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Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n!}\int_0^\infty \frac{x^n \sin\left(x+\frac{\pi}{4}\right)}{e^x}dx$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohammad Rostami-Afghanistan, Solution 2 by Syed Shahabudeen-India, Solution 3 by Muhammad Afzal-Pakistan, Solution 4 by Ajenikoko Gbolahan-Nigeria

Solution by Mohammad Rostami-Afghanistan

$$\Omega = \lim_{n\to\infty}\frac{1}{n!}\int_0^\infty \frac{x^n \sin\left(x+\frac{\pi}{4}\right)}{e^x}dx = \lim_{n\to\infty}\frac{1}{n!}\int_0^\infty \frac{x^n}{e^x}\cdot\frac{e^{ix+\frac{\pi}{4}i}-e^{-ix-\frac{\pi}{4}i}}{2i}dx =$$

$$= \lim_{n\to\infty}\frac{1}{n!}\left(\frac{e^{\frac{\pi}{4}i}}{2i}\int_0^\infty x^n e^{-(1-i)x}\,dx - \frac{e^{-\frac{pi}{4}i}}{2i}\int_0^\infty x^n e^{-(1+i)x}\,dx\right)\overset{(1\pm x)i=u}{=}$$

$$= \lim_{n\to\infty}\frac{1}{n!}\left(\frac{e^{\frac{\pi}{4}i}}{2i}\int_0^\infty \frac{u^{(n+1)-1}e^{-u}}{(1-i)^{n+1}}du - \frac{e^{-\frac{\pi}{4}i}}{2i}\int_0^\infty \frac{u^{(n+1)-1}e^{-u}}{(1+i)^{n+1}}du\right) =$$

$$= \lim_{n\to\infty}\frac{1}{\Gamma(n+1)}\cdot\Gamma(n+1)\left[\frac{e^{\frac{\pi}{4}i}}{2i(1-i)^{n+1}} - \frac{e^{-\frac{\pi}{4}i}}{2i(i+1)^{n+1}}\right] =$$

$$=\lim_{n\to\infty}\left[\frac{e^{\frac{\pi}{4}i}}{2i\left(\sqrt{2}e^{-\frac{\pi}{4}i}\right)^{n+1}}-\frac{e^{-\frac{\pi}{4}i}}{2i\left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^{n+1}}\right]=$$

$$=\lim_{n\to\infty}\frac{1}{\left(\sqrt{2}\right)^{n+1}}\left(\frac{e^{\frac{\pi}{2}i+\frac{\pi}{4}ni}}{2i}-\frac{e^{-\frac{\pi}{2}i-\frac{\pi}{4}ni}}{2i}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{\left(\sqrt{2}\right)^{n+1}}\left[\frac{\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)e^{\left(\frac{\pi}{4}n\right)i}}{2i}-\frac{\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)e^{\left(-\frac{\pi}{4}n\right)i}}{2i}\right]=$$

$$=\lim_{n\to\infty}\frac{1}{\left(\sqrt{2}\right)^{n+1}}\left(\frac{e^{\left(\frac{\pi}{4}n\right)i}+e^{\left(-\frac{\pi}{4}n\right)i}}{2}\right)=\lim_{n\to\infty}\frac{1}{\left(\sqrt{2}\right)^{n+1}}\cos\left(\frac{\pi}{4}n\right)=0$$

Solution 2 by Syed Shahabudeen-India

$$\Omega=\lim_{n\to\infty}\frac{1}{n!}\int_0^\infty\frac{x^n\sin\left(x+\frac{\pi}{4}\right)}{e^x}dx=\frac{1}{\sqrt{2}}\lim_{n\to\infty}\frac{1}{n!}\int_0^\infty\frac{x^n(\sin x+\cos x)}{e^x}dx=$$

$$=\frac{1}{\sqrt{2}}\lim_{n\to\infty}\frac{1}{n!}\left(Im\int_0^\infty x^ne^{-x(1-i)}\,dx+Re\int_0^\infty x^ne^{-x(1-i)}\,dx\right)=$$

$$=\frac{1}{\sqrt{2}}\lim_{n\to\infty}\frac{1}{n!}\left(ImM\left(e^{-x(1-i)}\right)+ReM\left(e^{-x(1-i)}\right)\right)\ ;(apply\ Mellin\ Transform)$$

$$=\frac{1}{\sqrt{2}}\lim_{n\to\infty}\left(Im(1-i)^{-(n+1)}+Re(1-i)^{-(n+1)}\right)=$$

$$=\lim_{n\to\infty}\frac{1}{\left(\sqrt{2}\right)^{n+2}}\left(Im\left(e^{i\frac{\pi(n+1)}{4}}\right)+Re\left(e^{i\frac{\pi(n+1)}{4}}\right)\right)=$$

$$=\lim_{n\to\infty}\frac{1}{\left(\sqrt{2}\right)^{n+2}}\left(\sin\left(\frac{\pi(n+1)}{4}\right)+\cos\left(\frac{\pi(n+1)}{4}\right)\right)=\lim_{n\to\infty}\frac{\cos\left(\frac{\pi n}{4}\right)}{\left(\sqrt{2}\right)^{n+1}}=0,\text{ because}$$

$$0\leftarrow-\frac{1}{\left(\sqrt{2}\right)^{n+1}}\le\frac{\cos\left(\frac{\pi n}{4}\right)}{\left(\sqrt{2}\right)^{n+1}}\le\frac{1}{\left(\sqrt{2}\right)^{n+1}}\to 0$$

