{\pi}{2}$$

$$\rightarrow \int_a^b (3 + 3 \cos x) dx \leq 6 \int_0^1 \frac{\sin x}{x} dx$$

$$\rightarrow 3(b-a) + 3(\sin b - \sin a) \leq 6 \int_a^b \frac{\sin x}{x} dx; (1)$$

$$\text{Next, let } g(x) = 6 \sin x - 4x - 2x \cos x, 0 \leq x \leq \frac{\pi}{2}$$

$$g'(x) = 6 \cos x - 4 - 2 \cos x + 2x \sin x = 2(2 \cos x - 2 + x \sin x)$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$g''(x) = -2x \sin x < 0, \forall 0 < x < \frac{\pi}{2} \rightarrow g'(x) \text{ decreasing on } \left(0, \frac{\pi}{2}\right)$$

$$\rightarrow g'(x) < g(0), \forall 0 < x < \frac{\pi}{2} \rightarrow g'(x) \text{ decreasing on } \left[0, \frac{\pi}{2}\right]$$

$$\rightarrow g'(x) < g'(0) = 0, \forall x \in \left[0, \frac{\pi}{2}\right] \rightarrow g(x) \text{ decreasing on } \left[0, \frac{\pi}{2}\right]$$

$$\rightarrow g(x) < g(0) = 0, \forall x \in \left[0, \frac{\pi}{2}\right] \rightarrow \frac{6 \sin x}{x} < 4 + 2 \cos x, 0 < x \leq \frac{\pi}{2}$$

$$6 \int_a^b \frac{\sin x}{x} dx < 4(b-a) + 2(\sin b - \sin a); (2)$$

From (1), (2) it follows that:

$$3(b-a) + 3(\sin b - \sin a) \leq 6 \int_a^b \frac{\sin x}{x} dx \leq 4(b-a) + 2(\sin b - \sin a)$$

**1475. If  $0 < a \leq b < \frac{\pi^3}{8}$  then:**

$$\int_{\sqrt[3]{a}}^{\sqrt[3]{b}} \sin x \cdot \sinh x dx \leq \frac{b-a}{3}$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Adrian Popa-Romania**

$$\int_{\sqrt[3]{a}}^{\sqrt[3]{b}} \sin x \cdot \sinh x dx \leq \frac{b-a}{3} \Leftrightarrow \int_{\sqrt[3]{a}}^{\sqrt[3]{b}} \sin x \cdot \sinh x dx \leq \int_{\sqrt[3]{a}}^{\sqrt[3]{b}} x^2 dx$$

$$\sin x \cdot \sinh x - x^2 \leq 0, \forall x \in \left(0, \frac{\pi^3}{8}\right); (1)$$

$$\text{Let } f(x) = \sin x \cdot \sinh x - x^2, \forall x \in \left(0, \frac{\pi^3}{8}\right).$$

$$f'(x) = \cos x \cdot \sinh x + \sin x \cosh x - 2x$$

$$f''(x) = 2 \cos x \cdot \cosh x - 2$$

$$f'''(x) = -2 \sin x \cdot \cosh x + 2 \cos x \cdot \sinh x$$

$$\frac{\pi^3}{8} \cong 3,87 \rightarrow \frac{\pi^3}{8} \in \left(\pi, \frac{3\pi}{2}\right)$$

$$f^{iv}(x) = -4 \sin x \cdot \sinh x.$$

$$\text{If } x \in (0, \pi) \rightarrow \sin x > 0, \sinh x = \frac{e^x - e^{-x}}{2} > 0.$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

If  $x \in \left(\pi, \frac{3\pi}{2}\right) \rightarrow \sin x < 0$  and  $\sinh x > 0$  then,  $\sin x \cdot \sinh x < 0$ .

**So, we must prove (1) in  $(0, \pi]$ .**

How,  $f^{iv}(x) < 0 \rightarrow f'''(x) \downarrow, f'''(x) = 0$  then,  $f'''(x) < 0$  then,  $f''(x) \downarrow$  and  $f''(x) = 0$ .

Hence,

$f''(x) < 0$  and  $f'(x) \downarrow$  and  $f'(0) = 0, f'(x) < 0$ .

**So,  $f(x) \downarrow$  and  $f(x) < f(0) = 0$ . Thus,**

$$\sin x \cdot \sinh x - x^2 \leq 0, \forall x \in \left(0, \frac{\pi}{8}\right); (1)$$

**1476. If  $0 < a \leq b < \frac{\pi}{2}$  then:**

$$\frac{8}{\pi^2} \int_a^b \log \left( \sec \left( \frac{\pi \sin x}{2} \right) \right) dx \leq b - a + \tan b - \tan a$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Adrian Popa-Romania**

$$\begin{aligned} \frac{8}{\pi^2} \int_a^b \log \left( \sec \left( \frac{\pi \sin x}{2} \right) \right) dx &\leq b - a + \tan b - \tan a \\ \Leftrightarrow \frac{8}{\pi^2} \int_a^b \log \left( \frac{1}{\cos \left( \frac{\pi \sin x}{2} \right)} \right) dx &\leq \int_a^b \left( 1 + \frac{1}{\cos^2 x} \right) dx \\ \frac{8}{\pi^2} \int_a^b -\log \left( \cos \left( \frac{\pi \sin x}{2} \right) \right) dx &\leq \int_a^b \left( 1 + \frac{1}{\cos^2 x} \right) dx \end{aligned}$$

We must to prove that:

$$-\frac{8}{\pi^2} \log \left( \cos \left( \frac{\pi \sin x}{2} \right) \right) \leq 1 + \frac{1}{\cos^2 x}, \forall x \in \left(0, \frac{\pi}{2}\right)$$

$$\begin{aligned} \because \log t \leq t - 1 \rightarrow -\log \left( \cos \left( \frac{\pi \sin x}{2} \right) \right) &\leq -\left( \cos \left( \frac{\pi \sin x}{2} \right) - 1 \right) = 1 - \cos \left( \frac{\pi \sin x}{2} \right) = \\ &= 2 \sin^2 \left( \frac{\pi \sin x}{4} \right) \end{aligned}$$

Now, we must to prove that:

$$\frac{8}{\pi^2} \cdot 2 \cdot \sin^2 \left( \frac{\pi \sin x}{4} \right) \leq 1 + \frac{1}{\cos^2 x}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

We know that:  $\sin a \leq a, \forall a \geq 0$  then,  $\sin^2\left(\frac{\pi \sin x}{4}\right) \leq \frac{\pi^2 \sin^2 x}{16}$ . Thus,

$$\frac{8}{\pi^2} \cdot 2 \cdot \sin^2\left(\frac{\pi \sin x}{4}\right) \leq \frac{8}{\pi^2} \cdot 2 \cdot \frac{\pi^2 \sin^2 x}{16} = \sin^2 x < 1 + \frac{1}{\cos^2 x}$$

Therefore,

$$\frac{8}{\pi^2} \int_a^b \log\left(\sec\left(\frac{\pi \sin x}{2}\right)\right) dx \leq b - a + \tan b - \tan a$$

**1477. If  $n \in \mathbb{N}, n \geq 2$  then:**

$$\frac{4}{n\pi} \sum_{k=2}^n \left( \int_{\frac{1}{k}}^{\frac{1}{x}} \frac{1}{x} \cdot \tan^{-1} x dx \right) \geq \log n$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Adrian Popa-Romania**

$$\begin{aligned} I &= \int_{\frac{1}{k}}^{\frac{1}{x}} \frac{1}{x} \cdot \tan^{-1} x dx \stackrel{\frac{1}{x}=t}{=} \int_{\frac{1}{k}}^k \frac{1}{t} \cdot \tan^{-1}\left(\frac{1}{t}\right) dt = \int_{\frac{1}{k}}^k \frac{1}{t} \cot^{-1} t dt \\ 2I &= \int_{\frac{1}{k}}^k \frac{1}{x} (\tan^{-1} x + \cot^{-1} x) dx = \frac{\pi}{2} \int_{\frac{1}{k}}^k \frac{1}{x} dx = \frac{\pi}{2} \left( \log k - \log \frac{1}{k} \right) = \pi \log k \\ \Rightarrow I &= \frac{\pi}{2} \cdot \log k \\ \frac{4}{n\pi} \cdot \sum_{k=2}^n \frac{\pi}{2} \log k &= \frac{4}{n\pi} \cdot \frac{\pi}{2} \cdot \sum_{k=2}^n \log k = \frac{2}{n} \cdot \sum_{k=2}^n \log k = \frac{2}{n} \cdot \log n! \stackrel{(?)}{\geq} \log n \\ \Rightarrow \log(n!)^2 &\geq \log(n^n) \Rightarrow (n!)^2 \geq n^n, \forall n \geq 2; (*) \end{aligned}$$

By Induction, suppose that  $(n!)^2 \geq n^n, \forall n \in \mathbb{N} \Rightarrow ((n+1)!)^2 \geq (n+1)^{n+1}$ .

$$\begin{aligned} \text{We have: } ((n+1)!)^2 &= (n! (n+1))^2 = (n!)^2 (n+1)^2 \stackrel{(?)}{\geq} (n+1)^n (n+1) | : (n+1) \Rightarrow \\ (n!)^2 (n+1) &\stackrel{(?)}{\geq} (n+1)^n \Leftrightarrow (n!)^2 (n+1) \geq \frac{(n+1)^n}{n^n} \cdot n^n \Leftrightarrow n+1 \geq \left(\frac{n+1}{n}\right)^n \end{aligned}$$

Which is true, because:

$$n \geq 2 \Rightarrow n+1 \geq 3 \text{ and } \left(\frac{n+1}{n}\right)^n < e < 3 \Rightarrow n+1 \geq \left(\frac{n+1}{n}\right)^n, \forall n > 2.$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Therefore,

$$\frac{4}{n\pi} \sum_{k=2}^n \left( \int_{\frac{1}{k}}^{\frac{1}{k-1}} \frac{1}{x} \cdot \tan^{-1} x \, dx \right) \geq \log n$$

**1478.**  $f: [a, a+1] \rightarrow \mathbb{R}$ ,  $f$  –continuous  $n \in \mathbb{N}, n \geq 2$ . Prove that exists

$c_1, c_2, \dots, c_{n-1} \in (a, a+1)$ , different in pairs such that:

$$\left| \int_a^{a+1} f(x) \, dx - \frac{f(c_1) + f(c_2) + \dots + f(c_{n-1})}{n} \right| \leq \int_a^{\frac{a+1}{2}} |f(x)| \, dx$$

*Proposed by Dan Radu Seclăman-Romania*

**Solution 1 by proposer**

For all  $n \in \mathbb{N}, n \geq 2$ , by Mean Value Theorem, exists:

$c_1 \in (a + \frac{1}{n}, a + \frac{2}{n}), \dots, c_{n-1} \in (a + \frac{n-1}{n}, a + 1)$ , different in pairs such that:

$$\begin{aligned} \int_a^{a+1} f(x) \, dx &= \int_a^{a+\frac{1}{n}} f(x) \, dx + \int_{a+\frac{1}{n}}^{a+\frac{2}{n}} f(x) \, dx + \dots + \int_{a+\frac{n-1}{n}}^{a+\frac{n}{n}} f(x) \, dx = \\ &= \int_a^{a+\frac{1}{n}} f(x) \, dx + \frac{f(c_1) + f(c_2) + \dots + f(c_{n-1})}{n} \end{aligned}$$

Hence,

$$\left| \int_a^{a+1} f(x) \, dx - \frac{f(c_1) + f(c_2) + \dots + f(c_{n-1})}{n} \right| = \left| \int_a^{a+\frac{1}{n}} f(x) \, dx \right| \leq \int_a^{a+\frac{1}{n}} |f(x)| \, dx ; (1)$$

But, for all  $n \in \mathbb{N}, n \geq 2$ , we have  $0 \leq a < a + \frac{1}{n} \leq a + \frac{1}{2}$  and then:

$$\int_a^{a+\frac{1}{n}} |f(x)| \, dx \leq \int_a^{\frac{a+1}{2}} |f(x)| \, dx ; (2)$$

From (1), (2) it follows the proposed problem.



