

*RMM - Calculus Marathon 1401 - 1500*

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor  
DANIEL SITARU

Available online  
[www.ssmrmh.ro](http://www.ssmrmh.ro)

ISSN-L 2501-0099



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

*Proposed by*

*Daniel Sitaru – Romania, Florică Anastase-Romania, Max Wong-Hong Kong,  
Srinivasa Raghava-AIRMC-India, Vasile Mircea Popa – Romania, Angad Singh-  
India, D.M. Bătinețu-Giurgiu-Romania, Neculai Stanciu – Romania, Naren  
Bhandari -Nepal, Asmat Qatea-Afghanistan, Arhgyadeep Chatterjee-India,  
Marian Ursărescu – Romania, Surjeet Singhania-India, Samir Cabiyev-  
Azerbaijan, Tobi Joshua-Nigeria, Akerele Olofin-Nigeria, Amrit Awasthi-  
India, Ty Halpen-USA, Ahmed Yackoube Chach-Mauritania, Simon Peter-  
Madagascar, Santiago Alvarez-Mexico, Ngulmun George Baite-India  
Ankush Kumar Parcha-India, Jaihon Obaidullah-Afghanistan, Izumi Ainsworth-  
Peru, Kaushik Mahanta-India, Muhammad Afzal-Pakistan, Dan Radu Seclăman-  
Romania, Serlea Kabay-Liberia, Mohamed Ahmed Nasery-Afghanistan  
Ruxandra Daniela Tonilă-Romania, Shivam Sharma-India*



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

*Solutions by*

*Daniel Sitaru – Romania, Adrian Popa-Romania, Mohammed Diai-Morocco,  
Ruxandra Daniela Tonilă-Romania, Ravi Prakash-India, Surjeet Singhania-  
India, Akerele Olofin-Nigeria, Kamel Benaicha-Algerie, Srinivasa Raghava-  
AIRMC-India, Ahmed Yackoube Chach-Mauritania, Max Wong-Hong Kong,  
Probal Chakraborty-India, Syed Shahabudeen-India, Angad Singh-India,  
Asmat Qatea-Afghanistan, Samar Das-India, Mikael Bernardo-Mozambique,  
Mohammad Rostami-Afghanistan, Muhammad Afzal-Pakistan, Naren  
Bhandari-Nepal, Amrit Awasthi-India, Serlea Kabay-Liberia, Igor Soposki-  
Macedonia, Yen Tung Chung-Taichung-Taiwan, Timson Azeez Folorunsho-  
Nigeria, Hemn Hsain-Egypt, Marian Ursărescu-Romania, Rana Ranino-Algerie,  
Ngulmun George Baite-India, Florică Anastase-Romania, Kaushik Mahanta-  
India, Zaharia Burghelea-Romania, Felix Marin-Romania, Santiago Alvarez-  
Mexico, Ngulmun George Baite-India, Izumi Ainsworth- Peru, Heimn Hsain-Iran  
Hussain Reza Zadah-Afghanistan, Remus Florin Stanca-Romania, Arslan  
Ahmed-Yemen, Fayssal Abdelli-Algerie, Mohammad Hamed Nasery-  
Afghanistan, Sire Ambrose-Albania, Ajentunmobi Abdulqooyum-Nigeria  
Dawid Bialek-Poland, Obaidullah Jaihon-Afghanistan, Dan Radu Seclăman-  
Romania, Ahmed Yackoube Chach-Mauritania, Ogwuche Moses-Nigeria  
Artan Ajredini-Presheva-Serbie*



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**1401. If  $0 < a < b < 1$ , then:**

$$\int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{b + a\sin x} dx \cdot \int_0^{\frac{\pi}{4}} \frac{1}{b + a\sin x} dx \geq \frac{\pi}{4} \left( \frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}} \right)$$

*Proposed by Florică Anastase-Romania*

**Solution 1 by Adrian Popa-Romania**

Let:  $f, g: [0, \frac{\pi}{4}] \rightarrow \mathbb{R}, f(x) = \frac{a + b\sin x}{b + a\sin x}, g(x) = \frac{1}{b + a\sin x}$  derivable with

$$f'(x) = \frac{(b^2 - a^2)\cos x}{(b + a\sin x)^2} > 0, g'(x) = -\frac{a\cos x}{(b + a\sin x)^2} < 0$$

$\rightarrow f$  is increasing and  $g$  decreasing

*Chebychev's*  $\Rightarrow \int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{b + a\sin x} dx \cdot \int_0^{\frac{\pi}{4}} \frac{1}{b + a\sin x} dx \geq \frac{\pi}{4} \int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{(b + a\sin x)^2} dx \quad (i)$

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{(b + a\sin x)^2} dx = \frac{b}{a} \int_0^{\frac{\pi}{4}} \frac{\frac{a^2}{b} - b + (b + a\sin x)}{(b + a\sin x)^2} dx \\ &= \frac{a^2 - b^2}{a} \int_0^{\frac{\pi}{4}} \frac{dx}{(b + a\sin x)^2} + \frac{b}{a} \int_0^{\frac{\pi}{4}} \frac{dx}{b + a\sin x} \quad (ii) \end{aligned}$$

$$\begin{aligned} \text{Let } t &= \frac{\cos x}{b + a\sin x} \rightarrow dt = -\frac{b}{a} \left( \frac{1}{b + a\sin x} + \frac{a^2 - b^2}{b(b + a\sin x)^2} \right) dx \rightarrow \\ t &= -\frac{b}{a} \int_0^{\frac{\pi}{4}} \frac{dx}{b + a\sin x} - \frac{a^2 - b^2}{b} \int_0^{\frac{\pi}{4}} \frac{dx}{(b + a\sin x)^2} \quad (iii) \end{aligned}$$

$$\text{From (ii), (iii) we get: } I = \frac{-\cos x}{b + a\sin x} \Big|_0^{\frac{\pi}{4}} = \frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}}$$

$$\text{So: } \int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{b + a\sin x} dx \cdot \int_0^{\frac{\pi}{4}} \frac{1}{b + a\sin x} dx \geq \frac{\pi}{4} \left( \frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}} \right)$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Solution 2 by Mohammed Diai-Rabat-Morocco**

$$\int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{b + a\sin x} dx \cdot \int_0^{\frac{\pi}{4}} \frac{1}{b + a\sin x} dx \geq \frac{\pi}{4} \left( \frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}} \right); (*)$$

$$\text{Let be } I = \int_0^{\frac{\pi}{4}} \frac{dx}{b + a\sin x}, J = \int_0^{\frac{\pi}{4}} \frac{\sin x}{b + a\sin x} dx, \alpha = \frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}}$$

$$\text{We have: } bI + aJ = \int_0^{\frac{\pi}{4}} \frac{dx}{b + a\sin x} + \int_0^{\frac{\pi}{4}} \frac{\sin x}{b + a\sin x} dx = \frac{\pi}{4}$$

$$\text{And } J = \left[ -\frac{\cos x}{b + a\sin x} \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{(-\cos x)(-\sin x)}{(b + a\sin x)^2} dx =$$

$$= \frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}} - a \int_0^{\frac{\pi}{4}} \frac{\cos^2 x}{(b + a\sin x)^2} dx = \alpha - aK, \text{ where } K = \int_0^{\frac{\pi}{4}} \frac{\cos^2 x}{(b + a\sin x)^2} dx$$

$$(*) \Leftrightarrow (aI + bJ)I \geq \frac{\pi}{4}\alpha \Leftrightarrow aI^2 + bIJ \geq \frac{\pi}{4}(J + aK)$$

$$\Leftrightarrow aI^2 + bIJ \geq (bI + aJ)J + \frac{\pi}{4}aK \Leftrightarrow I^2 - J^2 \geq \frac{\pi}{4}K \Leftrightarrow (I - J)(I + J) \geq \frac{\pi}{4}K$$

$$\Leftrightarrow \int_0^{\frac{\pi}{4}} \frac{1 - \sin x}{b + a\sin x} dx \cdot \int_0^{\frac{\pi}{4}} \frac{1 + \sin x}{b + a\sin x} dx \geq \frac{\pi}{4} \int_0^{\frac{\pi}{4}} \frac{(1 - \sin x)(1 + \sin x)}{(b + a\sin x)^2} dx$$

*The last inequality is true by Chebyshev's inequality since the functions have opposite monotony.*

*f:  $x \rightarrow \frac{1 + \sin x}{b + a\sin x}$  (f – increasing) and g:  $x \rightarrow \frac{1 - \sin x}{b + a\sin x}$  (g – decreasing)*

**Solution 3 by Ruxandra Daniela Tonilă-Romania**

$$\begin{aligned} \frac{\pi}{4} \left( \frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}} \right) &= \left( \frac{\pi}{4} - 0 \right) \left( \frac{1}{b} - \frac{\frac{\sqrt{2}}{2}}{b + \frac{a\sqrt{2}}{2}} \right) = \\ &= \left( \frac{\pi}{4} - 0 \right) \left( \frac{\cos 0}{b + a\sin 0} - \frac{\cos \frac{\pi}{4}}{b + a\sin \frac{\pi}{4}} \right) = \left( \frac{\cos 0}{b + a\sin 0} - \frac{\cos \frac{\pi}{4}}{b + a\sin \frac{\pi}{4}} \right) \left( - \int_0^{\frac{\pi}{4}} 1 dx \right); (1) \\ &\quad \left( \int f(x) dx \right)' = f(x), \forall f: R \rightarrow R, x \in R; (2) \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\text{Consider: } \frac{\cos x}{b + a \sin x} = \int f(x) dx; (3)$$

$$(2) \Leftrightarrow f(x) = \left( \frac{\cos x}{b + a \sin x} \right)' = \frac{-\sin x(b + a \sin x) - \cos x \cdot a \cos x}{(b + a \sin x)^2}$$

$$\Leftrightarrow f(x) = -\frac{a + b \sin x}{(b + a \sin x)^2}; (4)$$

*From (1), (2) and (4) it follows that:*

$$\frac{\pi}{4} \left( \frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}} \right) = \int_0^{\frac{\pi}{4}} 1 dx \cdot \int_0^{\frac{\pi}{4}} \frac{a + b \sin x}{(b + a \sin x)^2} dx$$

*Therefore, we have to prove that:*

$$\int_0^{\frac{\pi}{4}} \frac{a + b \sin x}{b + a \sin x} dx \cdot \int_0^{\frac{\pi}{4}} \frac{1}{b + a \sin x} dx \geq \int_0^{\frac{\pi}{4}} 1 dx \cdot \int_0^{\frac{\pi}{4}} \frac{a + b \sin x}{(b + a \sin x)^2} dx$$

$$\text{Let: } g: \left[0; \frac{\pi}{4}\right] \rightarrow \mathbb{R}, g(x) = \frac{(a + b \sin x)}{b + a \sin x} \text{ and } h: \left[0; \frac{\pi}{4}\right] \rightarrow \mathbb{R}, h(x) = \frac{1}{b + a \sin x}$$

*The inequality becomes:*

$$\begin{aligned} \int_0^{\frac{\pi}{4}} g(x) dx \cdot \int_0^{\frac{\pi}{4}} h(x) dx &\geq \int_0^{\frac{\pi}{4}} 1 dx \cdot \int_0^{\frac{\pi}{4}} g(x) h(x) dx \Leftrightarrow \\ - \int_0^{\frac{\pi}{4}} g(x) dx \cdot \int_0^{\frac{\pi}{4}} h(x) dx &< - \int_0^{\frac{\pi}{4}} 1 dx \cdot \int_0^{\frac{\pi}{4}} g(x) h(x) dx \Leftrightarrow \\ \int_0^{\frac{\pi}{4}} g(x) dx \cdot \int_0^{\frac{\pi}{4}} -h(x) dx &< \int_0^{\frac{\pi}{4}} 1 dx \cdot \int_0^{\frac{\pi}{4}} -g(x) h(x) dx \end{aligned}$$

$$\text{Let } h_1: \left[0; \frac{\pi}{4}\right] \rightarrow \mathbb{R}, \quad h_1(x) = 1 - g(x) \text{ and}$$

$$h_1'(x) = -\frac{(b^2 - a^2) \cos x}{(b + a \sin x)^2} < 0, \forall x \in \left[0, \frac{\pi}{4}\right]$$

$x$	$0$	$\frac{\pi}{4}$
$h_1'(x)$	-----	-----
$h_1(x)$	$h_1(0)$ ↘ ↘ ↘ ↘ ↘	$h_1\left(\frac{\pi}{4}\right)$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Leftrightarrow h_1(x) \geq h_1\left(\frac{\pi}{4}\right), \forall x \in \left[0, \frac{\pi}{4}\right]; (5)$$

$$h_1\left(\frac{\pi}{4}\right) = 1 - \frac{a + \frac{b\sqrt{2}}{2}}{b + \frac{a\sqrt{2}}{2}}$$

$$\frac{a + \frac{b\sqrt{2}}{2}}{b + \frac{a\sqrt{2}}{2}} \leq 1 \Leftrightarrow a + \frac{b\sqrt{2}}{2} \leq b + \frac{a\sqrt{2}}{2} \Leftrightarrow a + \frac{b\sqrt{2}}{2} - b - \frac{a\sqrt{2}}{2} \leq 0$$

$$\Leftrightarrow (b-a)\left(\frac{\sqrt{2}}{2} - 1\right) \leq 0, \text{ which is clearly true. Thus } h_1\left(\frac{\pi}{4}\right) = 1 - \frac{a + \frac{b\sqrt{2}}{2}}{b + \frac{a\sqrt{2}}{2}} \geq 0 \rightarrow$$

$$h_1\left(\frac{\pi}{4}\right) \geq 0 \Leftrightarrow h_1(x) \geq 0 \Leftrightarrow 1 - g(x) \geq 0 \Leftrightarrow g(x) \leq 1 \text{ and } h(x) = \frac{1}{b + a \sin x} > 0,$$

$$\forall x \in \left[0, \frac{\pi}{4}\right] \rightarrow -h(x) \leq -h(x)g(x)$$

$$g(x) \leq 1 \Leftrightarrow \int_0^{\frac{\pi}{4}} g(x) dx \leq \int_0^{\frac{\pi}{4}} 1 dx \text{ and } \int_0^{\frac{\pi}{4}} -h(x) dx \leq \int_0^{\frac{\pi}{4}} -h(x)g(x) dx \rightarrow \\ \int_0^{\frac{\pi}{4}} g(x) dx \cdot \int_0^{\frac{\pi}{4}} -h(x) dx \leq \int_0^{\frac{\pi}{4}} 1 dx \cdot \int_0^{\frac{\pi}{4}} -h(x)g(x) dx \\ \text{Therefore,}$$

$$\int_0^{\frac{\pi}{4}} \frac{a + b \sin x}{b + a \sin x} dx \cdot \int_0^{\frac{\pi}{4}} \frac{1}{b + a \sin x} dx \geq \frac{\pi}{4} \left( \frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}} \right)$$

**1402. Show that:**

$$\sum_{k=1}^n \frac{(-1)^k}{k^2} \binom{n-1}{k-1} = -\frac{H_n}{n}$$

for all  $n \in \mathbb{Z}^+$ , where  $H_n$  denotes the harmonic number.

*Proposed by Max Wong-Hong Kong*

**Solution 1 by Ravi Prakash-New Delhi-India**

$$\frac{(-1)^k}{k^2} \binom{n-1}{k-1} = \frac{(-1)^k}{nk^2} \frac{n!}{(k-1)! (n-k)!} = \frac{1}{n} \frac{(-1)^k}{k} \frac{n!}{k! (n-k)!}$$

$$\Rightarrow \sum_{k=1}^n \frac{(-1)^k}{k^2} \binom{n-1}{k-1} = \frac{1}{n} \sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} = \frac{1}{n} \sum_{k=1}^n (-1)^k \binom{n}{k} \int_0^1 x^{k-1}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \frac{1}{n} \int_0^1 \sum_{k=1}^n (-1)^k x^{k-1} \binom{n}{k} = \frac{1}{n} \int_0^1 \frac{1}{x} [(1-x)^n - 1] = \frac{1}{n} \int_0^1 \frac{x^n - 1}{1-x} \\
 &= -\frac{1}{n} \int_0^1 (1+x+x^2+\cdots+x^{n-1}) dx = -\frac{1}{n} \left[ x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{1}{n}x^n \right]_0^1 \\
 &= -\frac{1}{n} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) = -\frac{1}{n} H_n
 \end{aligned}$$

**Solution 2 by Surjeet Singhania-India**

$$\text{We know } H_n = \sum_{k=1}^n \frac{1}{k} = \sum_{k=1}^n \int_0^1 x^{k-1} dx = \int_0^1 \sum_{k=1}^n x^{k-1} dx$$

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx = -(1-x^n) \ln(1-x) \Big|_0^1 - n \int_0^1 x^{n-1} \ln(1-x) dx$$

Hence we get:

$$-\frac{H_n}{n} = \int_0^1 x^{n-1} \ln(1-x) dx$$

$$\text{Now } -\frac{H_n}{n} = \int_0^1 x^{n-1} \ln(1-x) dx = \int_0^1 (1-x)^{n-1} \ln(x) dx =$$

$$= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_0^1 x^k \ln(x) dx$$

$$\begin{aligned}
 &= -\sum_{k=1}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{(k+1)^2} = -\sum_{k=1}^n \binom{n-1}{k-1} \frac{(-1)^{k-1}}{k^2} = \sum_{k=1}^n \binom{n-1}{k-1} \frac{(-1)^k}{k^2} \\
 &\quad \sum_{k=1}^n \frac{(-1)^k}{k^2} \binom{n-1}{k-1} = -\frac{H_n}{n}
 \end{aligned}$$

**Solution 3 by Akerele Olofin-Nigeria**

Consider

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx = - \int_1^0 \frac{1-(1-u)^n}{u} du = \int_0^1 \frac{1-(1-u)^n}{u} du$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \int_0^1 \left( \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} u^{k-1} \right) du = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \int_0^1 u^{k-1} du \\
 &= \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \binom{n}{k} \Rightarrow H_n = \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \binom{n}{k}
 \end{aligned}$$

**Now,**

$$\begin{aligned}
 \sum_{k=1}^n \frac{(-1)^{k-1}}{k^2} \binom{n-1}{k-1} &= \sum_{k=1}^n \frac{(-1)^k}{k^2} \frac{\Gamma(n)}{\Gamma(n)\Gamma(n-k+1)} = \\
 &= \frac{1}{n} \sum_{k=1}^n \frac{(-1)^k}{k} \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} = \frac{1}{n} \sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} = -\frac{1}{n} \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \binom{n}{k} = -\frac{H_n}{n}
 \end{aligned}$$

**Finally,**

$$\sum_{k=1}^n \frac{(-1)^{k-1}}{k^2} \binom{n-1}{k-1} = -\frac{H_n}{n}$$

**Solution 4 by Kamel Benaicha-Algiers-Algerie**

$$\begin{aligned}
 S_n &= \sum_{k=1}^n \frac{(-1)^k}{k^2} \binom{n-1}{k-1} = \sum_{k=1}^n \frac{(-1)^k n! (n-1)!}{n(n-1)! k k! (n-k)!} = \frac{1}{n} \sum_{k=1}^n \frac{(-1)^k}{k} \binom{n}{k} \\
 &= \frac{1}{n} \int_0^1 \frac{1}{x} \left( \sum_{k=0}^n \binom{n}{k} (-1)^k \int_0^1 x^k \binom{n}{k} - 1 \right) dx \\
 &= \frac{1}{n} \int_0^1 \frac{1}{x} ((1-x)^n - 1) dx \stackrel{t=1-t}{=} -\frac{1}{n} \int_0^1 \frac{t^n - 1}{t-1} dt = -\frac{1}{n} \int_0^1 (1+t+t^2+\dots+t^{n-1}) dt = \\
 &= -\frac{1}{n} \left( 1 + \frac{1}{2} + \dots + \frac{1}{n} \right) = -\frac{H_n}{n} \\
 \therefore \sum_{k=1}^n \frac{(-1)^k}{k^2} \binom{n-1}{k-1} &= -\frac{H_n}{n}
 \end{aligned}$$

**1403.**

$$\sum_{n=1}^m (-1)^{1+2+3+\dots+n} (1+2+3+\dots+n) = \frac{1}{2} m(m+2) \cos\left(\frac{\pi m}{2}\right) - \frac{1}{2} (m+1) \sin\left(\frac{\pi m}{2}\right)$$

*Proposed by Srinivasa Raghava-AIRMC-India*

**Solution by proposer**



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$f_m = \sum_{n=1}^m (-1)^{1+2+3+\dots+n} (1+2+3+\dots+n) = \sum_{n=1}^m (-1)^{\frac{n(n+1)}{2}} \frac{n(n+1)}{2}$$

$$f_{m+1} - f_m = (-1)^{\frac{(m+1)(m+2)}{2}} \frac{(m+1)(m+2)}{2}$$

*From the difference equation the generation function follows:*

$$\sum_{m=1}^{\infty} f_m x^m = -\frac{x + 4x^2 + x^3}{(1+x^2)^3}$$

$$f_m = \lim_{\epsilon \rightarrow 1} \oint_{z=0} \frac{dz}{2\pi iz} z^{-m} \left[ -\frac{z + 4z^2 + z^3}{(\epsilon + z^2)^3} \right] = \lim_{\epsilon \rightarrow 1} \frac{1}{2} \partial_{\epsilon}^2 \oint_{z=0} \frac{dz}{2\pi iz} z^{-m} \left[ -\frac{z + 4z^2 + z^3}{(\epsilon + z^2)} \right] =$$

$$= \lim_{\epsilon \rightarrow 1} \left( -\frac{1}{2} \right) \partial_{\epsilon}^2 \left( \oint_{z=i\sqrt{\epsilon}} \frac{dz}{2\pi iz} z^{-m} \left[ -\frac{z + 4z^2 + z^3}{(\epsilon + z^2)} \right] + \oint_{z=-i\sqrt{\epsilon}} \frac{dz}{2\pi iz} z^{-m} \left[ -\frac{z + 4z^2 + z^3}{(\epsilon + z^2)} \right] \right)$$

$$= \lim_{\epsilon \rightarrow 1} \left( -\frac{1}{2} \right) \partial_{\epsilon}^2 \left( (i\sqrt{\epsilon})^{-m} \frac{1 + 4i\sqrt{\epsilon} - \epsilon}{2i\sqrt{\epsilon}} + (-i\sqrt{\epsilon})^{-m} \frac{1 - 4i\sqrt{\epsilon} - \epsilon}{-2i\sqrt{\epsilon}} \right) =$$

$$= \lim_{\epsilon \rightarrow 1} (i\sqrt{\epsilon})^{-m-1} \left[ (1 - (-1)^m) \frac{\epsilon + 3}{16\epsilon^2} + m \frac{2i(1 + (-1)^m)\sqrt{\epsilon} + (1 - (-1)^m)}{4\epsilon^2} \right] +$$

$$+ \lim_{\epsilon \rightarrow 1} (i\sqrt{\epsilon})^{-m-1} \left[ m^2 \frac{(1 + (-1)^m) + 4i\sqrt{\epsilon}(1 - (-1)^m) - \epsilon(1 - (-1)^m)}{16\epsilon^2} \right] =$$

$$= e^{-i\frac{\pi}{2}} \left[ \frac{\left( e^{-i\frac{\pi m}{2}} - e^{\frac{i\pi m}{2}} \right)}{4} + m \frac{2i \left( e^{-i\frac{\pi m}{2}} + e^{\frac{i\pi m}{2}} \right) + \left( e^{-i\frac{\pi m}{2}} - e^{\frac{i\pi m}{2}} \right)}{4} + m^2 \frac{i \left( e^{-i\frac{\pi m}{2}} + e^{\frac{i\pi m}{2}} \right)}{4} \right] =$$

$$= -\frac{1}{2} \sin\left(\frac{m\pi}{2}\right) + m \frac{1}{2} \left( 2\cos\left(\frac{m\pi}{2}\right) - \sin\left(\frac{m\pi}{2}\right) \right) + m^2 \frac{1}{2} \cos\left(\frac{m\pi}{2}\right)$$

**1404. Prove that:**

$$\Psi_1\left(\frac{1}{8}\right) - \Psi_1\left(\frac{3}{8}\right) - \Psi_1\left(\frac{5}{8}\right) + \Psi_1\left(\frac{7}{8}\right) = 4\pi^2\sqrt{2}$$

where  $\Psi_1(x)$  is the trigamma function.

*Proposed by Vasile Mircea Popa – Romania*

**Solution 1 by Akerele Olofin-Nigeria**

Consider



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\psi^{(0)}(1-x) - \psi^{(0)}(x) = \pi \cot(\pi x) \Rightarrow \psi^{(1)}(1-x) + \psi^{(1)}(x) = \pi^2 \csc^2(\pi x)$$

$$\text{From the last identity above when } x = \frac{1}{8} \Rightarrow \psi^{(1)}\left(\frac{1}{8}\right) + \psi^{(1)}\left(\frac{7}{8}\right) = \pi^2 \csc^2\left(\frac{\pi}{8}\right)$$

$$\text{when } x = \frac{5}{8}$$

$$\begin{aligned} \Rightarrow \psi^{(1)}\left(\frac{5}{8}\right) + \psi^{(1)}\left(\frac{3}{8}\right) &= \pi^2 \csc^2\left(\frac{5\pi}{8}\right) \Rightarrow \psi^{(1)}\left(\frac{1}{8}\right) - \psi^{(1)}\left(\frac{3}{8}\right) - \psi^{(1)}\left(\frac{5}{8}\right) + \psi^{(1)}\left(\frac{7}{8}\right) \\ &= \psi^{(1)}\left(\frac{1}{8}\right) + \psi^{(1)}\left(\frac{7}{8}\right) - \left( \psi^{(1)}\left(\frac{5}{8}\right) + \psi^{(1)}\left(\frac{3}{8}\right) \right) \\ &= \pi^2 \csc^2\left(\frac{\pi}{8}\right) - \pi^2 \csc^2\left(\frac{5\pi}{8}\right) = \pi^2 \left( \csc^2\left(\frac{\pi}{8}\right) - \csc^2\left(\frac{5\pi}{8}\right) \right) \\ &= \pi^2 (4 + 2\sqrt{2} - 4 + 2\sqrt{2}) = 4\pi^2\sqrt{2} \\ \Rightarrow \psi_1\left(\frac{1}{8}\right) + \psi_1\left(\frac{3}{8}\right) - \psi_1\left(\frac{5}{8}\right) - \psi_1\left(\frac{7}{8}\right) &= 4\pi^2\sqrt{2} \end{aligned}$$

**Solution 2 by Ahmed Yacoub Chach-Mauritania**

$$\mathcal{A} = \Psi_1\left(\frac{1}{8}\right) - \Psi_1\left(\frac{3}{8}\right) - \Psi_1\left(\frac{5}{8}\right) + \Psi_1\left(\frac{7}{8}\right) = 4\pi^2\sqrt{2}$$

D'une part:

$$\psi_1\left(\frac{1}{8}\right) + \psi_1\left(\frac{7}{8}\right) = \psi_1\left(\frac{1}{8}\right) + \psi_1\left(1 - \frac{1}{8}\right) = \frac{\pi^2}{\sin^2\left(\frac{\pi}{8}\right)} = \frac{2\pi^2}{1 - \cos\left(\frac{\pi}{4}\right)} = \frac{2\sqrt{2}\pi^2}{\sqrt{2} - 1}$$

D'autre Part:

$$\psi_1\left(\frac{3}{8}\right) + \psi_1\left(\frac{5}{8}\right) = \psi_1\left(\frac{3}{8}\right) + \psi_1\left(1 - \frac{3}{8}\right) = \frac{\pi^2}{\sin^2\left(\frac{3\pi}{8}\right)} = \frac{2\pi^2}{1 - \cos\left(\frac{3\pi}{4}\right)} = \frac{2\sqrt{2}\pi^2}{\sqrt{2} + 1}$$

finalement:

$$\mathcal{A} = \frac{2\sqrt{2}\pi^2}{\sqrt{2} - 1} - \frac{2\sqrt{2}\pi^2}{\sqrt{2} + 1} = 4\sqrt{2}\pi^2$$

**Solution 3 by Max Wong-Hong Kong**

$$\psi^{(1)}\left(\frac{1}{8}\right) - \psi^{(1)}\left(\frac{3}{8}\right) - \psi^{(1)}\left(\frac{5}{8}\right) + \psi^{(1)}\left(\frac{7}{8}\right)$$

By Euler's reflection formula

$$\Gamma(x)\Gamma(1-x) = \pi \csc \pi x \text{ for } 0 < x < 1$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\ln \Gamma(x) + \ln \Gamma(1-x) = \ln \pi + \ln \csc \pi x$$

$$\psi^{(0)}(x) + (-1)\psi^{(0)}(1-x) = \frac{-\pi \csc \pi x \cot \pi x}{\csc \pi x}$$

$$\psi^{(0)}(x) - \psi^{(0)}(1-x) = -\pi \cot \pi x$$

$$\psi^{(1)}(x) + \psi^{(1)}(1-x) = \pi^2 \csc^2 \pi x$$

$$\psi^{(1)}\left(\frac{1}{8}\right) - \psi^{(1)}\left(\frac{3}{8}\right) - \psi^{(1)}\left(\frac{5}{8}\right) + \psi^{(1)}\left(\frac{7}{8}\right)$$

$$= \pi^2 \left( \frac{1}{\sin^2 \frac{\pi}{8}} - \frac{1}{\sin^2 \frac{3\pi}{8}} \right) = \pi^2 \left( \frac{2}{1 - \cos \frac{\pi}{4}} - \frac{2}{1 - \cos \frac{3\pi}{4}} \right)$$

$$= \pi^2 \left( \frac{2}{1 - \frac{1}{2}\sqrt{2}} - \frac{2}{1 + \frac{1}{2}\sqrt{2}} \right) = \pi^2 \left( \frac{2\left(\frac{1}{2}\sqrt{2}\right) + 2\left(\frac{1}{2}\sqrt{2}\right)}{\frac{1}{2}} \right) = 4\pi^2\sqrt{2}$$

**Solution 4 by Probal Chakraborty-India**

We know the formulae  $\psi^m(z) = - \int_0^1 \frac{t^{z-1}}{1-t} (\log t)^m dt$

$$= \int_0^1 \left( -\frac{t^{-\frac{7}{8}}}{1-t} + \frac{t^{-\frac{5}{8}}}{1-t} + \frac{t^{-\frac{3}{8}}}{1-t} - \frac{t^{-\frac{1}{8}}}{1-t} \right) \log t dt$$

$$= \int_0^1 \left( \frac{t^{-\frac{5}{8}} + t^{-\frac{3}{8}} - t^{-\frac{7}{8}} - t^{-\frac{1}{8}}}{1-t} \right) \log t dt \Rightarrow z^8 = t \Rightarrow 8z^7 dz = dt$$

$$= \frac{\partial}{\partial a} \Big|_{a=0} \int_0^1 \left( \frac{t^{-\frac{5}{8}+a}}{1-t} + \frac{t^{-\frac{3}{8}+a}}{1-t} - \frac{t^{-\frac{7}{8}+a}}{1-t} - \frac{t^{-\frac{1}{8}+a}}{1-t} \right) dt$$

$$= \frac{\partial}{\partial a} \left[ \int_0^1 \frac{t^{-\frac{5}{8}+a} - 1 + t^{-\frac{3}{8}+a} - 1}{1-t} dt - \frac{t^{-\frac{7}{8}+a} - 1 + t^{-\frac{1}{8}+a} - 1}{1-t} dt \right]$$

$$= \frac{\partial}{\partial a} \Big|_{a=0} \left[ \pi \cot \left( \pi \left( \frac{5}{8} + a \right) \right) - \pi \cot \left( \pi \left( \frac{1}{8} + a \right) \right) \right]$$

$$= \left[ \pi^2 \csc^2 \frac{\pi}{8} - \pi^2 \csc^2 \frac{5\pi}{8} \right] = \pi^2 [4 + 2\sqrt{2} - 4 + 2\sqrt{2}] = 4\pi^2\sqrt{2}$$

$$\text{as } -\gamma + \int_0^1 \frac{1-t^z}{1-t} dt = \psi(z+1)$$



ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

**Solution 5 by Syed Shahabudeen-India**

$$\begin{aligned}
 \Omega &= \psi_1\left(\frac{1}{8}\right) - \psi_1\left(\frac{3}{8}\right) - \psi_1\left(\frac{5}{8}\right) + \psi_1\left(\frac{7}{8}\right) \\
 &= \frac{\partial}{\partial a} \left( \psi(a) - \psi(1-a) - \frac{1}{3}(\psi(3a) - \psi(1-3a)) \right) \\
 &= \frac{\partial}{\partial a} \left( -\pi \cot(a\pi) + \frac{1}{3}\pi \cot(3a\pi) \right) = \pi^2 \csc^2(a\pi) - \pi^2 \csc^2(3a\pi) \\
 &\quad \therefore \text{for } a = \frac{1}{8} \\
 \Omega &= \pi^2 \csc^2\left(\frac{\pi}{8}\right) - \pi^2 \csc^2\left(\frac{3\pi}{8}\right) = 4\pi^2\sqrt{2}
 \end{aligned}$$

**1405. If**

$$\phi(n) = \int_0^1 \frac{(x^n - 1) \ln(1-x)}{x \ln(x)} dx$$

**then prove that,**

$$\lim_{n \rightarrow 0} \left( \frac{2\phi(n)}{n^2} + \frac{\pi^2}{3n} \right) = \zeta(3)$$

*Proposed by Angad Singh-Pune-India*

**Solution 1 by proposer**

**Observe that,**

$$\phi(n) = - \int_0^1 \frac{(x^n - 1)}{\ln(x)} \sum_{k=1}^{\infty} \frac{x^{k-1}}{k} dx = - \sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 \frac{(x^{n+k-1} - x^{k-1})}{\ln(x)} dx$$

**If  $|n| < 1$  we have,**

$$\phi(n) = - \sum_{k=1}^{\infty} \frac{1}{k} \ln\left(\frac{n+k}{k}\right) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=1}^{\infty} \frac{\left(-\frac{n}{k}\right)^m}{m}$$

**hence**

$$\phi(n) = \sum_{m=1}^{\infty} \frac{(-n)^m}{m} \sum_{k=1}^{\infty} \frac{1}{k^{m+1}} = \sum_{m=1}^{\infty} \frac{(-n)^m \zeta(m+1)}{m}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

expanding  $\phi(n)$  we obtain,

$$\phi(n) = -\frac{\pi^2 n}{6} + \frac{\zeta(3)n^2}{2} - \frac{\zeta(4)n^3}{3} + \dots$$

finally,

$$\frac{2\phi(n)}{n^2} + \frac{\pi^2}{3n} = \zeta(3) - \frac{2\zeta(4)n}{3} + \dots$$

taking limit as  $n \rightarrow 0$  of both the sides completes the proof.