Solution 3 by Muhammad Afzal-Pakistan

$$\Omega=\lim_{n\to\infty}\frac{1}{n!}\int_0^\infty\frac{x^n\sin\left(x+\frac{\pi}{4}\right)}{e^x}dx\,;(1)$$

$$\omega = \int_0^\infty \frac{x^n \sin\left(x + \frac{\pi}{4}\right)}{e^x} dx = Im\left\{\int_0^\infty x^n e^{x(i-1)+\frac{i\pi}{4}} dx\right\} =$$

$$= Im\left\{e^{i\frac{\pi}{4}} \int_0^\infty x^n e^{x(i-1)} dx\right\} \overset{u=-x(i-1)}{=} Im\left\{\frac{e^{i\frac{\pi}{4}}}{(1-i)^{n-1}} \int_0^\infty u^n e^{-u} du\right\}$$

$$= \Gamma(n+1) Im\left\{e^{i\frac{\pi}{4}} \frac{1}{(1-i)^{n-1}}\right\} = n!\, Im\left\{e^{i\frac{\pi}{4}} \frac{(1+i)^{n-1}}{2^{n-1}}\right\} =$$

$$= \frac{n!}{2^{n-1}} Im\left\{e^{i\frac{\pi}{4}} 2^{\frac{n-1}{2}} e^{i\frac{\pi}{4}(n-1)}\right\} = \frac{n!}{2^{\frac{n-1}{2}}} Im\left(e^{in\frac{\pi}{4}}\right)$$

$$\Omega = \frac{n!}{2^{\frac{n-1}{2}}} \sin\left(n\frac{\pi}{4}\right) \overset{(1)}{\Rightarrow}$$

$$\Omega = \lim_{n\to\infty} \frac{1}{n!} \cdot \frac{n!}{2^{\frac{n-1}{2}}} \sin\left(\frac{n\pi}{4}\right) = \sqrt{2} \lim_{n\to\infty} \frac{\sin\left(\frac{n\pi}{4}\right)}{\sqrt{2^n}} = \sqrt{2} \lim_{n\to\infty} \frac{n\pi}{2} \cdot \frac{1}{\sqrt{2^n}} = 0$$

Solution 4 by Ajenikoko Gbolahan-Nigeria

$$\Omega = \lim_{n\to\infty} \frac{1}{n!} \int_0^\infty \frac{x^n \sin\left(x + \frac{\pi}{4}\right)}{e^x} dx = \lim_{n\to\infty} \frac{1}{n!} \cdot \frac{1}{\sqrt{2}} \int_0^\infty \frac{x^n \sin x + x^n \cos x}{e^x} dx =$$

$$= \lim_{n\to\infty} \frac{1}{n!} \cdot \frac{1}{\sqrt{2}} \int_0^\infty [Im(x^n e^{ix-x}) + Re(x^n e^{ix-x})] dx =$$

$$= \lim_{n\to\infty} \frac{1}{n!} \cdot \frac{1}{\sqrt{2}} \int_0^\infty \left[Im\left(x^n e^{-(1-i)x}\right) + Re\left(x^n e^{(1-i)x}\right)\right] dx =$$

$$= \lim_{n\to\infty} \frac{1}{n!} \cdot \frac{1}{\sqrt{2}} [Im\mathcal{L}\{x^n\}_{s=1-i} + Re\mathcal{L}\{x^n\}_{s=1-i}] =$$

$$= \lim_{n\to\infty} \frac{1}{n!} \cdot \frac{1}{\sqrt{2}} \left(Im\left(\frac{n!}{(1-i)^{n+1}}\right) + Re\left(\frac{n!}{(1-i)^{n+1}}\right)\right) =$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{2}} [\left(\frac{1}{\sqrt{2}}\right)^{n+1} \left(-\sin\left(\frac{\pi(-n-1)}{4}\right) + \cos\left(\frac{\pi(-n-1)}{4}\right)\right) =$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{2^{n+2}}} \left(-\sin\left(\frac{\pi(n-1)}{4}\right) + \cos\left(\frac{\pi(-n-1)}{4}\right)\right) = 0$$