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Solution 2 by Remus Florin Stanca-Romania**

We apply M.V.T. for function  $F(x) = \int f(x)dx$  on the intervals  $\left[a + \frac{k}{n}, a + \frac{k+1}{n}\right]$ ,  
 $n \in \{0, \dots, n-2\}$

$$\frac{\int_{a+\frac{k}{n}}^{a+\frac{k+1}{n}} f(x) dx}{n} = f(c_{k+1}), k \in \{0, 1, \dots, n-2\} \Rightarrow$$

$$\int_{a+\frac{k}{n}}^{a+\frac{k+1}{n}} f(x) dx = \frac{f(c_{k+1})}{n} \Rightarrow \sum_{k=1}^{n-2} \int_{a+\frac{k}{n}}^{a+\frac{k+1}{n}} f(x) dx = \frac{f(c_1) + \dots + f(c_{n-1})}{n} \Rightarrow$$

$$\left| \int_a^{a+1} f(x) dx - \frac{f(c_1) + f(c_2) + \dots + f(c_{n-1})}{n} \right| = \left| \int_a^{a+1} f(x) dx - \int_a^{a+\frac{n-1}{n}} f(x) dx \right| \leq \int_a^{a+\frac{1}{2}} |f(x)| dx; (1)$$

$$\text{Let } G(x) = \int_a^{a+x} f(t) dt \Rightarrow \int_a^{a+1} f(x) dx - \int_a^{a+\frac{n-1}{n}} f(x) dx = G(1) - G\left(\frac{n-1}{n}\right)$$

$G$  –continuous and derivable function so we apply M.V.T. on  $\left[\frac{n-1}{n}; 1\right]$

$$\frac{G(1) - G\left(\frac{n-1}{n}\right)}{\frac{1}{n}} = G'(x_0), x_0 \in \left[\frac{n-1}{n}; 1\right]$$

$$G(x) = F(a+x) - F(a) \Rightarrow G'(x) = f(a+x) \Leftrightarrow$$

$$\frac{G(1) - G\left(\frac{n-1}{n}\right)}{\frac{1}{n}} = f(a+x_0) \Rightarrow \int_a^{a+1} f(x) dx - \int_a^{a+\frac{n-1}{n}} f(x) dx = \frac{f(a+x_0)}{n} \stackrel{(1)}{\Leftrightarrow}$$

$$\left| \frac{f(a+x_0)}{n} \right| \leq \int_a^{a+\frac{1}{2}} |f(x)| dx \Leftrightarrow \frac{|f(a+x_0)|}{n} \leq \frac{|f(x_1)|}{2}, x_1 \in \left[a; a + \frac{1}{2}\right]$$

$$\exists M \in \mathbb{R} \text{ such that } |f(a+x_0)| \leq M \Leftrightarrow \left| \frac{f(a+x_0)}{n} \right| \leq \frac{M}{n}.$$

So, we need to prove that  $\exists n_\xi$  such that  $\forall n \geq n_\xi$ :

$$\frac{1}{n} < \xi, \forall \xi > 0 \text{ and } \xi = \frac{|f(x_1)|}{2M} \Leftrightarrow n \geq \frac{1}{\xi} \Leftrightarrow n_\xi = \left[ \frac{1}{\xi} \right] + 1 = \left[ \frac{2M}{|f(x_1)|} \right] + 1$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**So, for any  $\xi = \frac{|f(x_1)|}{2M}$  it is true that  $\exists n_\xi$  such that for any  $n \geq n_\xi$ :**

$$\frac{1}{n} \leq \frac{|f(x_1)|}{2M} \Rightarrow \exists c_1, c_2, \dots, c_n \text{ which satisfy the relationship.}$$

**1479. If  $0 < a \leq b < \frac{\pi}{2}$  then:**

$$\int_a^b \log\left(\frac{1-\sin x}{1+\sin x}\right) dx \geq (b-a) \log\left(\frac{1-\sin\left(\frac{a+b}{2}\right)}{1+\sin\left(\frac{a+b}{2}\right)}\right)$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Adrian Popa-Romania**

Let  $f: [a, b] \rightarrow \mathbb{R}, f(x) = \log\left(\frac{1-\sin x}{1+\sin x}\right); [a, b] \subset \left(0, \frac{\pi}{2}\right) \Rightarrow$

$$f'(x) = -\frac{2}{\cos x} < 0, \forall x \in \left(0, \frac{\pi}{2}\right) \Rightarrow f(x) \downarrow$$

$$f''(x) = -\frac{2 \sin x}{\cos^2 x} < 0, \forall x \in \left(0, \frac{\pi}{2}\right) \Rightarrow f \text{ -concave} \Rightarrow f\left(\frac{a+b}{2}\right) \geq \frac{f(a)+f(b)}{2}; (1)$$

We must to prove that:  $S_1 \geq S_2$

$$S_1 = \int_{\frac{a+b}{2}}^b \left( f\left(\frac{a+b}{2}\right) - f(x) \right) dx; S_2 = \int_a^{\frac{a+b}{2}} \left( f(x) - f\left(\frac{a+b}{2}\right) \right) dx$$

$$x = t + \frac{b-a}{2}, dx = dt. \begin{cases} x = \frac{a+b}{2} \\ x = b \end{cases} \Rightarrow \begin{cases} t = \frac{a+b}{2} - \frac{b-a}{2} = a \\ t = b - \frac{b-a}{2} = \frac{b+a}{2} \end{cases}$$

$$A_4 = \int_a^{\frac{a+b}{2}} \left( f\left(\frac{a+b}{2}\right) - f\left(t + \frac{b-a}{2}\right) \right) dx \stackrel{t \rightarrow x}{=} \int_a^b \left( f\left(\frac{a+b}{2}\right) - f\left(x + \frac{b-a}{2}\right) \right) dx$$

$$\Rightarrow f\left(\frac{a+b}{2}\right) - f\left(x + \frac{b-a}{2}\right) \stackrel{?}{\geq} f(x) - f\left(\frac{a+b}{2}\right) \Leftrightarrow$$

$$2f\left(\frac{a+b}{2}\right) \stackrel{?}{\geq} f(x) + f\left(x + \frac{b-a}{2}\right) \Leftrightarrow$$

$$2f\left(\frac{b-a}{2} + a\right) \geq f(x) + f\left(x + \frac{b-a}{2}\right) \Leftrightarrow$$

$$2f\left(\frac{b-a}{2} + a\right) \geq f(x) + f\left(x + \frac{b-a}{2}\right) \stackrel{?}{\geq} f\left(x + \frac{b-a}{2}\right) + f(x)$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$f - \text{concave} \Rightarrow f\left(\frac{(b-a)+2a}{2}\right) \geq \frac{f\left(\frac{b-a}{2}\right) + f\left(\frac{2a}{2}\right)}{2} \Rightarrow 2f\left(\frac{b-a}{2} + a\right) \geq f\left(\frac{b-a}{2}\right) + f(a)$$

**1480. If  $0 < a \leq b < 1$  then:**

$$\int_a^b x^{x-1} \cdot (1-x)^{1-x} dx \geq \log \sqrt{\frac{b}{a}}$$

*Proposed by Serlea Kabay-Liberia*

**Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco**

Let  $f(x) = x \log x, x \in (0, 1)$ .

We have :  $f''(x) = \frac{1}{x} > 0, \forall x \in (0, 1) \rightarrow f - \text{convex on } (0, 1)$

$$\stackrel{\text{Using Jensen}}{\Rightarrow} f(x) + f(1-x) \geq 2f\left(\frac{x+(1-x)}{2}\right) = 2f\left(\frac{1}{2}\right) = \log\left(\frac{1}{2}\right), \forall x \in (0, 1)$$

$$\leftrightarrow \log[x^x \cdot (1-x)^{1-x}] \geq \log\left(\frac{1}{2}\right) \leftrightarrow x^{x-1} \cdot (1-x)^{1-x} \geq \frac{1}{2x}, \forall x \in (0, 1)$$

$$\rightarrow \int_a^b x^{x-1} \cdot (1-x)^{1-x} dx \geq \int_a^b \frac{1}{2x} dx = \log \sqrt{\frac{b}{a}}$$

**1481. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \left( \lim_{x \rightarrow 2} \frac{\overbrace{\left( ((x)!)! \dots \right)!}^{n-\text{times}} - 2}{\underbrace{\left( ((x)!)! \dots \right)!}_{(n-1)-\text{times}} - 2} \right)$$

*Proposed by Mohamed Ahmed Nasery-Afghanistan*

**Solution by Amrit Awasthi-India**

Let's put:  $y = \underbrace{\left( ((x)!)! \dots \right)!}_{(n-1)-\text{times}}$ . Therefore as  $x \rightarrow 2$  also  $y \rightarrow 2$ , and we have:

$$\Omega = \lim_{y \rightarrow 2} \frac{\Gamma(y+1) - 2}{y-2} \stackrel{\left(\frac{0}{0}\right)}{=} \lim_{y \rightarrow 2} \Gamma'(y+1) = \lim_{y \rightarrow 2} \Gamma(y+1) \psi(y+1) =$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= \Gamma(3)\psi(3) = 2\left(\frac{3}{2} - \gamma\right) = 3 - 2\gamma$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \left( \lim_{x \rightarrow 2} \frac{\overbrace{\left( ((x)!)! \right)! \dots}^{n-times} - 2}{\underbrace{\left( ((x)!)! \right)! \dots}_{(n-1)-times} - 2} \right) = 3 - 2\gamma$$

**1482.** If  $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}, a_n, b_n \in \mathbb{R}_+^*, \forall n \in \mathbb{N}^*$  such that  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{n^2 \cdot a_n} = a \in \mathbb{R}_+^*$

and

$$\lim_{n \rightarrow \infty} \frac{b_{n+1}}{n^3 \cdot b_n} = b \in \mathbb{R}_+^* \text{ then find:}$$

$$\Omega = \lim_{n \rightarrow \infty} \left( \sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} - \sqrt[n]{\frac{b_n}{a_n}} \right)$$

*Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania*

*Solution by Marian Ursărescu-Romania*

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left( \sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} - \sqrt[n]{\frac{b_n}{a_n}} \right) = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{b_n}{a_n}} \left( \sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} \cdot \sqrt[n]{\frac{a_n}{b_n}} - 1 \right) = \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sqrt[n]{\frac{b_n}{a_n}} \cdot n \left( \sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} \cdot \sqrt[n]{\frac{a_n}{b_n}} - 1 \right); (1) \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sqrt[n]{\frac{b_n}{a_n}} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{b_n}{n^n \cdot a_n}} \stackrel{C-D}{=} \lim_{n \rightarrow \infty} \frac{b_{n+1}}{(n+1)^{n+1} \cdot a_{n+1}} \cdot \frac{n^n \cdot a_n}{b_n} = \\ &= \lim_{n \rightarrow \infty} \frac{b_{n+1}}{n^3 \cdot b_n} \cdot \frac{n^3 \cdot a_n}{a_{n+1}} \cdot \frac{n}{n+1} \cdot \left( \frac{n}{n+1} \right)^n = \frac{b}{ae}; (2) \end{aligned}$$

$$\lim_{n \rightarrow \infty} n \left( \sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} \cdot \sqrt[n]{\frac{a_n}{b_n}} - 1 \right) =$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 & n \left( e^{\log \sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}} \cdot \sqrt[n]{\frac{a_n}{b_n}}}} - 1 \right) \\
 &= \lim_{n \rightarrow \infty} \frac{n \left( e^{\log \sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}} \cdot \sqrt[n]{\frac{a_n}{b_n}}}} - 1 \right)}{\log \sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}} \cdot \sqrt[n]{\frac{a_n}{b_n}}}} \cdot \log \sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} \cdot \sqrt[n]{\frac{a_n}{b_n}} = \\
 &= \lim_{n \rightarrow \infty} \left[ \log \left( \sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} \right)^n \cdot \frac{a_n}{b_n} \right] = \log \left( \lim_{n \rightarrow \infty} \left( \frac{b_{n+1}}{a_{n+1}} \cdot \frac{a_n}{b_n} \cdot \sqrt[n+1]{\frac{a_{n+1}}{b_{n+1}}} \right) \right) \stackrel{(2)}{=} \\
 &= \log \left( b \cdot \frac{1}{a} \cdot 1 \cdot \frac{ae}{b} \right) = \log e = 1; (3)
 \end{aligned}$$