**Solution 2 by Asmat Qatea-Afghanistan**

$$\begin{aligned}
 \phi(n) &= \int_0^1 \frac{(x^n - 1) \ln(1-x)}{x \cdot \ln x} dx \Rightarrow \phi'(n) = \int_0^1 x^{n-1} \ln(1-x) dx = -\frac{H_n}{n} \\
 \lim_{n \rightarrow 0} \left( \frac{2\phi(n)}{n^2} + \frac{\pi^2}{3n} \right) &= \underbrace{\lim_{n \rightarrow 0} \left( \frac{6\phi(n) + \pi^2 n}{3n^2} \right)}_{\text{hopital}} = \lim_{n \rightarrow 0} \left( \frac{6\phi'(n) + \pi^2}{6n} \right) \\
 &= \underbrace{\lim_{n \rightarrow 0} \left( \frac{6 \left( -\frac{H_n}{n} \right) + \pi^2}{6n} \right)}_{\text{hopital}} = -\lim_{n \rightarrow 0} \left( \frac{H_n}{n} \right)' = -\underbrace{\lim_{n \rightarrow 0} \frac{H'_n \cdot n - H_n}{n^2}}_{\text{hopital}} \\
 &= -\lim_{n \rightarrow 0} \frac{(H'_n)' \cdot n + H'_n - H_n}{2n} = -\frac{1}{2} \lim_{n \rightarrow 0} (H'_n)' \\
 &= -\frac{1}{2} \lim_{n \rightarrow 0} (\psi_1(n+1))' = -\frac{1}{2} \lim_{n \rightarrow 0} \left( \sum_{k=0}^{\infty} \frac{1}{(k+n+1)^2} \right)' = \lim_{n \rightarrow 0} \left( \sum_{k=0}^{\infty} \frac{1}{(k+n+1)^3} \right) \\
 &= \sum_{k=0}^{\infty} \frac{1}{(k+1)^3} = \zeta(3)
 \end{aligned}$$

**1406. If  $(a_n)_{n \geq 1}$  is bounded positive real sequence, then compute**

$$\lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{n + a_k + k} \right)$$

*Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania*

**Solution 1 by Surjeet Singhania-India**

If  $(a_n)_{n \geq 1}$  be bounded real sequence  $\Rightarrow 0 < a_n \leq m$ , where  $m$  is any positive real number



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Let  $\varphi(n) = \sum_{k=1}^n \frac{1}{n+a_k+k}$ . So we need to find  $\lim_{n \rightarrow \infty} \varphi(n)$

Since  $0 < a_n \leq m \Rightarrow n+k \leq a_k+n+k \leq m+n+k$

$$\Rightarrow \frac{1}{n+k} \leq \frac{1}{a_k+n+k} \leq \frac{1}{m+n+k}$$

Take sum from  $k = 1$  to  $n$

$$\underbrace{\sum_{k=1}^n \frac{1}{n+k}}_{S_n} \leq \varphi(n) \leq \underbrace{\sum_{k=1}^n \frac{1}{n+k+m}}_{R_n}$$

$$\begin{aligned} R_n &= \sum_{k=1}^n \frac{1}{n+k+m} = \sum_{k=1}^n \int_0^1 x^{n+k+m-1} dx = \int_0^1 \frac{x^{n+m(1-x^n)}}{1-x} dx \\ &= \int_0^1 \frac{x^{n+m} - x^{2n+m}}{1-x} dx = H_{2n+m} - H_{n+m} \end{aligned}$$

Where  $H_2$  is Harmonic number

$$R_n = H_{2n+m} - H_{n+m}$$

$$\sim \gamma + \ln(2n+m) + O\left(\frac{1}{n}\right) - \gamma - \ln(n+m) + O\left(\frac{1}{n}\right) = \ln\left(\frac{2n+m}{n+m}\right) + O\left(\frac{1}{n^2}\right)$$

$$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \ln\left(\frac{2 + \frac{m}{n}}{1 + \frac{m}{n}}\right) = \ln(2)$$

$$\text{Also } \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1+\frac{k}{n}}$$

$$= \int_0^1 \frac{1}{1+x} dx = \ln(1+x) \Big|_0^1 = \ln(2) \Rightarrow \lim_{n \rightarrow \infty} S_n = \ln(2)$$

Hence by Sandwich theorem

$$\lim_{n \rightarrow \infty} \varphi(n) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+a_k+k} = \ln(2)$$

*Solution 2 by Samar Das-India*

Since  $(a_n)_{n \geq 1}$  is a bounded positive real sequence let lower bounded be  $l$  and upper bound is  $L$ . Therefore,  $n+k+L \geq n+k+a_n \geq n+k+l$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 & \Rightarrow \sum_{k=1}^n \frac{1}{n+k+L} \leq \sum_{k=1}^n \frac{1}{n+k+a_n} \leq \sum_{k=1}^n \frac{1}{n+k+l} \\
 & \Rightarrow \frac{1}{n} \sum_{k=1}^n \left( \frac{1}{1 + \frac{k}{n} + \frac{L}{n}} \right) \leq \frac{1}{n} \sum_{k=1}^n \left( \frac{1}{1 + \frac{k}{n} + \frac{a_n}{n}} \right) \leq \frac{1}{n} \sum_{k=1}^n \left( \frac{1}{1 + \frac{k}{n} + \frac{l}{n}} \right) \\
 & \quad \left( \lim_{n \rightarrow \infty} \frac{a_n}{n} = 0, \lim_{n \rightarrow \infty} \frac{l}{n} = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{L}{n} = 0 \right)
 \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + \frac{k}{n} + \frac{a_n}{n}} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left( \frac{1}{1 + \frac{k}{n}} \right) = \int_0^1 \frac{dx}{1+x} = \ln(1+x)|_0^1 = \ln 2$$

**1407. Find the limit**

$$\lim_{x \rightarrow 0^+} \left( 3 \log \left( \left| \sqrt{2} - \sqrt{1 + \sqrt{1 + 16x}} \right| \right) + \log(|1 - \sqrt{1 + 16x}|) - 4 \log x \right)$$

**Notation:**  $|.|$  denotes absolute value.

*Proposed by Naren Bhandari-Bajura-Nepal*

**Solution 1 by Asmat Qatea-Afghanistan**

$$\begin{aligned}
 & \lim_{x \rightarrow 0^+} \left( 3 \ln \left| \sqrt{2} - \sqrt{1 + \sqrt{1 + 16x}} \right| + \ln |1 - \sqrt{1 + 16x}| - 4 \ln x \right) \\
 & = \lim_{x \rightarrow 0^+} \left( \ln \left| \frac{\left( \sqrt{2} - \sqrt{1 + \sqrt{1 + 16x}} \right)^3 \cdot (1 - \sqrt{1 + 16x})}{x^4} \right| \right) \\
 & = \lim_{x \rightarrow 0^+} \left( \ln \left| \frac{\left( \sqrt{2} - \sqrt{1 + \sqrt{1 + 16x}} \cdot \frac{\sqrt{2} + \sqrt{1 + \sqrt{1 + 16x}}}{\sqrt{2} + \sqrt{1 + \sqrt{1 + 16x}}} \right)^3 \cdot (1 - \sqrt{1 + 16x}) \cdot \frac{(1 + \sqrt{1 + 16x})}{(1 + \sqrt{1 + 16x})}}{x^4} \right| \right)
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 &= \lim_{x \rightarrow 0^+} \left( \ln \left| \frac{\left( 1 - \sqrt{1 + 16x} \cdot \frac{1 + \sqrt{1 + 16x}}{1 + \sqrt{1 + 16x}} \right)^3 \cdot (-16)}{x^3 (\sqrt{2} + \sqrt{1 + \sqrt{1 + 16x}})^3 (1 + \sqrt{1 + 16x})} \right| \right) \\
 &= \lim_{x \rightarrow 0^+} \left( \ln \left| \frac{(-16x)^3 \cdot (-16)}{x^3 (\sqrt{2} + \sqrt{1 + \sqrt{1 + 16x}})^3 (1 + \sqrt{1 + 16x})^3 (1 + \sqrt{1 + 16x})} \right| \right) \\
 &\quad = \ln \left| \frac{16^4}{(\sqrt{2} + \sqrt{2})^3 (2)^4} \right| \\
 &\quad = \ln \left| \frac{8^4}{(2\sqrt{2})^3} \right| = \ln \left| \frac{2^9}{(2)^{\frac{3}{2}}} \right| = \frac{15}{2} \ln 2
 \end{aligned}$$

**Solution 2 by Mikael Bernardo-Mozambique**

$$\begin{aligned}
 &\lim_{x \rightarrow 0^+} \left( 3 \log \left( \left| \sqrt{2} - \sqrt{1 + \sqrt{1 + 16x}} \right| \right) + \log(|1 - \sqrt{1 + 16x}|) - 4 \log x \right) \\
 &\lim_{x \rightarrow 0^+} \left( 3 \log \left( \left| \sqrt{2} - \sqrt{1 + \sqrt{1 + 16x}} \right| \right) - 3 \log x + \log(|1 - \sqrt{1 + 16x}|) - \log x \right) \\
 &\lim_{x \rightarrow 0^+} \left( 3 \log \left( \frac{|\sqrt{2} - \sqrt{1 + \sqrt{1 + 16x}}|}{x} \right) + \log \left( \frac{|1 - \sqrt{1 + 16x}|}{x} \right) \right) \\
 &\therefore \lim_{x \rightarrow 0^+} (1 + nx) = \lim_{x \rightarrow 0^+} e^{nx} \\
 &\lim_{x \rightarrow 0^+} \left( 3 \log \left( \frac{|\sqrt{2} - \sqrt{1 + e^{16x}}|}{x} \right) + \log \left( \frac{|1 - \sqrt{e^{16x}}|}{x} \right) \right) \\
 &\lim_{x \rightarrow 0^+} \left( 3 \log \left( \frac{|\sqrt{2} - \sqrt{1 + e^{8x}}|}{x} \right) + \log \left( \frac{|1 - e^{8x}|}{x} \right) \right) \\
 &\quad 8x \rightarrow x \\
 &\lim_{x \rightarrow 0^+} \left( 3 \log \left( 8 \cdot \frac{|\sqrt{2} - \sqrt{1 + e^x}|}{x} \right) + \log \left( 8 \cdot \frac{|1 - e^x|}{x} \right) \right) \\
 &|\sqrt{2} - \sqrt{1 + e^x}| = \begin{cases} \sqrt{2} - \sqrt{1 + e^x}, \sqrt{2} - \sqrt{1 + e^x} \geq 0 \Rightarrow e^x \leq 1 \Rightarrow x \leq 0 \\ \sqrt{1 + e^x} - \sqrt{2}, \sqrt{2} - \sqrt{1 + e^x} < 0 \Rightarrow e^x > 1 \Rightarrow x > 0 \end{cases}
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$|1 - e^x| = \begin{cases} 1 - e^x, & 1 - e^x \geq 0 \Rightarrow e^x \leq 1 \Rightarrow x \leq 0 \\ e^x - 1, & 1 - e^x < 0 \Rightarrow e^x > 1 \Rightarrow x > 0 \end{cases}$$

for  $x > 0$

$$L = \lim_{x \rightarrow 0^+} \left( 3 \log \left( 8 \cdot \frac{\sqrt{1+e^x} - \sqrt{2}}{x} \right) + \log \left( 8 \cdot \frac{e^x - 1}{x} \right) \right)$$

$$L = \lim_{x \rightarrow 0^+} \left( 3 \log \left( 8 \cdot \frac{e^x - 1}{x} \cdot \frac{1}{\sqrt{1+e^x} + \sqrt{2}} \right) + \lim_{x \rightarrow 0^+} \log \left( 8 \cdot \frac{e^x - 1}{x} \right) \right)$$

$$L = 3 \log \left( \frac{8}{2\sqrt{2}} \right) + \log(8)$$

$$L = 3 \cdot \log \left( \frac{8}{\sqrt{2}} \right) = 3 \cdot \left( 3 - \frac{1}{2} \right) \log(2)$$

$$L = \frac{15}{2} \cdot \log(2)$$

**Solution 3 by Samar Das-India**

$$\begin{aligned} & \lim_{x \rightarrow 0^+} \left( 3 \log \left| \sqrt{2} - \sqrt{1 + \sqrt{1 + 16x}} \right| + \log |1 - \sqrt{1 + 16x}| - 4 \log x \right) \\ &= \lim_{x \rightarrow 0^+} \left( 3 \log \left| \frac{2 - 1 - \sqrt{1 + 16x}}{\sqrt{2} + \sqrt{1 + \sqrt{1 + 16x}}} \right| + \log |1 + 16x| - 4 \log x \right) \\ &= \lim_{x \rightarrow 0^+} \left( 3 \log |1 - \sqrt{1 + 16x}| + \log |1 - \sqrt{1 + 16x}| - 3 \log \left| \sqrt{2} + \sqrt{1 + \sqrt{1 + 16x}} \right| - 4 \log x \right) \\ &= \lim_{x \rightarrow 0^+} \left( 4 \log |1 - \sqrt{1 + 16x}| - 4 \log x - 3 \log \left| \sqrt{2} + \sqrt{1 + \sqrt{1 + 16x}} \right| \right) \\ &= \lim_{x \rightarrow 0^+} \left( 4 \log \left| \frac{1 - 1 - 16x}{1 + \sqrt{1 + 16x}} \right| - 4 \log x - 3 \log \left| \sqrt{2} + \sqrt{1 + \sqrt{1 + 16x}} \right| \right) \\ &= \lim_{x \rightarrow 0^+} \left( 4 \log 16x - 4 \log |1 + \sqrt{1 + 16x}| - 4 \log x - 3 \log \left| \sqrt{2} + \sqrt{1 + \sqrt{1 + 16x}} \right| \right) \\ &= \lim_{x \rightarrow 0^+} \left( 4 \log \left| \frac{16x}{x} \right| - 4 \log |1 + \sqrt{1 + 16x}| - 3 \log \left| \sqrt{2} + \sqrt{1 + \sqrt{1 + 16x}} \right| \right) \\ &= 4 \log 16 - 4 \log 2 - 3 \log(\sqrt{2} + \sqrt{2}) \\ &= 4 \log 8 - 3 \log(2\sqrt{2}) = \log \left( \frac{2^{12}}{2^2} \right) = \frac{15}{2} \log 2 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Solution 4 by Mohammad Rostami-Afghanistan**

$$\begin{aligned}
 \Omega &= \lim_{x \rightarrow 0^+} \left( 3 \log \left( \left| \sqrt{2} - \sqrt{1 + \sqrt{1 + 16x}} \right| \right) + \log(|1 - \sqrt{1 + 16x}|) - 4 \log x \right) \\
 \sqrt{1 + 16x} = u \Rightarrow 1 + 16x = u^2 \Rightarrow x &= \frac{u^2 - 1}{16}, \begin{cases} x > 0 \Rightarrow 16x > 0 \Rightarrow 1 + 16x > 1 \\ \sqrt{1 + 16x} > 1 \Rightarrow u > 1 \Rightarrow u \rightarrow 1^+ \end{cases} \\
 \Omega &= \lim_{u \rightarrow 1^+} \left[ \log \left( \frac{|\sqrt{2} - \sqrt{1+u}|^3 \cdot |1-u|}{(u^2-1)^4} \right) \right] + 4 \log 16 = \\
 &= \lim_{u \rightarrow 1^+} \left[ \log \left( \frac{|1-u|^3 \cdot |1-u|}{(\sqrt{2} + \sqrt{1+u})^3 \cdot (1-u)^4(1+u)^4} \right) \right] + 16 \log 2 = \\
 &= \log \frac{1}{(2\sqrt{2})^3 \cdot 2^4} + 16 \log 2 = -\log 2^{3+\frac{3}{2}+4} + 16 \log 2 \\
 &= \left( -\frac{17}{2} + 16 \right) \log 2 = \frac{15}{2} \log 2
 \end{aligned}$$

**Solution 5 by Muhammad Afzal-Pakistan**

$$\begin{aligned}
 &\lim_{x \rightarrow 0^+} \left[ 3 \ln \left| \sqrt{2} - \sqrt{1 + \sqrt{1 + 16x}} \right| + \ln |1 - \sqrt{1 + 16x}| - 4 \ln x \right] \\
 &\lim_{x \rightarrow 0^+} [3 \ln |\sqrt{2} - \sqrt{2+8x}| + \ln |8x| - \ln |x^4|], \lim_{x \rightarrow 0^+} \left[ 3 \ln |\sqrt{2}(1 - \sqrt{1 + 4x})| + \ln \left| \frac{8x}{x^4} \right| \right] \\
 &\lim_{x \rightarrow 0^+} 3 \left[ \ln \sqrt{2} + \ln |2x| + \ln \left| \frac{2}{x} \right| \right] \\
 &\lim_{x \rightarrow 0^+} 3 [\ln \sqrt{2} + \ln 4] \Rightarrow \frac{15}{2} \ln 2
 \end{aligned}$$

**1408. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \frac{(3n-1)! \cdot n^2}{(27)^n \cdot (n!)^3}$$

*Proposed by Asmat Qatea-Afghanistan*

**Solution 1 by Syed Shahabudeen-India**

$$\Omega = \lim_{n \rightarrow \infty} \frac{(3n-1)! \cdot n^2}{(27)^n \cdot (n!)^3}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{2\pi(3n-1)} \left(\frac{3n-1}{e}\right)^{3n-1} n^2}{3^{3n} (2\pi n)^{\frac{3}{2}} \left(\frac{n}{e}\right)^{3n}} \quad (\text{By Stirlings Formula})$$

$$= \frac{\sqrt{2\pi} e}{(2\pi)^{\frac{3}{2}}} \lim_{n \rightarrow \infty} \left( \frac{(3n-1)^{3n-\frac{1}{2}}}{3^{3n} n^{3n-\frac{1}{2}}} \right) = \frac{e}{2\pi} \lim_{n \rightarrow \infty} \frac{\left(1 - \frac{1}{3n}\right)^{3n}}{\left(3 - \frac{1}{n}\right)^{\frac{1}{2}}} = \frac{e}{2\pi} \frac{e^{-1}}{\sqrt{3}} = \frac{1}{2\pi\sqrt{3}}$$

**Solution 2 by Mohammed Diai-Rabat-Morocco**

**By Stirling Formula:**

$$\begin{aligned} n! &\stackrel{\infty}{\sim} \sqrt{2\pi n} n^n e^{-n}, (3n-1)! \stackrel{\infty}{\sim} \sqrt{2\pi(3n-1)} (3n-1)^{3n-1} e^{1-3n} \\ (3n-1)^{3n-1} &= (3n)^{3n-1} \left(1 - \frac{1}{3n}\right)^{3n-1} = (3n)^{3n-1} e^{(3n-1) \ln(1 - \frac{1}{3n})} = \\ &= (3n)^{3n-1} e^{(3n-1) \left(-\frac{1}{3n} + o\left(\frac{1}{n}\right)\right)} = (3n)^{3n-1} e^{-1 + \frac{1}{3n} + o(1)} \stackrel{\infty}{\sim} (3n)^{3n-1} e^{-1} \\ \frac{(3n-1)! n^2}{3^{3n} (n!)^3} &\stackrel{\infty}{\sim} \frac{\sqrt{2\pi(3n)} (3n)^{3n-1} e^{-1} e^{1-3n} n^2}{3^{3n} (\sqrt{2\pi n})^3 n^{3n} e^{-3n}} = \frac{\sqrt{3}}{6\pi} \end{aligned}$$

$$\text{Therefore } \Omega = \frac{\sqrt{3}}{6\pi}$$

**1409. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{e^n} \cdot e^{\int_0^\infty \left[ \frac{n}{e^x} \right] dx}, \quad [\cdot] - \text{floor function}$$

*Proposed by Arhgyadeep Chatterjee-India*

**Solution by Mohammad Rostami-Afghanistan**

$$\begin{aligned} I &= \int_0^\infty \left[ \frac{n}{e^x} \right] dx, \begin{cases} \frac{n}{e^x} = t \rightarrow n e^{-x} dx = dt \rightarrow -t dx = dt \\ dx = -\frac{dt}{t}; x \rightarrow \infty \Rightarrow t = 0 \text{ and } x = 0 \Rightarrow t = n \end{cases} \\ I &= \int_0^\infty \left[ \frac{n}{e^x} \right] dx = \int_n^0 [t] \frac{(-dt)}{t} = \int_0^n \frac{[t]}{t} dt = \\ &= \int_0^1 \frac{0}{t} dt + \int_1^2 \frac{1}{t} dt + \int_1^2 \frac{2}{t} dt + \dots + \int_{n-1}^n \frac{n-1}{n} dt = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{k}{t} dt = \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \sum_{k=1}^{n-1} [k \log k]_k^{k+1} = \sum_{k=1}^{n-1} k(\log(k+1) - \log k) = \\
 &= 1(\log 2 - \log 1) + 2(\log 3 - \log 2) + \cdots + (n-1)(\log n - \log(n-1)) = \\
 &= -(\log 2 + \log 3 + \cdots + \log(n-2) + \log(n-1) + (n-1)\log n) = \\
 &= -\log[(n-1)!] + \log n^{n-1} = \log \left[ \frac{n^{n-1}}{(n-1)!} \right] \\
 &I = \log \left[ \frac{n^{n-1}}{(n-1)!} \right] \rightarrow e^I = \frac{n^{n-1}}{(n-1)!}
 \end{aligned}$$

**Note:** if  $n \rightarrow \infty$ :  $\Gamma(n+1) \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ ;  $n! = \Gamma(n+1) = n \cdot \Gamma(n)$

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{e^n} \cdot e^{\int_0^\infty \left[\frac{n}{e^x}\right] dx} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{e^n} \cdot e^I = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{e^n} \cdot \frac{n^{n-1}}{(n-1)!} = \\
 &= \lim_{n \rightarrow \infty} \left[ \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \cdot \frac{1}{\sqrt{2\pi} \cdot n\Gamma(n)} \right] = \lim_{n \rightarrow \infty} \Gamma(n+1) \cdot \frac{1}{\sqrt{2\pi} \cdot \Gamma(n+1)} = \frac{1}{\sqrt{2\pi}}
 \end{aligned}$$

*Therefore,*

$$\Omega = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{e^n} \cdot e^{\int_0^\infty \left[\frac{n}{e^x}\right] dx} = \frac{1}{\sqrt{2\pi}}$$

**1410. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \sqrt[n]{\sum_{k=1}^n (2k-1) \binom{n}{k-1} \binom{n}{k}}$$

*Proposed by Marian Ursărescu – Romania*

**Solution 1 by Mohammed Diai-Rabat-Morocco**

$$1 \leq k \leq n \Rightarrow 1 \leq 2k-1 \leq 2n-1 \leq 2n$$

$$\Rightarrow \sum_{k=1}^n \binom{n}{k-1} \binom{n}{k} \leq \sum_{k=1}^n (2k-1) \binom{n}{k-1} \binom{n}{k} \leq 2n \sum_{k=1}^n \binom{n}{k-1} \binom{n}{k}$$

By Vandermonde Identity:

$$\sum_{k=1}^n \binom{n}{k-1} \binom{n}{k} = \binom{n+1}{2n} = \frac{(2n)!}{(n+1)!(n-1)!}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**It follows that:**

$$\exp\left(\frac{1}{n} \ln\left(\frac{(2n)!}{(n-1)!(n-1)!}\right)\right) \leq \left(\sum_{k=1}^n (2k-1) \binom{n}{k-1} \binom{n}{k}\right)^{\frac{1}{n}} \leq \\ \leq \exp\left(\frac{1}{n} \ln\left(2n \frac{(2n)!}{(n+1)!(n-1)!}\right)\right) \quad (*)$$

$$\frac{(2n)!}{(n+1)!(n-1)!} \sim \frac{2\sqrt{\pi} 2^{2n} n^{2n+\frac{1}{2}} e^{-2n}}{2\pi n n^{2n} e^{-2n}} = \frac{2^{2n}}{\sqrt{\pi n}} \quad \text{By Stirlying formula}$$

$$\frac{1}{n} \ln\left(\frac{(2n)!}{(n+1)!(n-1)!}\right) \sim \frac{2n \ln 2 - \frac{1}{2} \ln(\pi n)}{n} \sim 2 \ln 2$$

$$\frac{1}{n} \ln\left(2n \frac{(2n)!}{(n+1)!(n-1)!}\right) \sim \frac{\ln 2n + 2n \ln 2 - \frac{1}{2} \ln(\pi n)}{n} \sim 2 \ln 2$$

**By (\*) we can assume that:**

$$\Omega = \lim_{n \rightarrow \infty} \sqrt[n]{\sum_{k=1}^n (2k-1) \binom{n}{k-1} \binom{n}{k}} = \exp(2 \ln 2) = 4$$

**Solution 2 by Adrian Popa-Romania**

$$\Omega = \lim_{n \rightarrow \infty} \sqrt[n]{\sum_{k=1}^n (2k-1) C_n^{k-1} C_n^k} \\ C_n^{k-1} \cdot C_n^k < (2k-1) C_n^{k-1} C_n^k < (2n-1) C_n^{k-1} C_n^k \Rightarrow \\ \Rightarrow \sum_{k=1}^n C_n^{k-1} C_n^k < \sum_{k=1}^n (2k-1) C_n^{k-1} C_n^k < (2n-1) \sum_{k=1}^n C_n^{k-1} C_n^k \\ \sum_{k=1}^n C_n^{k-1} \cdot C_n^k = C_{2n}^{n+1}$$

$$\therefore \lim_{n \rightarrow \infty} \sqrt[n]{C_{2n}^{n+1}} \stackrel{C.D.}{=} \lim_{n \rightarrow \infty} \frac{C_{2n+2}^{n+2}}{C_{2n}^{n+1}} = \lim_{n \rightarrow \infty} \frac{\frac{(2n+2)!}{(n+2)! \cdot n!}}{\frac{(2n)!}{(n+1)!(n-1)!}} = \\ = \lim_{n \rightarrow \infty} \frac{(2n)!(2n+1)(2n+2)}{(n+1)!(n+1) \cdot (n-1)! \cdot n} \cdot \frac{(n+1)!(n-1)!}{(2n)!} = 4 \quad (1)$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\therefore \lim_{n \rightarrow \infty} \sqrt[n]{(2n-1)C_{2n}^{n+1}} \stackrel{C.D.}{=} \lim_{n \rightarrow \infty} \frac{(2n+1)C_{2n+2}^{n+2}}{(2n-1)C_{2n}^{n+1}} = 4 \quad (2)$$

From (1) and (2)  $\Rightarrow \Omega = 4$ .

**1411. If  $f(x) = \frac{\sinh x}{2 \cosh 2x+1}$  for all  $x \in (0, \infty)$  and  $I = \int_0^\infty x^2 f(x) dx$ , then**

**prove:**

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{4}{n} \left( \frac{9I}{7} - \sum_{k=1}^n \frac{1}{k^3} \right)$$

where  $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$  is Riemann zeta function.

*Proposed by Narendra Bhandari-Bajura-Nepal*

**Solution by proposer**

We shall be using the following result for all  $n, m \geq 0$

$$I(n, m) = \int_0^1 x^n \log^m x \, dx = (-1)^m \frac{m!}{(n+1)^{m+1}}; \quad (1)$$

We prove result (1) by making substitution  $x = e^y \rightarrow dx = e^y dy$

$$I(n, m) = \int_{-\infty}^0 e^{y(n+1)} y^m dy \stackrel{u=-(n+1)y}{\cong} \int_0^{\infty} \frac{e^{-u} u^m}{(-(n+1)^m (n+1))} du$$

Since  $\Gamma(m+1) = \int_0^{\infty} e^{-u} u^m du$  (definition of gamma function) and hence

$$I(n, m) = (-1)^m \frac{\Gamma(m+1)}{(n+1)^{m+1}} = (-1)^m \frac{m!}{(n+1)^{m+1}}$$

We call the integral  $I = \int_0^{\infty} \frac{x^2 \sinh x}{2 \cosh 2x+1} dx$  and since  $2 \cosh 2x = 3 + 4 \sinh^2 x$  and thus

$$I = \int_0^{\infty} \frac{x^2 \sinh x}{3 + 4 \sinh^2 x} dx; \quad (2)$$

Further we know that  $\sinh x = \frac{e^x - e^{-x}}{2}$ . Plugging in (1) therefore, gives us

$$I = \frac{1}{2} \int_0^{\infty} \frac{x^2 (e^x - e^{-x})}{3 + (e^x - e^{-x})^2} dx \stackrel{e^{-x}=y}{\cong} \frac{1}{2} \int_0^1 \frac{\log^2 y (1 - y^2)}{3y^2 + (1 - y^2)^2} dy = \frac{1}{2} \int_0^1 \frac{\log^2 y (1 - y^2)}{y^4 + y^2 + 1} dy$$

Now we make substitution  $y^2 = t \rightarrow dy = \frac{dt}{2\sqrt{t}}$  and the latter integral becomes:



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$I = \frac{1}{4} \int_0^1 \frac{\log^2(\sqrt{t})(1-t)}{\sqrt{t}(t^2+t+1)} dt = \frac{1}{16} \int_0^1 \frac{\log^2 t(1-t)}{\sqrt{t}(t^2+t+1)} dt = \frac{1}{16} \int_0^1 \frac{(1-t)^2 \log^2 t}{(1-t^3)\sqrt{t}} dt$$

Since  $x \in (0, 1)$  and we exploit the geometric series of  $\sum_{n=0}^{\infty} x^{3n} = \frac{1}{1-x^3}$  and interchanging summation and integral sign we have:

$$\frac{1}{16} \sum_{n=0}^{\infty} \int_0^1 \frac{\log^2 t(1-t)^2 t^{3n}}{\sqrt{t}} dt = \frac{1}{16} \sum_{n=0}^{\infty} \int_0^1 \log^2 t(1-2t+t^2) t^{3n-\frac{1}{2}} dt$$

Using the result (1) our integral

$$I = \frac{2!}{16} \sum_{n=0}^{\infty} \left( \frac{1}{\left(3n + \frac{1}{2}\right)^3} - \frac{2}{\left(3n + \frac{3}{2}\right)^3} + \frac{1}{\left(3n + \frac{5}{2}\right)^3} \right) =$$

Since for all  $n > 0$ ,  $\psi_n(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}$  and plugging  $n = 2$  we have:

$$= \frac{1}{8 \cdot 54} \left( 2\psi_2\left(\frac{1}{2}\right) - \psi_2\left(\frac{5}{6}\right) - \psi_2\left(\frac{1}{6}\right) \right); (3)$$

Since  $\psi_2\left(\frac{1}{2}\right) = -14\zeta(3)$ ,  $\psi_2\left(\frac{1}{6}\right) = -182\zeta(2) - 4\sqrt{3}\pi^2$  and  $\psi_2\left(\frac{5}{6}\right) = -182\zeta(3) +$

$4\sqrt{3}\pi^2$  and putting back these value to (3) we have:

$$\frac{9}{7} \int_0^\infty \frac{x^2 \sinh x}{2 \cosh 2x + 1} dx = \frac{9}{7} \cdot \frac{336}{432} \zeta(3) = \zeta(3)$$

Now we need to evaluate:

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \zeta(3) - \sum_{k=1}^n \frac{1}{k^3} \right) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n(n+k)^3} = S$$

Interchanging  $n, k$  wit  $k, n$  doesn't change the overall summation due to symmetry, so

$$2S = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n(n+k)^3} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k(n+k)^3} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{nk(n+k)^2}$$

Due to result (1) we can express the summand  $\frac{1}{(n+k)^2} = - \int_0^1 x^{n+k-1} \log x dx$  and hence

$$2S = - \int_0^1 \frac{\log x}{x} \left( \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^n}{n} \frac{x^k}{k} \right) dx = - \int_0^1 \frac{\log x \log^2(1-x)}{1-x} dx \stackrel{x \rightarrow 1-x}{\cong}$$



## ROMANIAN MATHEMATICAL MAGAZINE

$$\frac{\text{www.ssmrmh.ro}}{= - \int_0^1 \frac{\log^2 x \log(1-x)}{1-x} dx}$$

We solve the last integral using *Feynman technique* so we set

$$J(a) = - \int_0^1 \frac{\log^2 x \log(1-ax)}{1-x} dx, \quad J(1) = -2s, J(0) = 0$$

Differentiating with respect to  $a$  we have:

$$\begin{aligned} J'(a) &= - \int_0^1 \frac{x \log^2 x}{(1-x)(1-ax)} dx = \int_0^1 \frac{x \log^2 x}{1-a} \left( \frac{a}{1-ax} - \frac{1}{1-x} \right) dx = \\ &= \frac{1}{1-a} \sum_{n=0}^{\infty} \int_0^1 (a^{n+1} x^{n+1} \log^2 x - x^{n+1} \log^2 x) dx \end{aligned}$$

Due to result (1) the last integral reduces to

$$\frac{2}{1-a} \sum_{n=0}^{\infty} \left( \frac{a^{n+1}}{(n+2)^3} - \frac{1}{(n+2)^3} \right) = \frac{2}{1-a} \left( \frac{Li_3(a) - a}{a} - (\zeta(3) - 1) \right)$$

Therefore,

$$\begin{aligned} -2S &= 2 \int_0^1 \left( \frac{Li_3(a)}{a} - \frac{Li_3(a) - \zeta(3)}{1-a} \right) da = 2\zeta(4) + 2 \int_0^1 \frac{Li_3(a) - \zeta(3)}{1-a} da \stackrel{IBP}{=} \\ -2S &= 2\zeta(4) + 2 \int_0^1 \frac{\log(1-a) Li_2(a)}{a} da = 2\zeta(4) + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 a^{n-1} \log^2 a da \end{aligned}$$

Since  $\int_0^1 x^{n-1} \log x dx = -\frac{H_n}{n}$  and hence

$$= 2\zeta(4) - 2 \sum_{n=1}^{\infty} \frac{H_n}{n^3} = 2\zeta(4) - \frac{5}{2}\zeta(4) = -\frac{1}{2}\zeta(4)$$

And  $4S = \zeta(4)$  also due to Euler's identity

$$2 \sum_{k=1}^{\infty} \frac{H_k}{k^3} = 5\zeta(4) - \zeta^2(2) = \frac{5}{2}\zeta(4)$$

And hence

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{4}{n} \left( \zeta(3) - \sum_{k=1}^n \frac{1}{k^3} \right)$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**1412.**  $x_1 = \frac{1}{2}, 2x_{n+1}^2 + \sqrt{1 - x_n^2} = 1, n \geq 1$ . Find:

$$\Omega = \lim_{n \rightarrow \infty} (2^n \cdot x_n)$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Adrian Popa-Romania**

From mathematical induction principle, we have:

$$\begin{aligned}
 P(n): x_n &= \sin\left(\frac{\pi}{2^{n-1} \cdot 6}\right), \forall n \in N, n \geq 1 \\
 2x_{n+1}^2 + \sqrt{1 - x_n^2} &= 1 \Rightarrow 2x_{n+1}^2 + \sqrt{\sin\left(\frac{\pi}{2^{n-1} \cdot 6}\right)} = 1 \Rightarrow \\
 2x_{n+1}^2 &= 1 - \cos\left(\frac{\pi}{2^{n-1} \cdot 6}\right) = 2 \sin^2\left(\frac{\pi}{2^n \cdot 6}\right) \\
 \Rightarrow x_{n+1} &= \sin\left(\frac{\pi}{2^n \cdot 6}\right) \\
 x_1 = \sin\frac{\pi}{6} &\Rightarrow 2x_2^2 + \sqrt{1 - \sin^2\frac{\pi}{6}} = 1 \Rightarrow 2x_2^2 = 2 \sin^2\frac{\pi}{12} \Rightarrow x_2 = \sin\frac{\pi}{12} \\
 2x_3^2 &= \sqrt{1 - \sin^2\frac{\pi}{12}} \Rightarrow 2x_3^2 = 1 - \cos\frac{\pi}{2} = 2 \sin^2\frac{\pi}{24} \Rightarrow x_3 = \sin\frac{\pi}{24}
 \end{aligned}$$

From mathematical induction principle, we have:

$$x_n = \sin\left(\frac{\pi}{2^{n-1} \cdot 6}\right) = \sin\left(\frac{3\pi}{3 \cdot 2^n}\right)$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} (2^n \cdot x_n) = \lim_{n \rightarrow \infty} 2^n \cdot \sin\left(\frac{3\pi}{3 \cdot 2^n}\right) = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{3\pi}{3 \cdot 2^n}\right)}{\frac{1}{2^n}} = \frac{\pi}{3}$$

**Solution 2 by Mohammad Rostami-Kabul-Afghanistan**

$$\begin{aligned}
 x_n = \sin \alpha &\rightarrow 2x_{n+1}^2 + \cos \alpha = 1 \rightarrow x_{n+1}^2 = \frac{1 - \cos \alpha}{2} = \sin^2 \frac{\alpha}{2} \\
 \rightarrow x_{n+1} &= \sin \frac{\alpha}{2} \rightarrow 2x_{n+2}^2 + \cos \frac{\alpha}{2} \rightarrow x_{n+2}^2 = \frac{1 - \cos \frac{\alpha}{2}}{2} = \sin^2 \frac{\alpha}{4} \rightarrow x_{n+2} = \sin \frac{\alpha}{4} \\
 x_n &= \sin \frac{\alpha}{2^0}, x_{n+1} = \sin \frac{\alpha}{2^1}, \dots, x_{n+k} = \sin \frac{\alpha}{2^k}
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$n+k=t \rightarrow k=t-n \rightarrow x_t = \sin \frac{\alpha}{2^{t-n}}$$

$$t=1 \rightarrow x_1 = \frac{1}{2} = \sin \frac{\pi}{6} = \sin \frac{\alpha}{2 \cdot 2^{-n}} \rightarrow \frac{\pi}{2 \cdot 3} = \frac{\alpha}{2 \cdot 2^{-n}} \rightarrow \frac{\alpha}{2^{-n}} = \frac{\pi}{3}$$

$$\rightarrow x_t = \sin \left( \frac{1}{2^t} \cdot \frac{\alpha}{2^{-n}} \right) = \sin \left[ \frac{\pi}{3} \cdot \left( \frac{1}{2^t} \right) \right] \xrightarrow{t \rightarrow n} x_n = \sin \left[ \frac{\pi}{3} \left( \frac{1}{2^n} \right) \right]$$

$$\Omega = \lim_{n \rightarrow \infty} (2^n \cdot x_n) = \lim_{n \rightarrow \infty} \left( 2^n \cdot \sin \left[ \frac{\pi}{3} \left( \frac{1}{2^n} \right) \right] \right) =$$

$$\left( \because \frac{\pi}{3} \left( \frac{1}{2^n} \right) = u \rightarrow 2^n = \frac{\pi}{3} \cdot \frac{1}{u}; n \rightarrow \infty \Rightarrow u \rightarrow 0 \right)$$

$$= \lim_{u \rightarrow 0} \left( \frac{\pi}{3} \cdot \frac{1}{u} \sin u \right) = \frac{\pi}{3} \lim_{u \rightarrow 0} \frac{\sin u}{u} = \frac{\pi}{3}$$

**Solution 3 by Asmat Qatea- Afghanistan**

$$\therefore \sqrt[n]{2 - \sqrt{2 + \cdots + \sqrt{2 + 2x}}} = 2 \sin \left( \frac{\cos^{-1} x}{2^n} \right)$$

$$\therefore \sqrt[n]{2 - \sqrt{2 + \cdots + \sqrt{2 + 2x}}} = 2 \cos \left( \frac{\cos^{-1} x}{2^n} \right)$$

$$x_n^2 = \frac{1}{2} - \frac{1}{2} \sqrt{1 - x_{n+1}^2} \rightarrow x_n = \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{1 - x_{n+1}^2}}$$

$$x_n = \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{1 - \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - x_{n-2}^2} \right)}} = \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 - x_{n-2}^2}}}$$

$$x_n = \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 - \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - x_{n-3}^2} \right)}}} = \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 - x_{n-3}^2}}}}$$

$$x_n = \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \cdots + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 - x_{n-(n-1)}^2}}} =$$

$$\begin{aligned}
 &= \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \cdots + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{4}}}}} = \frac{1}{2} \sqrt{2 - \sqrt{2 + \cdots + \sqrt{2 + \sqrt{2 + 2 \cdot \frac{\sqrt{3}}{2}}}}} \\
 &= \sin\left(\frac{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)}{2^{n-1}}\right) = \sin\left(\frac{\pi}{2^{n-1}}\right)
 \end{aligned}$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} (2^n \cdot x_n) = \lim_{n \rightarrow \infty} 2^n \cdot \sin\left(\frac{\frac{\pi}{6}}{2^{n-1}}\right) = \frac{\pi}{3}$$

**1413. If  $1 < a \leq b$  then:**

$$\int_a^b \int_a^b \int_a^b \frac{\log x \cdot \log(yz)}{\log y \cdot \log z} dx dy dz \geq 2(b-a)^3$$

*Proposed by Asmat Qatea-Afghanistan*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 &\int_a^b \int_a^b \int_a^b \frac{\log x \cdot \log(yz)}{\log y \cdot \log z} dx dy dz = \int_a^b \int_a^b \int_a^b \frac{\log x \cdot (\log y + \log z)}{\log y \cdot \log z} dx dy dz = \\
 &= \int_a^b \int_a^b \int_a^b \frac{\log x \cdot \log y}{\log y \cdot \log z} dx dy dz + \int_a^b \int_a^b \int_a^b \frac{\log x \cdot \log z}{\log y \cdot \log z} dx dy dz = \\
 &= \int_a^b \int_a^b \int_a^b \frac{\log x}{\log z} dx dy dz + \int_a^b \int_a^b \int_a^b \frac{\log x}{\log y} dx dy dz = \\
 &= 2 \int_a^b \log x dx \cdot \int_a^b \frac{1}{\log x} dx \cdot \int_a^b dx = 2(b-a) \int_a^b \log x dx \cdot \int_a^b \frac{1}{\log x} dx = \\
 &= 2(b-a)^2 \left( \frac{1}{b-a} \int_a^b \log x dx \right) \cdot \int_a^b \frac{1}{\log x} dx \stackrel{\text{HM-AM INTEGRAL FORM}}{\underset{\cong}{\geq}}
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\geq 2(b-a)^2 \left( \frac{b-a}{\int_a^b \frac{1}{\log x} dx} \right) \cdot \int_a^b \frac{1}{\log x} dx = 2(b-a)^3$$

**Equality holds for  $a = b$ .**

**1414.**

If  $\phi = \phi(x, m) = \sum_{k=1}^n \frac{e^{-kx^2}}{m - \log(k^2)}$  then:

$$\int_0^\infty x^5 \log(x) \phi(3 - 2\gamma) dx = \frac{\zeta(3)}{4}$$

*Proposed by Angad Singh-Pune-India*

**Solution 1 by Amrit Awasthi-India**

Consider the function:  $f(x) = \log(x)e^{-kx}$ . The Mellin transform of  $f(x)$  is given by:

$$\begin{aligned} & \frac{\Gamma(s)\psi(s) - \Gamma(s)\log(k)}{k^s} \\ \rightarrow \int_0^\infty x^{s-1} \log(x) e^{-kx} dx &= \frac{\Gamma(s)\psi(s) - \Gamma(s)\log(k)}{k^s}; (*) \end{aligned}$$

Now, consider the integral

$$\begin{aligned} I &= \int_0^\infty x^5 \log(x) e^{-kx^2} dx \stackrel{(x^2=t, xdx=\frac{1}{2}dt)}{\cong} \int_0^\infty \frac{t^2 \log(\sqrt{t}) e^{-kt} dt}{2} = \frac{1}{4} \int_0^\infty x^2 \log(x) e^{-kx} dx = \\ &\stackrel{(s=3), (*)}{=} \frac{1}{4} \cdot \frac{\Gamma(3)\psi(3) - \Gamma(3)\log(k)}{k^3} = \frac{1}{4k^3} (3 - 2\gamma - 2\log k) \end{aligned}$$

$$\text{or } \int_0^\infty \frac{x^5 \log(x) e^{-kx^2} dx}{(3 - 2\gamma - \log(k^2))} = \frac{1}{4k^3}$$

Now, summing both sides from  $k = 1$  to  $k = \infty$ , we get:

$$\int_0^\infty x^5 \log(x) \sum_{k=1}^\infty \frac{e^{-kx^2}}{3 - 2\gamma - \log(k^2)} dx = \frac{1}{4} \sum_{k=1}^\infty \frac{1}{k^3}$$

Therefore,



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\int_0^\infty x^5 \log(x) \phi(3 - 2\gamma) dx = \frac{\zeta(3)}{4}$$

**Solution 2 by Syed Shahabudeen-India**

$$\begin{aligned}
 \Omega &= \int_0^\infty x^5 \log(x) \phi(3 - 2\gamma) dx = \int_0^\infty x^5 \log(x) \sum_{k=1}^\infty \frac{e^{-kx^2}}{3 - 2\gamma - \log(k^2)} dx = \\
 &= \sum_{k=1}^\infty \frac{e^{-kx^2}}{3 - 2\gamma - \log(k^2)} \int_0^\infty x^5 \log(x) e^{-kx^2} dx \\
 A &= \int_0^\infty x^5 \log(x) e^{-kx^2} dx \underset{t=kx^2}{=} \frac{1}{4k^3} \left( \int_0^\infty t^2 \log(t) e^{-t} dt - \log(k) \int_0^\infty t^2 e^{-t} dt \right) = \\
 &= \frac{1}{4k^3} (\psi(3)\Gamma(3) - \log(k)\Gamma(3)) = \frac{1}{4k^3} (3 - 2\gamma - \log(k^3)) \\
 \Omega &= \sum_{k=1}^n \frac{1}{3 - 2\gamma - \log(k^2)} \int_0^\infty x^5 \log(x) e^{-kx^2} dx = \frac{1}{4} \sum_{k=1}^\infty \frac{1}{k^3} = \frac{\zeta(3)}{4}
 \end{aligned}$$

**Solution 3 by Asmat Qatea-Afghanistan**

$$\begin{aligned}
 \Omega &= \int_0^\infty x^5 \log(x) \phi(3 - 2\gamma) dx = \int_0^\infty x^5 \log(x) e^{-kx^2} \sum_{k=1}^\infty \frac{1}{3 - 2\gamma - \log(k^2)} dx = \\
 &= \sum_{k=1}^\infty \frac{1}{3 - 2\gamma - \log(k^2)} \int_0^\infty x^5 \log(x) e^{-kx^2} dx = \\
 &= \sum_{k=1}^\infty \frac{1}{3 - 2\gamma - \log(k^2)} \cdot \frac{1}{4k^3} (3 - 2\gamma - 2\log(k)) = \frac{\zeta(3)}{4} \\
 \Gamma(n) &= \int_0^\infty t^{n-1} e^{-t} dt \stackrel{(t=kx^2, k \geq 1)}{\cong} \int_0^\infty (kx^2)^{n-1} e^{-kx^2} 2kx dx \\
 \Gamma'(x) &= \int_0^\infty (kx^2)^{n-1} \log(kx^2) e^{-kx^2} 2kx dx = \\
 &= \log(k) \int_0^\infty (kx^2)^{n-1} e^{-kx^2} 2kx dx + 2 \int_0^\infty (kx^2)^{n-1} \log(x) e^{-kx^2} (2kx) dx \\
 \Gamma(n)\psi(n) &= \log(k) \cdot \Gamma(n) + 4k^n \int_0^\infty x^{2n-1} \log(x) e^{-kx^2} dx
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned} \rightarrow \int_0^\infty x^{2n-1} \log(x) e^{-kx^2} dx &= \frac{1}{4k^n} (\Gamma(n)\psi(n) - \log(k)\Gamma(n)) \xrightarrow{n=3} \\ \int_0^\infty x^5 \log(x) e^{-kx^2} dx &= \frac{1}{4k^3} (3 - 2\gamma - 2\log(k)) \end{aligned}$$

**Solution 4 by Serlea Kabay-Liberia**

$$\begin{aligned} \phi(3 - 2\gamma) &= \phi(x, 3 - 2\gamma) = \sum_{k=1}^{\infty} \frac{1}{3 - 2\gamma - \log(k^2)} \underbrace{\int_0^\infty x^5 \log(x) e^{-kx^2} dx}_I \\ I(a) &:= \int_0^\infty x^a \log(x) e^{-kx^2} dx, I'(a)|_{a=5} = I \xrightarrow{u=kx^2} \\ I(a) &= \int_0^\infty (\sqrt{u})^a e^{-u} \frac{du}{2\sqrt{k^{a+1}u}} = \frac{1}{2\sqrt{k^{a+1}u}} \int_0^\infty u^{\frac{a-1}{2}} e^{-u} du = \frac{\Gamma(\frac{a+1}{2})}{2k^{\frac{a+1}{2}}} \\ I'(a) &= -\frac{\Gamma(\frac{a+1}{2})}{4k^{\frac{a+1}{2}}} \left( \log k - \psi\left(\frac{a+1}{2}\right) \right) \leftrightarrow I = -\frac{\Gamma(3)}{4k^3} (\log(k) - \psi(3)) \\ I &= -\frac{2}{4k^3} \left( \log(k) - \frac{3}{2} + \gamma \right) = \frac{1}{4k^3} (3 - 2\gamma - \log(k^2)) \end{aligned}$$

Now,

$$\begin{aligned} \Omega &= \int_0^\infty x^5 \log(x) \phi(3 - 2\gamma) dx = \sum_{k=1}^{\infty} \frac{1}{3 - 2\gamma - \log(k^2)} \left( \frac{1}{4k^3} (3 - 2\gamma - \log(k^2)) \right) = \\ &= \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^3} \end{aligned}$$

Therefore,

$$\int_0^\infty x^5 \log(x) \phi(3 - 2\gamma) dx = \frac{\zeta(3)}{4}$$

**Solution 5 by Mohammad Rostami-Afghanistan**

$$\begin{aligned} \Omega &= \int_0^\infty x^5 \frac{\partial}{\partial a} \Big|_{a=0} x^a \sum_{k=1}^{\infty} \frac{e^{-kx^2}}{3 - 2\gamma - \log(k^2)} dx = \\ &= \sum_{k=1}^{\infty} \frac{1}{3 - 2\gamma - \log(k^2)} \cdot \frac{\partial}{\partial a} \Big|_{a=0} \int_0^\infty x^{5+a} e^{-kx^2} dx \stackrel{(kx^2=u)}{\cong} \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \sum_{k=1}^{\infty} \frac{1}{3 - 2\gamma - \log(k^2)} \cdot \frac{\partial}{\partial a} \Big|_{a=0} \int_0^{\infty} \left(\frac{u}{k}\right)^{\frac{5+a}{2}} e^{-u} \frac{du}{2k} \left(\frac{k}{u}\right)^{\frac{1}{2}} = \\
 &= \sum_{k=1}^{\infty} \frac{1}{3 - 2\gamma - \log(k^2)} \cdot \frac{\partial}{\partial a} \Big|_{a=0} \int_0^{\infty} u^{\frac{6+a}{2}-1} e^{-u} du = \\
 &= \sum_{k=1}^{\infty} \frac{1}{3 - 2\gamma - \log(k^2)} \cdot \left[ \frac{1}{2} k^{-3-\frac{a}{2}} \cdot \Gamma\left(3 + \frac{a}{2}\right) \right] \Big|_{a=0} = \\
 &= \sum_{k=1}^{\infty} \frac{1}{3 - 2\gamma - \log(k^2)} \left[ -\frac{1}{4} \log(k) \cdot k^{-3-\frac{a}{2}} \cdot \Gamma\left(3 + \frac{a}{2}\right) + \frac{1}{4} k^{-3-\frac{a}{2}} \Gamma'\left(3 + \frac{a}{2}\right) \right] \Big|_{a=0} = \\
 &= \sum_{k=1}^{\infty} \frac{1}{3 - 2\gamma - \log(k^2)} \left[ -\frac{\Gamma(3) \log k}{4k^3} + \frac{\Gamma(3) \psi(3)}{4k^3} \right] = \\
 &= \sum_{k=1}^{\infty} \frac{1}{3 - 2\gamma - \log(k^2)} \left[ \frac{-\log(k^2) + 2\left(\frac{3}{2} - \gamma\right)}{4k^3} \right] = \\
 &= \sum_{k=1}^{\infty} \frac{1}{3 - 2\gamma - \log(k^2)} \cdot \frac{3 - 2\gamma - \log(k^2)}{4k^3} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{\zeta(3)}{4}
 \end{aligned}$$

Therefore,

$$\int_0^{\infty} x^5 \log(x) \phi(3 - 2\gamma) dx = \frac{\zeta(3)}{4}$$

### **Solution 6 by proposer**

Using the definition of gamma function and some manipulations, it can be shown that,

$$\int_0^{\infty} x^{n-1} e^{-ax^2} \log(x) dx = \frac{\Gamma\left(\frac{n}{2}\right)}{4a^{\frac{n}{2}}} \left( \psi\left(\frac{n}{2}\right) - \log(a) \right), \text{ where } a, n > 0$$

Thus,

$$\int_0^{\infty} x^{n-1} \log(x) \frac{e^{-ax^2}}{\psi\left(\frac{n}{2}\right) - \log(a)} dx = \frac{\Gamma\left(\frac{n}{2}\right)}{4a^{\frac{n}{2}}}$$

Substituting  $n = 6$ , replacing  $a$  by  $k$  and summing up both the sides from  $k = 1$  to  $k = \infty$ ,

we have:



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\int_0^\infty x^5 \log(x) \sum_{k=1}^\infty \frac{e^{-kx^2}}{\psi(3) - \log(k)} dx = \frac{\zeta(3)}{2}$$

Since  $\psi(n) = -\gamma + H_{n-1}$  for all  $n \in \mathbb{N}$ , we have:

$$\int_0^\infty x^5 \log(x) \sum_{k=1}^\infty \frac{e^{-kx^2}}{3 - 2\gamma - \log(k^2)} dx = \frac{\zeta(3)}{4}$$

**1415. Find without any software:**

$$\Omega = \int \frac{(x^2 + 6x + 15) \sin x}{x^4 + 12x^3 + 54x^2 + 108x + 81} dx$$

*Proposed by Daniel Sitaru – Romania*

**Solution 1 by Adrian Popa-Romania**

$$\begin{aligned} \Omega &= \int \frac{(x^2 + 6x + 15) \sin x}{x^4 + 12x^3 + 54x^2 + 108x + 81} dx = \\ &= \int \frac{(x^2 + 6x + 9 + 6) \sin x}{(x+3)^4} dx = \int \frac{[(x+3)^2 + 6] \sin x}{(x+3)^4} dx = \\ &= \underbrace{\int \frac{\sin x}{(x+3)^2} dx}_{y_1} + 6 \underbrace{\int \frac{\sin x}{(x+3)^4} dx}_{y_2} \end{aligned}$$

$$y_2 = 6 \int \frac{\sin x}{(x+3)^4} dx$$

$$f = \sin x; g' = \frac{1}{(x+3)^4}$$

$$f' = \cos x; g = \frac{1}{3} \cdot \frac{1}{(x+3)^3}$$

$$y_2 = -\frac{2 \sin x}{(x+3)^3} + 2 \int \frac{\cos x}{(x+3)^3} dx; f = \cos x; g' = \frac{1}{(x+3)^3}$$

$$f' = -\sin x; g = -\frac{1}{2} \cdot \frac{1}{(x+3)^2}$$

$$y_2 = -\frac{2 \sin x}{(x+3)^3} - \frac{\cos x}{(x+3)^2} - \underbrace{\int \frac{\sin x}{(x+3)^2} dx}_{y_1}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\Omega = y_1 - \frac{2 \sin x}{(x+3)^3} - \frac{\cos x}{(x+3)^2} - y_1$$

$$\Omega = -\frac{2 \sin x}{(x+3)^3} - \frac{\cos x}{(x+3)^2} + c$$

**Solution 2 by Igor Soposki-Skopje-Macedonia**

$$\int \frac{(x^2 + 6x + 15) \sin x}{x^4 + 12x^3 + 54x^2 + 108x + 81} dx = \int \frac{[(x+3)^2 + 6] \sin x}{(x+3)^4}$$

$$= \begin{cases} u = \sin x \\ du = \cos x dx \end{cases}$$

$$dv = \frac{(x+3)^2 + 6}{(x+3)^4} dx$$

$$v = \int \frac{(x+3)^2 + 6}{(x+3)^4} dx = \begin{cases} x+3 = t \\ dx = dt \end{cases} =$$

$$= \int \frac{t^2 + 6}{t^4} = \int \frac{1}{t^2} dt + 6 \int \frac{1}{t^4} dt = -\frac{1}{t} - \frac{2}{t^3} =$$

$$= -\frac{1}{(x+3)} - \frac{2}{(x+3)^3} = uv - \int v \cdot du = -\frac{(x+3)^2 + 2}{(x+3)^3}$$

$$= -\sin x \cdot \frac{(x+3)^2 + 2}{(x+3)^3} + \underbrace{\int \frac{(x+3)^2 + 2}{(x+3)^3} \cos x dx}_{=I_1}$$

$$I_1 = \begin{cases} u = \cos x \\ du = -\sin x dx \end{cases}$$

$$dv = \frac{(x+3)^2 + 2}{(x+3)^3} dx$$

$$v = \int \frac{(x+3)^2 + 2}{(x+3)^3} dx = \begin{cases} x+y = t \\ dx = t \end{cases}$$

$$= \int \frac{t^2 + 2}{t^3} dt = \int \frac{1}{t} dt + 2 \int \frac{dt}{t^3} = \ln t - \frac{1}{t^2} = \ln(x+3) - \frac{1}{(x+3)^2}$$

$$= u \cdot v - v \cdot du = \cos x \left[ \ln(x+3) - \frac{1}{(x+3)^2} \right] + \underbrace{\int \left( \ln(x+3) - \frac{1}{(x+3)^2} \right) \sin x dx}_{I_2}$$

$$I_2 = \int \ln(x+3) \cdot \sin x dx - \underbrace{\int \frac{\sin x}{(x+3)^2} dx}_{=I_3}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$I_3 = \int \frac{\sin x}{(x+3)^2} dx = \begin{cases} u = \sin x & dv = \frac{dx}{(x+3)^2} \\ du = \cos x dx & v = -\frac{1}{(x+3)} \end{cases} =$$

$$= -\frac{\sin x}{(x+3)} + \underbrace{\int \frac{\cos x}{(x+3)} dx}_{I_4}$$

$$I_4 = \int \frac{\cos x}{(x+3)} dx = \begin{cases} u = \cos x & dv = \frac{dx}{x+3} \\ du = -\sin x dx & v = \ln(x+3) \end{cases} =$$

$$= \cos x \cdot \ln(x+3) + \int \ln(x+3) \cdot \sin x dx$$

$$I_3 = -\frac{\sin x}{(x+3)} + \cos x \cdot \ln(x+3) + \int \ln(x+3) \cdot \sin x dx$$

$$I_2 = \int \ln(x+3) \cdot \sin x dx + \frac{\sin x}{(x+3)} - \cos x \cdot \ln(x+3) - \int \ln(x+3) \sin x dx \\ = +\frac{\sin x}{(x+3)} - \cos x \cdot \ln(x+3)$$

$$I_1 = \cos x \cdot \ln(x+3) - \frac{\cos x}{(x+3)^2} + \frac{\sin x}{(x+3)} - \cos x \ln(x+3)$$

$$I = -\frac{\sin x [(x+3)^2 + 2]}{(x+3)^3} - \frac{\cos x}{(x+3)^2} + \frac{\sin x}{(x+3)} = \\ = \frac{-(x+3)^2 \sin x - 2 \sin x - \cos x (x+3) + \sin x (x+3)^2}{(x+3)^3} = \\ = -\frac{(x+3) \cos x + 2 \sin x}{(x+3)^3} + c$$