From (1), (2), (3) it follows that:

$$\Omega = \lim_{n \rightarrow \infty} \left( \sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} - \sqrt[n]{\frac{b_n}{a_n}} \right) = \frac{b}{ae}$$

**1483. If  $n \in \mathbb{N}$  then prove that:**

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+3)(k+5) \dots (k+2n+1)} = \frac{1}{2^n \cdot n!} \left| \sum_{k=0}^n \binom{n}{k} (-1)^k H_{2k+1} \right|$$

**$H_n$  – Harmonic Number.**

*Proposed by Asmat Qatea-Afghanistan*

**Solution by Ravi Prakash-New Delhi-India**

$$\begin{aligned}
 & \frac{1}{(k+1)(k+3)(k+5) \dots (k+2n+1)} = \\
 &= \sum_{r=0}^n \frac{1}{(-2r-1)(-2r-1+3) \dots (-2r-1+2r-1) \dots (-2r-1+2n+1)(k+2r+1)} \\
 &= \sum_{r=0}^n \frac{(-1)^r}{2^n(r!)(n-r)!} \cdot \frac{1}{k+2r+1} = \sum_{r=0}^n \frac{1}{n! \cdot 2^n} (-1)^r \binom{n}{k} \cdot \frac{1}{k+2r+1} = \\
 &= \frac{1}{n! \cdot 2^n} \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{k} \left[ \frac{1}{k+2r+1} - \frac{1}{k+2r+3} \right]
 \end{aligned}$$

Thus,



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 & \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+3)(k+5) \dots (k+2n+1)} = \\
 & = \frac{1}{n! \cdot 2^n} \sum_{r=1}^{n-1} \binom{n-1}{r} \left\{ \sum_{k=1}^{\infty} \left( \frac{1}{k+2r+1} - \frac{1}{k+2r+3} \right) \right\} = \\
 & = \frac{1}{n! \cdot 2^n} \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} \left( \frac{1}{2r+2} + \frac{1}{2r+3} \right) = \\
 & = \frac{1}{2^n \cdot n!} \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} (H_{2r+3} - H_{2r+1}) = \\
 & = -\frac{1}{2^n \cdot n!} \sum_{r=0}^{n-1} (-1)^r \left[ \binom{n-1}{r} H_{2r+1} - \binom{n-1}{r} H_{2r+3} \right] = \\
 & = -\frac{1}{2^n \cdot n!} \left[ \binom{n-1}{0} H_1 - \binom{n-1}{0} H_3 - \binom{n-1}{1} H_3 + \binom{n-1}{1} H_5 + \binom{n-1}{2} H_5 + \dots \right. \\
 & \quad \left. + (-1)^n \binom{n-1}{n-1} H_{2n+1} \right] = \\
 & = \frac{1}{2^n \cdot n!} \sum_{r=0}^n (-1)^r \binom{n}{r} H_{2r+1}
 \end{aligned}$$

**1484. For  $|z| \leq \frac{1}{16}$  prove the following generating function:**

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{\operatorname{sgn}^n(z)}{n} \binom{4n}{2n} z^n = \\
 & = 4 \log 2 - \log \left( 1 + \sqrt{1 - \operatorname{sgn}(z) 16z} \right) - 2 \log \left( \sqrt{2} + \sqrt{1 + \sqrt{1 - \operatorname{sgn}(z) 16z}} \right)
 \end{aligned}$$

*Proposed by Narendra Bhandari-Bajura-Nepal*

**Solution 1 by Kamel Benaicha-Algiers-Algerie**

$$\begin{aligned}
 \Omega(z) & \stackrel{|z| \leq \frac{1}{16}}{=} \sum_{n=1}^{\infty} \frac{\operatorname{sgn}^n(z)}{n} \binom{4n}{2n} z^n = \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{4n}}{n \cdot 2^{4n-1}} \cdot \frac{(4n)!}{((2n)!)^2} = \\
 & = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(16z \operatorname{sgn}(z))^n}{n} \cdot \frac{\Gamma(2n + \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(2n + 1)} =
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{t}\sqrt{1-t}} \sum_{n=0}^{\infty} \frac{1}{n} (16 sgn(z) t^2)^n dt = \\
 &= -\frac{1}{\pi} \int_0^1 \frac{\log(1 - 16z sgn(z)t^2)}{\sqrt{t}\sqrt{1-t}} dt \stackrel{u=\sqrt{t}}{\cong} -\frac{2}{\pi} \int_0^1 \frac{\log(1 - 16z sgn(z)u^4)}{\sqrt{1-u^2}} du = \\
 &\stackrel{u=\sin\theta}{\cong} -\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \log(1 - 16z sgn(z) \sin^4 \theta) d\theta = \\
 &= -\frac{2}{\pi} \left[ \int_0^{\frac{\pi}{2}} \log(1 - 4\sqrt{z sgn(z)} \sin^2 \theta) d\theta + \int_0^{\infty} \log(1 + 4\sqrt{z sgn(z)} \sin^2 \theta) d\theta \right] = \\
 &= -\frac{2}{\pi} \left[ \int_0^{\frac{\pi}{2}} \log(\cos^2 \theta + (1 - 4\sqrt{z sgn(z)}) \sin^2 \theta) d\theta + \int_0^{\frac{\pi}{2}} \log(\cos^2 \theta + (1 + 4\sqrt{z sgn(z)}) \sin^2 \theta) d\theta \right] \\
 &= -2 \left( \log\left(\frac{1 + \sqrt{1 - 4\sqrt{z sgn(z)}}}{2}\right) + \log\left(\frac{1 + \sqrt{1 + 4\sqrt{z sgn(z)}}}{2}\right) \right) \\
 &\therefore \sum_{n=1}^{\infty} \frac{sgn^n(z)}{n} \binom{4n}{2n} z^n = \\
 &= 4 \log 2 - 2 \left( \log\left(1 + \sqrt{1 - 4\sqrt{z sgn(z)}}\right) + \log\left(1 + \sqrt{1 + 4\sqrt{z sgn(z)}}\right) \right); (I) \\
 \text{Put: } x &= \sqrt{z sgn(z)}, \text{ so } f(x) = 2 \left( \log\left(1 + \sqrt{1 - 4\sqrt{x}}\right) + \log\left(1 + \sqrt{1 + 4\sqrt{x}}\right) \right) = \\
 &= 2 \log\left(1 + \sqrt{1 - 4\sqrt{x}} + \sqrt{1 + 4\sqrt{x}} + \sqrt{1 - 16x}\right) \\
 f(x) &= 2 \log\left(1 + \sqrt{1 - 16x} + \frac{(\sqrt{1 - 4\sqrt{x}} + \sqrt{1 + 4\sqrt{x}})^2}{\sqrt{1 - 4\sqrt{x}} + \sqrt{1 + 4\sqrt{x}}}\right) = \\
 &= 2 \log\left(1 + \sqrt{1 - 16x} + \frac{2(1 + \sqrt{1 - 16x})}{\sqrt{1 - 4\sqrt{x}} + \sqrt{1 + 4\sqrt{x}}}\right) = \\
 &= 2 \log(1 + \sqrt{1 - 16x}) + 2 \log\left(2 + \sqrt{1 - 4\sqrt{x}} + \sqrt{1 + 4\sqrt{x}}\right) - \log\left(\sqrt{1 - 4\sqrt{x}} + \sqrt{1 + 4\sqrt{x}}\right)^2 \\
 &= 2 \log(1 + \sqrt{1 - 16x}) + 2 \log\left((\sqrt{2})^2 + \sqrt{(\sqrt{1 - 4\sqrt{x}} + \sqrt{1 + 4\sqrt{x}})^2}\right) -
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 & -\log \left( 2 + 2\sqrt{1 + \sqrt{1 - 16x}} \right) = \\
 & = \log(1 + \sqrt{1 - 16x}) + 2 \log \left( (\sqrt{2})^2 + \sqrt{2} \sqrt{(1 + \sqrt{1 - 16x})} \right) - \log 2 = \\
 & = \log(1 + \sqrt{1 - 16x}) + 2 \log \left( \sqrt{2} + \sqrt{1 + \sqrt{1 - 16x}} \right); (II)
 \end{aligned}$$

From (I), (II) it follows that:

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{sgn^n(z)}{n} \binom{4n}{2n} z^n = \\
 & = 4 \log 2 - \log \left( 1 + \sqrt{1 - sgn(z)16z} \right) - 2 \log \left( \sqrt{2} + \sqrt{1 + \sqrt{1 - sgn(z)16z}} \right)
 \end{aligned}$$

Note:  $sgn(z)z \geq 0, \Gamma \left( n + \frac{1}{2} \right) = \frac{\sqrt{\pi}\Gamma(2n)}{2^{2n-1}\Gamma(n)}$

$$\int_0^{\frac{\pi}{2}} \log(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) d\alpha = \pi \log \left( \frac{|a| + |b|}{2} \right); (a, b) \neq (0, 0)$$

**Solution 2 by Felix Marin-Romania**

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{sgn^n(z)}{n} \binom{4n}{2n} \Big|_{|z| < \frac{1}{16}} = 2 \sum_{n=1}^{\infty} \underbrace{\frac{(-4\sqrt{|z|})^{2n}}{2n}}_{\binom{4n}{2n}} \left( -\frac{1}{2} \right) (-4)^{2n} = \\
 & = 2 \sum_{n=2}^{\infty} \frac{(-4\sqrt{|z|})^n}{n} \binom{-\frac{1}{2}}{n} \frac{1 + (-1)^n}{2} = f(-4\sqrt{z}) + f(4\sqrt{z})
 \end{aligned}$$

$$\begin{aligned}
 \text{Where, } f(\xi) &= \sum_{n=2}^{\infty} \binom{-\frac{1}{2}}{n} \frac{\xi^n}{n} = \sum_{n=2}^{\infty} \binom{-\frac{1}{2}}{n} \int_0^{\xi} t^{n-1} dt = \int_0^{\xi} \left[ \sum_{n=2}^{\infty} \binom{-\frac{1}{2}}{n} t^n \right] \frac{dt}{t} = \\
 &= \int_0^{\xi} \frac{(1+t)^{-\frac{1}{2}} - 1 - \frac{t}{2}}{t} dt \stackrel{t=x^2-1}{=} \int_1^{\sqrt{1+\xi}} \left( x - \frac{2}{1+x} \right) dx = \\
 &= \frac{1}{2} \xi + 2 \log 2 - 2 \log(1 + \sqrt{1 + \xi})
 \end{aligned}$$

Therefore,



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sum_{n=1}^{\infty} \frac{sgn^n(z)}{n} \binom{4n}{2n} \Big|_{|z|<\frac{1}{16}} = 4 \log 2 - 2 \log \left( 1 + \sqrt{1 - 4\sqrt{|z|}} \right) - 2 \log \left( 1 + \sqrt{1 + 4\sqrt{|z|}} \right)$$

**1485.**

$$\text{If } a, b \in \mathbb{R}, a + b = 1, e_n = \left(1 + \frac{1}{n}\right)^n, c_n = -\log n + \sum_{k=1}^n \frac{1}{k}$$

**Then find:**

$$\Omega = \lim_{n \rightarrow \infty} \left( (n+1)^a \sqrt[n+1]{((n+1)! c_n)^b} - n^a \sqrt[n]{(n! e_n)^b} \right)$$

*Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania*

**Solution by Mikael Bernardo-Mozambique**

$$\begin{aligned}
 \text{Let: } x_n &= (n+1)^a \sqrt[n+1]{((n+1)! c_n)^b} - n^a \sqrt[n]{(n! e_n)^b} = \\
 &= n^a \cdot \sqrt[n]{(n! e_n)^b} \cdot (u_n - 1) = n^a \cdot \sqrt[n]{(n! e_n)^b} \cdot \frac{u_n - 1}{\log u_n} \cdot \log u_n^n = \\
 &= \left( \frac{\sqrt[n]{n! e_n}}{n} \right)^b \cdot \frac{u_n - 1}{\log u_n} \cdot \log u_n^n, \forall n \geq 2, \text{ where } u_n = \left( \frac{n+1}{n} \right)^a \cdot \left( \frac{\sqrt[n+1]{(n+1)! c_n}}{\sqrt[n]{n! e_n}} \right)^b \\
 \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n]{n! e_n}}{n} \right)^b &= \left( \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right) \cdot \frac{\sqrt[n]{n!}}{n} \right)^b = \left( \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \right)^b = \left( \frac{1}{e} \right)^b; (1) \\
 \lim_{n \rightarrow \infty} u_n &= \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^a \cdot \left( \frac{\sqrt[n+1]{(n+1)! c_n}}{\sqrt[n+1]{(n+1)!}} \cdot \frac{n}{\sqrt[n]{n!}} \cdot \sqrt[n+1]{c_n} \cdot \frac{n+1}{n} \right)^b = \\
 &= 1 \cdot \left( \frac{1}{e} \cdot e \cdot 1 \cdot 1 \right) = 1; (2) \\
 \lim_{n \rightarrow \infty} \frac{u_n - 1}{\log u_n} &= 1; (3) \\
 \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^{an} \cdot \left( \frac{(n+1)! c_n}{n! e_n} \cdot \frac{1}{\sqrt[n+1]{(n+1)!} \cdot \sqrt[n+1]{c_n}} \right)^b = \\
 &= e^a \cdot \left( \lim_{n \rightarrow \infty} \frac{\gamma}{e} \cdot \frac{n+1}{\sqrt[n+1]{(n+1)!}} \cdot 1 \right) = e^a \cdot \left( \frac{\gamma}{e} \cdot e \right)^b = e \left( \frac{\gamma}{e} \right)^b; (4)
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \left( (n+1)^a \sqrt[n+1]{((n+1)! c_n)^b} - n^a \sqrt[n]{(n! e_n)^b} \right) = \left( \frac{1}{e} \right)^b \left( 1 + b \cdot \log \left( \frac{\gamma}{e} \right) \right)$$

**1486.**

$$e_n = \left( 1 + \frac{1}{n} \right)^n, \gamma_n = -\log n + \sum_{k=1}^n \frac{1}{k}$$

**Find:**

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!! \gamma_n}} - \frac{n^2}{\sqrt[n]{(2n-1)!! e_n}} \right)$$

*Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania*

*Solution by Mikael Bernardo-Mozambique*

$$\begin{aligned} \text{Let } v_n &= \frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!! \gamma_n}} - \frac{n^2}{\sqrt[n]{(2n-1)!! e_n}} = \\ &= \frac{n^2}{\sqrt[n]{(2n-1)!! e_n}} (u_n - 1) = \frac{n^2}{\sqrt[n]{(2n-1)!! e_n}} \cdot \frac{u_n - 1}{\log u_n} \cdot \log(u_n^n), \\ \text{where } u_n &= \left( \frac{n+1}{n} \right)^2 \cdot \frac{\sqrt[n]{(2n-1)!! e_n}}{\sqrt[n+1]{(2n+1)!! \gamma_n}} \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n^2}{\sqrt[n]{(2n-1)!! e_n}} &= \lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{\frac{(2n)!}{2^n \cdot n!} \cdot \left( 1 + \frac{1}{n} \right)^n}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!} \cdot 2n}{\sqrt[n]{(2n)!}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{((n!) \cdot 2n)^n}{(2n)!}} \stackrel{c-D}{=} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)! \cdot (2n+2)^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n! \cdot (2n)^n} = \lim_{n \rightarrow \infty} \frac{(n+1) \cdot e \cdot (2n+2)}{(2n+2)(2n+1)} = \frac{e}{2}; (1) \\ \lim_{n \rightarrow \infty} u_n &= \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^2 \cdot \frac{\sqrt[n]{(2n-1)!! e_n}}{\sqrt[n+1]{(2n+1)!! \gamma_n}} = \\ &= 1 \cdot \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n]{(2n-1)!!}}{n} \cdot \frac{n+1}{\sqrt[n+1]{(2n+1)!!}} \cdot \left( 1 + \frac{1}{n} \right) \cdot \frac{1}{\sqrt[n+1]{\gamma_n}} \cdot \frac{n}{n+1} \right) = \frac{2}{e} \cdot \frac{e}{2} \cdot 1 \cdot 1 \cdot 1 = 1 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\text{So, } \lim_{n \rightarrow \infty} \frac{u_n - 1}{\log u_n} = 1; (2)$$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} u_n^n &= \lim_{n \rightarrow \infty} \left( \left(1 + \frac{1}{n}\right)^{2n} \cdot \frac{(2n-1)!! e_n}{(2n+1)!! \gamma_n} \cdot \sqrt[n+1]{(2n+1)!! \gamma_n} \right) = \\
 &= \frac{e^3}{\gamma} \cdot \lim_{n \rightarrow \infty} \left( \frac{\frac{(2n)!}{2^n \cdot n!}}{\frac{(2n+2)!}{2^{n+1} \cdot (n+1)!}} \cdot (n+1) \cdot \frac{\sqrt[n+1]{(2n+1)!!}}{n+1} \cdot \sqrt[n+1]{\gamma_n} \right) = \\
 &= \frac{e^3}{\gamma} \cdot \lim_{n \rightarrow \infty} \left( \frac{(2n)! \cdot 2^{n+1} \cdot (n+1)!}{(2n+2)!} \cdot \frac{2(n+1)}{e} \cdot 1 \right) = \\
 &= \frac{e^3}{\gamma} \cdot \lim_{n \rightarrow \infty} \left( \frac{2(n+1)^2}{(2n+2)(2n+1)} \right) = \frac{2e^2}{\gamma} \cdot \frac{1}{2} \cdot \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^2}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right)} = \frac{e^2}{\gamma}; (3)
 \end{aligned}$$

From (1), (2), (3) it follows that:

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!! \gamma_n}} - \frac{n^2}{\sqrt[n]{(2n-1)!! e_n}} \right) = \frac{e}{2} \cdot 1 \cdot \log \left( \frac{e^2}{\gamma} \right) = \frac{e}{2} \left( 2 + \log \left( \frac{1}{\gamma} \right) \right)$$

**1487. Prove that:**

$$\lim_{n \rightarrow \infty} H_n = \lim_{n \rightarrow \infty} \left( \frac{1+2+\dots+n}{2!} - \frac{1^3+2^3+\dots+n^3}{4!} + \frac{1^5+2^5+\dots+n^5}{6!} - \dots + \dots \right) - \frac{1}{2} \log(2 - 2 \cos 1)$$

*Proposed by Amrit Awasthi-India*

**Solution by proposer**

We know that:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots - \dots + \dots$$

$$\frac{\cos x}{x} = \frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \dots - \dots + \dots$$

Taking summation from  $x = 1$  to  $x = n$  and making ' $n$ ' approach infinity, we have:

$$\lim_{n \rightarrow \infty} \sum_{x=1}^n \frac{\cos x}{x} = \lim_{n \rightarrow \infty} \left( \sum_{x=1}^n \frac{1}{x} - \sum_{k=1}^n \frac{x}{2!} + \sum_{k=1}^n \frac{x^3}{4!} - \sum_{k=1}^n \frac{x^5}{6!} + \dots - \dots + \dots \right)$$

Or we have:



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\lim_{n \rightarrow \infty} H_n = \lim_{n \rightarrow \infty} \left( \sum_{x=1}^n \frac{\cos x}{x} + \sum_{k=1}^n \frac{x}{2!} - \sum_{k=1}^n \frac{x^3}{4!} + \sum_{k=1}^n \frac{x^5}{6!} + \dots - \dots + \dots \right); (*)$$

**Now, consider**

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

Adding, we have:  $2 \cos x = e^{ix} + e^{-ix}$ .

Dividing both sides with ' $x$ ' and summing from  $x = 1$  to  $x = \infty$

$$\begin{aligned} 2 \sum_{x=1}^{\infty} \frac{\cos x}{x} &= \sum_{x=1}^{\infty} \frac{e^{ix} + e^{-ix}}{x} = -\log(1 - e^i) - \log(1 - e^{-i}) = \\ &= -\log \left( 1 - \underbrace{(e^i + e^{-i})}_{2 \cos 1} + 1 \right) = -\log(2 - 2 \cos 1) \\ \lim_{n \rightarrow \infty} \sum_{x=1}^n \frac{\cos x}{x} &= -\frac{1}{2} \log(2 - 2 \cos 1) \end{aligned}$$

Plugging the value in (\*) it follows that:

$$\begin{aligned} \lim_{n \rightarrow \infty} H_n &= \lim_{n \rightarrow \infty} \left( \frac{1+2+\dots+n}{2!} - \frac{1^3+2^3+\dots+n^3}{4!} + \frac{1^5+2^5+\dots+n^5}{6!} - \dots + \dots \right) \\ &\quad - \frac{1}{2} \log(2 - 2 \cos 1) \end{aligned}$$

**1488. If  $a_1 = 0$  and  $a_k = (-1)^k(k-2)! - (k-1)a_{k-1}$ , then**

$$\sum_{k=1}^{\infty} \frac{a_k}{(k+1)!} = \log^2 2 - 2 \log 2 + 1 \text{ and}$$

$$\sum_{k=1}^{\infty} a_k = \frac{e}{2} \left( \gamma^2 + \frac{\pi^2}{6} \right) - \frac{1}{2} \int_0^1 e^x \log^2(1-x) dx$$

where the first one converges and the second one diverges.

*Proposed by Angad Singh-India*

**Solution by proposer**

If  $|m| < 1, x \in \mathbb{R}$  then,



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$(1+m)^x = e^{x \log(1+m)} = \sum_{k=0}^{\infty} \frac{\log^k(1+m)}{k!} x^k \text{ also } (1+m)^x = \sum_{k=0}^{\infty} \binom{x}{k} m^k$$

**Comparing the coefficient of  $x$  in both the series, we have**

$$\log(1+m) = m - \frac{m^2}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \dots$$

**Similarly, comparing the coefficient of  $x^2$  in both the series, we have**

$$\frac{\log^2(1+m)}{2} = \frac{m^2}{2} - \frac{m^3}{2} + \frac{11m^4}{24} + \dots \text{ thus,}$$

$$\frac{\log^2(1+m)}{2} = \sum_{k=1}^{\infty} \frac{a_k}{k!} m^k, \text{ where } a_1 = 0 \text{ and } a_k = (-1)^k (k-2)! - (k-1)a_{k-1}.$$

**Now,**

$$\sum_{k=1}^{\infty} \frac{a_k}{(k+1)!} = \int_0^1 \frac{\log^2(1+m)}{2} dm = \log^2 2 - 2 \log 2 + 1$$

**Similarly,**

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{a_k}{k!} \int_0^{\infty} e^{-m} m^k dm = \frac{1}{2} \int_0^{\infty} e^{-m} \log^2(1+m) dm$$

**It is known that,**

$$\int_0^{\infty} e^{-m} m^{k-1} \log^2 m dm = \psi'(k)\Gamma(k) + \psi(k)\Gamma'(k) = \Gamma(k) (\psi'(k) + \psi^2(k))$$

**Substituting  $k = 1$ , we have,**

$$\int_0^{\infty} e^{-m} \log^2 m dm = \psi'(1) + \psi^2(1) = \gamma^2 + \frac{\pi^2}{6}$$

**Since,  $\psi(1) = -\gamma$  and  $\psi'(1) = \frac{\pi^2}{6}$ , now observe that,**

$$\int_0^{\infty} e^{-m} \log^2 m dm = \int_0^1 e^{m-1} \log^2(1-m) dm + \int_0^{\infty} e^{-m-1} \log^2(1+m) dm$$