**Solution 3 by Akerele Olofin-Nigeria**

$$\text{Now } \omega = \int \frac{(x^2+6x+15) \sin x}{x^4+12x^3+54x^2+108x+81} dx$$

$$\Rightarrow \omega = \int \frac{(x^2+6x+15) \sin x}{(x+3)^4} dx = \underbrace{\int \frac{\sin x}{(x+3)^2} dx}_{\lambda_1} + 6 \int \frac{\sin x}{(x+3)^4} dx \\ \Rightarrow \lambda_1 = \int \frac{\sin x}{(x+3)^2} dx \left\{ \begin{array}{l} u = x+3 \\ \Rightarrow du = dx \end{array} \right.$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 \Rightarrow \lambda_1 &= \int \frac{\sin(u-3)}{u^2} du \stackrel{IBP}{=} -\frac{\sin(u-3)}{u} + \int \underbrace{\frac{\cos(u-3)}{u}}_{\lambda_3} du \\
 \Rightarrow \lambda_3 &= \int \frac{\cos(u-3)}{u} du = \sin(3) \int \frac{\sin(u)}{u} du + \\
 &+ \cos(3) \int \frac{\cos(u)}{u} du = \sin(3) si(u) + \cos(3) ci(u) \\
 \Rightarrow \lambda_1 &= \sin(3) si(x+3) + \cos(3) ci(x+3) - \frac{\sin(x)}{x+3} \\
 \Rightarrow \lambda_2 & \text{ (By the same technique)} = 6 \int \frac{\sin(x)}{(x+3)^4} dx \\
 \Rightarrow \lambda_2 &= -\sin(3) si(x+3) - \cos(3) ci(x+3) + \frac{(x^2+6x+7)}{(x+3)^3} \sin x + \frac{\cos(x)}{(x+3)^2} \\
 \Rightarrow \omega &= \int \frac{\sin(x)}{(x+3)^2} dx + 6 \int \frac{\sin(x)}{(x+3)^4} dx = \lambda_1 + \lambda_2 \\
 &= \sin(3) si(x+3) + \cos(3) ci(x+3) - \frac{\sin(x)}{x+3} - \sin(3) si(x+3) \\
 &- \cos(3) ci(x+3) + \frac{(x^2+6x+7)}{(x+3)^3} \sin(x) - \frac{\cos(x)}{(x+3)^2} + c \\
 \Rightarrow \omega &= \frac{(x^2+6x+7)}{(x+3)^3} \sin(x) - \frac{\sin(x)}{x+3} - \frac{\cos(x)}{(x+3)^2} + c \\
 \Rightarrow \omega &= -\frac{2 \sin(x)}{(x+3)^3} - \frac{\cos(x)}{(x+3)^2} + c
 \end{aligned}$$

**Solution 4 by Samar Das-India**

$$\begin{aligned}
 \Omega &= \frac{(x^2+6x+15) \sin x dx}{x^4+12x^3+54x^2+108x+81} \\
 x^2+6x+15 &\mid x^4+12x^3+54x^2+108x+81 \\
 &\underline{x^4+6x^3+15x^2} \\
 &6x^3+39x^2+108x+81 \\
 &\underline{6x^3+36x^2+90x} \\
 &3x^2+18x+81 \\
 &\underline{3x^2+18x+45}
 \end{aligned}$$

36

$$\Rightarrow \Omega = \int \frac{(\sin x) dx}{(x^2+6x+3+\frac{36}{x^2+6x+15})} \quad (1)$$

$$\text{Now, } x^2+6x+3+\frac{36}{x^2+6x+15}=x^2+6x+15+\frac{36}{x^2+6x+15}-12$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \left( a + \frac{36}{a} - 12 \right) \quad (\text{Let } a = x^2 + 6x + 15) = \frac{(a-6)^2}{a} \\
 \therefore \frac{\sin x}{(x^2 + 6x + 3) + \frac{36}{x^2 + 6x + 15}} &= \frac{a \sin x}{(a-6)^2} = \frac{(a-6) \sin x}{(a-6)^2} \\
 &= \left( \frac{1}{a-6} + \frac{6}{(a-6)^2} \right) \sin x = \left( \frac{1}{x^2+6x+9} + \frac{6}{(x^2+6x+9)^2} \right) \sin x \quad (2) \\
 \stackrel{(1),(2)}{\Rightarrow} \Omega &= \int \left( \frac{1}{(x+3)^2} + \frac{6}{(x+3)^2} \right) \sin x \, dx \\
 &= \int \frac{\sin x}{(x+3)^2} \, dx + 6 \int \frac{\sin x}{(x+3)^4} \, dx \\
 &= \int \frac{\sin x}{(x+3)^2} \, dx + 6 \underbrace{\left\{ \overbrace{\sin x \int (x+3)^{-4} \, dx}^{IBP} - \int \frac{\cos x (x+3)^{-3}}{-3} \, dx \right\}}_{-3} \\
 &= \int \frac{\sin x}{(x+3)^2} \, dx + 6 \left( \frac{(\sin x)(x+3)^{-3}}{-3} + \frac{1}{3} \int (\cos x)(x+3)^{-3} \, dx \right) \\
 &= \int \frac{\sin x}{(x+3)^2} \, dx + (-2) \frac{\sin x}{(x+3)^3} + 2 \left\{ (\cos x) \int (x+3)^{-3} \, dx - \int \frac{(-\sin x)(x+3)^{-2}}{-2} \, dx \right\} \\
 &= \int \frac{\sin x}{(x+3)^2} \, dx - \frac{2 \sin x}{(x+3)^3} + \left( \frac{-\cos x}{(x+3)^2} \right) - \int \frac{\sin x}{(x+3)^2} \, dx \\
 &= - \left\{ \frac{2 \sin x}{(x+3)^3} + \frac{\cos x}{(x+3)^2} \right\} + C
 \end{aligned}$$

**Solution 5 by Yen Tung Chung-Taichung-Taiwan**

$$\begin{aligned}
 \int \frac{(x^2 + 6x + 15) \sin x}{x^4 + 12x^3 + 54x^2 + 108x + 81} \, dx &= \underbrace{\int \frac{((x+3)^2 + 6) \sin x}{(x+3)^4} \, dx}_{\text{let } y=x+3} = \\
 &= \int \frac{(y^2 + 6) \sin(y-3)}{y^4} \, dy \\
 &= \int \frac{\sin(y-3)}{y^2} \, dy + \int \frac{6 \sin(y-3)}{y^4} \, dy = \int \frac{\sin(y-3)}{y^2} \, dy - \\
 &\quad - \frac{\sin(y-3) + y \cos(y-3)}{y^3} - \int \frac{\sin(y-3)}{y^2} \, dy \\
 &= - \frac{\sin(y-3) + y \cos(y-3)}{y^3} + C = - \frac{\sin x + (x+3) \cos x}{(x+3)^3} + C
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 \text{Where } \int \frac{6 \sin(y-3)}{y^4} dy &= -\frac{\sin(y-3)}{y^3} + \int \frac{2 \cos(y-3)}{y^3} dy \left( \begin{array}{l} u = \sin(y-3) \\ du = \cos(y-3)dy \end{array} \right. \\
 &\quad \left. dv = \frac{6}{y^4} dy \quad v = -\frac{2}{y^3} \right) \\
 &= -\frac{\sin(y-3)}{y^3} + \left( -\frac{\cos(y-3)}{y^2} - \int \frac{\sin(y-3)}{y^2} dy \right) \\
 &\quad \left( \begin{array}{l} u = \cos(y-3) \\ du = -\sin(y-3)dy \end{array} \right. \\
 &\quad \left. dv = \frac{2}{y^3} dy \quad v = -\frac{1}{y^2} \right) \\
 &= -\frac{\sin(y-3) + y \cos(y-3)}{y^3} - \int \frac{\sin(y-3)}{y^2} dy
 \end{aligned}$$

**Solution 6 by Syed Shahabudeen-Kerala-India**

$$\begin{aligned}
 \Omega &= \int \frac{(x^2 + 6x + 15) \sin x}{(x^4 + 12x^3 + 54x^2 + 108x + 81)} dx \\
 &= \int \frac{((x+3)^2 + 6)}{(x+3)^4} \sin(x) dx = \int \left( \frac{1}{(x+3)^2} + \frac{6}{(x+3)^4} \right) \sin(x) dx \\
 &= \operatorname{Im} \left( \underbrace{\int \left( \frac{e^{ix}}{(x+3)^2} \right) dx}_A + \underbrace{6 \int \frac{e^{ix}}{(x+3)^4} dx}_B \right)
 \end{aligned}$$

Here

$$\begin{aligned}
 A &= \int \frac{e^{ix}}{(x+3)^2} dx \quad (\text{by IBP}) = \frac{-ie^{ix}}{(x+3)^2} - 2i \int \frac{e^{ix}}{(x+3)^3} dx \\
 B &= \int \frac{e^{ix}}{(x+3)^4} dx \quad (\text{by IBP}) = \frac{-1}{3} \frac{e^{ix}}{(x+3)^3} + \frac{i}{3} \int \frac{e^{ix}}{(x+3)^3} dx \\
 \therefore \Omega &= \operatorname{Im}(A + 6B) \\
 &= \operatorname{Im} \left( -e^{ix} \left( \frac{i}{(x+3)^2} + \frac{2}{(x+3)^3} \right) \right) = - \left( \frac{\cos x}{(x+3)^2} + \frac{2 \sin x}{(x+3)^3} \right)
 \end{aligned}$$

**Solution 7 by Timson Azeez Folorunsho-Nigeria**

$$\begin{aligned}
 \Omega &= \int \frac{(x^2 + 6x + 15) \sin x}{x^4 + 12x^3 + 54x^2 + 108x + 81} dx = \int \frac{(x^2 + 6x + 15) \sin x}{(x^2 + 6x + 3)(x^2 + 6x + 15) + 36} dx \\
 &= \int \frac{(x^2 + 6x + 15) \sin x}{(x^2 + 6x + 15 - 12)(x^2 + 6x + 15) + 36} dx =
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 &= \int \frac{(x^2 + 6x + 15)\sin x}{(x^2 + 6x + 15)^2 - 12(x^2 + 6x + 15) + 36} dx = \\
 &= \int \frac{(x^2 + 6x + 15)\sin x}{(x^2 + 6x + 15 - 6)^2} dx = \int \frac{(x^2 + 6x + 9 + 6)\sin x}{(x^2 + 6x + 12)^2} dx = \\
 &= \int \frac{(x^2 + 6x + 9)\sin x + 6\sin x}{(x + 3)^4} dx = \int \left( \frac{(x + 3)^2 \sin x}{(x + 3)^4} + \frac{6\sin x}{(x + 3)^4} \right) dx = \\
 &= \int ((x + 3)^{-2} \sin x + 6(x + 3)^{-2} \sin x) dx = \\
 &= \int (x + 3)^{-2} \sin x dx - 2(x + 3)^{-3} \sin x + 2 \int (x + 3)^{-3} \cos x dx = \\
 &= -\frac{2\sin x}{(x + 3)^3} - \frac{\cos x}{(x + 3)^2}
 \end{aligned}$$

**1416. Suppose  $\alpha, \beta, \gamma$  be the roots of  $x^3 - 5x + 7$ , then evaluate the sum**

$$\sum_{\alpha, \beta, \gamma} \frac{x^3}{4x^3 - 18x^2 - 10x + 37}$$

*Proposed by Surjeet Singhania-India*

**Solution by proposer**

Observe that:

$$\left| \frac{-5z + 7}{z^3} \right| \leq \frac{5|z| + 7}{|z|^3} = \frac{5 \cdot 3 + 7}{3^3} = \frac{22}{27} < 1$$

So,  $|-5z + 7| < |z^3|$  for  $|z| = 3$ .

By Rouche's Theorem:  $z^3 - 5z + 7$  has all roots inside  $|z| = 3$ . Assume:

$$f(z) = \frac{z^3}{(z - 6)(z^3 - 5z + 7)} = \frac{z^3}{z^4 - 6z^3 - 5z^2 + 37z - 52}$$

$f(z)$  – is Meromorphic function, hence sum of all residue of  $f(z)$  at singularities of  $f$  including residue at infinity is 0. All poles are of order 1, hence:

$$\text{Res}_{z=\alpha} f(z) = \frac{\alpha^3}{4\alpha^3 - 18\alpha^2 - 10\alpha + 37}$$

Poles of  $f$  are  $\alpha, \beta, \gamma, 6$  and  $\infty$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sum_{x=\alpha, \beta, \gamma} \operatorname{Res}_{z=x} f(z) + \operatorname{Res}_{z=6} f(z) + \operatorname{Res}_{z=\infty} f(z) = 0$$

$$\sum_{\alpha, \beta, \gamma} \frac{z^3}{4z^3 - 18z^2 - 10z + 37} = -\operatorname{Res}_{z=\infty} f(z) - \operatorname{Res}_{z=6} f(z) = 1 - \frac{216}{193} = -\frac{23}{193}$$

Therefore,

$$\sum_{\alpha, \beta, \gamma} \frac{z^3}{4z^3 - 18z^2 - 10z + 37} = -\frac{23}{193}$$

**1417. Find the real values of  $a$  and  $b$  such that:**

$$\frac{a}{b} \int_0^\pi \sin x \left( 1 + \frac{\sqrt{5}}{3} \cos x \right)^5 dx = F_{12}$$

**Notation:**  $F_n$  denotes  $n^{\text{th}}$  Fibonacci number

*Proposed by Naren Bhandari-Bajura-Nepal*

**Solution 1 by Max Wong-Hong Kong**

$$\begin{aligned} & \int_0^\pi \sin \left( 1 + \frac{\sqrt{5}}{3} \cos x \right)^5 dx. \text{ Let } u = 1 + \frac{\sqrt{5}}{3} \cos x. \quad du = -\frac{\sqrt{5}}{3} \sin x dx \\ &= \int_{1+\frac{\sqrt{5}}{3}}^{1-\frac{\sqrt{5}}{3}} \left( -\frac{\sqrt{5}}{3} \right)^{-1} u^5 du = -\frac{3}{\sqrt{5}} \frac{1}{6} [u^6]_{1+\frac{\sqrt{5}}{3}}^{1-\frac{\sqrt{5}}{3}} \\ &= -\frac{1}{2\sqrt{5}} \left( \left( 1 - \frac{\sqrt{5}}{3} \right)^6 - \left( 1 + \frac{\sqrt{5}}{3} \right)^6 \right) = +\frac{1}{2 \cdot 3^6 \sqrt{5}} \left( (3 + \sqrt{5})^6 - (3 - \sqrt{5})^6 \right) \\ &= \frac{2^5}{3^6 \sqrt{5}} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^6 - \left( \frac{3 - \sqrt{5}}{2} \right)^6 \right) = \frac{2^5}{3^6 \sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{12} - \left( \frac{1 - \sqrt{5}}{2} \right)^{12} \right) \stackrel{\text{Binet}}{=} \frac{2^5}{3^6} F_{12} \\ & \frac{a}{b} \left( \frac{2^5}{3^6} \right) F_{12} = F_{12}, \quad \frac{a}{b} = \frac{3^6}{2^5} = \frac{729}{32} \\ & (a, b) = (729x, 32x), x \in \mathbb{R} \setminus \{0\} \end{aligned}$$

**Solution 2 by Syed Shahabudeen-India**

$$F_{12} = \frac{a}{b} \int_0^\pi \sin x \left( 1 + \frac{\sqrt{5}}{3} \cos x \right)^5 dx = \frac{a}{b} \left( \frac{1}{3^5 \sqrt{5}} \right) \int_{3-\sqrt{5}}^{3+\sqrt{5}} t^5 dt$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \frac{a}{b} \left( \frac{1}{3^5 6\sqrt{5}} \right) \left( (3 + \sqrt{5})^6 - (3 - \sqrt{5})^6 \right) \\
 &= \frac{a}{b} \left( \frac{2^6}{3^5 6\sqrt{5}} \right) \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{12} - \left( \frac{1 - \sqrt{5}}{2} \right)^{12} \right) \left( \because \left( \frac{3 \pm \sqrt{5}}{2} \right) = \left( \frac{1 \pm \sqrt{5}}{2} \right)^2 \right) \\
 &= \frac{a}{b} \left( \frac{2^5}{3^6} \right) F_{12} \text{ (by binet formula)} \Rightarrow \frac{a}{b} = \frac{3^6}{2^5} \therefore a = 729 \text{ and } b = 32
 \end{aligned}$$

**Solution 3 by Henn Hsain-Egypt**

$$\begin{aligned}
 \Omega &= \frac{a}{b} \int_0^\pi \sin(x) \left( 1 + \frac{\sqrt{5}}{3} \cos(x) \right)^5 dx = F_{12} \\
 \Omega &= -\frac{a}{b} \left( \frac{3}{\sqrt{5}} \right)^5 \frac{1}{6} \left( 1 + \frac{\sqrt{5}}{3} \cos(x) \right)^6 \Big|_0^\pi \\
 \Omega &= -\frac{a}{2b} \left( \frac{1}{\sqrt{5}} \right) \left( \left( 1 - \frac{\sqrt{5}}{3} \right)^6 - \left( 1 + \frac{\sqrt{5}}{3} \right)^6 \right), \Omega = \frac{a}{2b} \left( \frac{1}{\sqrt{5}} \right) \left( \left( 1 + \frac{\sqrt{5}}{3} \right)^6 - \left( 1 - \frac{\sqrt{5}}{3} \right)^6 \right) \\
 \Omega &= \frac{2^5 a}{3^6 \sqrt{5} b} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^6 - \left( \frac{3 - \sqrt{5}}{2} \right)^6 \right), \quad \Omega = \frac{2^5 a}{3^6 \sqrt{5} b} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{12} - \left( \frac{1 - \sqrt{5}}{2} \right)^{12} \right) \\
 F_{2n+1} &= \frac{(\phi)^{2n+1} - \left( \frac{1}{\phi} \right)^{2n+1}}{\sqrt{5}}, \quad \Omega = \frac{2^5 a}{3^6 b} F_{12}, \quad \frac{2^5 a}{3^6 b} F_{12} = F_{12}, \frac{a}{b} = \frac{3^6}{2^5} \quad a, b \in \mathbb{R}^+
 \end{aligned}$$

**1418. Find:**

$$\Omega(n) = \int \frac{x^2 + n^2 x + 1}{x^4 - nx^2 + 1} dx, n \in [0, 2)$$

*Proposed by Samir Cabiyev-Azerbaijan*

**Solution 1 by Samar Das-India**

$$\begin{aligned}
 \Omega(n) &= \int \frac{x^2 + n^2 x + 1}{x^4 - nx^2 + 1} dx \quad (\because n \in [0, 2)) \\
 &= \int \frac{(x^2 + 1)dx}{(x^4 + 1 - nx^2)} + \int \frac{n^2 x \, dx}{x^4 + 1 - nx^2} \\
 &= \int \frac{\left( 1 + \frac{1}{x^2} \right) dx}{\left( x^2 + \frac{1}{x^2} - n \right)} + \frac{n^2}{2} \int \frac{dy}{y^2 + 1 - ny} \quad (\text{let } y = x^2)
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \int \frac{\left(1 + \frac{1}{x^2}\right) dx}{\left(x - \frac{1}{x}\right)^2 + 2 - n} + \frac{n^2}{2} \int \frac{dy}{\left(y - \frac{n}{2}\right)^2 + \left(\frac{\sqrt{4-n^2}}{2}\right)^2} \\
 &= \frac{1}{\sqrt{2-n}} \tan^{-1} \left( \frac{x - \frac{1}{x}}{\sqrt{2-n}} \right) + \frac{n^2}{2} \times \frac{2}{\sqrt{4-n^2}} \tan^{-1} \left( \frac{\left(y - \frac{n}{2}\right)}{\sqrt{4-n^2}} \times 2 \right) + c \\
 &\stackrel{y=x^2}{=} \frac{1}{\sqrt{2-n}} \tan^{-1} \left( \frac{x - \frac{1}{x}}{\sqrt{2-n}} \right) + \frac{n^2}{\sqrt{4-n^2}} \tan^{-1} \left( \frac{2x^2 - n}{\sqrt{4-n^2}} \right) + c
 \end{aligned}$$

**Solution 2 by Ravi Prakash-New Delhi-India**

$$\begin{aligned}
 \Omega(n) &= \int \frac{x^2 + n^2 x + 1}{x^4 - nx^2 + 1} dx, n \in [0, 2) = I_1 + n^2 I_2 \text{ where } I_1 = \int \frac{x^2 + 1}{x^4 - nx^2 + 1} dx \\
 &\quad \text{and } I_2 = \int \frac{x dx}{x^4 - nx^2 + 1} \\
 I_1 &= \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2} - n} dx = \int \frac{d\left(x - \frac{1}{x}\right)}{\left(x - \frac{1}{x}\right)^2 + 2 - n} = \frac{1}{\sqrt{2-n}} \tan^{-1} \left( \frac{x - \frac{1}{x}}{\sqrt{2-n}} \right) \\
 &= \frac{1}{\sqrt{2-n}} \tan^{-1} \left( \frac{x^2 - 1}{x\sqrt{2-n}} \right) \\
 &\quad \text{For } I_2, \text{ put } x^2 = t \\
 I_2 &= \frac{1}{2} \int \frac{dt}{t^2 - nt + 1} = \frac{1}{2} \int \frac{dt}{\left(t - \frac{n}{2}\right)^2 + 1 - \left(\frac{n}{2}\right)^2} = \frac{2}{\sqrt{4-n^2}} \tan^{-1} \left( \frac{\left(t - \frac{n}{2}\right)(2)}{\sqrt{4-n^2}} \right) \\
 &\quad \text{Thus,} \\
 \Omega(n) &= \frac{1}{\sqrt{2-n}} \tan^{-1} \left( \frac{x^2 - 1}{x\sqrt{2-n}} \right) + \frac{2n^2}{\sqrt{4-n^2}} \tan^{-1} \left( \frac{2x^2 - n}{\sqrt{4-n^2}} \right) + c
 \end{aligned}$$

**Solution 3 by Timson Azees Folorunsho-Nigeria**

$$\begin{aligned}
 \Omega(n) &= \underbrace{\int \frac{(x^2 + 1) dx}{x^4 - nx^2 + 1}}_{I_1} + \underbrace{\int \frac{n^2 x}{x^4 - nx^2 + 1} dx}_{I_2} \\
 I_1 &= \int \frac{(x^2 + 1) dx}{x^4 - nx^2 + 1} \Rightarrow \int \frac{\left(1 + \frac{1}{x^2}\right) dx}{x^2 - n + \frac{1}{x^2}} \\
 I_1 &= \int \frac{\left(1 + \frac{1}{x^2}\right) dx}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{2-n})^2} \text{ put } y = x - \frac{1}{x}; dy = \left(1 + \frac{1}{x^2}\right) dx
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$I_1 = \int \frac{dy}{y^2 + (\sqrt{2-n})^2} \Rightarrow \frac{1}{\sqrt{2-n}} \tan^{-1} \left( \frac{y}{\sqrt{2-n}} \right)$$

$$I_1 = \frac{1}{\sqrt{2-n}} \tan^{-1} \left( \frac{x^2 - 1}{x\sqrt{2-n}} \right)$$

$$I_2 = \int \frac{n^2 x dx}{x^4 - nx^2 + 1} \text{ put } u = x^2; \frac{dy}{2} = x dx$$

$$I_2 = \frac{1}{2} \int \frac{n^2 du}{u^2 - nu + 1} \Rightarrow \frac{n^2}{2} \int \frac{dy}{\left(y - \frac{n}{2}\right)^2 + \left(\sqrt{1 - \frac{n^2}{4}}\right)^2}$$

$$I_2 = \frac{n^2}{\sqrt{4-n^2}} \tan^{-1} \left( \frac{2y-n}{\sqrt{4-n^2}} \right) \Rightarrow \frac{n^2}{\sqrt{4-n^2}} \tan^{-1} \left( \frac{2x^2-n}{\sqrt{4-n^2}} \right)$$

$$\Omega(n) = I_1 + I_2 + C$$

$$\Omega(n) = \frac{1}{\sqrt{2-n}} \tan^{-1} \left( \frac{x^2 - 1}{x\sqrt{2-n}} \right) + \frac{n^2}{\sqrt{4-n^2}} \tan^{-1} \left( \frac{2x^2-n}{\sqrt{4-n^2}} \right) + C$$

**1419.**

If  $(\gamma_n)_{n \geq 1}$ ,  $\gamma_n = -\log n + \sum_{k=1}^n \frac{1}{k}$  with  $\lim_{n \rightarrow \infty} \gamma_n = \gamma$ ,

$\gamma$  is Euler – Mascheroni constant. Then compute:

$$\Omega = \lim_{n \rightarrow \infty} (\sin \gamma_n - \sin \gamma) \sqrt[n]{n!}$$

*Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania*

**Solution 1 by Marian Ursărescu-Romania**

$$\Omega = \lim_{n \rightarrow \infty} (\sin \gamma_n - \sin \gamma) \sqrt[n]{n!} = \lim_{n \rightarrow \infty} (\sin \gamma_n - \sin \gamma) n \cdot \frac{\sqrt[n]{n!}}{n}; (1)$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n!}{n^n}} \stackrel{(C-D)}{=} \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n = \frac{1}{e}; (2)$$

$$\lim_{n \rightarrow \infty} (\sin \gamma_n - \sin \gamma) n = \lim_{n \rightarrow \infty} \frac{(\sin \gamma_n - \sin \gamma)}{\frac{1}{n}} \stackrel{(LC-S)}{=} \lim_{n \rightarrow \infty} \frac{\sin \gamma_{n+1} - \sin \gamma_n}{\frac{1}{n+1} - \frac{1}{n}} =$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= \lim_{n \rightarrow \infty} \frac{2 \sin\left(\frac{\gamma_{n+1} - \gamma_n}{2}\right) \cos\left(\frac{\gamma_{n+1} + \gamma_n}{2}\right)}{-\frac{1}{n(n+1)}} = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{\gamma_{n+1} - \gamma_n}{2}\right)}{\gamma_{n+1} - \gamma_n} \cdot \cos\left(\frac{\gamma_{n+1} + \gamma_n}{2}\right) \cdot \frac{\gamma_{n+1} - \gamma_n}{-\frac{1}{n(n+1)}}; (3)$$

$$\lim_{n \rightarrow \infty} \cos\left(\frac{\gamma_{n+1} + \gamma_n}{2}\right) = \cos \gamma; (4)$$

$$\lim_{n \rightarrow \infty} \frac{\sin\left(\frac{\gamma_{n+1} - \gamma_n}{2}\right)}{\gamma_{n+1} - \gamma_n} = 1; (5)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\gamma_{n+1} - \gamma_n}{-\frac{1}{n+1}} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1} - \log(n+1) + \log n}{-\frac{1}{n(n+1)}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1} - \log \frac{n+1}{n}}{-\frac{1}{n(n+1)}} = \\ &= \lim_{n \rightarrow \infty} \frac{1 - (n+1) \log\left(1 + \frac{1}{n}\right)}{-\frac{1}{n(n+1)}} \stackrel{(1=x)}{=} \lim_{x \rightarrow 0} \frac{1 - \left(\frac{1}{x} + 1\right) \log(1+x)}{-x} = \\ &= \lim_{x \rightarrow 0} \frac{x + (x+1) \log(x+1)}{-x^2} \stackrel{(l'H)}{=} \lim_{x \rightarrow 0} \frac{1 - \log(x+1) - 1}{-2x} = \\ &= \lim_{x \rightarrow 0} \frac{\log(1+x)}{2x} = \frac{1}{2}; (6) \end{aligned}$$

From (1)~(6) it follows that:

$$\Omega = \lim_{n \rightarrow \infty} (\sin \gamma_n - \sin \gamma)^{\sqrt[n]{n!}} = \frac{1}{2e} \cos \gamma$$

**Solution 2 by Mikael Bernardo-Mozambique**

$$\begin{aligned} \because \sin x - \sin y &= 2 \sin \frac{x-y}{2} \cos \frac{x+y}{2} \\ \Omega &= \lim_{n \rightarrow \infty} \left( 2 \sin \frac{\gamma_n - \gamma}{2} \cos \frac{\gamma_n + \gamma}{2} \right) \cdot \sqrt[n]{n!} = \\ &\quad \because n! \cong \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \\ &= \cos \gamma \lim_{n \rightarrow \infty} \left( 2 \sin \frac{\gamma_n - \gamma}{2} \cdot \sqrt[n]{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} \right) = \cos \gamma \lim_{n \rightarrow \infty} \left( \frac{\sin \frac{\gamma_n - \gamma}{2}}{\frac{\gamma_n - \gamma}{2}} \right) (\gamma_n - \gamma) \sqrt[n]{\sqrt{n} \cdot \frac{n}{e}} \\ &\quad \lim_{n \rightarrow \infty} \sqrt[n]{\sqrt{n}} \stackrel{(C-D)}{=} \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{\sqrt{n}} = 1 \\ \Omega &= \frac{1}{e} \cos \gamma \cdot 1 \cdot 1 \cdot \lim_{n \rightarrow \infty} (\gamma_n - \gamma) \cdot n \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$H_n \sim \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} \left( 1 + o\left(\frac{1}{n^2}\right) \right)$$

$$\lim_{n \rightarrow \infty} (\gamma_n - \gamma) \cdot n = \lim_{n \rightarrow \infty} (H_n - \log n - \gamma) \cdot n = \frac{1}{2}$$

Therefore,

$$\Omega = \lim_{n \rightarrow \infty} (\sin \gamma_n - \sin \gamma) \sqrt[n]{n!} = \frac{1}{2e} \cos \gamma$$

**1420. Prove that:**

$$\lim_{s \rightarrow 1} \int_0^\infty e^{-st} \left( \sum_{n=0}^{\infty} \frac{t^n \sin \frac{\pi n}{4}}{n!} \right) dt = \frac{i-1}{(\sqrt{2} + (i-1))^2}$$

*Proposed by Tobi Joshua-Nigeria*

**Solution 1 by Syed Shahabudeen-India**

$$\begin{aligned} \lim_{s \rightarrow 1} \int_0^\infty e^{-st} \left( \sum_{n=0}^{\infty} \frac{t^n \sin \frac{\pi n}{4}}{n!} \right) dt &= \lim_{s \rightarrow 1} \Im m \int_0^\infty e^{-st} \left( \exp \left( te^{\frac{i\pi}{4}} \right) \right) dt \stackrel{\text{Laplace transform}}{\cong} \\ &= \lim_{s \rightarrow 1} \Im m \mathcal{L} \left\{ \exp \left( te^{\frac{i\pi}{2}} \right) \right\} = \Im m \left( \frac{1}{s - e^{\frac{i\pi}{4}}} \right) = \Im m \frac{\sqrt{2}}{\sqrt{2} - (i+1)} = \\ &= \Im m \frac{\sqrt{2} \left( \sqrt{2} + (1+i) \right)}{2(1-i)} = \Im m \frac{\sqrt{2} + i(2+\sqrt{2})}{2\sqrt{2}} = \frac{1}{2} + \frac{1}{\sqrt{2}} = \frac{4i(1+\sqrt{2})}{8i} = \\ &= \frac{(i-1) \left( \sqrt{2} - (i-1) \right)^2}{((\sqrt{2} + (i-1))(\sqrt{2} - (i-1))^2} = \frac{i-1}{(\sqrt{2} + (i-1))^2} \end{aligned}$$

**Solution 2 by Akerele Olofin-Nigeria**

$$LHS = \frac{i-1}{(\sqrt{2} + (i-1))^2} = \frac{1 + \sqrt{2}}{2}$$

$$\text{Let: } \Omega = \int_0^\infty e^{-st} \left( \sum_{n=0}^{\infty} \frac{t^n \sin \frac{\pi n}{4}}{n!} \right) dt = \sum_{n=0}^{\infty} \frac{\sin \left( \frac{\pi n}{4} \right)}{n!} \int_0^\infty e^{-st} t^n dt =$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \frac{\sin\left(\frac{\pi n}{4}\right)}{n!} \cdot \frac{n!}{s^{n+1}} = \operatorname{Im} \left( \frac{1}{s} \sum_{n=0}^{\infty} \frac{\exp\left(\frac{in\pi}{4}\right)}{s^n} \right) = \operatorname{Im} \left( \frac{1}{s} \sum_{n=0}^{\infty} \left( \frac{\exp\left(\frac{i\pi}{4}\right)}{s} \right)^n \right) = \\
 &= \operatorname{Im} \left( \frac{1}{s - \exp\left(\frac{i\pi}{4}\right)} \right) = \operatorname{Im} \left( \frac{1}{s - \cos\frac{\pi}{4} - i \sin\frac{\pi}{4}} \right) = \\
 &= \operatorname{Im} \left( \frac{s - \cos\frac{\pi}{4} + i \sin\frac{\pi}{4}}{s^2 - \sqrt{2}s + 1} \right) = \frac{1}{\sqrt{2}(s^2 - \sqrt{2}s + 1)}
 \end{aligned}$$

Therefore,

$$\lim_{s \rightarrow 1} \Omega(s) = \lim_{s \rightarrow 1} \frac{1}{\sqrt{2}(s^2 - \sqrt{2}s + 1)} = \frac{1}{2\sqrt{2} - 2} = \frac{1 + \sqrt{2}}{2} = \frac{i - 1}{(\sqrt{2} + (i - 1))^2}$$

**Solution 3 by Mohammad Rostami-Afghanistan**

$$\begin{aligned}
 \Omega &= \lim_{s \rightarrow 1} \int_0^\infty e^{-st} \left( \sum_{n=0}^{\infty} \frac{t^n \sin \frac{\pi n}{4}}{n!} \right) dt = \frac{i - 1}{(\sqrt{2} + (i - 1))^2} \\
 \frac{i - 1}{(\sqrt{2} + (i - 1))^2} &= \frac{i - 1}{2 + 2\sqrt{2}(i - 1) + (i - 1)^2} = \frac{i - 1}{2 + 2\sqrt{2}i - 2\sqrt{2} + i^2 - 2i + 1} = \\
 &= \frac{i - 1}{(2\sqrt{2} - 2)i - (2\sqrt{2} - 2)} = \frac{i - 1}{(2\sqrt{2} - 2)(i - 1)} = \frac{1}{2\sqrt{2} - 1} = \frac{\sqrt{2} + 1}{2} \\
 \rightarrow \Omega &= \frac{\sqrt{2} + 1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \Omega &= \lim_{s \rightarrow 1} \int_0^\infty e^{-st} \left( \sum_{n=0}^{\infty} \frac{t^n \sin \frac{\pi n}{4}}{n!} \right) dt = \lim_{s \rightarrow 1} \sum_{n=0}^{\infty} \frac{\sin \frac{\pi n}{4}}{n!} \int_0^\infty e^{-st} t^n dt \stackrel{st=u}{=} \\
 &= \lim_{s \rightarrow 1} \sum_{n=0}^{\infty} \frac{\sin \frac{\pi n}{4}}{n!} \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s} = \lim_{s \rightarrow 1} \sum_{n=0}^{\infty} \frac{\sin \frac{n\pi}{4}}{n!} \cdot \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^{(n+1)-1} du = \\
 &= \lim_{s \rightarrow 1} \sum_{n=0}^{\infty} \frac{\sin \frac{n\pi}{4}}{\Gamma(n)} \cdot \frac{\Gamma(n+1)}{s^{n+1}} = \sum_{n=0}^{\infty} \sin \frac{n\pi}{4} =
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= 0 + \left( \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} \right) + 0 + \left( -\frac{\sqrt{2}}{2} - 1 - \frac{\sqrt{2}}{2} \right) + 0 + \left( \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} \right) + 0 + \\
 &\quad + \left( -\frac{\sqrt{2}}{2} - 1 - \frac{\sqrt{2}}{2} \right) + \dots = \frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + 1}{2} \rightarrow \Omega = \frac{\sqrt{2} + 1}{2}
 \end{aligned}$$

**1421. Prove that:**

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sin^{2n+1} x}{x} dx = \frac{\pi^3}{12} - \pi \log^2 2$$

*Proposed by Narendra Bhandari-Bajura-Nepal*

**Solution 1 by proposer**

Let  $F(m, n) = \sum_{k=0}^n (-1)^k \binom{m}{k}$ ,  $m \geq 1$  now and have Pascal identity:

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1} \text{ and therefore,}$$

$$\begin{aligned}
 F(m, n) &= \sum_{k=0}^n (-1)^k \left[ \binom{m-1}{k} + \binom{m-1}{k-1} \right] = \sum_{k=0}^n (-1)^k \left[ \binom{m-1}{k} - (-1) \binom{m-1}{k-1} \right] = \\
 &= \sum_{k=0}^n \left[ (-1)^k \binom{m-1}{k} - (-1)^{k+1} \binom{m-1}{k-1} \right] = (-1)^n \binom{m-1}{n}
 \end{aligned}$$

We obtained the final result observing the telescoping sum and therefore,

$$F(m, n) = \sum_{k=0}^n (-1)^k \binom{n}{k} = (-1)^n \binom{m-1}{n} \text{ and if } m = 2n+1 \text{ then}$$

$$F(2n+1, n) = (-1)^n \binom{2n}{n}.$$

Now we recall the power reduction formula of:

$$\sin^{2n+1} x = \frac{(-1)^n}{4^n} \sum_{k=0}^n \binom{2n+1}{k} \sin((2n+1-2k)x) = \frac{(-1)^n}{4^n} \sum_{k=0}^n \binom{2n+1}{k} \sin(ax),$$

where  $a = 2n+1-2k > 0$  and it is well known that:

$$\int_0^{\infty} \frac{\sin(ax)}{x} dx = \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}, \text{ the original integral reduces to}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\int_0^\infty \frac{\sin^{2n+1} x}{x} dx = \frac{(-1)^n}{4^n} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \int_0^\infty \frac{\sin(ax)}{x} dx = \frac{\pi}{2 \cdot 4^n} \binom{2n}{n}$$

*and we are left to evaluate the sum*

$$S = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{4^n \cdot n^2} \binom{2n}{n}.$$

*Since the generating function of central binomial coefficients is*

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}} \text{ for } |x| < \frac{1}{4}.$$

*Dividing by x and hence on integration from 0 to y we have*

$$\sum_{n=1}^{\infty} \frac{1}{n} \binom{2n}{n} y^n = \int_0^y \left( \frac{1}{x\sqrt{1-4x}} - \frac{1}{x} \right) dx = 2 \log \left( \frac{2}{1 + \sqrt{1-4y}} \right)$$

*Further we divide by y and integrate it to get*

$$\sum_{n=1}^{\infty} \frac{y^n}{n^2} \binom{2n}{n} = 2 \int \log \left( \frac{2}{1 + \sqrt{1-4y}} \right) \frac{dy}{y} = 2 \log \log y - 2 \int \frac{\log(1 + \sqrt{1-4y})}{y} dy$$

*Now, substituting z =  $\sqrt{1-4y} \rightarrow y = \frac{1-z^2}{4}$  and the integral we have*

$$-2 \int \frac{z \log(1+z)}{1-z^2} dz = \int \left( \frac{\log(1+z)}{1+z} + \frac{\log(1+z)}{z-1} \right) dz$$

*Applying by parts in formal integral it is easy to show*

$$F = \log^2(1+z) - \int \frac{\log(1+z)}{1+z} dz \rightarrow F = \frac{1}{2} \log^2(1+z) \text{ and for latter}$$

*integral set z - 1 = u such that we have:*

$$\int \frac{\log(1+z)}{1+z} dz = \log 2 \log u + \int \frac{\log(1+\frac{u}{2})}{u} du = \log 2 \log u - Li_2\left(-\frac{u}{2}\right) \text{ undo}$$

*the substitution and evaluating for constant of integral at y = 0 we have then*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} \binom{2n}{n} y^n &= 3 \log^2 2 + 2 Li_2\left(\frac{2y}{1 + \sqrt{1-4y}}\right) - \\ &- \log^2(1 + \sqrt{1-4y}) - 2 \log 2 \log\left(\frac{1 - \sqrt{1-4y}}{y}\right), \text{ now setting } y = \frac{1}{4} \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{\pi}{2} \sum_{k=0}^{\infty} \frac{1}{4^n \cdot n^2} \binom{2n}{n} = \frac{\pi}{2} (\zeta(2) - 2 \log^2(2)) = \frac{\pi^3}{12} - \pi \log^2 2, \text{ where we used}$$

$$Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2 2}{2}$$

**Solution 2 by Surjeet Singhania-India**

$$\text{Denote: } X_n = \sin^{2n+1} x = \frac{i(-1)^{n+1}}{2^{2n+1}} (e^{ix} - e^{-ix})^{2n+1}$$

$$X_n = \frac{(-1)^{n+1}}{2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k e^{2nix+ix-2ikx}; \quad (1)$$

$$\text{Also, } X_n = \frac{(-1)^{(n+1)i}}{2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k e^{2ikx-2inx-2ix}; \quad (2)$$

**Adding (1), (2) we get:**

$$X_n = \sin^{2n+1} x = \frac{(-1)^n}{2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k \sin(2nx + x - 2kx)$$

$$\text{Denote: } I_n = \int_0^\infty \frac{\sin^{2n+1} x}{x} dx =$$

$$= \frac{(-1)^n}{2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k \int_0^\infty \frac{\sin\{(2m+1-2k)x\}}{x} dx$$

$$\text{We know that: } \int_0^\infty \frac{\sin(mx)}{x} dx = \begin{cases} \frac{\pi}{2}; & \text{if } m > 0 \\ -\frac{\pi}{2}; & \text{if } m < 0 \end{cases}$$

$$\begin{aligned} I_n &= \frac{(-1)^n}{2^{2n+1}} \sum_{k=0}^n \binom{2n+1}{k} (-1)^k \left(\frac{\pi}{2}\right) + \frac{(-1)^n}{2^{2n+1}} \sum_{k=n+1}^{2n+1} \binom{2n+1}{k} (-1)^k \left(-\frac{\pi}{2}\right) = \\ &= \frac{\pi(-1)^n}{2^{2n+1}} \underbrace{\sum_{k=0}^n \binom{2n+1}{k} (-1)^k}_{J_n} = \frac{\pi(-1)^n}{2^{2n+1}} J_n \end{aligned}$$

$$J_n = \sum_{k=0}^n (-1)^k \binom{2n+1}{k} = \frac{1}{2\pi i} \oint_{|z|=\epsilon} \frac{(1+z)^{2n+1}}{z} \sum_{k=0}^n \frac{(-1)^k}{z^k} dz$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 J_n &= \frac{1}{2\pi i} \oint_{|z|=\epsilon} (1+z)^{2n} \left( 1 - \left( -\frac{1}{z} \right)^{n+1} \right) dz = (-1)^n \binom{2n}{n} \\
 I_n &= \int_0^\infty \frac{\sin^{2n+1} x}{x} dx = \frac{\pi}{2(2n+1)} \binom{2n}{n} \\
 \sum_{n=1}^{\infty} \frac{I_n}{n^2} &= -\frac{\pi}{2} \int_0^1 \frac{\log x}{x} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^n}{4^n} dx = \frac{\pi}{2} \int_0^1 \frac{\log x}{x} \left( 1 - \frac{1}{\sqrt{1-x}} \right) dx = \\
 &= \frac{\pi}{2} \int_0^1 \frac{\log(1-x)}{1-x} \left( 1 - \frac{1}{\sqrt{x}} \right) dx = -\pi \int_0^1 \frac{\log(1-x^2)}{1+x} dx = \\
 &= -\frac{\pi \log^2 2}{2} - \pi \int_0^1 \frac{\log x}{2-x} dx = -\frac{\pi}{2} \log^2 2 - \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \int_0^1 x^{k-1} \log x dx = \\
 &= -\frac{\pi}{2} \log^2 2 + \pi \sum_{k=1}^{\infty} \frac{1}{2^k k^2} = \pi Li_2 \left( \frac{1}{2} \right) - \frac{\pi}{2} \log^2 2 = \frac{\pi^3}{12} - \pi \log^2 2
 \end{aligned}$$

*Therefore,*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sin^{2n+1} x}{n^2} \frac{dx}{x} = \frac{\pi^3}{12} - \pi \log^2 2$$

**1422. Prove that:**

$$\begin{aligned}
 &\int_0^1 \frac{(x^3 + 1) \tan^{-1} x + x \log x + (x^2 + 1) \log(1+x)}{1+x^2} dx = \\
 &= \frac{1}{96} (\pi^2 + 12\pi(2 + \log 2) + 48(\log 16 - G - 3))
 \end{aligned}$$

-where  $G$  –Catalan constant.

*Proposed by Akerele Olofin-Nigeria*

**Solution by Syed Shahabudeen-India**

$$\begin{aligned}
 &\int_0^1 \frac{(x^3 + 1) \tan^{-1} x + x \log x + (x^2 + 1) \log(1+x)}{1+x^2} dx = \\
 &= \int_0^1 \frac{x^3 \tan^{-1} x}{1+x^2} dx + \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx + \int_0^1 \frac{x \log x}{1+x^2} dx + \int_0^1 \log(1+x) dx \\
 &= A + B + C + D
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 A &= \int_0^1 \frac{x^3 \tan^{-1} x}{1+x^2} dx \stackrel{\tan^{-1} x = \theta}{=} \int_0^{\frac{\pi}{4}} \theta \tan^3 \theta d\theta \stackrel{IBP}{=} \\
 &= \frac{\pi}{4} \left( 1 - \frac{1}{2} \log 2 \right) - \int_0^{\frac{\pi}{4}} \left( \frac{1}{2} \sec^2 \theta + \log(\cos \theta) \right) d\theta = \\
 &= \frac{\pi}{4} \left( 1 - \frac{1}{2} \log 2 \right) - \left( \frac{1}{2} + \int_0^{\frac{\pi}{4}} \log(\cos \theta) d\theta \right)
 \end{aligned}$$

Apply Fourier series of  $\log(\cos \theta)$ , it follows that,

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \log(\cos \theta) d\theta &= - \left( \int_0^{\frac{\pi}{4}} \left( \sum_{k=1}^{\infty} \frac{(-1)^k \cos(2k\pi)}{k} + \log 2 \right) d\theta \right) = \\
 &= - \left( -\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^2} + \frac{\pi}{4} \log 2 \right) = \frac{G}{2} - \frac{\pi}{4} \log 2 \\
 A &= \frac{\pi}{4} - \frac{1}{2} - \frac{G}{2} + \frac{\pi}{8} \log 2 \\
 B &= \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx = \frac{\pi^2}{32} \\
 C &= \int_0^1 \frac{x \log x}{1+x^2} dx = \frac{1}{4} \left( \frac{1}{4} \left( \psi^1(1) - \psi^1\left(\frac{1}{2}\right) \right) \right) = -\frac{\pi^2}{48} \\
 D &= \int_0^1 \log(1+x) dx = \log 4 - 1 \\
 \Omega &= A + B + C + D = \frac{1}{96} \left( \pi^2 + 12\pi(2 + \log 2) + 48(\log 16 - G - 3) \right)
 \end{aligned}$$

**1423. Find:**

$$\Omega = \int_0^\infty \frac{\sqrt{x} \cdot \tan^{-1} x}{x^2 + 1} dx$$

*Proposed by Vasile Mircea Popa-Romania*

**Solution by Rana Ranino-Setif-Algerie**

$$\Omega = \int_0^\infty \frac{\sqrt{x} \cdot \tan^{-1} x}{x^2 + 1} dx = \int_0^1 \frac{1}{y^2 - 1} \int_0^\infty \left( \frac{x^{-\frac{1}{2}}}{1+x^2} - \frac{x^{-\frac{1}{2}}}{1+x^2 y^2} \right) dx dy$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\int_0^\infty \frac{x^{-\frac{1}{2}}}{1+x^2} dx \stackrel{t=x^2}{=} \frac{1}{2} \int_0^\infty \frac{t^{\frac{1}{4}-1}}{1+t} dt = \frac{\pi}{2 \sin \frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$$

$$\int_0^\infty \frac{x^{-\frac{1}{2}}}{1+x^2y^2} dx \stackrel{t=x^2y^2}{=} \frac{1}{2} y^{-\frac{1}{2}} \int_0^\infty \frac{t^{\frac{1}{4}-1}}{1+t} dt = \frac{\pi}{\sqrt{2}} y^{-\frac{1}{2}}$$

$$\begin{aligned} \Omega &= \frac{\pi}{\sqrt{2}} \int_0^1 \frac{1-y^{-\frac{1}{2}}}{y^2-1} dy = \frac{\pi}{\sqrt{2}} \sum_{n=0}^{\infty} \int_0^1 \left( y^{2n-\frac{1}{2}} - y^{2n} \right) dy = \\ &= \frac{\pi}{2\sqrt{2}} \sum_{n=0}^{\infty} \left( \frac{1}{n+\frac{1}{4}} - \frac{1}{n+\frac{1}{2}} \right) = \frac{\pi}{2\sqrt{2}} \underbrace{\left( \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{4}\right) \right)}_{\frac{\pi}{2} + \log 2} \end{aligned}$$

Therefore,

$$\Omega = \int_0^\infty \frac{\sqrt{x} \cdot \tan^{-1} x}{x^2+1} dx = \frac{\pi}{2\sqrt{2}} \left( \frac{\pi}{2} + \log 2 \right)$$

**1424. Prove that:**

$$\int_0^\infty \frac{x-x^3}{(4x-4x^3)^2+(2-6x^2+x^4)^2} dx = \frac{1}{16\sqrt{2}} (\pi - \log \left( \frac{\sqrt{2}+1}{\sqrt{2}-1} \right))$$

*Proposed by Angad Singh-India*

**Solution by proposer**

Consider the complex line integral

$$J = \oint_C \frac{dz}{1+z^4}$$

Where  $R$  tends to infinity and  $C$  is a rectangle having vertices  $0+0i, 1+0i, 1+iR$  and  $0+iR$  traversed anti-clockwise. Hence,

$$J = \int_0^1 \frac{1}{1+x^4} dx + \int_0^R \frac{1}{1+(1+iy)^4} idy + \int_1^0 \frac{1}{1+(x+iR)^4} dx + \int_R^0 \frac{1}{1+(iy)^4} dy$$

$$\text{Let } I_1 = \int_0^1 \frac{1}{1+x^4} dx, I_2 = \int_0^R \frac{1}{1+(1+iy)^4} idy$$

$$I_3 = \int_1^0 \frac{1}{1+(x+iR)^4} dx, I_4 = \int_R^0 \frac{1}{1+(iy)^4} dy$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Now, let us find the real part of all the integrals listed above.**

**Using the residue theorem, we have:**

$$J = 2\pi i \cdot \frac{1}{2i\sqrt{2}(1+i)} = \frac{\pi}{2\sqrt{2}}(1-i), \text{ thus}$$

$$\operatorname{Re}(J) = \frac{\pi}{2\sqrt{2}}$$

Similarly,  $I_1$  can also be easily using standard methods to get,

$$I_1 = \int_0^1 \frac{1}{1+x^4} dx = \frac{\pi}{4\sqrt{2}} + \frac{1}{4\sqrt{2}} \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

**Observe that:**

$$\operatorname{Re}(I_2) = \int_0^R \frac{4x - 4x^3}{(4x - 4x^3)^2 + (2 - 6x^2 + x^4)^2} dx$$

**Also it can be shown that:**

$$|I_3| = \left| \int_1^0 \frac{1}{1+(x+iR)^4} dx \right| \leq \int_0^1 \frac{1}{|1+(x+iR)^4|} dx \leq \frac{1}{R^4 - 1}$$

Since,  $\lim_{R \rightarrow \infty} \frac{1}{R^4 - 1} = 0$ , thus  $I_3$  vanishes, hence  $\operatorname{Re}(I_3) \rightarrow 0$  as  $R \rightarrow \infty$ .

**Finally,**

$$I_4 = \int_R^0 \frac{1}{1+(iy)^4} dy = -i \int_0^R \frac{1}{1+y^4} dy, \text{ thus } \operatorname{Re}(I_4) = 0.$$

**Equating all the real parts, we get:**

$$\operatorname{Re}(J) = \operatorname{Re}(I_1) + \operatorname{Re}(I_2) + \operatorname{Re}(I_3) + \operatorname{Re}(I_4)$$

$$\frac{\pi}{2\sqrt{2}} = \frac{\pi}{4\sqrt{2}} + \frac{1}{4\sqrt{2}} \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) + \int_0^R \frac{4x - 4x^3}{(4x - 4x^3)^2 + (2 - 6x^2 + x^4)^2} dx$$

**After the manipulation an as  $R \rightarrow \infty$ , we complete the proof.**

**1425.**

$$\text{If } \Phi(n) = \int_0^1 \frac{Li_n(x) \log^3(1-x)}{x} dx$$

**Prove that  $\Phi(n)$  can be expressed as:**



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\Phi(n) = - \sum_{k=1}^n \frac{(H_k)^3 + 3H_k H_k^{(2)} + 2H_k^{(3)}}{k^{n+1}}$$

where  $H_n^{(m)}$  is the  $n^{th}$  generalized harmonic number of weight  $m$ .

*Proposed by Akerele Olofin-Nigeria*

**Solution by Ngulmun George Baite-India**

$$\Phi(n) = \int_0^1 \frac{Li_n(x) \log^3(1-x)}{x} dx$$

$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} - \text{polylogarithm function.}$$

$$\phi(n) = \int_0^1 \sum_{k=1}^{\infty} \frac{x^k}{k^n} \cdot \frac{\log^3(1-x)}{x} dx = \sum_{k=1}^{\infty} \frac{1}{k^n} \int_0^1 x^{k-1} \log^3(1-x) dx$$

$$\phi(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} I_k; (1)$$

$$I_k = \int_0^1 x^{k-1} \log^3(1-x) dx =$$

$$= \frac{1}{k} \int_0^1 (x^k - 1)' \log^3(1-x) dx = \left[ \frac{1}{k} (x^k - 1) \log^3(1-x) \right]_0^1$$

$$- \frac{3}{k} \int_0^1 \frac{1-x^k}{1-x} \log^3(1-x) dx$$

$$I_k = - \frac{3}{k} \int_0^1 \sum_{m=1}^k x^{m-1} \log^2(1-x) dx = - \frac{3}{k} \sum_{m=1}^k \int_0^1 x^{m-1} \log^2(1-x) dx$$

$$= - \frac{3}{k} \sum_{m=1}^k J_m; (2)$$

$$J_m = \int_0^1 x^{m-1} \log^2(1-x) dx = \frac{1}{m} \int_0^1 (x^m - 1)' \log^2(1-x) dx =$$

$$= \left[ \frac{1}{m} (x^m - 1) \log^2(1-x) \right]_0^1 - \frac{2}{m} \int_0^1 \frac{1-x^m}{1-x} \log(1-x) dx =$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= -\frac{2}{m} \int_0^1 \sum_{p=1}^m x^{p-1} \log(1-x) dx = -\frac{2}{m} \sum_{p=1}^m \int_0^1 x^{p-1} \log(1-x) dx = \\
 &= \frac{2}{m} \sum_{p=1}^m \frac{H_p}{p} = \frac{(H_m)^2 + H_m^{(2)}}{m}
 \end{aligned}$$

$$\begin{aligned}
 \text{From (2): } I_k &= -\frac{3}{k} \sum_{m=1}^k \frac{(H_m)^2 + H_m^{(2)}}{m} = -\frac{3}{k} \cdot \frac{1}{3} \left\{ H_k^3 + 3H_k H_k^{(2)} + 2H_k^{(3)} \right\} = \\
 &= -\frac{H_k^3 + 3H_k H_k^{(2)} + 2H_k^{(3)}}{k^{n+1}}
 \end{aligned}$$

$$\phi(n) = \int_0^1 \frac{\text{Li}_n(x) \log^3(1-x)}{x} dx = - \sum_{k=1}^n \frac{(H_k)^3 + 3H_k H_k^{(2)} + 2H_k^{(3)}}{k^{n+1}}$$

**1426. Find without any software:**

$$\Omega = \int \frac{\log x}{\log^2 x + (2 - 2ex) \log x + 2e^2 x^2 - 2ex + 1} dx$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Asmat Qatea-Afghanistan**

$$\begin{aligned}
 \Omega &= \int \frac{\log x}{\log^2 x + (2 - 2ex) \log x + 2e^2 x^2 - 2ex + 1} dx = \\
 &= \int \frac{\log x}{\log^2 x + 2(1-ex) \log x + e^2 x^2 + (ex-1)^2} dx = \\
 &= \int \frac{\log x}{(\log x - ex + 1)^2 + e^2 x^2} dx = \\
 &= \int \frac{\log x}{(\log x - ex + 1)^2 \left( 1 + \left( \frac{ex}{\log x - ex + 1} \right)^2 \right)} dx = (*)
 \end{aligned}$$

$$\left( \text{Let: } \frac{ex}{\log x - ex + 1} = u \rightarrow \frac{e \log x}{(\log x - ex + 1)^2} dx = du \right)$$

$$(*) = \frac{1}{e} \int \frac{du}{1 + u^2} = \frac{1}{e} \tan^{-1} u + C$$

Therefore,



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}\Omega &= \int \frac{\log x}{\log^2 x + (2 - 2ex) \log x + 2e^2x^2 - 2ex + 1} dx = \\ &= \frac{1}{e} \tan^{-1} \left( \frac{ex}{\log x - ex + 1} \right) + C\end{aligned}$$

**Solution 2 by Syed Shahabudeen-India**

$$\begin{aligned}\Omega &= \int \frac{\log x}{\log^2 x + (2 - 2ex) \log x + 2e^2x^2 - 2ex + 1} dx \stackrel{x=e^t}{=} \\ &= \int \frac{te^t}{t^2 + (2 - 2e^{t+1})t + 2e^{2+2t} - 2e^{t+1} + 1} dt = \\ &= \int \frac{te^t}{(t+1)^2 - 2e^{t+1}(t+1) + 2e^{2+2t}} dt = \int \frac{te^t}{(t+1 - e^{t+1})^2 + e^{2+2t}} dt = \\ &= \int \frac{te^t}{(t+1 - e^{t+1}) \left( 1 + \frac{e^{2+2t}}{(t+1 - e^{t+1})^2} \right)} dt = (*) \\ \text{Let: } m &= \frac{e^{t+1}}{t+1 - e^{t+1}} \\ (*) &= \frac{1}{e} \int \frac{dm}{1+m^2} = \frac{1}{e} \tan^{-1} m + C\end{aligned}$$

Therefore,

$$\begin{aligned}\Omega &= \int \frac{\log x}{\log^2 x + (2 - 2ex) \log x + 2e^2x^2 - 2ex + 1} dx = \\ &= \frac{1}{e} \tan^{-1} \left( \frac{ex}{\log x - ex + 1} \right) + C\end{aligned}$$

**1427.**

*If  $0 < a < b \leq \frac{\pi}{2}$ , then:*

$$\int_a^b \left( \frac{\int_0^x \frac{\sin t}{1 + \cos t} dt}{\int_0^x \log \left( \frac{1 + \sin t}{1 + \cos t} \right) dt} \right) dx \leq \left( a - \frac{a^2}{b} \right) \cdot \frac{\log 2}{\int_0^a \log \left( \frac{1 + \sin t}{1 + \cos t} \right) dt}$$

*Proposed by Florică Anastase-Romania*

**Solution 1 by Ruxandra Daniela Tonilă-Romania**



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\int_0^x \frac{\sin t}{1 + \cos t} dt = - \int_0^x \frac{-\sin t}{1 + \cos t} dt = - \int_0^x \frac{(1 + \cos t)'}{1 + \cos t} dt = -\log(1 + \cos t)|_0^x = \log 2 - \log(1 + \cos x) > 0, \forall x \in [a, b]; (1)$$

Let  $h_1: [0, \frac{\pi}{4}] \rightarrow \mathbb{R}, h_1(t) = \frac{1 + \sin t}{1 + \cos t}, h_1'(t) = \frac{1 + \sin t + \cos t}{(1 + \cos t)^2} > 0, \forall t \in [0, \frac{\pi}{2}]$

$t$	0	$\frac{\pi}{2}$
$h_1'(t)$	+	+
$h_1(t)$	$h_1(0)$	$\nearrow$

$$\text{So, } h_1(0) \leq h_1(t) \leq h_1\left(\frac{\pi}{2}\right), \forall t \in [0, \frac{\pi}{2}]$$

$$\Leftrightarrow \frac{1}{2} \leq \frac{1 + \sin t}{1 + \cos t} \leq 2, \forall t \in [0, \frac{\pi}{2}]$$

$$\Leftrightarrow \log\left(\frac{1}{2}\right) \leq \log\left(\frac{1 + \sin t}{1 + \cos t}\right) \leq \log 2, \forall t \in [0, \frac{\pi}{2}]$$

$$\Leftrightarrow \int_0^x \log\left(\frac{1}{2}\right) dt \leq \int_0^x \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt \leq \int_0^x \log 2 dt$$

$$\Leftrightarrow x \log\left(\frac{1}{2}\right) \leq \int_0^x \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt \leq x \log 2$$

$$\frac{1}{x \log 2} \leq \frac{1}{\int_0^x \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt} \leq \frac{1}{x \log\left(\frac{1}{2}\right)}; (2)$$

$$\Leftrightarrow \frac{\int_0^x \frac{\sin t}{1 + \cos t} dt}{x \log 2} \leq \frac{\int_0^x \frac{\sin t}{1 + \cos t} dt}{\int_0^x \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt} \leq \frac{\int_0^x \frac{\sin t}{1 + \cos t} dt}{x \log\left(\frac{1}{2}\right)}$$

$$\Leftrightarrow \frac{\log 2 - \log(1 + \cos x)}{x \log 2} \leq \frac{\int_0^x \frac{\sin t}{1 + \cos t} dt}{\int_0^x \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt} \leq \frac{\log(1 + \cos x) - \log 2}{x \log 2}$$

Now,

$$\frac{\int_0^x \frac{\sin t}{1 + \cos t} dt}{\int_0^x \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt} \leq \frac{\log(1 + \cos x) - \log 2}{x \log 2}$$

$$\Leftrightarrow \int_a^b \frac{\int_0^x \frac{\sin t}{1 + \cos t} dt}{\int_0^x \log \left( \frac{1 + \sin t}{1 + \cos t} \right) dt} dx \leq \int_a^b \frac{\log(1 + \cos x) - \log 2}{x \log 2} dx$$

Let:  $I = \int_a^b \frac{\int_0^x \frac{\sin t}{1 + \cos t} dt}{\int_0^x \log \left( \frac{1 + \sin t}{1 + \cos t} \right) dt} dx$ , hence

$$I \leq \int_a^b \frac{\log(1 + \cos x) - \log 2}{x \log 2} dx$$

$$\text{But: } \int_a^b \frac{\log(1 + \cos x) - \log 2}{x \log 2} dx = \int_a^b \frac{\log(1 + \cos x)}{x \log 2} dx - \int_a^b \frac{1}{x} dx = \\ = \frac{1}{\log 2} \int_a^b \frac{\log(1 + \cos x)}{x} dx - \int_a^b \frac{1}{x} dx$$