**Thus,**

$$\frac{1}{2} \int_0^{\infty} e^{-m} \log^2(1+m) dm = \frac{e}{2} \int_0^{\infty} e^{-m} \log^2 m dm - \frac{1}{2} \int_0^1 e^m \log^2(1-m) dm$$

**which completes the proof.**



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**1489. Find:**

$$\Omega = \lim_{\substack{x \rightarrow 1 \\ x > 1}} \left( \frac{x^{x^{x^{\dots}}} - 1}{x - 1} \right)^{\frac{1}{\sqrt{x-1}}}$$

*Proposed by Mohammad Hamed Nasery-Afghanistan*

**Solution by Adrian Popa-Romania**

$$\begin{aligned}
 \Omega &= \lim_{\substack{x \rightarrow 1 \\ x > 1}} \left( \frac{x^{x^{x^{\dots}}} - 1}{x - 1} \right)^{\frac{1}{\sqrt{x-1}}} = \lim_{\substack{x \rightarrow 1 \\ x > 1}} \left[ \left( \frac{x^{x^{x^{\dots}}} - 1}{x - 1} \right)^{\frac{x-1}{x^{x^{x^{\dots}}} - 1}} \right]^{\frac{x^{x^{x^{\dots}}} - 1}{(x-1)\sqrt{x-1}}} = \\
 &= e^{\lim_{x \rightarrow 1} \frac{x^{x^{x^{\dots}}} - 1}{(x-1)\sqrt{x-1}}}; \quad x^{x^{x^{\dots}}} = a(x), a(x) = x^{a(x)} \rightarrow a'(x) = (e^{a(x) \log x})' = \\
 &\quad = x^{a(x)} \cdot \left( a'(x) \log x + \frac{a(x)}{x} \right) \\
 \rightarrow a''(x) &= (x^{a(x)})' \left( a'(x) \log x + \frac{a(x)}{x} \right) + x^{a(x)} \left( a''(x) \log x + \frac{a'(x)}{x} + \frac{x a'(x) - a(x)}{x^2} \right) \\
 \lim_{x \rightarrow 1} \frac{x^{x^{x^{\dots}}} - x}{(x-1)^{\frac{3}{2}}} &= \lim_{x \rightarrow 1} \frac{x^{x^{x^{\dots}}} \left( a'(x) \log x + \frac{a(x)}{x} \right) - 1}{\frac{3}{2}(x-1)^{\frac{1}{2}}} = \\
 &= \lim_{x \rightarrow 1} \frac{x^{x^{x^{\dots}}} \left( a'(x) \log x + \frac{a(x)}{x} \right)^2 + x^{x^{x^{\dots}}} \left( a''(x) \log x + \frac{a'(x)}{x} + \frac{x a'(x) - a(x)}{x^2} \right)}{\frac{3}{4}\sqrt{x-1}} = 1
 \end{aligned}$$

**1490.**  $H_n = \psi(n+1) - \gamma; n \in \mathbb{R}_+, Li_2(n) = \sum_{k=1}^{\infty} \frac{n^k}{k^2}; |n| < 1.$

**Find:**

$$\Omega = \lim_{n \rightarrow \frac{1}{2}} \frac{\log \left( 1 + \sqrt{H_n} - \sqrt{\frac{H_1}{2}} \right)}{\sqrt[3]{Li_2(n)} - \sqrt[3]{Li_2\left(\frac{1}{2}\right)}}$$

*Proposed by Mikael Bernardo-Mozambique*



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Solution by Asmat Qatea-Afghanistan**

$$\Omega = \lim_{n \rightarrow \frac{1}{2}} \frac{\log\left(1 + \sqrt{H_n} - \sqrt{H_{\frac{1}{2}}}\right)_{L'H}}{\sqrt[3]{Li_2(n)} - \sqrt[3]{Li_2\left(\frac{1}{2}\right)}} \stackrel{\psi_1\left(\frac{3}{2}\right)}{=} \frac{\frac{\psi_1\left(\frac{3}{2}\right)}{2 \cdot \sqrt{\psi\left(\frac{3}{2}\right) + \gamma}}}{\frac{2 \log 2}{3 \cdot \sqrt[3]{\left(Li_2\left(\frac{1}{2}\right)\right)^2}}} = \frac{3 \cdot \psi_1\left(\frac{3}{2}\right) \cdot \sqrt[3]{\left(Li_2\left(\frac{1}{2}\right)\right)^2}}{2 \log 2 \cdot \sqrt{\psi\left(\frac{3}{2}\right) + \gamma}}$$

Note:  $\psi(n+1) = H_n - \gamma \Rightarrow (H_n)' = \psi_1(n+1)$

$$\psi_1\left(\frac{3}{2}\right) = \frac{\pi^2}{2} - 4, \quad \psi\left(\frac{3}{2}\right) = 2 - \gamma - \log 4$$

$$Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2 2}{2}$$

$$Li_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \Rightarrow (Li_2(z))' = \sum_{k=1}^{\infty} \frac{z^{k-1}}{k} = \frac{1}{z} \sum_{k=1}^{\infty} \frac{z^k}{k} = -\frac{\log(1-z)}{z}$$

**1491. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \frac{\sqrt{1! + \sqrt{2! + \sqrt{3! + \dots + n!}}}}{H_n}$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Asmat Qatea-Afghanistan**

$$\begin{aligned} S &= \sqrt{1! + \sqrt{2! + \sqrt{3! + \dots + n!}}} \\ \sqrt{a + \sqrt{a + \dots + \sqrt{a}}} &= p \Rightarrow \sqrt{a+p} = p \Rightarrow p^2 - p - a = 0 \\ \Rightarrow p &= \frac{1 + \sqrt{1 + 4a}}{2} \Rightarrow a = 1 \Rightarrow p = \frac{1 + \sqrt{5}}{2} = \phi \end{aligned}$$

Hence,



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$R = \sqrt{2 + \sqrt{2^{2^1} + \sqrt{2^{2^2} + \dots + \sqrt{2^{2^n}}}}} = \sqrt{2} \cdot \left( \sqrt{1 + \sqrt{1 + \dots + \sqrt{1}}} \right) = \sqrt{2}\phi$$

$2^{2^{n-1}} \stackrel{(?)}{\geq} n! \Rightarrow n \in [1, \infty) \Rightarrow 2^{2^{n-1}}$  and  $n!$  are continuous on the given interval:

$$\lim_{n \rightarrow \infty} \frac{2^{2^{n-1}}}{n!} \quad (\text{apply d'Alembert's ratio test})$$

$$a_n = \frac{2^{2^{n-1}}}{n!} \Rightarrow a_{n+1} = \frac{2^{2^n}}{(n+1)!}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{2^{2^n}}{(n+1)!}}{\frac{2^{2^{n-1}}}{n!}} = \lim_{n \rightarrow \infty} \frac{2^{2^n-1}}{n+1} = \infty$$

$$\Rightarrow 2^{2^{n-1}} > n!$$

$n$	$2^{2^{n-1}}$	$n!$
1	2	1
2	4	2
3	16	6
4	256	24
5	$2^{16}$	120

Since  $2^{2^n}$  is strictly greater than  $n!$  therefore,  $S$  is convergent ( $S < R$ )

$$1 < S < \sqrt{2} \cdot \phi$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} \frac{\sqrt{1! + \sqrt{2! + \sqrt{3! + \dots + n!}}}}{H_n} = 0$$

**1492. Prove that:**

$$\frac{1}{1^5} + \frac{1}{1^5 + 2^5} + \frac{1}{1^5 + 2^5 + 3^5} + \dots = 60 - 4\pi^2 + 8\pi\sqrt{3} \tan\left(\frac{\pi\sqrt{3}}{2}\right)$$

*Proposed by Asmat Qatea-Afghanistan*

**Solution 1 by Ahmed Yackoube Chach-Mauritania**

$$\Omega = \frac{1}{1^5} + \frac{1}{1^5 + 2^5} + \frac{1}{1^5 + 2^5 + 3^5} + \dots = \sum_{k=1}^{\infty} \frac{1}{\sum_{k=1}^n k^5} = \sum_{k=1}^{\infty} \frac{12}{k^2(k+1)^2(2k^2+2k-1)}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= 12 \sum_{n=1}^{\infty} \left( -\frac{1}{n^2} + \frac{4}{2n^2 + 2n - 1} - \frac{1}{(n+1)^2} \right) = 48 \sum_{n=1}^{\infty} \frac{1}{2n^2 + 2n - 1} + 12 - 4\pi^2; (*) \\
 \sum_{n=1}^{\infty} \frac{1}{2n^2 + 2n - 1} &= 1 + \sum_{n=0}^{\infty} \frac{1}{2n^2 + 2n - 1} = 1 + \sum_{n=0}^{\infty} \frac{2}{4n^2 + 4n + 1 - 3} = \\
 &= 1 + \sum_{n=0}^{\infty} \frac{2}{(2n+1)^2 - (\sqrt{3})^2} = 1 + \sum_{n=0}^{\infty} \frac{2}{(2n+1+\sqrt{3})(2n+1-\sqrt{3})} = \\
 &= 1 + \frac{1}{2} \cdot \sum_{n=0}^{\infty} \frac{1}{\left(n + \frac{1+\sqrt{3}}{2}\right)\left(n + \frac{1-\sqrt{3}}{2}\right)} = 1 + \frac{1}{2} \cdot \frac{\psi_0\left(\frac{1+\sqrt{3}}{2}\right) - \psi_0\left(\frac{1-\sqrt{3}}{2}\right)}{\frac{1+\sqrt{3}}{2} - \frac{1-\sqrt{3}}{2}} = \\
 &= 1 - \frac{1}{2} \cdot \frac{\psi_0\left(1 - \frac{1+\sqrt{3}}{2}\right) - \psi_0\left(\frac{1+\sqrt{3}}{2}\right)}{\sqrt{3}} = 1 - \frac{\pi}{2\sqrt{3}} \cot\left(\pi \cdot \frac{1+\sqrt{3}}{2}\right)
 \end{aligned}$$

**Replacing in (\*) it follows that:**

$$\Omega = 48 \sum_{n=1}^{\infty} \frac{1}{2n^2 + 2n - 1} + 12 - 4\pi^2 = 12 - 4\pi^2 + 48 + 48 \cdot \frac{\pi}{2\sqrt{3}} \tan\left(\frac{\pi\sqrt{3}}{2}\right)$$

$$\text{Therefore: } \Omega = 60 - 4\pi^2 + 8\pi\sqrt{3} \tan\left(\frac{\pi\sqrt{3}}{2}\right)$$

**Solution 2 by Ogwuche Moses-Nigeria**

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{12}{n^2(n+1)^2(2n^2+2n-1)} &= 12 \left\{ \sum_{n=1}^{\infty} \left( \frac{4}{2n^2+2n-1} - \frac{1}{(n+1)^2} - \frac{1}{n^2} \right) \right\} = \\
 &= 12 \left\{ 4 \sum_{n=1}^{\infty} \frac{1}{2n^2+2n-1} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} - \sum_{n=1}^{\infty} \frac{1}{n^2} \right\} = 12(A - B - C) \\
 C &= \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}; \quad B = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} - 1 \\
 A &= \sum_{n=1}^{\infty} \frac{1}{2n^2+2n-1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2+n-\frac{1}{2}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2} =
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \frac{1}{2\sqrt{3}} \left\{ \left( \psi\left(1 + \frac{1+\sqrt{3}}{2}\right) - \psi\left(\frac{1+\sqrt{3}}{2}\right)\right) - \left( \psi\left(1 + \frac{1-\sqrt{3}}{2}\right) - \psi\left(\frac{1-\sqrt{3}}{2}\right)\right) \right. \\
 &\quad \left. + \left( \psi\left(1 - \frac{1-\sqrt{3}}{2}\right) - \psi\left(\frac{1-\sqrt{3}}{2}\right)\right) \right\}
 \end{aligned}$$