Let:  $h_2: \left[0, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$ ,  $h_2(x) = \log(1 + \cos x)$ ,  $h'_2(x) = -\frac{\sin x}{1 + \cos x} < 0$

$x$	0	$\frac{\pi}{2}$
$h'_2(x)$	—	—
$h_2(x)$	$h_2(0) \searrow \searrow \searrow h_2\left(\frac{\pi}{2}\right)$	

$$\Leftrightarrow \log\left(1 + \cos \frac{\pi}{2}\right) \leq h_2(x) \leq \log(1 + \cos 0^\circ) | \cdot \frac{1}{x}$$

$$0 \leq \frac{\log(1 + \cos x)}{x} \leq \frac{\log 2}{x} \Leftrightarrow$$

$$0 \leq \int_a^b \frac{\log(1 + \cos x)}{x} dx \leq \log 2 \int_a^b \frac{1}{x} dx \Leftrightarrow$$

$$0 \leq \frac{1}{\log 2} \int_a^b \frac{\log(1 + \cos x)}{x} dx \leq \int_a^b \frac{1}{x} dx \Leftrightarrow$$

$$-\int_a^b \frac{1}{x} dx \leq \frac{1}{\log 2} \int_a^b \frac{\log(1 + \cos x)}{x} dx - \int_a^b \frac{1}{x} dx \leq 0$$

Therefore,

$$I \leq \int_a^b \frac{\log(1 + \cos x) - \log 2}{x \log 2} dx \leq 0$$

So, it is enough to prove that:



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\left(a - \frac{a^2}{b}\right) \cdot \frac{\log 2}{\int_0^a \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt} \geq 0$$

From (2) we have:

$$\frac{1}{alog\left(\frac{1}{2}\right)} \geq \frac{1}{\int_0^a \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt} \geq \frac{1}{alog2}$$

$$\text{and } \left(a - \frac{a^2}{b}\right) = a^2 \left(\frac{1}{a} - \frac{1}{b}\right) > 0, (\because a < b)$$

Thus, we get:

$$\frac{a^2 \left(\frac{1}{a} - \frac{1}{b}\right) \log 2}{\int_0^a \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt} \geq \frac{a^2 \left(\frac{1}{a} - \frac{1}{b}\right) \log 2}{alog2} = a \left(\frac{1}{a} - \frac{1}{b}\right) > 0$$

$$\Rightarrow \left(a - \frac{a^2}{b}\right) \cdot \frac{\log 2}{\int_0^a \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt} \geq 0 \geq I$$

$$\Leftrightarrow \int_a^b \frac{\int_0^x \frac{\sin t}{1 + \cos t} dt}{\int_0^x \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt} dx \leq \left(a - \frac{a^2}{b}\right) \cdot \frac{\log 2}{\int_0^a \log\left(\frac{1 + \sin t}{1 + \cos t}\right) dt}$$

**Solution 2 by proposer**

First:

$$\int_0^{\frac{\pi}{2}} \frac{\sin t}{1 + \cos t} dt = \int_0^{\frac{\pi}{2}} \frac{(-\cos t)'}{1 + \cos t} dt = \log 2$$

Let functions:  $f, g: [0, \frac{\pi}{2}] \rightarrow \mathbb{R}, f(x) = \frac{\sin x}{1 + \cos x},$

$$F(x) = \int_0^x f(t) dt, g(x) = \log\left(\frac{1 + \sin x}{1 + \cos x}\right) \text{ and } G(x) = \int_0^x g(t) dt$$

How:  $F''(x) = f'(x) = \frac{1}{1 + \cos x} > 0, G''(x) = g'(x) = \frac{1 + \cos x + \sin x}{(1 + \sin x)(1 + \cos x)} > 0,$

$\forall x \in [0, \frac{\pi}{2}] \rightarrow F, G \text{ is convex} \rightarrow \forall \tau \in [0, 1] \text{ and } p, q \in \mathbb{R} \text{ such that:}$

$$F((1 - \tau)p + \tau q) \leq (1 - \tau)F(p) + \tau F(q), \text{ for } p = 0, q = x_2, \tau = \frac{x_1}{x_2}, x_1 < x_2 \rightarrow$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\frac{F(x_1)}{x_1} < \frac{F(x_2)}{x_2} \rightarrow \frac{F(x)}{x} \text{ is increasing}$$

(analogous  $\frac{G(x)}{x}$  is increasing  $\rightarrow \frac{x}{G(x)}$  decreasing)

*Applying Chebyshev's inequality, we get:*

$$\begin{aligned} \int_a^b \frac{F(x)}{G(x)} dx &= \int_a^b \frac{F(x)}{x} \cdot \frac{x}{G(x)} dx \leq \frac{1}{b-a} \cdot \int_a^b \frac{F(x)}{x} dx \cdot \int_a^b \frac{x}{G(x)} dx \leq \\ &\leq (b-a) \cdot \frac{F(b)}{b} \cdot \frac{a}{G(a)} \leq \frac{a}{b} \cdot \frac{F\left(\frac{\pi}{2}\right)}{G(a)} = \frac{a}{b} \cdot \frac{\log 2}{\int_0^a \log\left(\frac{1+\sin t}{1+\cos t}\right) dt} \end{aligned}$$

*Therefore,*

$$\int_a^b \left( \frac{\int_0^x \frac{\sin t}{1+\cos t} dt}{\int_0^x \log\left(\frac{1+\sin t}{1+\cos t}\right) dt} \right) dx \leq \left( a - \frac{a^2}{b} \right) \cdot \frac{\log 2}{\int_0^a \log\left(\frac{1+\sin t}{1+\cos t}\right) dt}$$

**1428. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \left( n - \frac{1}{3!} \sum_{k=1}^n k^2 + \frac{1}{5} \sum_{k=1}^n k^4 - \frac{1}{7!} \sum_{k=1}^n k^6 + \frac{1}{9!} \sum_{k=1}^n k^8 \dots \right)$$

*Proposed by Amrit Awasthi-India*

**Solution by proposer**

We know by Taylor expansion that,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \dots - \dots$$

That implies,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots + \dots - \dots$$

Or

$$\frac{\sin 1}{1} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots - \dots + \dots$$

$$\frac{\sin 2}{2} = 1 - \frac{2^2}{3!} + \frac{2^4}{5!} - \frac{2^6}{7!} + \dots - \dots + \dots$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\frac{\sin 3}{3} = 1 - \frac{3^2}{3!} + \frac{3^4}{5!} - \frac{3^6}{7!} + \dots - \dots + \dots$$

$$\frac{\sin n}{n} = 1 - \frac{n^2}{3!} + \frac{n^4}{5!} - \frac{n^6}{7!} + \dots - \dots + \dots$$

**Adding, and taking the as limit n approaches infinity**

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\sin k}{k} = \lim_{n \rightarrow \infty} \left( n - \frac{1}{3!} \sum_{k=1}^n k^2 + \frac{1}{5} \sum_{k=1}^n k^4 - \frac{1}{7!} \sum_{k=1}^n k^6 + \frac{1}{9!} \sum_{k=1}^n k^8 \dots \right)$$

**Now, let's evaluate the sum on left hand side. It can be rewritten as,**

$$\sum_{k=1}^{\infty} \frac{\sin k}{k} = \sum_{k=0}^{\infty} \frac{\sin k}{k} - \lim_{k \rightarrow 0} \frac{\sin k}{k} = -1 + \sum_{k=0}^{\infty} \frac{\sin k}{k}$$

**Using Abel-Palana summation formula the sum can be rewritten as**

$$\sum_{k=0}^{\infty} \frac{\sin k}{k} = \underbrace{\int_0^{\infty} \frac{\sin x}{x} dx}_{\frac{\pi}{2}} + \frac{1}{2} \underbrace{\lim_{x \rightarrow 0} \frac{\sin x}{x}}_1 + i \int_0^{\infty} \left[ \underbrace{\frac{\sin(ix)}{ix}}_0 - \underbrace{\frac{\sin(-ix)}{ix}}_0 \right] \frac{dx}{e^{2\pi x} - 1}$$

**Therefore,**

$$\sum_{k=0}^{\infty} \frac{\sin k}{k} = -1 + \sum_{k=0}^{\infty} \frac{\sin k}{k} = -1 + \frac{\pi + 1}{2} = \frac{\pi - 1}{2}$$

**So,**

$$\Omega = \lim_{n \rightarrow \infty} \left( n - \frac{1}{3!} \sum_{k=1}^n k^2 + \frac{1}{5} \sum_{k=1}^n k^4 - \frac{1}{7!} \sum_{k=1}^n k^6 + \frac{1}{9!} \sum_{k=1}^n k^8 \dots \right) = \frac{\pi - 1}{2}$$

**1429. Find without any software:**

$$\Omega = \int_0^1 \log(1-x) \cdot \cos(4 \cos^{-1} x) dx$$

*Proposed by Max Wong-Hong Kong*

**Solution by Mohammad Rostami-Kabul –Afghanistan**

$$\Omega = \int_0^1 \log(1-x) \cdot \cos(4 \cos^{-1} x) dx =$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \int_0^1 \frac{\partial}{\partial a} \Big|_{a=0} (1-x)^a \cdot (8x^4 - 8x^2 + 1) dx = \\
 &= \frac{\partial}{\partial a} \Big|_{a=0} \left[ 8 \int_0^1 x^{5-1} (1-x)^{a+1-1} dx - 8 \int_0^1 x^{3-1} (1-x)^{a+1-1} dx + \int_0^1 x^{1-1} (1-x)^{a+1-1} dx \right] = \\
 &= \frac{\partial}{\partial a} \Big|_{a=0} [8B(5, a+1) - 8B(3, a+1) + B(1, a+1)] = \\
 &= \frac{\partial}{\partial a} \Big|_{a=0} \left[ 8 \frac{\Gamma(5)\Gamma(a+1)}{\Gamma(a+6)} - 8 \frac{\Gamma(3)\Gamma(a+1)}{\Gamma(a+4)} + \frac{\Gamma(1)\Gamma(a+1)}{\Gamma(a+2)} \right] = \\
 &= \frac{\partial}{\partial a} \Big|_{a=0} \left[ (8)(24) \frac{\Gamma(a+1)}{\Gamma(a+6)} - (8)(2) \frac{\Gamma(a+1)}{\Gamma(a+4)} + \frac{\Gamma(a+1)}{\Gamma(a+2)} \right] = \\
 &= (8)(24) \left[ \frac{\Gamma'(a+1)\Gamma(a+6) - \Gamma(a+1)\Gamma'(a+6)}{\Gamma^2(a+6)} \right]_{a=0} - \\
 &\quad - 16 \left[ \frac{\Gamma'(a+1)\Gamma(a+4) - \Gamma(a+1)\Gamma'(a+4)}{\Gamma^2(a+4)} \right]_{a=0} \\
 &\quad + \left[ \frac{\Gamma'(a+1)\Gamma(a+2) - \Gamma(a+1)\Gamma'(a+2)}{\Gamma^2(a+2)} \right]_{a=0} \\
 &= (8)(24) \left[ \frac{\Gamma'(1)\Gamma(6) - \Gamma(1)\Gamma'(6)}{\Gamma^2(6)} \right] - \\
 &\quad - 16 \left[ \frac{\Gamma'(1)\Gamma(4) - \Gamma(1)\Gamma'(4)}{\Gamma^2(4)} \right] + \left[ \frac{\Gamma'(1)\Gamma(2) - \Gamma(1)\Gamma'(2)}{\Gamma^2(2)} \right] = \\
 &= (8)(24) \left[ \frac{\Psi(1) - \Psi(6)}{(5)(24)} \right] - 16 \left[ \frac{\Psi(1) - \Psi(4)}{6} \right] + \left[ \frac{\Psi(1) - \Psi(2)}{1} \right] = \\
 &= \frac{8}{5} [-\gamma - (H_5 - \gamma)] - \frac{8}{3} [-\gamma - (H_3 - \gamma)] + [-\gamma - (H_1 - \gamma)] = \\
 &= -\frac{8}{5} H_5 + \frac{8}{3} H_3 - H_1 = -\frac{8}{5} \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \right) + \frac{8}{3} \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right) - 1 = \frac{53}{225}
 \end{aligned}$$

Therefore,

$$\Omega = \int_0^1 \log(1-x) \cdot \cos(4 \cos^{-1} x) dx = \frac{53}{225}$$

**1430. Prove that:**

$$\int_0^{\frac{\pi}{6}} \left( \sum_{n=1}^{\infty} \frac{\cos(2nx)}{n} \right)^2 dx = \frac{7\pi^3}{216}$$

*Proposed by Ahmed Yackoube Chach-Mauritania*

**Solution 1 by Kaushik Mahanta-India**

$$I = \int_0^{\frac{\pi}{6}} \left( \sum_{n=1}^{\infty} \frac{\cos(2nx)}{n} \right)^2 dx = \int_0^{\frac{\pi}{6}} \left( \sum_{n=1}^{\infty} \frac{e^{i(2nx)} + e^{-i(2nx)}}{2n} \right)^2 dx =$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 &= \int_0^{\frac{\pi}{6}} \left[ \frac{1}{2} \left( \sum_{n=1}^{\infty} \frac{e^{i(2nx)}}{n} + \sum_{n=1}^{\infty} \frac{e^{-i(2nx)}}{n} \right) \right]^2 dx = \\
 &= \int_0^{\frac{\pi}{6}} \left[ -\frac{1}{2} (\log(1 - e^{2ix}) + \log(e^{-2ix}(-1 + e^{2ix}))) \right]^2 dx = \\
 &= \int_0^{\frac{\pi}{6}} \left[ -\frac{1}{2} \log((1 - e^{2ix}) \cdot e^{-2ix}(-(1 - e^{2ix}))) \right]^2 dx = \\
 &= \int_0^{\frac{\pi}{6}} \left[ \frac{1}{2} \log(-(1 - e^{2ix})(e^{-ix})^2) \right]^2 dx = \int_0^{\frac{\pi}{6}} [\log i + \log(-2i \sin x)]^2 dx = \\
 &= \int_0^{\frac{\pi}{6}} \left[ \log|i| + \frac{\pi i}{2} + \log 2 + \log|-i| - \frac{\pi i}{2} + \log(2 \sin x) \right]^2 dx = \int_0^{\frac{\pi}{6}} \log^2(2 \sin x) dx
 \end{aligned}$$

**For solving  $I$ , we will now employ the method using by Random Variable in MSE.**

$$\begin{aligned}
 I &= \int_0^{\frac{\pi}{6}} \log^2(2 \sin x) dx = \\
 &= \int_0^{\frac{\pi}{6}} \left( x - \frac{\pi}{2} \right)^2 dx + Re \int_0^{\frac{\pi}{6}} \log^2(1 - e^{2ix}) dx = \\
 &= \frac{19\pi^3}{648} + Re \int_C \frac{\log^2(1 - z)}{2iz} dz = \frac{19\pi^3}{648} + \frac{1}{2} Im \int_C \frac{\log^2(1 - z)}{2} dz
 \end{aligned}$$

Where  $C$  – unit circle from  $z = 1$  to  $z = e^{\frac{i\pi}{3}}$ .

$$\begin{aligned}
 Im \int_C \frac{\log^2(1 - z)}{2} dz &= Im \int_0^{e^{\frac{i\pi}{3}}} \frac{\log^2(1 - z)}{z} dz \stackrel{IPB}{=} \\
 &= [Im \log^2(1 - z) \log z]_1^{e^{\frac{i\pi}{3}}} + 2Im \int_1^{e^{\frac{i\pi}{3}}} \frac{\log(1 - z) \log z}{1 - z} dz = \\
 &= Im \log^2 \left( e^{-\frac{i\pi}{3}} \right) \log \left( e^{\frac{i\pi}{3}} \right) + [2Im \log(1 - z) Li_2(1 - z)]_1^{e^{\frac{i\pi}{3}}} + 2Im \int_1^{e^{\frac{i\pi}{3}}} \frac{Li_2(1 - z)}{1 - z} dz \\
 &= -\frac{\pi^3}{27} - \frac{2\pi}{3} \sum_{n=1}^{\infty} \frac{\cos \left( \frac{n\pi}{3} \right)}{n^2} + 2 \sum_{n=1}^{\infty} \frac{\cos \left( \frac{n\pi}{3} \right)}{n^2}
 \end{aligned}$$

Integrating both sides of Fourier series,



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} = \frac{\pi - \theta}{2}; 0 < \theta < 2\pi$$

We get,

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2} = \frac{\theta^2}{4} - \frac{\pi\theta}{2} + \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^4} = \frac{\theta^3}{12} - \frac{\pi\theta^2}{4} + \frac{\pi^2\theta}{6}$$

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n^2} = \frac{\pi^2}{36}, \quad \sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n^2} = \frac{5\pi^3}{162}$$

$$I = \frac{19\pi^3}{648} + \frac{1}{2} \left[ -\frac{\pi^3}{27} - \frac{28}{3} \cdot \frac{\pi^2}{36} + 2 \cdot \frac{5\pi^3}{162} \right] = \frac{7\pi^3}{216}$$

**Solution 2 by Akerele Olofin-Nigeria**

$$\text{Since: } \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} = \frac{\pi - \theta}{2}; 0 < \theta < 2\pi$$

$$\int_{\pi}^x \sum_{n=1}^N \sin(nt) dt = \int_{\pi}^x \frac{\cos \frac{t}{2} - \cos\left(N + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} dt; x \in (0, 2\pi)$$

$$\sum_{n=1}^N \frac{\cos(nx)}{n} = \sum_{n=1}^N \frac{(-1)^n}{n} - \int_{\pi}^x \frac{\cos \frac{t}{2} - \cos\left(N + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} dt =$$

$$= \sum_{n=1}^N \frac{(-1)^n}{n} - \log\left(\sin \frac{x}{2}\right) + \int_{\pi}^x \frac{\cos\left(N + \frac{1}{2}\right)t}{2 \sin \frac{t}{2}} dt$$

In the above integral it approaches 0 as  $N \rightarrow \infty$ , we get:

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n} = -\log 2 - \log\left(\sin \frac{x}{2}\right); (0 < x < 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\cos(2nx)}{n} = -\log 2 - \log(\sin x) = -\log(2 \sin x)$$

$$\int_0^{\frac{\pi}{6}} \left( \sum_{n=1}^{\infty} \frac{\cos(2nx)}{n} \right)^2 dx = \int_0^{\frac{\pi}{6}} (-\log(2 \sin x))^2 dx = \int_0^{\frac{\pi}{6}} (\log(2 \sin x))^2 dx =$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= \int_0^{\frac{\pi}{6}} (\log^2 2 + 2 \log 2 \log(\sin x) + \log^2(\sin x)) dx$$

Since we are familiar with the Fourier series expansion of  $\log(\sin x)$

$$\begin{aligned} & \int_0^{\frac{\pi}{6}} \left( \sum_{n=1}^{\infty} \frac{\cos(2nx)}{n} \right)^2 dx = \\ &= \log^2 2 \cdot \frac{\pi}{6} + 2 \log 2 \left\{ -\frac{1}{72} i(-36Li_2(\sqrt[3]{-i}) + \pi^2 - 12i\pi \log 2) \right\} + \\ &+ \frac{1}{432} \left( 14\pi^3 + 7\pi \log^2 2 + 9\sqrt{3} \log 2 \left( 3\psi^{(1)}\left(\frac{1}{3}\right) - \psi^{(1)}\left(\frac{5}{6}\right) \right) \right) \end{aligned}$$

Therefore,

$$\int_0^{\frac{\pi}{6}} \left( \sum_{n=1}^{\infty} \frac{\cos(2nx)}{n} \right)^2 dx = \frac{19\pi^3}{648} + \frac{1}{2} \left[ -\frac{\pi^3}{27} - \frac{28}{3} \cdot \frac{\pi^2}{36} + 2 \cdot \frac{5\pi^3}{162} \right] = \frac{7\pi^3}{216}$$

**1431. Find without any software:**

$$\Omega = \int_0^1 \frac{\ln(x+1) - \ln(x+3)}{(x+3)^2} dx$$

*Proposed by Simon Peter-Madagascar*

**Solution by Daniel Sitaru-Romania**

$$\begin{aligned} \Omega &= \int_0^1 \frac{\ln(x+1) - \ln(x+3)}{(x+3)^2} dx = - \int_0^1 \left( \frac{1}{x+3} \right)' \ln\left(\frac{x+1}{x+3}\right) dx = \\ &= -\frac{1}{4} \ln\frac{1}{2} + \frac{1}{3} \ln\frac{1}{3} + 2 \int_0^1 \frac{1}{x+3} \cdot \frac{2}{\frac{x+1}{x+3}} dx = \\ &= -\frac{1}{4} \ln\frac{1}{2} + \frac{1}{3} \ln\frac{1}{3} + 2 \left( \int_0^1 \frac{dx}{4(x+1)} - \int_0^1 \frac{dx}{4(x+3)} - \int_0^1 \frac{dx}{2(x+3)^2} \right) = \\ &= -\frac{1}{4} \ln\frac{1}{2} + \frac{1}{3} \ln\frac{1}{3} + \int_0^1 \frac{dx}{2(x+1)} - \int_0^1 \frac{dx}{2(x+3)} - \int_0^1 \frac{dx}{(x+3)^2} = \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= -\frac{1}{4} \ln \frac{1}{2} + \frac{1}{3} \ln \frac{1}{3} + \frac{1}{2} \ln 2 - \frac{1}{4} \ln 4 + \frac{1}{2} \ln 3 + \frac{1}{4} - \frac{1}{3} = \\
 &= \left( \frac{1}{4} + \frac{1}{2} - 1 \right) \ln 2 + \left( -\frac{1}{3} + \frac{1}{2} \right) \ln 3 - \frac{1}{12} = \frac{\ln 3}{6} - \frac{\ln 2}{4} - \frac{1}{12}
 \end{aligned}$$

**1432. Find a closed form:**

$$\Omega = \int_0^\infty \frac{x \log(1+x)}{x^4 + 1} dx$$

*Proposed by Vasile Mircea Popa-Romania*

**Solution 1 by Zaharia Burgheloa-Romania**

$$\Omega = \int_0^\infty \frac{x \log(1+x)}{x^4 + 1} dx = \int_0^1 \frac{x \log(x+1)}{x^4 + 1} dx + \int_1^\infty \frac{x \log(x+1)}{x^4 + 1} dx =$$

$$\begin{aligned}
 &= \int_0^1 \frac{x \log x}{x^4 + 1} dx + \int_0^1 \frac{x \log \left(\frac{1}{x} + 1\right)}{x^4 + 1} dx + \int_1^\infty \frac{x \log(x+1)}{x^4 + 1} dx \stackrel{x \rightarrow \frac{1}{x}}{\cong} \\
 &= \int_0^1 \frac{x \log x}{x^4 + 1} dx + 2 \int_1^\infty \frac{x \log(x+1)}{x^4 + 1} dx = I + 2J
 \end{aligned}$$

$$J = \frac{1}{2}((J+K) + (J-K)), K = \int_1^\infty \frac{x \log(x-1)}{x^4 + 1} dx$$

$$\begin{aligned}
 I &= \int_0^1 \frac{x \log x}{x^4 + 1} dx \stackrel{x^2 \rightarrow x}{\cong} \frac{1}{4} \int_0^1 \frac{\log x}{x^2 + 1} dx = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{2n} \log x dx = \\
 &= \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^2} = -\frac{G}{4}
 \end{aligned}$$

$$\begin{aligned}
 J + K &= \int_1^\infty \frac{x \log(x^2 - 1)}{x^4 + 1} dx \stackrel{x^2 \rightarrow x}{\cong} \frac{1}{2} \int_1^\infty \frac{\log(x-1)}{x^2 + 1} dx \stackrel{\frac{x+1}{x-1} \rightarrow x}{\cong} \frac{1}{2} \int_1^\infty \frac{\log\left(\frac{2}{x-1}\right)}{x^2 + 1} dx \\
 &\rightarrow J + K = \frac{1}{4} \int_1^\infty \frac{\log 2}{x^2 + 1} dx = \frac{\pi}{16} \log 2
 \end{aligned}$$

$$J - K = \int_1^\infty \frac{x \log\left(\frac{x+1}{x-1}\right)}{x^4 + 1} dx \stackrel{\frac{x-1}{x+1} \rightarrow x}{\cong} \int_0^1 \frac{(x^2 - 1) \log x}{x^4 + 6x^2 + 1} dx \stackrel{IBP}{\cong}$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$\begin{aligned}
 &= \frac{1}{2} \int_0^1 \frac{\tan^{-1}((\sqrt{2}+1)x) - \tan^{-1}((\sqrt{2}-1)x)}{x} dx = \\
 &= \frac{1}{2} \int_0^{\sqrt{2}+1} \frac{\tan^{-1}x}{x} dx - \frac{1}{2} \int_0^{\sqrt{2}-1} \frac{\tan^{-1}x}{x} dx = \frac{1}{2} \int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{\tan^{-1}\left(\frac{1}{x}\right)}{x} dx \\
 &\rightarrow J - K = \frac{\pi}{8} \int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{1}{x} dx = \frac{\pi}{4} \log(1 + \sqrt{2})
 \end{aligned}$$

Therefore,

$$\Omega = \int_0^\infty \frac{x \log(1+x)}{x^4 + 1} dx = \frac{\pi}{16} \log 2 + \frac{\pi}{4} \log(1 + \sqrt{2}) - \frac{G}{4}$$

**Solution 2 by Felix Marin-Romania**

$$\begin{aligned}
 \Omega &= \int_0^\infty \frac{x \log(1+x)}{x^4 + 1} dx = \Re \left\{ 2\pi i \lim_{z \rightarrow p} \left[ (z-p) \frac{z \log(z+1)}{z^4 + 1} \right] - \int_0^\infty \frac{iy \log(iy+1)}{i^4 y^4 + 1} idy \right\} \\
 &= \frac{1}{4} \pi \log(2 + \sqrt{2}) - \Re \int_0^\infty \frac{y \log(iy+1)}{y^4 + 1} dy = \\
 &= \frac{1}{8} \pi \log 2 + \frac{1}{4} \pi \log(1 + \sqrt{2}) - \frac{1}{2} \int_0^\infty \frac{y \log(y^2 + 1)}{y^4 + 1} dy = \\
 &\stackrel{y^2 \rightarrow y}{=} \frac{1}{8} \pi \log 2 + \frac{1}{4} \pi \log(1 + \sqrt{2}) - \underbrace{\frac{1}{4} \int_0^\infty \frac{\log(y+1)}{y^2 + 1} dy}_J \\
 J &= \int_0^\infty \frac{\log(y+1)}{y^2 + 1} dy = \int_0^1 \frac{\log(y+1)}{y^2 + 1} dy + \int_1^\infty \frac{\log\left(\frac{1}{y} + 1\right)}{\frac{1}{y^2} + 1} \left(-\frac{dy}{y^2}\right) = \\
 &= - \underbrace{\int_0^1 \frac{\log y}{y^2 + 1} dy}_G + 2 \underbrace{\int_0^1 \frac{\log(y+1)}{y^2 + 1} dy}_{y=\tan \theta} = G + 2 \int_0^{\frac{\pi}{4}} \log(\tan \theta + 1) d\theta \\
 \int_0^{\frac{\pi}{4}} \log(\tan \theta + 1) d\theta &= \int_0^{\frac{\pi}{4}} \log\left(\tan\left(\frac{\pi}{4} - \theta\right) + 1\right) d\theta = \int_0^{\frac{\pi}{4}} [\log 2 - \log(\tan \theta + 1)] d\theta \\
 &\rightarrow \int_0^{\frac{\pi}{4}} \log(\tan \theta + 1) d\theta = \frac{\pi}{8} \log 2 \rightarrow J = \int_0^\infty \frac{\log(y+1)}{y^2 + 1} dy = G + \frac{\pi}{4} \log 2
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Therefore,

$$\Omega = \int_0^\infty \frac{x \log(1+x)}{x^4 + 1} dx = \frac{\pi}{16} \log 2 + \frac{\pi}{4} \log(1 + \sqrt{2}) - \frac{G}{4}$$

**1433.**

If  $f(x) - 2f\left(\frac{1}{1-x}\right) = x$ , compute  $\Omega = \int_2^3 f(x) dx$

*Proposed by Santiago Alvarez-Mexico*

**Solution 1 by Asmat Qatea-Afghanistan**

$$f(x) - 2f\left(\frac{1}{1-x}\right) = 1; (I), \text{ put } x \rightarrow \frac{1}{1-x} \text{ then}$$

$$f\left(\frac{1}{1-x}\right) - 2f\left(\frac{x-1}{x}\right) = \frac{1}{1-x}; (II), \text{ put } x \rightarrow \frac{1}{1-x} \text{ then}$$

$$f\left(\frac{x-1}{x}\right) - 2f(x) = \frac{x-1}{x}; (III)$$

*Adding first two relations, we get:*

$$f(x) - 4f\left(\frac{x-1}{x}\right) = x + \frac{2}{1-x}$$

$$\text{From } (I) + 2(II) + 4(III) \rightarrow -7f(x) = x + \frac{2}{1-x} + 4 \cdot \frac{(x-1)}{x}$$

$$f(x) = \frac{1}{7} \left( \frac{2}{x-1} + \frac{4}{x} - x - 4 \right) \rightarrow \int_2^3 f(x) dx = \frac{1}{4} \left( \log\left(\frac{81}{4}\right) - \frac{13}{2} \right)$$

**Solution 2 by Mohammed Rostami-Afghanistan**

$$f(x) - 2f\left(\frac{1}{1-x}\right) = x; \text{ putting } x = \frac{1}{1-t} \rightarrow \frac{1}{1-x} = \frac{1}{1-\frac{1}{1-t}} = \frac{t-1}{t}$$

$$f\left(\frac{1}{1-t}\right) - 2f\left(\frac{t-1}{t}\right) = \frac{1}{1-t} \rightarrow f\left(\frac{1}{1-x}\right) - 2f\left(\frac{x-1}{x}\right) = \frac{1}{1-x}$$

$$t = \frac{1}{1-u} \rightarrow \frac{1}{1-t} = \frac{u-1}{u} \text{ and } \frac{t-1}{t} = u$$

$$f\left(\frac{u-1}{u}\right) - 2f(u) = \frac{u-1}{u} \rightarrow$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$f\left(\frac{x-1}{x}\right) - 2f(x) = \frac{x-1}{x}$  and from  $f\left(\frac{1}{1-x}\right) - 2f\left(\frac{x-1}{x}\right) = \frac{1}{1-x}$ , we get:

$$f(x) - 4f\left(\frac{x-1}{x}\right) = x + \frac{2}{1-x}$$

Now, from  $f\left(\frac{x-1}{x}\right) - 2f(x) = \frac{x-1}{x}$  and  $f(x) - 4f\left(\frac{x-1}{x}\right) = x + \frac{2}{1-x}$ , we get:

$$-7f(x) = 4\left(\frac{x-1}{x}\right) + x + \frac{2}{1-x}$$

$$f(x) = -\frac{4}{7} + \frac{4}{7x} - \frac{x}{7} + \frac{2}{7(x-1)}$$

$$\text{Hence, } \Omega = \int_2^3 \left( -\frac{4}{7} + \frac{4}{7x} - \frac{x}{7} + \frac{2}{7(x-1)} \right) dx = -\frac{13}{14} + \frac{2}{7} \log\left(\frac{9}{2}\right)$$