We know that:  $\psi(1+x) - \psi(x) = \frac{1}{x}$  and  $\psi(1-x) - \psi(x) = \pi \cdot \cot(\pi x)$ .

$$\begin{aligned}
 A &= \frac{1}{2\sqrt{3}} \left\{ \frac{2}{1+\sqrt{3}} - \frac{2}{1-\sqrt{3}} + \pi \cdot \cot\left(\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)\pi\right) \right\} = \\
 &= \frac{1}{2\sqrt{3}} \left\{ 2\sqrt{3} + \pi \cdot \tan\left(\frac{\pi\sqrt{3}}{2}\right) \right\} = 1 + \frac{\pi}{6}\sqrt{3} \cdot \tan\left(\frac{\pi\sqrt{3}}{2}\right)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{1}{1^5} + \frac{1}{1^5 + 2^5} + \frac{1}{1^5 + 2^5 + 3^5} + \dots &= 12(4A - B - C) = \\
 &= 12[4\left(1 + \frac{\pi}{6}\sqrt{3} \cdot \tan\left(\frac{\pi\sqrt{3}}{2}\right)\right) - \left(\frac{\pi^2}{6} - 1\right) - \frac{\pi^2}{6}] = \\
 &= 12\left[5 - \frac{\pi^2}{3} + \frac{2\pi\sqrt{3}}{3}\tan\left(\frac{\pi\sqrt{3}}{2}\right)\right] = 60 - 4\pi^2 + 8\pi\sqrt{3}\tan\left(\frac{\pi\sqrt{3}}{2}\right)
 \end{aligned}$$

**1493.**

$$x_1 = 1, x_2 = \frac{3}{2}, x_{n+2} = \frac{(n+2)! \cdot (5x_n - 2x_{n+1}) + 5n + 13}{3(n+2)!}$$

Find:

$$\Omega = \lim_{n \rightarrow \infty} 2^n x_n \left( \left( \tan^{-1} \frac{1}{n^2} + \tan^{-1} \frac{1}{(n+1)^2} \right) \sum_{k=2}^n k(k-1) \binom{n}{k} \right)^{-1}$$

*Proposed by Ruxandra Daniela Tonilă-Romania*

**Solution 1 by Adrian Popa-Romania**

$$x_{n+2} = \frac{(5x_n - 2x_{n+1})}{3} + \frac{5n + 13}{3(n+2)!} \Leftrightarrow$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$3x_{n+2} + 2x_{n+1} - 5x_n = \frac{5(n+2) + 3}{(n+2)!} \Leftrightarrow$$

$$3x_{n+2} + 2x_{n+1} - 5x_n = \frac{5}{(n+1)!} + \frac{3}{(n+2)!}$$

$$n = 1: 3x_3 + 2x_2 - 5x_1 = \frac{5}{2!} + \frac{3}{3!}$$

$$n = 2: 3x_4 + 2x_3 - 5x_2 = \frac{5}{3!} + \frac{3}{4!}$$

⋮

$$n = n-2: 3x_n + 2x_{n-1} - 5x_{n-2} = \frac{5}{(n-1)!} + \frac{3}{n!}$$

Adding these relations, it follows that:

$$5x_{n-1} + 3x_n - 3x_2 - 5x_1 = 5 \sum_{k=3}^n \frac{1}{(k-1)!} + 3 \sum_{k=3}^n \frac{1}{k!} \Rightarrow$$

$$5x_{n-1} + 3x_n = 5 \left( \sum_{k=3}^n \frac{1}{(k-1)!} + x_1 \right) + 3 \left( \sum_{k=3}^n \frac{1}{k!} + x_2 \right) \Rightarrow x_n = \sum_{k=3}^n \frac{1}{k!} + \frac{3}{2}$$

$$\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \Rightarrow e = 1 + 1 + \frac{1}{2} + \sum_{k=3}^{\infty} \frac{1}{k!} \Rightarrow \lim_{n \rightarrow \infty} x_n = e - 1$$

$$\Omega = \lim_{n \rightarrow \infty} \frac{2^n \cdot x_n}{\tan^{-1} \left( \frac{\frac{1}{n^2} + \frac{1}{(n+1)^2}}{1 - \frac{1}{n^2(n+1)^2}} \right) \cdot \sum_{k=2}^n k(k-1) \binom{n}{k}}$$

$$\because \frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1} \Rightarrow k \binom{n}{k} = n \binom{n-1}{k-1} \Rightarrow S = \sum_{k=2}^n k(k-1) \binom{n}{k}$$

$$= \sum_{k=2}^n n(k-1) \binom{n-1}{k-1}$$

$$\frac{k-1}{n-1} \binom{n-1}{k-1} = \binom{n-2}{k-2} = (n-1) \binom{n-2}{k-2} \Rightarrow S = n(n-1) \cdot 2^{n-2}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 \Rightarrow \Omega &= \lim_{n \rightarrow \infty} \frac{2^n \cdot x_n}{\tan^{-1} \left( \frac{\frac{1}{n^2} + \frac{1}{(n+1)^2}}{1 - \frac{1}{n^2(n+1)^2}} \right) \cdot n(n-1) \cdot 2^{n-2}} = \\
 &\lim_{n \rightarrow \infty} \frac{2^n \cdot x_n \cdot \frac{(n+1)^2 + n^2}{n^2(n+1)^2 - 1}}{\tan^{-1} \left( \frac{(n+1)^2 + n^2}{n^2(n+1)^2 - 1} \right) \cdot \frac{(n+1)^2 + n^2}{n^2(n+1)^2 - 1} \cdot n(n-1) \cdot 2^{n-2}} = \\
 &= \lim_{n \rightarrow \infty} \frac{2^n x_n \cdot 4}{(2n^2 + 2n + 1)(n^2 - n) \cdot 2^n} = \lim_{n \rightarrow \infty} \frac{4x_n}{2} = 2(e-1)
 \end{aligned}$$

**Solution 2 by Akerele Olofin-Nigeria**

$$\begin{aligned}
 x_{n+2} &= \frac{(5x_n - 2x_{n+1})}{3} + \frac{5n+13}{3(n+2)!} \Leftrightarrow \\
 3x_{n+2} + 2x_{n+1} - 5x_n &= \frac{5(n+2)+3}{(n+2)!} \Leftrightarrow \\
 3x_{n+2} + 2x_{n+1} - 5x_n &= \frac{5}{(n+1)!} + \frac{3}{(n+2)!} \\
 n = 1: \quad 3x_3 + 2x_2 - 5x_1 &= \frac{5}{2!} + \frac{3}{3!} \\
 n = 2: \quad 3x_4 + 2x_3 - 5x_2 &= \frac{5}{3!} + \frac{3}{4!} \\
 &\vdots \\
 n = n-2: \quad 3x_n + 2x_{n-1} - 5x_{n-2} &= \frac{5}{(n-1)!} + \frac{3}{n!}
 \end{aligned}$$

**Adding these relations, it follows that:**

$$\begin{aligned}
 5x_{n-1} + 3x_n - 3x_2 - 5x_1 &= 5 \sum_{k=3}^n \frac{1}{(k-1)!} + 3 \sum_{k=3}^n \frac{1}{k!} \Rightarrow \\
 5x_{n-1} + 3x_n &= 5 \left( \sum_{k=3}^n \frac{1}{(k-1)!} + x_1 \right) + 3 \left( \sum_{k=3}^n \frac{1}{k!} + x_2 \right) \Rightarrow x_n = \sum_{k=3}^n \frac{1}{k!} + \frac{3}{2} \\
 \lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} \left( \sum_{k=3}^n \frac{1}{k!} + \frac{3}{2} \right) = \lim_{n \rightarrow \infty} \left( \sum_{k=0}^n \frac{1}{k!} \right) - 1 = e - 1
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned} \because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \Rightarrow e = 1 + 1 + \frac{1}{2} + \sum_{k=3}^{\infty} \frac{1}{k!} \\ \Rightarrow \lim_{n \rightarrow \infty} x_n = e - 1; (*) \end{aligned}$$

$$\begin{aligned} \because \frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1} \Rightarrow k \binom{n}{k} = n \binom{n-1}{k-1} \Rightarrow S = \sum_{k=2}^n k(k-1) \binom{n}{k} \\ = \sum_{k=2}^n n(k-1) \binom{n-1}{k-1} \end{aligned}$$

$$\begin{aligned} \frac{k-1}{n-1} \binom{n-1}{k-1} = \binom{n-2}{k-2} = (n-1) \binom{n-2}{k-2} \Rightarrow S = n(n-1) \cdot 2^{n-2}; (**) \\ \Rightarrow \Omega = \lim_{n \rightarrow \infty} \frac{2^n \cdot x_n}{\tan^{-1} \left( \frac{\frac{1}{n^2} + \frac{1}{(n+1)^2}}{1 - \frac{1}{n^2(n+1)^2}} \right) \cdot n(n-1) \cdot 2^{n-2}} = \\ \lim_{n \rightarrow \infty} \frac{2^n \cdot x_n \cdot \frac{(n+1)^2 + n^2}{n^2(n+1)^2 - 1}}{\tan^{-1} \left( \frac{(n+1)^2 + n^2}{n^2(n+1)^2 - 1} \right) \cdot \frac{(n+1)^2 + n^2}{n^2(n+1)^2 - 1} \cdot n(n-1) \cdot 2^{n-2}} = \\ = \lim_{n \rightarrow \infty} \frac{2^n x_n \cdot 4}{\frac{(2n^2 + 2n + 1)(n^2 - n)}{n^2(n+1)^2 - 1} \cdot 2^n} = 2 \lim_{n \rightarrow \infty} x_n = 2(e - 1) \end{aligned}$$

**1494.** If  $(x_n)_{n \geq 0}$  verify  $x_0 = x_1 = 1$  and  $x_n = x_{n-1} - x_{n-2}$ , then prove that  $(x_n)_{n \geq 0}$  is periodic and find the general term of sequence.

*Proposed by Neculai Stanciu-Romania*

*Solution by Ravi Prakash-New Delhi-India*

Characteristic equation of  $x_n = x_{n-1} - x_{n-2}$  is  $t^2 - t + 1 = 0 \Rightarrow t = -\omega, -\omega^2$

$$x_n = A(-\omega)^2 + B(-\omega^2)^n$$

When  $n \in \{0, 1\} \Rightarrow \begin{cases} A + B = 1 \\ -A\omega - B\omega^2 = 1 \end{cases} \Rightarrow A = -\frac{1}{\omega-1}, B = \frac{\omega}{\omega-1}$ . Thus,

$$x_n = \frac{(-1)^{n+1}}{\omega-1} (\omega^n - \omega^{2n+1})$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

As  $x_{n+6} = x_n, \forall n \in \mathbb{N} \Rightarrow (x_n)_{n \geq 0}$  –is periodic.

**1495. Prove that:**

$$\sum_{n=1}^{\infty} \sum_{k=1}^n \int_0^{\infty} \frac{dx}{x} \frac{\sin x}{(n^4 \sin^2 x + k^4 \cos^2 x)} = \frac{7\pi^5}{720}$$

*Proposed by Narendra Bhandari-Bajura-Nepal*

**Solution by Rana Ranino-Setif-Algerie**

Using Lobachevsky's Formula:  $f(x) = f(\pi - x) = f(\pi + x)$

$$\begin{aligned} & \therefore \int_0^{\infty} f(x) \frac{\sin x}{x} dx = \int_0^{\frac{\pi}{2}} f(x) dx \\ \Omega &= \sum_{n=1}^{\infty} \sum_{k=1}^n \int_0^{\frac{\pi}{2}} \frac{dx}{(n^4 \sin^2 x + k^4 \cos^2 x)} \stackrel{t=\tan x}{=} \sum_{n=1}^{\infty} \sum_{k=1}^n \int_0^{\infty} \frac{dx}{n^4 t^2 + k^4} = \\ &= \frac{\pi}{2} \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{1}{n^2 k^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^n \frac{1}{k^2} \\ \Omega &= \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} = \frac{\pi}{2} \left( \frac{7}{4} \zeta(4) \right) = \frac{7\pi^5}{720} \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \sum_{k=1}^n \int_0^{\infty} \frac{dx}{x} \frac{\sin x}{(n^4 \sin^2 x + k^4 \cos^2 x)} = \frac{7\pi^5}{720}$$

**1496. Prove that:**

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \prod_{k=1}^m \left( 1 + \int_0^{\infty} \tanh \left( \frac{x}{n} \right) \frac{e^{-x} dx}{x} \right)^{\frac{n}{k}} \frac{1}{m} = e^{\gamma}$$

$e$  –is called Euler's number and  $\gamma$  –is Euler-Mascheroni constant.