### *Solution 3 by Muhammad Afzal-Pakistan*

$$f(x) - 2f\left(\frac{1}{1-x}\right) = x; \quad (1)$$

$$\text{Let } m = \frac{1}{1-x} \rightarrow x = 1 - \frac{1}{m}$$

$$f\left(1 - \frac{1}{m}\right) - 2f(m) = 1 - \frac{1}{m} \text{ or } f\left(1 - \frac{1}{x}\right) - 2f(x) = 1 - \frac{1}{x}; \quad (2)$$

$$\text{From } 2 \cdot (1) + (2) \rightarrow f\left(1 - \frac{1}{x}\right) - 4f\left(\frac{1}{1-x}\right) = 2x - \frac{1}{x} + 1; \quad (3)$$

$$\text{Put } x = 1 - \frac{1}{m} \rightarrow f\left(1 - \frac{m}{m-1}\right) - 4f(m) = 2 - \frac{2}{m} - \frac{m}{m-1} + 1$$

$$f\left(\frac{1}{1-m}\right) - 4f(m) = 3 - \frac{2}{m} - \frac{m}{m-1} \text{ or } f\left(\frac{1}{1-x}\right) - 4f(x) = 3 - \frac{2}{x} - \frac{x}{x-1}$$

$$2f\left(\frac{1}{1-x}\right) - 8f(x) = 6 - \frac{4}{x} - \frac{2x}{x-1}; \quad (4)$$

$$\text{Adding (1) with (4), we get } -7f(x) = 6 + x - 4 \cdot \frac{1}{x} - \frac{2x}{x-1}$$

$$-7 \int_2^3 f(x) dx = -\frac{1}{7} \left( \frac{13}{2} - \log\left(\frac{81}{4}\right) \right)$$

### *Solution 4 by proposer*

$$f(x) - 2f\left(\frac{1}{1-x}\right) = x; \quad (1)$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$f\left(\frac{x-1}{x}\right) - 2f(x) = \frac{x-1}{x}; \quad (2) \text{ and } f\left(\frac{1}{1-x}\right) - 2f\left(\frac{x-1}{x}\right) = \frac{1}{1-x}; \quad (3)$$

$$\text{Thus,} \quad -4f(x) + f\left(\frac{1}{1-x}\right) = \frac{2x-2}{x} + \frac{1}{1-x}$$

$$-8f(x) + f(x) = \frac{4x-4}{x} + \frac{2}{1-x} + x$$

$$-7f(x) = \frac{4x-4}{x} + \frac{2}{1-x} + x$$

$$\text{Then: } -7 \int_2^3 f(x) dx = \int_2^3 \left( \frac{4x-4}{x} + \frac{2}{1-x} + x \right) dx = \frac{2}{7} \log\left(\frac{9}{2}\right) - \frac{13}{14}$$

**1434.**

If  $I(z) = \int_0^1 \int_0^1 \frac{dxdy}{(1-xyz)(1+x)(1+y)}$ ,  $\forall |z| < 1$ , then prove:

$$I(z) = \frac{1}{1-z} \left[ Li_2(z) - 2Li_2\left(\frac{1+z}{2}\right) + \zeta(2) \right], \text{ where}$$

$Li_2(z)$  – Dilogarithm function,  $\zeta(z)$  – Rieman's zeta function

*Proposed by Ngulmun George Baite-India*

*Solution by proposer*

Note that  $\frac{1}{n+k} = \int_0^1 x^{n+k-1} dx$ , we write

$$\log 2 - \overline{H_n} = (-1)^n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{n+k} = (-1)^n \int_0^1 x^n \left[ \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1} \right] dx = \\ = (-1)^n \int_0^1 \frac{x^n}{1+x} dx, \quad \text{for } |z| < 1$$

$$\sum_{n=0}^{\infty} (\overline{H_n} - \log 2)^2 z^n = \sum_{n=0}^{\infty} \left( \int_0^1 \frac{x^n}{1+x} dx \right)^2 z^n = \\ = \sum_{n=0}^{\infty} \left( \int_0^1 \frac{x^n}{1+x} dx \right) \left( \int_0^1 \frac{y^n}{1+y} dy \right) z^n = \sum_{n=0}^{\infty} \int_0^1 \int_0^1 \frac{x^n y^n z^n}{(1+x)(1+y)} dxdy = \\ = \int_0^1 \int_0^1 \left( \sum_{n=0}^{\infty} x^n y^n z^n \right) \frac{dxdy}{(1+x)(1+y)}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 \sum_{n=0}^{\infty} (\overline{H_n} - \log 2)^2 z^n &= \int_0^1 \int_0^1 \frac{dxdy}{(1-xyz)(1+x)(1+y)} = I(z) \\
 I(z) &= \sum_{n=0}^{\infty} (\overline{H_n} - \log 2)^2 z^n = \sum_{n=0}^{\infty} \left( \overline{H_n^2} - 2\log 2 \overline{H_n} + \log^2 2 \right) z^n = \\
 &= \sum_{n=0}^{\infty} \overline{H_n^2} z^n - 2\log 2 \sum_{n=0}^{\infty} H_n z^n + \log^2 2 \sum_{n=0}^{\infty} z^n = \\
 &= \frac{1}{1-z} \left[ Li_2(z) + 2\log 2 \log(1+z) + 2Li_2\left(\frac{1}{2}\right) - 2Li_2\left(\frac{1+z}{2}\right) \right] - 2\log 2 \frac{\log(1+z)}{1-z} + \log^2 2 \left( \frac{1}{1-z} \right) = \\
 &= \frac{1}{1-z} \left[ Li_2(z) - 2Li_2\left(\frac{1+z}{2}\right) + \zeta(2) \right], \text{ where } \overline{H_n} = \sum_{k=1}^n \frac{(-1)^{k-1}}{k}
 \end{aligned}$$

**1435. Prove that:**

$$\int_{-\infty}^{\infty} \frac{\log(t+1)}{t^2+1} dt = \frac{\pi}{2} \left( \log 2 + \frac{\pi}{2} i \right)$$

*Proposed by Simon Peter-Madagascar*

**Solution by Rana Ranino-Setif-Algerie**

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{\log(t+1)}{t^2+1} dt &= \int_{-\infty}^0 \frac{\log(t+1)}{t^2+1} dt + \int_0^{\infty} \frac{\log(t+1)}{t^2+1} dt = \\
 &= \int_0^{\infty} \frac{\log(1-t)}{t^2+1} dt + \int_0^{\infty} \frac{\log(t+1)}{t^2+1} dt \\
 \int_0^{\infty} \frac{\log(1-t)}{t^2+1} dt &\stackrel{t=\tan\theta}{=} \int_0^{\frac{\pi}{2}} \log(1-\tan^2\theta) d\theta \\
 &= \underbrace{\int_0^{\frac{\pi}{2}} \log(\cos 2\theta) d\theta}_I - 2 \underbrace{\int_0^{\frac{\pi}{2}} \log(\cos \theta) d\theta}_{-\frac{\pi}{2} \log 2}
 \end{aligned}$$

$$\begin{aligned}
 I &= \frac{1}{2} \int_0^{\pi} \log(\cos \theta) d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \log(\cos \theta) d\theta + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \log(\cos \theta) d\theta = \\
 &= -\frac{\pi}{4} \log 2 + \frac{1}{2} \int_0^{\frac{\pi}{2}} \log \left( \cos \left( \theta + \frac{\pi}{2} \right) \right) d\theta =
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= -\frac{\pi}{4} \log 2 + \frac{1}{2} \int_0^{\frac{\pi}{2}} \log(-\sin \theta) d\theta = -\frac{\pi}{4} \log 2 + \frac{\pi}{4} \log(-1) + \frac{1}{2} \int_0^{\frac{\pi}{2}} \log(\sin \theta) d\theta = \\
 &= -\frac{\pi}{2} \log 2 + \frac{i\pi^2}{4}
 \end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty} \frac{\log(t+1)}{t^2+1} dt = -\frac{\pi}{2} \log 2 + \frac{i\pi^2}{4} + \pi \log 2 = \frac{\pi}{2} + \frac{i\pi^2}{4}$$

$$\int_{-\infty}^{\infty} \frac{\log(t+1)}{t^2+1} dt = \frac{\pi}{2} \left( \log 2 + \frac{\pi}{2} i \right)$$

**1436. Prove that:**

$$\int_0^1 \int_0^1 \frac{\sin^{-1}(\sqrt{1-x}\sqrt{y}) \cos^{-1}(\sqrt{1-x}\sqrt{y})}{\sqrt{1-y}\sqrt{xy-y+1}} dy dx = 8 \log 2 - \frac{7}{2} \zeta(3)$$

*Proposed by Srinivasa Raghava-AIRMC-India*

**Solution by Rana Ranino-Setif-Algerie**

$$\begin{aligned}
 \Omega &= \int_0^1 \int_0^1 \frac{\sin^{-1}(\sqrt{1-x}\sqrt{y}) \cos^{-1}(\sqrt{1-x}\sqrt{y})}{\sqrt{1-y}\sqrt{xy-y+1}} dy dx; \left( \text{set: } x = 1 - \frac{\sin^2 \theta}{\sin^2 \varphi} \right) \\
 \Omega &= 4 \int_0^{\frac{\pi}{2}} \int_0^{\varphi} \frac{\theta \left( \frac{\pi}{2} - \theta \right) \sin \theta}{\sin \varphi} d\theta d\varphi = \\
 &= 4 \int_0^{\frac{\pi}{2}} \frac{\varphi^2 \cos \varphi - \frac{\pi}{2} \varphi \cos \varphi - 2 \cos \varphi + \frac{\pi}{2} \sin \varphi - 2\varphi \sin \varphi + 2}{\sin \varphi} d\varphi \\
 \Omega &= 4 \underbrace{\int_0^{\frac{\pi}{2}} \varphi^2 \cot \varphi d\varphi}_{I_1} - 2\pi \underbrace{\int_0^{\frac{\pi}{2}} \varphi \cot \varphi d\varphi}_{I_2} + \underbrace{\int_0^{\frac{\pi}{2}} \left( \frac{\pi}{2} - 2\varphi \right) d\varphi}_{I_3} + \underbrace{8 \int_0^{\frac{\pi}{2}} \frac{1 - \cos \varphi}{\sin \varphi} d\varphi}_{I_4} \\
 I_1 &\stackrel{IBP}{=} 4[\varphi^2 \log(\sin \varphi)]_0^{\frac{\pi}{2}} - 8 \int_0^{\frac{\pi}{2}} \varphi \log(\sin \varphi) d\varphi = -8 \int_0^{\frac{\pi}{2}} \varphi \log(\sin \varphi) d\varphi \\
 \text{Recall } \log(\sin \varphi) &= -\log 2 - \sum_{k=1}^{\infty} \frac{\cos 2k\varphi}{k}
 \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$I_1 = 8 \log 2 \int_0^{\frac{\pi}{2}} \varphi d\varphi + 8 \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\frac{\pi}{2}} \varphi \cos(2k\varphi) d\varphi \stackrel{t=2k\varphi}{=} \pi^2 \log 2 + 2 \sum_{k=1}^{\infty} \frac{1}{k^3} \int_0^{k\pi} t \cos t dt$$

$$I_1 = \pi^2 \log 2 + 2 \sum_{k=1}^{\infty} \frac{1}{k^3} [t \sin t + \cos t]_0^{k\pi} = \pi^2 \log 2 + 2 \sum_{k=1}^{\infty} \frac{1}{k^3} (\cos(k\pi) - 1)$$

$$I_1 = \pi^2 \log 2 + 2 \sum_{k=1}^{\infty} \left( \frac{(-1)^k}{k^3} - \frac{1}{k^3} \right) = \pi^2 \log 2 - 2(\eta(3) + \zeta(3)) = \pi^2 \log 2 - \frac{7}{2} \zeta(3)$$

$$I_2 = -2\pi \int_0^{\frac{\pi}{2}} \varphi \cot \varphi d\varphi \stackrel{IBP}{=} -2\pi [\varphi \log(\sin \varphi)]_0^{\frac{\pi}{2}} + 2\pi \int_0^{\frac{\pi}{2}} \log(\sin \varphi) d\varphi = \\ = -\pi^2 \log 2$$

$$I_3 = \int_0^{\frac{\pi}{2}} \left( \frac{\pi}{2} - 2\varphi \right) d\varphi = \frac{\pi^2}{4} - \frac{\pi^2}{4} = 0$$

$$I_4 = 8 \int_0^{\frac{\pi}{2}} \frac{2 \sin^2 \left( \frac{\varphi}{2} \right)}{\sin \varphi} d\varphi = 8 \int_0^{\frac{\pi}{2}} \tan \left( \frac{\varphi}{2} \right) d\varphi = - \left[ 16 \log(\cos \left( \frac{\varphi}{2} \right)) \right]_0^{\frac{\pi}{2}} = 8 \log 2$$

$$\Omega = \pi^2 \log 2 - \frac{7}{3} \zeta(3) - \pi^2 \log 2 + 8 \log 2 = 8 \log 2 - \frac{7}{2} \zeta(3)$$

Therefore,

$$\int_0^1 \int_0^1 \frac{\sin^{-1}(\sqrt{1-x}\sqrt{y}) \cos^{-1}(\sqrt{1-x}\sqrt{y})}{\sqrt{1-y}\sqrt{xy-y+1}} dy dx = 8 \log 2 - \frac{7}{2} \zeta(3)$$

**1437. If  $0 < a \leq b$  then:**

$$\int_a^b 2^{\frac{1}{\sqrt{x}}} dx \geq 3(b-a) + \frac{2^a - 2^b}{\log 4}$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Asmat Qatea-Afghanistan**

$$\int_a^b 2^{\frac{1}{\sqrt{x}}} dx \geq 3(b-a) + \frac{2^a - 2^b}{\log 4} \leftrightarrow \int_a^b 2^{\frac{1}{\sqrt{x}}} dx \geq \int_a^b \left( 3 - \frac{2^x}{2} \right) dx$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\leftrightarrow \int_a^b \left( 2^{\frac{1}{\sqrt{x}}} + \frac{2^x}{2} - 3 \right) dx \geq 0 \leftrightarrow f(x) = \frac{2^x}{2} + 2^{\frac{1}{\sqrt{x}}} - 3 \geq 0, x > 0$$

$$\begin{aligned} f(x) &= \frac{2^x}{2} + 2^{\frac{1}{\sqrt{x}}} - 3 = \frac{2^x + 2^{\frac{1}{\sqrt{x}}} + 2^{\frac{1}{\sqrt{x}}}}{2} - 3 \stackrel{AM-GM}{\geq} \frac{3}{2} \sqrt[3]{2^{x+\frac{2}{\sqrt{x}}}} - 3 = \\ &= 3 \sqrt[3]{2^{x+\frac{2}{\sqrt{x}}-1}} - 3 = 3 \sqrt[3]{2^{\frac{x\sqrt{x}+2-\sqrt{x}}{\sqrt{x}}}} - 3 = \\ &= 3 \sqrt[3]{2^{\frac{(\sqrt{x}+2)(\sqrt{x}-1)^2}{\sqrt{x}}}} - 3 \geq 3 \sqrt[3]{2^0} - 3 = 3 - 3 = 0 \end{aligned}$$

**Equality holds for simultaneous**

$$2^x = 2^{\frac{1}{\sqrt{x}}}, \sqrt{x} - 1 = 0 \rightarrow x = 1$$

In initial integral inequality, equality holds for  $a = b$ .

**1438.**

If  $A(n) = \sum_{m=1}^{3n} \frac{1+2+3+\dots+m}{\cos\left(\frac{m\pi}{3}\right)}$  then prove that:

$$\sum_{n=1}^{\infty} A(n)x^n = \frac{7}{8(x-1)} - \frac{23}{8(x+1)} - \frac{3}{4(x+1)^2} + \frac{9}{2(x+1)^3}$$

*Proposed by Srinivasa Raghava-AIRMC-India*

**Solution 1 by Asmat Qatea-Afghanistan**

$$\begin{aligned} A(n) &= \sum_{m=1}^{3n} \frac{m(m+1)}{2 \cos\left(\frac{m\pi}{3}\right)}, A(n+1) = \sum_{m=1}^{3n+3} \frac{m(m+1)}{2 \cos\left(\frac{m\pi}{3}\right)} = \\ &= \sum_{m=1}^{3n} \frac{m(m+1)}{2 \cos\left(\frac{m\pi}{3}\right)} + \sum_{m=3n+1}^{3n+3} \frac{m(m+1)}{2 \cos\left(\frac{m\pi}{3}\right)} \\ A(n+1) - A(n) &= \sum_{m=3n+1}^{3n+3} \frac{m(m+1)}{2 \cos\left(\frac{m\pi}{3}\right)} = \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

$$= \frac{(3n+1)(2n+2)}{2 \cos\left(\frac{(3n+1)\pi}{3}\right)} + \frac{(3n+2)(3n+3)}{2 \cos\left(\frac{(3n+2)\pi}{3}\right)} + \frac{(3n+3)(3n+4)}{2 \cos\left(\frac{(3n+3)\pi}{3}\right)}$$

$$A(n+1) - A(n) = -(-1)^n \cdot \frac{9n^2 + 33n + 20}{2}$$

$$\sum_{n=1}^m (A(n+1) - A(n)) = -\frac{1}{2} \left( 9 \sum_{n=1}^m (-1)^n n^2 + 33 \sum_{n=1}^m (-1)^n n + 20 \sum_{n=1}^m (-1)^n \right)$$

$$A(m+1) - A(1) =$$

$$= -\frac{1}{2} \left( 9 \cdot (-1)^m \cdot \frac{m(m+1)}{2} + 33 \cdot \frac{(2m+1)(-1)^m - 1}{4} + 20 \cdot \frac{(-1)^m - 1}{2} \right)$$

$$A(m+1) = -(-1)^m \cdot \frac{18m^2 + 84m + 73}{8} - \frac{7}{8}$$

$$A(m) = (-1)^m \cdot \frac{18m^2 + 48m + 7}{8} - \frac{7}{8}$$

$$A(n) = (-1)^n \cdot \frac{18n^2 + 48n + 7}{8} - \frac{7}{8}$$

$$A(n) = \frac{1}{8} (18n^2 \cdot (-1)^n + 48n \cdot (-1)^n + 7 \cdot (-1)^n - 7)$$

$$\sum_{n=1}^{\infty} (-1)^n x^n = -\frac{x}{x+1}, \sum_{n=1}^{\infty} (-1)^n n x^n = -\frac{x}{(x+1)^2}, \sum_{n=1}^{\infty} (-1)^n n^2 x^n = \frac{x^2 - x}{(1+x)^3}$$

$$\sum_{n=1}^{\infty} A(n) x^n = \frac{1}{8} \left( 18 \cdot \frac{x^2 - x}{(1+x)^3} - 48 \cdot \frac{x}{(1+x)^2} - \frac{7x}{1+x} - \frac{7x}{1-x} \right)$$

**Therefore,**

$$\sum_{n=1}^{\infty} A(n) x^n = \frac{7}{8(x-1)} - \frac{23}{8(x+1)} - \frac{3}{4(x+1)^2} + \frac{9}{2(x+1)^3}$$

**Solution 2 by Ravi Prakash-New Delhi-India**

$$\text{Let } b_r = \frac{1+2+3+\cdots+r}{\cos \frac{r\pi}{3}}, B(n) = A(2n) \text{ and } C(n) = A(2n-1)$$

$$B(n) = A(2n) = \sum_{r=1}^{6n} b_r = \sum_{r=0}^{n-1} (b_{6r+1} + b_{6r+2} + \cdots + b_{6r+r})$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 & \text{But: } b_{6r+1} + b_{6r+2} + \cdots + b_{6r+r} = \\
 & = \frac{(6r+1)(6r+2)}{2 \cdot \frac{1}{2}} + \frac{(6r+2)(6r+3)}{2 \cdot \left(-\frac{1}{2}\right)} + \frac{(6r+3)(6r+4)}{2 \cdot (-1)} + \frac{(6r+5)(6r+6)}{2 \cdot \left(\frac{1}{2}\right)} + \\
 & + \frac{(6r+6)(6r+7)}{2 \cdot (1)} = (36r^2 + 18r + 2) - (36r^2 + 30r + 6) + (36r^2 + 66r + 30) - \\
 & - (36r^2 + 54r + 20) + (18r^2 + 39r + 21) - (18r^2 + 21r + 6) = 18r + 21
 \end{aligned}$$

Thus,

$$\begin{aligned}
 B(n) &= \sum_{r=0}^{n-1} (18r + 21) = 9(n-1)n + 21n = 9n^2 + 12n = \frac{9}{4}(2n)^2 + 6(2n) \\
 C(n) &= A(2n-1) = A(2n) - (b_{6n-2} + b_{6n-1} + b_{6n}) = \\
 &= 9n^2 + 21n - \left( \frac{(6n-2)(6n-1)}{2 \cdot \left(-\frac{1}{2}\right)} + \frac{(6n-1)(6n)}{2 \cdot \left(\frac{1}{2}\right)} + \frac{(6n)(6n+1)}{2} \right) = \\
 &= 9n^2 - 3n + 2 = -\left[ \frac{9}{4}(2n-1)^2 + 6(2n-1) + \frac{7}{4} \right]
 \end{aligned}$$

Hence,

$$A(n) = \begin{cases} \frac{9}{4}n^2 + 6n, & \text{if } n \text{ is even} \\ -\frac{9}{4}n^2 - 6n - \frac{7}{4}, & \text{if } n \text{ is odd} \end{cases}$$

Now,

$$\begin{aligned}
 \sum_{n=1}^{\infty} A(n)x^n &= \frac{9}{4} \sum_{n=1}^{\infty} (-1)^n n^2 x^n + 6 \sum_{n=1}^{\infty} (-1)^n n x^n - \frac{7}{4} \sum_{n=1}^{\infty} x^{2n-1} = \\
 &= \frac{9}{4} \cdot \frac{x^2 - x}{(1+x)^3} + 6 \cdot \frac{-x}{(1+x)^2} - \frac{7}{4} \cdot \frac{x}{1-x^2} = \\
 &= \frac{7}{8(x-1)} - \frac{23}{8(x+1)} - \frac{3}{4(x+1)^2} + \frac{9}{2(x+1)^3}
 \end{aligned}$$

**1439.**

$$\Omega = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n 2^k \right) \left( \prod_{k=1}^n \cos \left( \frac{(n+1-2k)\pi}{2n} \right) \right)$$

*Proposed by Asmat Qatea-Afghanistan*

**Solution 1 by Izumi Ainsworth-Lima-Peru**

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}} \rightarrow$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\prod_{k=1}^n \sin\left(\frac{(2k-1)\pi}{2n}\right) \frac{\prod_{k=1}^{n-1} \sin\left(\frac{2k\pi}{2n}\right)}{\prod_{k=1}^{n-1} \sin\left(\frac{2k\pi}{n}\right)} = \frac{\prod_{k=1}^{2n-1} \sin\left(\frac{k\pi}{2n}\right)}{\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right)} \rightarrow \frac{\frac{2n}{2^{2n-1}}}{\frac{n}{2^{n-1}}} = \frac{1}{2^{n-1}}; (*)$$

**Now, we have:**

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n 2^k \right) \left( \prod_{k=1}^n \cos \left( \frac{(n+1-2k)\pi}{2n} \right) \right) = \\ &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n 2^k \right) \left( \prod_{k=1}^n \cos \left( \frac{\pi}{2} - \frac{(2k-1)\pi}{2n} \right) \right) = \\ &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n 2^k \right) \left( \prod_{k=1}^n \sin \left( \frac{(2k-1)\pi}{2n} \right) \right) \stackrel{(*)}{=} \lim_{n \rightarrow \infty} (2^{n+1} - 2) \cdot \frac{1}{2^{n-1}} = 4 - \lim_{n \rightarrow \infty} \frac{4}{2^n} = 4 \end{aligned}$$

**Solution 2 by Florică Anastase-Romania**

$$\text{Let: } P_n = \prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$$

The roots of the binomial equation  $x^n - 1 = 0$  are  $x_j = \cos 2k\pi + i \sin 2k\pi$ ,

$$j = 1, 2, \dots, n, k = 0, 1, \dots, n-1.$$

So, we have:

$$\begin{aligned} x^n - 1 &= (x - x_1)(x - x_2) \cdot \dots \cdot (x - x_j) \cdot \dots \cdot (x - x_n) \\ &= (x - 1) \left( x - \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n} \right) \left( x - \cos \frac{4\pi}{n} - i \sin \frac{4\pi}{n} \right) \cdot \dots \\ &\quad \cdot \left( x - \cos \frac{2(n-1)\pi}{n} - i \sin \frac{2(n-1)\pi}{n} \right); \quad (1) \end{aligned}$$

From  $\frac{x^n-1}{x-1} = x^{n-1} + x^{n-2} + \dots + 1$  and  $x = 1$ , the relation (1) becomes:

$$\begin{aligned} &\left( x - \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n} \right) \left( x - \cos \frac{4\pi}{n} - i \sin \frac{4\pi}{n} \right) \cdot \dots \\ &\quad \cdot \left( x - \cos \frac{2(n-1)\pi}{n} - i \sin \frac{2(n-1)\pi}{n} \right) = n; \quad (2) \end{aligned}$$

Using the relations  $1 - \cos a = 2 \sin^2 \frac{a}{2}$ ;  $\sin 2a = 2 \sin a \cdot \cos a \Rightarrow (2)$  becomes:

$$\left( 2 \sin^2 \frac{\pi}{n} - 2 i \sin \frac{\pi}{n} \cos \frac{\pi}{n} \right) \left( 2 \sin^2 \frac{2\pi}{n} - 2 i \sin \frac{2\pi}{n} \cos \frac{2\pi}{n} \right) \cdot \dots$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\dots \cdot \left( 2\sin^2 \frac{(n-1)\pi}{n} - 2i\sin \frac{(n-1)\pi}{n} \cos \frac{(n-1)\pi}{n} \right) = n; \quad (3)$$

*Multiplying each factor with  $i$  and extracting  $2^{n-1}$ , we get:*

$$\begin{aligned} P_n &= \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}} \\ \Omega &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n 2^k \right) \left( \prod_{k=1}^n \cos \left( \frac{(n+1-2k)\pi}{2n} \right) \right) = \\ &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n 2^k \right) \left( \prod_{k=1}^n \cos \left( \frac{\pi}{2} - \frac{(2k-1)\pi}{2n} \right) \right) = \\ &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n 2^k \right) \left( \prod_{k=1}^n \sin \left( \frac{(2k-1)\pi}{2n} \right) \right) = 4 \cdot \lim_{n \rightarrow \infty} \frac{2^{n-1} - \frac{1}{2}}{2^{n-1}} = 4 \end{aligned}$$

**1440. Evaluate the cute sum without using any “Special function”:**

$$\Omega = \sum_{n=1}^{\infty} \frac{64^n}{n^3 \binom{2n}{n}^3}$$

*Proposed by Tobi Joshua-Nigeria*

**Solution by Akerele Olofin-Nigeria**

$$\begin{aligned} \Omega &= \sum_{n=1}^{\infty} \frac{64^n}{n^3 \binom{2n}{n}^3} \\ \Omega &= \sum_{n=1}^{\infty} \frac{64^n}{n^3 \binom{2n}{n}^3} = \sum_{n=1}^{\infty} \frac{64^n (\sqrt{\pi})^3 \Gamma(n+1)^3}{n^3 \Gamma\left(n+\frac{1}{2}\right)^3} = \sum_{n=0}^{\infty} \frac{64^{n+1} (\sqrt{\pi})^3 \Gamma(n+2)^3}{(n+1)^3 \Gamma\left(n+\frac{3}{2}\right)^3} \\ &= 64(\sqrt{\pi})^3 \sum_{n=0}^{\infty} \frac{64^n \Gamma(n+2) \Gamma(n+2) \Gamma(n+2)}{(n+1)(n+1)(n+1) \Gamma\left(n+\frac{3}{2}\right) \Gamma\left(n+\frac{3}{2}\right) \Gamma\left(n+\frac{3}{2}\right)} \end{aligned}$$

**Recall**

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_n z^n}{\prod_{i=1}^q (b_i)_n n!}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Then

$$\begin{aligned}
 \Omega &= 64(\sqrt{\pi})^3 \sum_{n=0}^{\infty} \frac{64^n \Gamma(n+1) \Gamma(n+1) \Gamma(n+1) \Gamma(n+1)}{\Gamma(n+\frac{3}{2}) \Gamma(n+\frac{3}{2}) \Gamma(n+\frac{3}{2})} \frac{1}{n!} \\
 &= \frac{64(\sqrt{\pi})^3}{\Gamma(\frac{3}{2}) \Gamma(\frac{3}{2}) \Gamma(\frac{3}{2})} \sum_{n=0}^{\infty} \frac{64^n \Gamma(n+1) \Gamma(n+1) \Gamma(n+1) \Gamma(n+1) \Gamma(\frac{3}{2}) \Gamma(\frac{3}{2}) \Gamma(\frac{3}{2})}{\Gamma(n+\frac{3}{2}) \Gamma(n+\frac{3}{2}) \Gamma(n+\frac{3}{2})} \frac{1}{n!} \\
 &= \frac{(64)(8)(\sqrt{\pi})^3}{(\sqrt{\pi})^3} \left( {}_4F_2 \left( 1, 1, 1, 1; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 64 \right) \right) \\
 &= (64)(8) {}_4F_2 \left( 1, 1, 1, 1; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 64 \right) \\
 &= 8 \left[ {}_4F_2 \left( 1, 1, 1, 1; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 1 \right) \right] \\
 \Rightarrow \Omega &= \sum_{n=1}^{\infty} \frac{64^n}{n^3 \binom{2n}{n}} = 8 \left[ {}_4F_2 \left( 1, 1, 1, 1; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 1 \right) \right]
 \end{aligned}$$

where,

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_n z^n}{\prod_{i=1}^q (b_i)_n n!} \quad (\text{denotes Hypergeometric function})$$

**1441.** Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = (\Gamma(x+1))^{\frac{1}{x}}$ ,  $g(x) = x^{1+\frac{1}{x}}$ , where  $\Gamma$  is the

**gamma function.** Compute:

$$\Omega = \lim_{x \rightarrow \infty} \sum_{k=0}^{2m} (-1)^k \cdot \binom{2m}{k} \cdot \frac{(f(x))^k + (g(x))^k}{(f(x) + g(x))^k}$$

*Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania*

**Solution by Mikael Bernardo-Mozambique**

$$\Omega = \lim_{x \rightarrow \infty} \sum_{k=0}^{2m} (-1)^k \cdot \binom{2m}{k} \cdot \frac{(f(x))^k + (g(x))^k}{(f(x) + g(x))^k} =$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$= \lim_{x \rightarrow \infty} \sum_{k=0}^{2m} (-1)^k \cdot \binom{2m}{k} \cdot \frac{\left(\Gamma(x+1)\right)^{\frac{k}{x}} + x^{k+\frac{k}{x}}}{\left(\left(\Gamma(x+1)\right)^{\frac{1}{x}} + x^{1+\frac{1}{x}}\right)^k} = \lim_{n \rightarrow \infty} \sum_{k=0}^{2m} (-1)^k \cdot \binom{2m}{k} \cdot \frac{\left(\sqrt[x]{x!}\right)^k + x^k \left(\sqrt[x]{x!}\right)^k}{\left(\sqrt[x]{x!} + x^{\frac{1}{x}}\right)^k}$$

$$\because \lim_{x \rightarrow \infty} \sqrt[x]{x} = 1; \quad x! \approx \left(\frac{x}{e}\right)^x$$

$$\begin{aligned} \Omega &= \lim_{x \rightarrow \infty} \sum_{k=0}^{2m} (-1)^k \cdot \binom{2m}{k} \cdot \frac{\left(\frac{x}{e}\right)^k + x^k}{\left(\frac{x}{e} + x\right)^k} = \lim_{x \rightarrow \infty} \sum_{k=0}^{2m} (-1)^k \cdot \binom{2m}{k} \cdot \frac{x^k \left(\left(\frac{1}{e}\right)^k + 1\right)}{x^k \left(\frac{1}{e} + 1\right)^k} = \\ &= \sum_{k=0}^{2m} (-1)^k \cdot \binom{2m}{k} \cdot \left(\left(\frac{1}{e+1}\right)^k + \left(\frac{e}{e+1}\right)^k\right) \end{aligned}$$

$$\text{Let } \frac{1}{e+1} = t, \frac{e}{e+1} = u, \text{ then}$$

$$\begin{aligned} \Omega &= \sum_{k=0}^{2m} \binom{2m}{k} \cdot (-t)^k + \sum_{k=0}^{2m} \binom{2m}{k} (-u)^k = (1-t)^{2m} + (1-u)^{2m} = \\ &= \left(1 - \frac{1}{e+1}\right)^{2m} + \left(1 - \frac{e}{e+1}\right)^{2m} = \left(\frac{e}{e+1}\right)^{2m} + \left(\frac{1}{e+1}\right)^{2m} = \frac{e^{2m} + 1}{(e+1)^{2m}} \end{aligned}$$