*Proposed by Narendra Bhandari-Bajura-Nepal*

**Solution 1 by Surjeet Singhania-India**

Consider Laplace transformation  $\mathcal{L}(\tanh x) = \int_0^{\infty} e^{-sx} \tanh x dx$ .



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 \mathcal{L}(\tanh x) &= \int_0^\infty e^{-sx} \left( \frac{e^x - e^{-x}}{e^x + e^{-x}} \right) dx = \sum_{p=1}^{\infty} (-1)^{p-1} \int_0^\infty (e^{-sx-2px+2x} - e^{-sx-2px}) dx = \\
 &= \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{s+2p-2} - \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{s+2p} = A - B. \\
 B &= \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{s+2p} = \sum_{p=1}^{\infty} \left( -\frac{1}{4p+s} + \frac{1}{4p} + \frac{1}{4p-2+s} - \frac{1}{4p} \right) = \\
 &= -\frac{s-2}{4} \sum_{p=1}^{\infty} \frac{1}{p(4p+s-2)} + \frac{s}{4} \sum_{p=1}^{\infty} \frac{1}{p(4p+s)} = -\frac{1}{4} \psi^{(0)}\left(\frac{s+2}{4}\right) + \frac{1}{4} \psi^{(0)}\left(\frac{s+4}{4}\right)
 \end{aligned}$$

Similarly,

$$A = -\frac{1}{4} \psi^{(0)}\left(\frac{s}{4}\right) + \frac{1}{4} \psi^{(0)}\left(\frac{s+4}{4}\right), \text{ since } \psi^{(0)}(z) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z-1)}$$

$$\text{Hence, } \mathcal{L}(\tanh x) = -\frac{1}{s} - \frac{1}{2} \psi^{(0)}\left(\frac{s}{4}\right) + \frac{1}{2} \psi^{(0)}\left(\frac{s+2}{4}\right)$$

$$\mathcal{L}_s\left(\frac{\tanh x}{x}\right) = \int_s^\infty \mathcal{L}_p(\tanh x) dp = 2 \log \left| \frac{\Gamma\left(\frac{p+2}{4}\right)}{\sqrt{p} \Gamma\left(\frac{p}{4}\right)} \right|_s^\infty = 2 \log \left( \frac{\sqrt{s} \Gamma\left(\frac{s}{4}\right)}{2 \Gamma\left(\frac{s+2}{4}\right)} \right)$$

Hence,

$$\mathcal{L}_s\left(\frac{\tanh x}{x}\right) = 2 \log \left( \frac{\sqrt{s} \Gamma\left(\frac{s}{4}\right)}{2 \Gamma\left(\frac{s+2}{4}\right)} \right)$$

$$\int_0^\infty \tanh x \frac{e^{-nx} dx}{x} = \int_0^\infty \tanh\left(\frac{x}{n}\right) \frac{e^{-x} dx}{x} = 2 \log \left( \frac{\sqrt{s} \Gamma\left(\frac{s}{4}\right)}{2 \Gamma\left(\frac{s+2}{4}\right)} \right)$$

We can expand it about Infinity

$$\int_0^\infty \tanh\left(\frac{x}{n}\right) \frac{e^{-x} dx}{x} = \underbrace{\frac{1}{n} - \frac{2}{3n^2} + o\left(\frac{1}{n^5}\right)}_{\text{Laurent series}}$$

Now,

$$\lim_{n \rightarrow \infty} \prod_{k=1}^m \left( 1 + \int_0^\infty \tanh\left(\frac{x}{n}\right) \frac{e^{-x} dx}{x} \right)^{\frac{n}{k}} \frac{1}{m} = \lim_{n \rightarrow \infty} \exp \left( \frac{n}{k} \log \left( 1 + \frac{1}{n} - \frac{2}{3n^2} + o\left(\frac{1}{n^5}\right) \right) \right)$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\lim_{n \rightarrow \infty} \prod_{k=1}^m \left( 1 + \int_0^\infty \tanh\left(\frac{x}{n}\right) \frac{e^{-x} dx}{x} \right)^{\frac{n}{k}} \frac{1}{m} = \exp\left(\frac{1}{k}\right)$$

Therefore,

$$\begin{aligned} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \prod_{k=1}^m \left( 1 + \int_0^\infty \tanh\left(\frac{x}{n}\right) \frac{e^{-x} dx}{x} \right)^{\frac{n}{k}} \frac{1}{m} &= \lim_{m \rightarrow \infty} \prod_{k=1}^m \exp\left(\frac{1}{k}\right) \frac{1}{m} = \\ &= \lim_{m \rightarrow \infty} \exp\left(-\log m + \sum_{k=1}^m \frac{1}{k}\right) = e^\gamma \end{aligned}$$

**Solution 2 by proposer**

Let function  $\lambda_n(x) = \tanh\left(\frac{x}{n}\right) \frac{e^{-x}}{x}$ ,  $\forall x \in (0, \infty)$  and since with the change of variable

$x \rightarrow nx$  we have:

$$\int_0^\infty \lambda_n(x) dx = \int_0^\infty \tanh(x) x^{-1} e^{-nx} dx$$

We observe that  $|\lambda_n(x)| \leq e^{-nx} = x_n(x)$  since the point wise limit of  $\lambda_n(x)$ ,  $\forall x \in (0, \infty)$  is  $e^{-nx}$  as  $n \rightarrow \infty$  and  $\int_0^\infty x_n(x) dx = \frac{1}{n}$  and due to Lebesgue dominating convergence theorem.

$$\lim_{n \rightarrow \infty} \int_0^\infty \lambda_n(x) dx = \lim_{n \rightarrow \infty} \int_0^\infty x_n(x) dx = 0; (1)$$

And thus we have by (1):

$$\begin{aligned} \lim_{m \rightarrow \infty} \prod_{k=1}^m \exp\left(\frac{1}{k}\right) \frac{1}{m} &= \\ &= \lim_{m \rightarrow \infty} \exp(H_m - \log m) = e^\gamma \end{aligned}$$

**Solution 3 by Mikael Bernardo-Mozambique**

$$\text{Let } L = \lim_{n \rightarrow \infty} \prod_{k=1}^m \left( 1 + \int_0^\infty \tanh\left(\frac{x}{n}\right) \frac{e^{-x} dx}{x} \right)^n \stackrel{x \rightarrow x}{=} \exp \left\{ \lim_{n \rightarrow \infty} n \cdot \int_0^\infty \tanh(x) \frac{e^{-nx} dx}{x} \right\}$$

By Maz identity:  $I = \int_0^\infty h(t) G(t) dt = \int_0^\infty H(s) g(s) ds$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 L &= \exp\left\{\lim_{n \rightarrow \infty} n \cdot \int_0^{\infty} \mathcal{L}(e^{-nx} \cdot \tanh(x)) \cdot \mathcal{L}^{-1}\left(\frac{1}{x}\right) dx\right\} \\
 &\because \mathcal{L}(e^{-nx} \tanh(x)) = -\frac{1}{s+n} - \frac{1}{2} \psi\left(\frac{s+n}{4}\right) + \frac{1}{2} \psi\left(\frac{s+n+2}{4}\right) \\
 L &= \exp\left\{\lim_{n \rightarrow \infty} n \int_0^{\infty} \left(-\frac{1}{s+n} - \frac{1}{2} \psi\left(\frac{s+n}{4}\right) + \frac{1}{2} \psi\left(\frac{s+n+2}{4}\right)\right) \cdot 1 ds\right\} = \\
 &= \exp\left\{\lim_{n \rightarrow \infty} n \cdot \left[-\log(s+n) - 2 \log\left(\Gamma\left(\frac{s+n}{4}\right)\right) + 2 \log\left(\Gamma\left(\frac{s+n+2}{4}\right)\right)\right]_0^{\infty}\right\} = \\
 &= \exp\left\{\lim_{n \rightarrow \infty} 2n \cdot \left[\log\left(\frac{\Gamma\left(\frac{s+n+2}{4}\right)}{\sqrt{s+n} \cdot \Gamma\left(\frac{s+n}{4}\right)}\right)\right]_0^{\infty}\right\} \\
 \lim_{s \rightarrow \infty} \log\left(\frac{\Gamma\left(\frac{s+n+2}{4}\right)}{\sqrt{s+n} \cdot \Gamma\left(\frac{s+n}{4}\right)}\right) &= \lim_{s \rightarrow \infty} \log\left(\frac{\Gamma\left(\frac{s+2}{4}\right)}{\sqrt{s} \cdot \Gamma\left(\frac{s}{4}\right)}\right) = \lim_{s \rightarrow \infty} \log\left(\frac{\left(\frac{s}{4}\right)^{\frac{1}{2}-0}}{\sqrt{s}}\right) = \log\left(\frac{1}{2}\right) \\
 &\because \frac{\Gamma(n+a)}{\Gamma(n+b)} \cong n^{a-b} \left(1 + \frac{(a+b-1)(a-b)}{2n} + o\left(\frac{1}{n^2}\right)\right) \\
 L &= \exp\left\{\lim_{n \rightarrow \infty} 2n \cdot \log\left(\frac{\sqrt{n} \cdot \Gamma\left(\frac{n}{4}\right)}{2 \cdot \Gamma\left(\frac{n+2}{4}\right)}\right)\right\} = \\
 &= \exp\left\{\lim_{n \rightarrow \infty} 2n \cdot \log\left(\frac{\sqrt{n} \cdot \left(\frac{n}{4}\right)^{-\frac{1}{2}}}{2} \left(1 + \frac{\left(\frac{1}{2}-1\right)\left(-\frac{1}{2}\right)}{2 \cdot \frac{n}{4}} + o\left(\frac{1}{n^2}\right)\right)\right)\right\} = \\
 &= \exp\left\{\lim_{n \rightarrow \infty} 2n \cdot \log\left(1 + \frac{1}{2n} + o\left(\frac{1}{n^2}\right)\right)\right\} = \exp\left\{\lim_{n \rightarrow \infty} \frac{\log\left(1 + \frac{1}{2n} + o\left(\frac{1}{n^2}\right)\right)}{\frac{1}{2n}}\right\} = e
 \end{aligned}$$

Therefore,

$$\Omega = \lim_{m \rightarrow \infty} \prod_{k=1}^m e^{\frac{1}{k}} \frac{1}{m} = e^{\lim_{m \rightarrow \infty} H_m - \log m} = e^{\gamma}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**1497. Prove that:**

$$\sum_{k=1}^{\infty} \left( \frac{1}{k(k+2)(k+4) \dots (k+2n)} - \frac{1}{(k+1)(k+3) \dots (k+2n+1)} \right) = \frac{\sqrt{\pi}}{2^{n+1} \left(n + \frac{1}{2}\right)!}$$

*Proposed by Asmat Qatea-Afghanistan*

**Solution by Felix Marin-Romania**

$$\begin{aligned} & \sum_{k=1}^{\infty} \left( \frac{1}{k(k+2)(k+4) \dots (k+2n)} - \frac{1}{(k+1)(k+3) \dots (k+2n+1)} \right) = \\ &= \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+3) \dots (k+2n+1)} - \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+3) \dots (k+2n+1)} = \\ &= \frac{1}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1)} = \frac{1}{\prod_{k=0}^n (2k+1)} = \frac{1}{2^{n+1} \prod_{k=0}^n \left(k + \frac{1}{2}\right)} = \frac{1}{2^{n+1} \left(\frac{1}{2}\right)^{\frac{n+1}{2}}} = \\ &= \frac{\Gamma\left(\frac{1}{2}\right)}{2^{n+1} \Gamma\left(\frac{1}{2} + n + 1\right)} = \frac{\sqrt{\pi}}{2^{n+1} \left(n + \frac{1}{2}\right)!} \end{aligned}$$

**1498.**

$$\Omega = \lim_{n \rightarrow \infty} \log \left\{ \frac{\sqrt[n^2]{n\$}}{\sqrt{n}} \right\} = -\frac{a}{b} \lim_{n \rightarrow 0} {}_2F_1(1, 1; 2; -n)$$

**Find the value of  $\log_2(b-a)$ ,  $\forall b > a$ , \$ – suprefactorial,**

**${}_2F_1(a, b; c; x)$  – Gauss hypergeometric function.**

*Proposed by Muhammad Afzal-Pakistan*

**Solution by Amrit Awasthi-India**

Let  $P(n)$  – be the superfactorial of  $n$ , then  $P(n) = n! (n-1)! (n-2)! \dots 3! 2! 1!$ .

This can be rewritten as  $P(n) = 1^n 2^{n-1} 3^{n-2} \dots (n-1)^2 n^1$ .

$$P(n) = \prod_{k=1}^n k^{n-k+1}$$

Taking  $\log$  both sides and rearranging



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\log P(n) = (n+1) \sum_{k=1}^n \log k - \sum_{k=1}^n k \log k$$

$$\begin{aligned}\log P(n) &= \log(\Gamma(n+1))^{n+1} - \log H(n), \text{ where } H(n) \\ &= \prod_{k=1}^n k^k \text{ is the hyperfactorial of } n\end{aligned}$$

$$P(n) = \frac{(\Gamma(n+1))^{n+1}}{H(n)}, \quad \text{where } G(n) - \text{Barnes } G - \text{function.}$$

Then,  $P(n) = G(n+1)\Gamma(n+1)$ . Thus,

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n^2]{P(n)}}{\sqrt{n}} = \lim_{n \rightarrow \infty} \left[ \frac{1}{n^2} (\log G(n) + \log \Gamma(n+1)) - \frac{1}{2} \log n \right]$$

Now, using Stirling approximation we have:  $\log(G(n+1)) \cong n^2 \left( \frac{1}{2} \log n - \frac{3}{4} \right)$  and  $\log \Gamma(n+1) \cong n \log n$ . Putting in the equation and solving we have:

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n^2]{P(n)}}{\sqrt{n}} = -\frac{3}{4}$$

We know that  ${}_2F_1(1, 1; 2; -n) = \frac{\log(1+n)}{n}$  therefore when  $n$  approaches zero using L'Hopital's rule we have,

$$-\frac{a}{b} \lim_{n \rightarrow 0} {}_2F_1(1, 1; 2; -n) = -\frac{a}{b} \lim_{n \rightarrow 0} \frac{\log(1+n)}{n} = -\frac{a}{b}$$

$$\text{Hence, } -\frac{3}{4} = -\frac{a}{b} \Rightarrow (a, b) \in \{(3k, 4k) | k \in \mathbb{N}\}$$

$$\log_2(b-a) = \log_2 1 = 0.$$

**1499. If  $f(x) = \tanh(n\pi x)$  and  $g(x) = x^2 + 1$  for all  $x \in [0, \infty)$ ,**

$$\chi_{n,m} = n^m - \Gamma\left(\frac{1}{n^m}\right) \text{ and}$$

**$\lambda(k, j) = j! (k-j)!$  where  $m$  is fixed positive real number, then prove that:**

$$\lim_{n \rightarrow \infty} \prod_{k=0}^{\infty} \prod_{j=0}^k \left( 1 + \int_0^{\infty} \frac{f(x)}{g^2(x)} \frac{dx}{n} \right)^{\frac{2n\chi_{n,m}}{\lambda(k,j)}} = e^{\gamma e^2}$$

**Where  $e, \gamma$  –denotes Euler-number and Euler-Mascheroni constant respectively.**

*Proposed by Narendra Bhandari-Nepal, Kaushik Mahanta-India,*

*Surjeet Singhania-India, Shivam Sharma-India*



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Solution by Artan Ajredini-Presheva-Serbie**

Let  $h_n(x) = \frac{x \tanh(n\pi x)}{n(x^2+1)^2}$ , with  $x \in [0, \infty)$ . We have:  $|h_n(x)| \leq \frac{x}{n(x^2+1)^2} = \phi(x)$  and

$$\int_0^\infty \phi(x) dx = \frac{1}{2n}$$

Also,  $\lim_{n \rightarrow \infty} h_n(x) = 0, \forall x \in [0, \infty)$ . Thus, by Lebesgue Dominated Convergence Theorem we

have:

$$\lim_{n \rightarrow \infty} \int_0^\infty h_n(x) dx = 0; \quad (1)$$

Let  $u_n(x) = \frac{2x \tanh(n\pi x)}{(x^2+1)^2} \chi_{n,m}$ . Also, using by Lebesgue Dominated Convergence Theorem

we get:

$$\lim_{n \rightarrow \infty} \int_0^\infty u_n(x) dx = \gamma; \quad (2)$$

Since  $|u_n(x)| \leq \frac{2x}{(x^2+1)^2} \chi_{n,m} = \phi(x)$ , (also  $\int_0^\infty \phi(x) dx = \chi_{n,m}$  and  $\lim_{n \rightarrow \infty} u_n(x) = \gamma$ )

$$\left( \text{using } \lim_{n \rightarrow \infty} \left( n^m - \Gamma\left(\frac{1}{n^m}\right) \right) = \gamma \right)$$

Now, by (1), (2) we have:

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \prod_{k=0}^{\infty} \prod_{j=0}^k \left( 1 + \int_0^\infty \frac{f(x)}{g^2(x)} \frac{dx}{n} \right)^{\frac{2n\chi_{n,m}}{\lambda(k,j)}} = \prod_{k=0}^{\infty} \prod_{j=0}^k e^{\frac{\gamma}{\lambda(k,j)}} \\ &= \exp \left( \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\gamma}{j!(k-j)!} \right); \quad (3) \end{aligned}$$

Applying Cauchy Product Theorem in (3) we obtain:

$$L = e^{\gamma e^2}.$$

**1500. Given two sequences of positive real numbers  $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$  such that**



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = e \text{ and } b_n = \sqrt[n]{\frac{n!}{\sum_{k=1}^n \frac{k^k}{n^n}}}. \text{ Find:}$$

$$\Omega = \lim_{n \rightarrow \infty} \left( \frac{a_{n+1} b_{n+1}}{n+1} - \frac{a_n b_n}{n} \right)$$

*Proposed by Ty Halpen-Florida-SUA*

**Solution by Ruxandra Daniela Tonilă-Romania**

Suppose that exist  $l \in \mathbb{R}$  such that  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} = l \Rightarrow \lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0$

impossible. Thus,  $\lim_{n \rightarrow \infty} a_n = +\infty$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{b_n}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n!}{\sum_{k=1}^n \frac{k^k}{n^n}}} \cdot \frac{1}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n \cdot n!}{\sum_{k=1}^n k^k} \cdot \frac{1}{n^n}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n!}{\sum_{k=1}^n k^k}} \stackrel{C-D}{=} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{\sum_{k=1}^{n+1} k^k} \cdot \frac{\sum_{k=1}^n k^k}{n!} = \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n k^k}{\sum_{k=1}^{n+1} k^k} \cdot (n+1) = \\ &= \lim_{n \rightarrow \infty} \frac{(1^2 + 2^2 + \dots + n^2)(n+1)}{1^2 + 2^2 + \dots + (n+1)^2} = \lim_{n \rightarrow \infty} \frac{(n+1)n^n}{(n+1)^{n+1}} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n = \frac{1}{e}; \quad (1) \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \frac{a_{n+1} b_{n+1}}{n+1} - \frac{a_n b_n}{n} \right) &= \lim_{n \rightarrow \infty} \frac{a_n b_n}{n} \left( \frac{a_{n+1} b_{n+1}}{a_n b_n} - 1 \right) = \\ &= \lim_{n \rightarrow \infty} \frac{a_n}{n} \cdot \frac{b_n}{n} \cdot n \left( \frac{a_{n+1}}{n+1} \cdot \frac{b_{n+1}}{n+1} \cdot \frac{n}{b_n} - 1 \right) = \\ &= \lim_{n \rightarrow \infty} \frac{a_n}{n} \cdot \lim_{n \rightarrow \infty} \frac{b_n}{n} \cdot \lim_{n \rightarrow \infty} n \cdot \frac{e^{\log\left(\frac{a_{n+1}}{a_n} \cdot \frac{b_{n+1}}{b_n} \cdot \frac{n}{b_n}\right)} - 1}{\log\left(\frac{a_{n+1}}{a_n} \cdot \frac{b_{n+1}}{b_n} \cdot \frac{n}{b_n}\right)} \cdot \log\left(\frac{a_{n+1}}{a_n} \cdot \frac{b_{n+1}}{b_n} \cdot \frac{n}{b_n}\right) = \\ &= \lim_{n \rightarrow \infty} \frac{a_n}{n} \cdot \lim_{n \rightarrow \infty} \frac{b_n}{n} \cdot \lim_{n \rightarrow \infty} n \log\left(\frac{a_{n+1}}{a_n} \cdot \frac{b_{n+1}}{b_n} \cdot \frac{n}{b_n}\right); \quad (2) \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} \stackrel{C-D}{=} \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{n+1 - n} = e; \quad (3)$$

$$\lim_{n \rightarrow \infty} n \log\left(\frac{a_{n+1}}{a_n} \cdot \frac{b_{n+1}}{b_n} \cdot \frac{n}{b_n}\right) = \log\left(\lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \cdot \frac{b_{n+1}}{b_n} \cdot \frac{n}{b_n}\right)^n\right) =$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= \log \left( \lim_{n \rightarrow \infty} \left( \frac{a_{n+1}}{a_n} \right)^n \cdot \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n \cdot \lim_{n \rightarrow \infty} \left( \frac{b_{n+1}}{b_n} \right)^n \right); (4)$$

$$\lim_{n \rightarrow \infty} \left( \frac{a_{n+1}}{a_n} \right)^n = \lim_{n \rightarrow \infty} \left( 1 + \frac{a_{n+1} - a_n}{a_n} \right)^n =$$

$$= \lim_{n \rightarrow \infty} \left[ \left( 1 + \frac{a_{n+1} - a_n}{a_n} \right)^{\frac{a_n}{a_{n+1}-a_n}} \right]^{n \cdot \frac{a_{n+1}-a_n}{a_n}} = e^{\lim_{n \rightarrow \infty} (a_{n+1}-a_n) \cdot \lim_{n \rightarrow \infty} \left( \frac{n}{a_n} \right)} \stackrel{(3)}{=} e^{e \cdot \frac{1}{e}} = e; (5)$$

$$\lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n = \frac{1}{e}; (6)$$

$$\lim_{n \rightarrow \infty} \left( \frac{b_{n+1}}{n+1} \right)^n \stackrel{c-d}{=} \lim_{n \rightarrow \infty} \left( \frac{b_{n+1}}{n+1} \cdot \frac{n}{b_n} \cdot \frac{n+1}{n} \right)^n = \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^n = e; (7)$$

**From (4)~(7) it follows that**

$$\lim_{n \rightarrow \infty} n \log \left( \frac{a_{n+1}}{a_n} \cdot \frac{b_{n+1}}{n+1} \cdot \frac{n}{b_n} \right) = \log \left( e \cdot \frac{1}{e} \cdot e \right) = 1; (8)$$

**From (1), (2), (8) it follows that:**

$$\lim_{n \rightarrow \infty} \left( \frac{a_{n+1} \cdot b_{n+1}}{n+1} - \frac{a_n \cdot b_n}{n} \right) = e \cdot \frac{1}{e} \cdot 1 = 1$$



ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

*It's nice to be important but more important it's to be nice.*

*At this paper works a TEAM.*

*This is RMM TEAM.*

*To be continued!*

*Daniel Sitaru*