**1442.** Let  $(b_n)_{n \geq 2}$ ,  $b_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}$  be Bătinetu's sequence and

$$\omega_n = 1 - \frac{\binom{n}{1}}{3} + \frac{\binom{n}{2}}{5} - \dots + \frac{(-1)^n \binom{n}{n}}{2n+1}$$

**Find:**

$$\Omega = \lim_{n \rightarrow \infty} \left( 1 + \frac{\sqrt[n]{\omega_n}}{n!} \right)^{\frac{n!}{b_n^n}}$$

*Proposed by Florică Anastase-Romania*

**Solution 1 by proposer**

$$b_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}} = \frac{n^2}{\sqrt[n]{n!}} \cdot (u_n - 1),$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$u_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{\sqrt[n]{n!}}{n^2}, \forall n \geq 2$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \cdot \frac{n+1}{\sqrt[n]{(n+1)!}} \cdot \frac{\sqrt[n]{n!}}{n} \right) = 1 \cdot e \cdot \frac{1}{e}$$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \left( \frac{n}{\sqrt[n]{n!}} \cdot \frac{u_n - 1}{\log u_n} \cdot \log u_n \right) =$$

$$= e \cdot 1 \cdot \lim_{n \rightarrow \infty} \left( \log \left( \left( \frac{n+1}{n} \right)^{2n} \cdot \frac{n! \cdot \sqrt[n+1]{(n+1)!}}{(n+1)!} \right) \right)$$

$$= e \cdot \log \left( \lim_{n \rightarrow \infty} e_n^2 \cdot \frac{n+1}{\sqrt[n+1]{(n+1)!}} \right) = e \cdot \log \left( e^2 \cdot \frac{1}{e} \right) = e$$

$$(1-x^2)^n = \binom{n}{0} - \binom{n}{1}x^2 + \binom{n}{2}x^4 - \dots + (-1)^n \binom{n}{n}x^{2n}$$

$$\begin{aligned} I_n &= \int_0^1 (1-x^2)^n \cdot x' dx = (1-x^2)^n \cdot x \Big|_0^1 + 2n \int_0^1 (1-x^2)^{n-1} \cdot x^2 dx = \\ &= -2n \int_0^1 (1-x^2-1)(1-x^2)^{n-1} dx = -2n \int_0^1 (1-x^2)^n dx + 2n \int_0^1 (1-x^2)^{n-1} dx = \\ &= -2nI_n + 2nI_{n-1} \Rightarrow I_n = \frac{2^{2n} \cdot (n!)^2}{(2n+1)!} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\omega_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{2^{2n} \cdot (n!)^2}{(2n+1)!}} \stackrel{\text{Criterionul C-D'Alembert}}{\cong} \lim_{n \rightarrow \infty} \frac{2^{2(n+1)}((n+1)!)^2}{(2n+3)!} \cdot \frac{(2n+1)!}{2^{2n}(n!)^2} = 1$$

$$\Rightarrow \Omega = \lim_{n \rightarrow \infty} \left( 1 + \frac{\sqrt[n]{\omega_n}}{n!} \right)^{\frac{n!}{b_n^n}} = e^{\lim_{n \rightarrow \infty} \frac{\sqrt[n]{\omega_n} \cdot n!}{n^e}} = e^0 = 1$$

**Solution 2 by Asmat Qatea-Afghanistan**

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \left( \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}} \right) = \lim_{n \rightarrow \infty} ((n+1)e - ne) = e$$

$$\therefore \lim_{n \rightarrow \infty} \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} = \lim_{n \rightarrow \infty} \frac{n^2}{\sqrt[n]{n!}} = e$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\omega_n = \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{2k+1}, \quad \omega_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k x^{2k+1}}{2k+1} \rightarrow \omega_n = \omega_n(1), \omega_n(0) = 0$$

$$\omega'_n(x) = \sum_{k=0}^n \binom{n}{k} (-x^2)^k \rightarrow \omega'_n(x) = (1-x^2)^n$$

$$\int_0^1 \omega'_n(x) dx = \int_0^1 (1-x^2)^n dx \rightarrow \omega_n(1) - \omega_n(0) = \int_0^1 (1-x^2)^n dx$$

$$\omega_n(1) = \int_0^1 (1-x^2)^n dx \stackrel{x=\sin t}{\cong} \int_0^{\frac{\pi}{2}} \cos^{2n+1} t dt; (*)$$

$$= \frac{1}{2} B\left(\frac{1}{2}, n+1\right) = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \rightarrow \omega_n = \frac{\sqrt{\pi} \cdot n!}{2 \left(n+\frac{1}{2}\right)!}$$

$$\lim_{n \rightarrow \infty} (\omega_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{\pi} \cdot n!}{2 \left(n+\frac{1}{2}\right)!} \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{\pi}}{2} \cdot n^{-\frac{1}{2}} \right)^{\frac{1}{n}} = 1; (**)$$

$$\Omega = \lim_{n \rightarrow \infty} \left( 1 + \frac{\sqrt[n]{\omega_n}}{n!} \right)^{\frac{n!}{b_n}} = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n!} \right)^{\frac{n!}{e^n}} = 1$$

**Note:**

$$B(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$\lim_{n \rightarrow \infty} \frac{(n+a)!}{n!} = n^a; (**)$$

**Solution 3 by Narendra Bhandari-Bajura-Nepal**

$$\omega_n = \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{2k+1} = \int_0^1 \left( \sum_{k=0}^n (-1)^k \binom{n}{k} x^{2k} \right) dx = \int_0^1 (1-x^2)^n dx$$

**And with  $x^2 \rightarrow x$ , we get:**

$$\omega_n = \int_0^1 x^{-\frac{1}{2}} (1-x)^n dx = \frac{1}{2} \cdot \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \cdot 2^{n+1} \cdot \frac{n! \sqrt{\pi}}{(2n+1)!! \sqrt{\pi}}$$

**Simplification gives us the closed form for  $\omega_n = \frac{4^n (n!)^2}{(2n+1)!} = \frac{4^n}{2n+1} \binom{2n}{n}^{-1}$**



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Since  $0 < \omega_n < 1$  as  $\frac{4^n}{2n+1} \binom{2n}{n}^{-1} \sim \frac{\sqrt{\pi n}}{2n+1} < 1$  and hence  $\frac{n\sqrt{\omega_n}}{n!} \rightarrow 0$  as  $n \rightarrow \infty$ .**

**Further with the use of Stirling approximation  $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)$ , the limit of  $b_n = e$  for all  $n$**

**positive integer and thus  $\frac{n!}{b_n^n} \rightarrow \infty$  as  $n \rightarrow \infty$ .**

**This shows limit attains the form  $1^\infty$ . So,**

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left( 1 + \frac{n\sqrt{\omega_n}}{n!} \right)^{\frac{n!}{b_n^n}} = \exp \left( \lim_{n \rightarrow \infty} \frac{n\sqrt{\omega_n}}{b_n^n} \right) \sim \exp \left( \lim_{n \rightarrow \infty} \frac{1}{e^n} \sqrt[n]{\frac{\sqrt{\pi n}}{2n+1}} \right) \\ &= \exp \left( \lim_{n \rightarrow \infty} \frac{1}{e^n} \right) = 1 \end{aligned}$$

**Solution 4 by Mikael Bernardo-Mozambique**

$$\begin{aligned} \omega_n &= \sum_{k=0}^n \frac{(-1)^k \binom{n}{k}}{2k+1} = \int_0^1 \sum_{k=1}^n \binom{n}{k} (-1)^k x^{2k} dx = \int_0^1 (1-x^2)^n dx \stackrel{x=\sqrt{u}}{\cong} \\ &= \frac{1}{2} \int_0^1 u^{\frac{1}{2}-1} (1-u)^{n+\frac{1}{2}-1} du = \frac{1}{2} B\left(\frac{1}{2}, n+\frac{1}{2}\right) = \\ &= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}+1\right)} = \frac{\sqrt{\pi} n!}{2 \left(n+\frac{1}{2}\right) \Gamma\left(n+\frac{1}{2}\right)} = (*) \\ &\quad \because \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} \cdot \prod_{k=1}^n (2k-1) \end{aligned}$$

$$(*) = \frac{\sqrt{\pi} n!}{2 \left(n+\frac{1}{2}\right) \frac{\sqrt{\pi}}{2^n} \cdot \prod_{k=1}^n (2k-1)} = \frac{2^n \cdot n!}{(2n+1) \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}$$

$$\Omega = \lim_{n \rightarrow \infty} \left( 1 + \frac{n\sqrt{\omega_n}}{n!} \right)^{\frac{n!}{b_n^n}} = \lim_{n \rightarrow \infty} \left[ \left( 1 + \frac{n\sqrt{\omega_n}}{n!} \right)^{\frac{n!}{n\sqrt{\omega_n}}} \right]^{\frac{n\sqrt{\omega_n}}{b_n^n}} = \exp \left\{ \lim_{n \rightarrow \infty} \frac{n\sqrt{\omega_n}}{b_n^n} \right\}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} b_n &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \lim_{n \rightarrow \infty} \frac{n^2}{\sqrt[n]{n!}} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \lim_{n \rightarrow \infty} \frac{(n+1)^2 - n^2}{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}} = \\ &= \lim_{n \rightarrow \infty} \frac{n^2}{\sqrt[n]{n!}}; \because n! \cong \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \end{aligned}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 \Omega &\stackrel{c-d}{=} \exp \left\{ \lim_{n \rightarrow \infty} \frac{\omega_{n+1}}{\omega_n} \cdot \frac{1}{b_n^n} \right\} = \\
 &= \exp \left\{ \lim_{n \rightarrow \infty} \frac{2^{n+1} \cdot (n+1)!}{(2n+3) \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \cdot (2n+1)} \cdot \frac{(2n+1) \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n \cdot n!} \cdot \frac{1}{\sqrt[n]{n!}} \right\} \\
 &= \exp \left\{ 2 \lim_{n \rightarrow \infty} \frac{n \cdot n!}{n \left( 2 + \frac{3}{n} \right)} \cdot \frac{1}{n!} \cdot \frac{n!}{e^n n^{2n}} \right\} = \\
 &= \exp \left\{ \lim_{n \rightarrow \infty} \frac{\sqrt{2n\pi} \left( \frac{n}{e} \right)^n}{n^{2n}} \right\} = \exp \left\{ \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{e^n n^n} \right\} = \exp \left\{ \lim_{n \rightarrow \infty} \frac{1}{e^n n^{n-\frac{1}{2}}} \right\} = e^0 = 1
 \end{aligned}$$

**Solution 5 by Adrian Popa-Romania**

$$\begin{aligned}
 \lim_{n \rightarrow \infty} b_n &= \lim_{n \rightarrow \infty} \frac{(n+1)^2 - n^2}{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}} = \lim_{n \rightarrow \infty} (a_{n+1} - a_n), \text{ where } a_n = \frac{n^2}{\sqrt[n]{n!}} \\
 \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} \cdot \frac{\sqrt[n]{n!}}{n^2} = \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{n+1}{\sqrt[n+1]{(n+1)!}} \cdot \frac{\sqrt[n]{n!}}{n} \\
 \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n!}{n^n}} \stackrel{c-d}{=} \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n = \frac{1}{e} \\
 &\rightarrow \lim_{n \rightarrow \infty} \frac{n+1}{\sqrt[n+1]{(n+1)!}} = e \rightarrow \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = e \cdot \frac{1}{e} = 1 \\
 \lim_{n \rightarrow \infty} \left( \frac{a_{n+1}}{a_n} \right)^n &= \lim_{n \rightarrow \infty} \left( \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} \cdot \frac{\sqrt[n]{n!}}{n^2} \right)^n = \lim_{n \rightarrow \infty} \left[ \frac{(n+1)^2}{n^2} \cdot \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \right]^n = \\
 &= \lim_{n \rightarrow \infty} \left( \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \right)^n = \lim_{n \rightarrow \infty} \frac{n!}{[n! (n+1)]^{\frac{n}{n+1}}} = \lim_{n \rightarrow \infty} \frac{(n!)^{1-\frac{n}{n+1}}}{(n+1)^{\frac{n}{n+1}}} = \\
 &= \lim_{n \rightarrow \infty} \frac{(n!)^{\frac{1}{n}} \cdot n}{n(n+1)^{\frac{n}{n+1}}} = e \rightarrow \lim_{n \rightarrow \infty} b_n = e \log e = e.
 \end{aligned}$$

**Now, we find the sum:**

$$\omega_n = \sum_{k=0}^n \frac{(-1)^k \binom{n}{k}}{2k+1}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 \int_0^1 (1-x^2)^n dx &= \int_0^1 \left( \binom{n}{0} - x^2 \binom{n}{1} + x^4 \binom{n}{2} - \dots + (-1)^n x^{2n} \binom{n}{n} \right) dx = \\
 &= x \binom{n}{0} \Big|_0^1 - \frac{x^3}{3} \binom{n}{1} \Big|_0^1 + \frac{x^5}{5} \binom{n}{2} \Big|_0^1 + \dots + \frac{(-1)^n x^{2n+1}}{2n+1} \binom{n}{n} \Big|_0^1 = \sum_{k=0}^n \frac{(-1)^k}{2k+1} \binom{n}{k} \\
 I_{2n+1} &= \int_0^1 (1-x^2)^n dx \stackrel{x=\sin t}{=} \int_0^{\frac{\pi}{2}} \cos^{2n} t \cdot \cos t dt = \\
 &= [\sin t \cdot \cos^{2n} t]_0^{\frac{\pi}{2}} + 2n \int_0^{\frac{\pi}{2}} \cos^{2n-1} t \cdot \sin^2 t dt = \\
 &= 2n \int_0^{\frac{\pi}{2}} \cos^{2n-1} t (1 - \cos^2 t) dt = 2n \int_0^{\frac{\pi}{2}} \cos^{2n-1} t dt - 2n \int_0^{\frac{\pi}{2}} \cos^{2n+1} t dt \rightarrow \\
 I_{2n+1} &= 2nI_{2n-1} - 2nI_{2n+1} \rightarrow (2n+1)I_{2n+1} = 2nI_{2n-1} \\
 \frac{I_{2n+1}}{I_{2n-1}} &= \frac{2n}{2n+1},
 \end{aligned}$$

*Putting  $n = 1, 2, 3, \dots, n$ , and multiplying these relations, we get:*

$$\begin{aligned}
 \frac{I_{2n+1}}{I_1} &= \frac{(2n)!!}{(2n+1)!!}, I_1 = \int_0^{\frac{\pi}{2}} \cos t dt = \sin t \Big|_0^{\frac{\pi}{2}} = 1 \\
 \int_0^1 (1-x^2)^n dx &= \frac{(2n)!!}{(2n+1)!!} \rightarrow \sum_{k=0}^n \frac{(-1)^k}{2k+1} \binom{n}{k} = \frac{(2n)!!}{(2n+1)!!} = \frac{4^n \cdot (n!)^2}{(2n+1)!}
 \end{aligned}$$

*Now, we find the proposed limit*

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sqrt[n]{\frac{4^n \cdot (n!)^2}{(2n+1)!}} &= 4 \cdot \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(n!)^2}{(2n+1)!}} \stackrel{C-D}{=} 4 \cdot \lim_{n \rightarrow \infty} \frac{((n+1)!)^2}{(2n+3)!} \cdot \frac{(2n+1)!}{(n!)^2} = \\
 &= 4 \cdot \lim_{n \rightarrow \infty} \frac{(n!)^2 (n+1)^2 (2n+1)!}{(n!)^2 (2n+1)^2 (2n+2)(2n+3)} = 4 \cdot \frac{1}{4} = 1
 \end{aligned}$$

$$\Omega = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n!} \cdot \sqrt[n]{\frac{4^n \cdot (n!)^2}{(2n+1)!}} \right)^{\frac{n!}{e^n}} = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n!} \right)^{\frac{n!}{e^n}} = \lim_{n \rightarrow \infty} \left[ \left( 1 + \frac{1}{n!} \right)^{\frac{n!}{e^n}} \right]^{\frac{1}{e^n}} = e^0 = 1$$

**Solution 6 by Heimn Hsain-Iran**

$$b_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$n! \approx \left(\frac{n}{e}\right)^n, (2n+1)! \approx \left(\frac{n+1}{e}\right)^{n+1}$$

$$b_n = \frac{e(n+1)^2}{n+1} - \frac{en^2}{n} = e((n+1) - n) = e$$

$$\omega_n = \sum_{k=0}^n \frac{(-1)^k}{2k+1} \binom{n}{k}$$

$$\omega_n = \int_0^1 \omega_n(x) dx = \sum_{k=0}^n \binom{n}{k} (-1)^k x^{2k} = \int_0^1 (1-x^2)^n dx = B\left(\frac{1}{2}, n+1\right)$$

$$\omega_n = \frac{\sqrt{\pi} \cdot n!}{2 \left(n + \frac{1}{2}\right)} \rightarrow 1, \text{when } n \rightarrow \infty$$

$$\Omega = \lim_{n \rightarrow \infty} \left( 1 + \frac{\sqrt[n]{\omega_n}}{n!} \right)^{\frac{n!}{b_n^n}} = e^{\lim_{n \rightarrow \infty} \frac{n!}{b_n^n} \left( 1 + \frac{\sqrt[n]{\omega_n}}{n!} - 1 \right)} = e^{\lim_{n \rightarrow \infty} \frac{\sqrt[n]{\omega_n}}{e^n}} = e^{\lim_{n \rightarrow \infty} \frac{1}{e^n}} = e^0 = 1$$

**1443. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{n-1} \frac{n^7 \cdot k^3 - k^7 \cdot n^3}{n^{11}(\log n - \log k)} \right)$$

*Proposed by Asmat Qatea-Afghanistan*

**Solution 1 by Hussain Reza Zadah-Afghanistan**

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{n-1} \frac{n^7 \cdot k^3 - k^7 \cdot n^3}{n^{11}(\log n - \log k)} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{k=1}^{n-1} \frac{\left(\frac{k}{n}\right)^3 - \left(\frac{k}{n}\right)^7}{\log\left(\frac{n}{k}\right)} = \int_0^1 \frac{x^7 - x^3}{\log x} dx = \\ &= \int_0^1 \int_3^7 x^n dx dn = \int_3^7 \frac{1}{n+1} dn = [\log(n+1)]_3^7 = \log 8 - \log 4 = \log 2 \end{aligned}$$

**Solution 2 by Kaushik Mahanta-India**

$$\Omega = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{n-1} \frac{n^7 \cdot k^3 - k^7 \cdot n^3}{n^{11}(\log n - \log k)} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{k=1}^{n-1} \frac{\frac{n^7 k^3}{n^{10}} - \frac{k^7 n^3}{n^{10}}}{\left(-\log\left(\frac{k}{n}\right)\right)} =$$



## ROMANIAN MATHEMATICAL MAGAZINE

[www.ssmrmh.ro](http://www.ssmrmh.ro)

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{k=1}^{n-1} \frac{\left(\frac{k}{n}\right)^7 - \left(\frac{k}{n}\right)^3}{\log\left(\frac{k}{n}\right)} = \int_0^1 \frac{x^7 - x^3}{\log x} dx = (*)$$

$$\text{Recall } \int_0^1 \frac{x^a - 1}{\log x} dx = \log(a+1)$$

$$(*) = \int_0^1 \frac{x^7 - 1}{\log x} dx - \int_0^1 \frac{x^3 - 1}{\log x} dx = \log(7+1) - \log(3+1) = \log 2$$

**Solution 3 by Adrian Popa-Romania**

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{n-1} \frac{n^7 \cdot k^3 - k^7 \cdot n^3}{n^{11}(\log n - \log k)} \right) = \lim_{n \rightarrow \infty} \left( \frac{1}{n} \cdot \sum_{k=1}^{n-1} \frac{\left(\frac{k}{n}\right)^7 - \left(\frac{k}{n}\right)^3}{\log\left(\frac{k}{n}\right)} \right) = \\ &= \int_0^1 \frac{x^7}{\log x} dx - \int_0^1 \frac{x^3}{\log x} dx = \int_0^1 \frac{x^7 - x^3}{\log x} dx = \int_0^1 \int_3^7 x^a da dx = (*) \\ &\quad \because \frac{x^7 - x^3}{\log x} = \frac{x^a}{\log x} \Big|_{a=3}^{a=7} = \int_3^7 x^a da \\ (*) &= \int_3^7 \int_0^1 x^a dx da = \int_3^7 \frac{1}{a+1} da = \log(a+1) \Big|_3^7 = \log 2 \end{aligned}$$

**Solution 4 by Samar Das-India**

$$\begin{aligned} \Omega &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{n-1} \frac{n^7 \cdot k^3 - k^7 \cdot n^3}{n^{11}(\log n - \log k)} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{k=1}^{n-1} \frac{\frac{n^7 k^3}{n^{10}} - \frac{k^7 n^3}{n^{10}}}{\left(-\log\left(\frac{k}{n}\right)\right)} = \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{k=1}^{n-1} \frac{\left(\frac{k}{n}\right)^7 - \left(\frac{k}{n}\right)^3}{\log\left(\frac{k}{n}\right)} = \int_0^1 \frac{x^7 - x^3}{\log x} dx = \int_0^1 \int_3^7 x^a da dx = \\ &= \int_3^7 \frac{1}{a+1} da = \log(a+1) \Big|_3^7 = \log 2 \end{aligned}$$

**Solution 5 by Syed Shahabudeen-India**

$$\Omega = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{n-1} \frac{n^7 \cdot k^3 - k^7 \cdot n^3}{n^{11}(\log n - \log k)} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{k=1}^{n-1} \frac{\frac{n^7 k^3}{n^{10}} - \frac{k^7 n^3}{n^{10}}}{\left(-\log\left(\frac{k}{n}\right)\right)} =$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{k=1}^{n-1} \frac{\left(\frac{k}{n}\right)^7 - \left(\frac{k}{n}\right)^3}{\log\left(\frac{k}{n}\right)} = \int_0^1 \frac{x^7 - x^3}{\log x} dx = \left(\frac{\partial}{\partial a}\right)_{a=3} \int_0^1 (x^{a+4} - x^a) dx = \\
 &= \left(\frac{\partial}{\partial a}\right)_{a=3} \left( \frac{1}{a+5} - \frac{1}{a+1} \right) = \left[ \log \frac{a+5}{a+1} \right]_{a=3} = \log 2
 \end{aligned}$$

**Solution 6 by Remus Florin Stanca-Romania**

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{n-1} \frac{n^7 \cdot k^3 - k^7 \cdot n^3}{n^{11}(\log n - \log k)} \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \frac{n^7 k^3 \left( 1 - \left(\frac{k}{n}\right)^7 \left(\frac{n}{k}\right)^3 \right)}{n^{11} \log\left(\frac{n}{k}\right)} = \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \frac{1}{n} \left(\frac{k}{n}\right)^3 \cdot \frac{1 - \left(\frac{k}{n}\right)^4}{\log\left(\frac{n}{k}\right)} = \int_0^1 \frac{x^3 - x^7}{-\log x} dx = \int_0^1 \frac{x^7 - x^3}{\log x} dx \stackrel{\log x = t}{=} \\
 &= \int_{-\infty}^0 \frac{e^{7t} - e^{3t}}{t} e^t dt = \int_{-\infty}^0 \frac{e^{8t} - e^{4t}}{t} dt = \int_0^\infty \frac{e^{-4t} - e^{-8t}}{t} dt = \int_0^\infty \mathcal{L}(e^{-4t} - e^{-8t}) dt
 \end{aligned}$$

Where  $\mathcal{L}(f(x))$  – is the Laplace transform of  $f$ .

$$I = \int_0^\infty \left( \frac{1}{p+4} - \frac{1}{p+8} \right) dp = \log 2$$

**Solution 7 by Arslan Ahmed-Yemen**

$$\begin{aligned}
 \Omega &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^{n-1} \frac{n^7 \cdot k^3 - k^7 \cdot n^3}{n^{11}(\log n - \log k)} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{k=1}^{n-1} \frac{\frac{n^7 k^3}{n^{10}} - \frac{k^7 n^3}{n^{10}}}{\left(-\log\left(\frac{k}{n}\right)\right)} = \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{k=1}^{n-1} \frac{\left(\frac{k}{n}\right)^7 - \left(\frac{k}{n}\right)^3}{\log\left(\frac{k}{n}\right)} = \int_0^1 \frac{x^7 - x^3}{\log x} dx = (*)
 \end{aligned}$$

$$\int_0^1 \frac{x^a - x^b}{\log x} dx = \log \frac{a+1}{b+1}, (*) = \int_0^1 \frac{x^7 - x^3}{\log x} dx = \log \frac{7+1}{3+1} = \log 2$$

**1444. Find:**

$$\Omega = \frac{1}{\Gamma(n)[\psi_0^3(n) + 3\psi_0(n)\psi_1(n) + \psi_2(n)]} \lim_{k \rightarrow \infty} \sum_{m=0}^k \int_m^\infty (x-m)^{n-1} \log^3(x-m) e^{-x} dx$$

*Proposed by Ankush Kumar Parcha-India*



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

**Solution by Mohammad Rostami-Kabul-Afghanistan**

$$\begin{aligned}
 I &= \int_m^\infty (x-m)^{n-1} \log^3(x-m) e^{-x} dx \stackrel{x-m=u}{\cong} \int_0^\infty u^{n-1} \log^3 u e^{-u-m} du = \\
 &= \int_0^\infty u^{n-1} \cdot \frac{\partial^3}{\partial a^3} \Big|_{a=0} u^a \cdot e^{-u-m} du = e^{-m} \frac{\partial^3}{\partial a^3} \Big|_{a=0} \int_0^\infty u^{(n+a)-1} e^{-u} du = \\
 &= e^{-m} \frac{\partial^3}{\partial a^3} \Big|_{a=0} \Gamma(n+a) = e^{-m} \{[\Gamma'(n+a)]'\}'_{a=0} = \\
 &= e^{-m} \{[\psi_0(n+a)\Gamma(n+a)]'\}'_{a=0} = \\
 &= e^{-m} \{\psi_1(n+a)\Gamma(n+a) + \Gamma'(n+a)\psi_0(n+a)\}'_{a=0} = \\
 &= e^{-m} \{\Gamma(n+a)[\psi_1(n+a) + \psi_0^2(n+a)]\}'_{a=0} = \\
 &= e^{-m} \{\Gamma(n+a)\psi_0(n+a)[\psi_1(n+a) + \psi_0^2(n+a)] \\
 &\quad + \Gamma(n+a)[\psi_2(n+a) + 2\psi_1(n+a)\psi_0(n+a)]\}'_{a=0} = \\
 &= \Gamma(n)[\psi_0^3(n) + 3\psi_0(n)\psi_1(n) + \psi_2(n)]
 \end{aligned}$$

**Therefore,**

$$\begin{aligned}
 \Omega &= \frac{1}{\Gamma(n)[\psi_0^3(n) + 3\psi_0(n)\psi_1(n) + \psi_2(n)]} \lim_{k \rightarrow \infty} \sum_{m=0}^k \int_m^\infty (x-m)^{n-1} \log^3(x-m) e^{-x} dx \\
 &= \sum_{m=0}^\infty \left(\frac{1}{e}\right)^m = \frac{1}{1 - \frac{1}{e}} = \frac{e}{e-1}
 \end{aligned}$$

**1445. Find:**

$$\Omega = \lim_{n \rightarrow \infty} \int_{e^{H_n}}^{e^{H_{n+1}}} \frac{n^7}{(x+n)^7 - x^7 - n^7} dx$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Mikael Bernardo-Mozambique**

$$\Omega = \lim_{n \rightarrow \infty} \int_{e^{H_n}}^{e^{H_{n+1}}} \frac{n^7}{(x+n)^7 - x^7 - n^7} dx = \lim_{n \rightarrow \infty} \int_{e^{H_n}}^{e^{H_{n+1}}} \frac{dx}{\left(1 + \frac{x}{n}\right)^7 - \left(\frac{x}{n}\right)^7 - 1}$$



## ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

$$\text{Let: } f(x) = \frac{1}{\left(1 + \frac{x}{n}\right)^7 - \left(\frac{x}{n}\right)^7 - 1}; x \neq 0$$

$\exists c_n \in [e^{H_n}, e^{H_{n+1}}]$ , such that:

$$\begin{aligned} \int_{e^{H_n}}^{e^{H_{n+1}}} \frac{dx}{\left(1 + \frac{x}{n}\right)^7 - \left(\frac{x}{n}\right)^7 - 1} &= (e^{H_{n+1}} - e^{H_n}) \cdot \frac{1}{\left(1 + \frac{c_n}{n}\right)^7 - \left(\frac{c_n}{n}\right)^7 - 1} \\ \Omega &= \lim_{n \rightarrow \infty} \frac{e^{H_{n+1}} - e^{H_n}}{\left(1 + \frac{c_n}{n}\right)^7 - \left(\frac{c_n}{n}\right)^7 - 1} = \lim_{n \rightarrow \infty} \frac{e^{H_{n+1}-H_n} - 1}{H_{n+1} - H_n} \cdot \frac{e^{H_n}(H_{n+1} - H_n)}{\left(1 + \frac{c_n}{n}\right)^7 - \left(\frac{c_n}{n}\right)^7 - 1} \\ &\quad \because H_{n+1} - H_n = \frac{1}{n+1} \\ \Omega &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1} - 1}{\frac{1}{n+1}} \cdot \frac{e^{H_n}}{(n+1)\left(\left(1 + \frac{c_n}{n}\right)^7 - \left(\frac{c_n}{n}\right)^7 - 1\right)} \\ H_n &\cong \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} \left(1 + o\left(\frac{1}{n^2}\right)\right) \\ e^{H_n} &\cong n \cdot e^{\gamma + \frac{1}{2n} - \frac{1}{12n^2} \left(1 + o\left(\frac{1}{n^2}\right)\right)} \\ \Omega &= \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{e^{\gamma + \frac{1}{2n} - \frac{1}{12n^2} \left(1 + o\left(\frac{1}{n^2}\right)\right)}}{\left(1 + \frac{c_n}{n}\right)^7 - \left(\frac{c_n}{n}\right)^7 - 1} = \frac{e^\gamma}{(1 + e^\gamma)^7 - e^{7\gamma} - 1} \end{aligned}$$

$\gamma$  – Euler Mascheroni Constant.

**1446.**

$$\Omega(n, k) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2) \cdot \dots \cdot (n+k)}$$

If  $a, b, c > 0, abc = 1$  then:

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} > \frac{3}{2\Omega(n, k)}$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Asmat Qatea-Afghanistan**